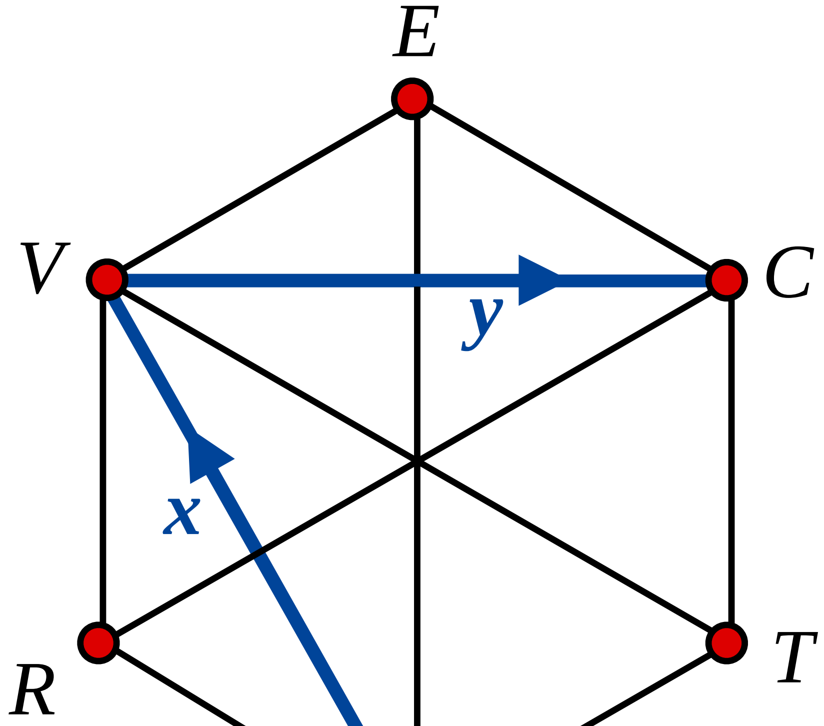
Vectors



Arrows on a mission

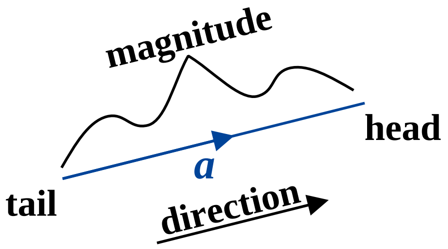
V E C T O R S I

Chapter 1

GCSE and **A-Level Pure Mathematics**
Vectors I

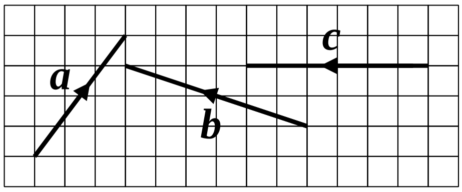
1.1 Arrows on paper

A vector can be thought of as being an arrow on the page.
The arrow has a fixed length, called its magnitude, and acts in a fixed direction.
Its location, however is not fixed. It is free to be **TRANSLATED** about the page.



The direction of a vector is from its tail to its head and its magnitude is the length between the tail and the head.

1.1.1 Example



Vector $\boldsymbol{a}$ is described mathematically by using the fact that to pass from its tail to its head one journeys 3 squares across and 4 squares up.

Thus: $$\boldsymbol{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Alternatively: $$\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$$

In the alternative description of vector $\boldsymbol{a}$ the $\boldsymbol{i}$ and the $\boldsymbol{j}$ are vectors of magnitude 1.
Vectors with a magnitude of 1 are called unit vectors.
$\boldsymbol{i}$ is a unit vector in the positive x-axis direction.
$\boldsymbol{j}$ is a unit vector in the positive y-axis direction.
So even if vector $\boldsymbol{a}$ is written as $\boldsymbol{a} = 4\boldsymbol{j} + 3\boldsymbol{i}$, it would still be “3 across and 4 up”.
When written as a column vector, however, swapping the 3 and the 4 is disastrous as the upper number is always the horizontal x-axis instruction and the lower number is always the vertical y-axis instruction.

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \neq \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

1.1.2 Exercise

Write down two mathematical descriptions for each of the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$.

[4 marks]

1.2 Factorising a vector

It is often helpful to pull out any common integer term from a vector.

1.2.1 Example

To factorise vector $\boldsymbol{g}$ where $\boldsymbol{g} = \begin{pmatrix} 25 \\ 15 \end{pmatrix}$

notice that both 25 and 15 divide by 5. $\therefore\ \boldsymbol{g} = 5\begin{pmatrix} 5 \\ 3 \end{pmatrix}$

1.2.2 Exercise

Factorise out any common term from each of these vectors.

$$\boldsymbol{m} = \begin{pmatrix} 20 \\ 55 \end{pmatrix} \qquad \boldsymbol{n} = 12\,\boldsymbol{i} + 8\,\boldsymbol{j}$$

[2 marks]

1.3 Parallel Vectors

To show that two vectors are parallel, show that they have the same direction.

1.3.1 Example

$$\boldsymbol{s} = \begin{pmatrix} 16 \\ 20 \end{pmatrix} \qquad \boldsymbol{t} = \begin{pmatrix} 28 \\ 35 \end{pmatrix}$$

To show that vectors $\boldsymbol{s}$ and $\boldsymbol{t}$ are parallel, first, factorise out the common factors,

$$\boldsymbol{s} = 4\begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad \boldsymbol{t} = 7\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

It's now obvious that they have the same direction !
One vector has simply got more magnitude (length of the vector) than the other, but in the same direction.

1.3.2 Exercise

Show that the following vectors are parallel,

$$\boldsymbol{a} = \begin{pmatrix} 12 \\ 18 \end{pmatrix} \qquad \boldsymbol{z} = \begin{pmatrix} 16 \\ 24 \end{pmatrix}$$

[3 marks]

1.4 Vector Arithmetic

Vectors can be added to, or subtracted from, each other.
They can also be multiplied by a number.

1.4.1 Example

Vectors $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{e}$ are : $\boldsymbol{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\quad \boldsymbol{c} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ $\quad \boldsymbol{e} = \begin{pmatrix} -9 \\ 3 \end{pmatrix}$

Here is the calculation to work out the vector $2\,\boldsymbol{a} + \boldsymbol{c} - \boldsymbol{e}$;

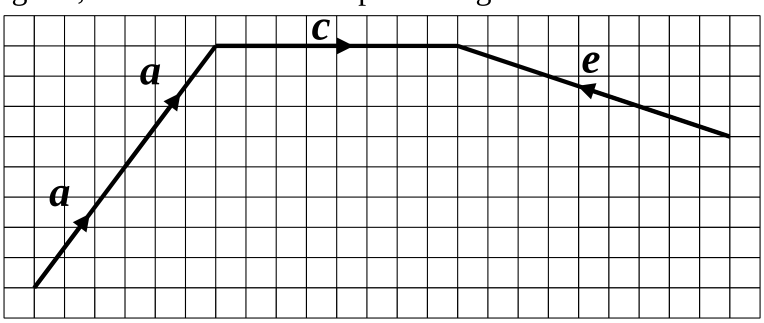
$$2\,\boldsymbol{a} + \boldsymbol{c} - \boldsymbol{e} = 2 \times \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 8 \\ 0 \end{pmatrix} - \begin{pmatrix} -9 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times 3 + 8 - (-9) \\ 2 \times 4 + 0 - 3 \end{pmatrix}$$

$$= \begin{pmatrix} 23 \\ 5 \end{pmatrix}$$

The calculation can be explained using geometry.
Addition corresponds to joining vectors up “head to tail”.
Subtraction corresponds to joining vectors up “head to head”, (or “tail to tail).
So the above calculation corresponds to the following diagram.

On the diagram, draw in the vector representing the answer.



[1 mark]

1.4.2 Exercise

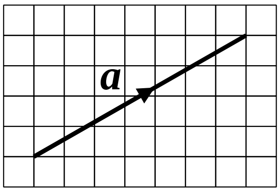
If vectors $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{e}$ are once again,

$$\boldsymbol{a} = 3\,\boldsymbol{i} + 4\,\boldsymbol{j} \qquad \boldsymbol{c} = 8\,\boldsymbol{i} \qquad \boldsymbol{e} = -9\,\boldsymbol{i} + 3\,\boldsymbol{j}$$

Calculate the vector $\boldsymbol{a} + 2\,\boldsymbol{c} - \boldsymbol{e}$

[3 marks]

1.5 The Magnitude of a vector



A vector can be used to represent many aspects of the physical world: a force, a velocity or an acceleration, for example. So it's confusing to talk about a vector's ***length*** because forces, velocities and accelerations are not lengths.
A new word is needed for the length of a vector.
The word is **MAGNITUDE**.

1.5.1 Example

$$\boldsymbol{a} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

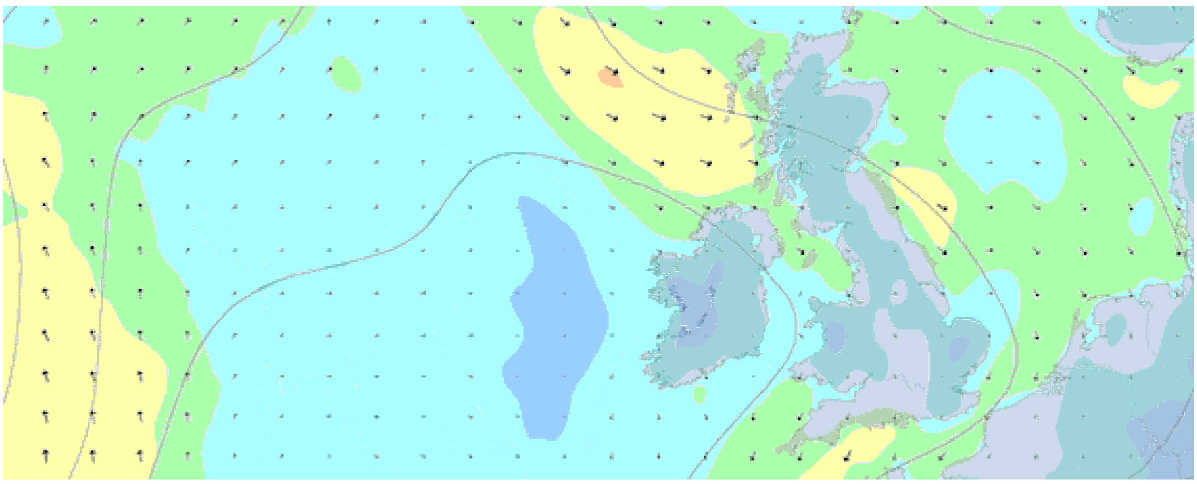
$$\begin{aligned} |\boldsymbol{a}| &= \sqrt{7^2 + 4^2} \\ &= \sqrt{49 + 16} \\ &= \sqrt{65} \end{aligned}$$

$|\boldsymbol{a}|$ is read as “the magnitude of vector ***a***”

1.5.2 Exercise

Find $|\boldsymbol{p}|$ giving the answer is surd form where $\boldsymbol{p} = \begin{pmatrix} 6 \\ 11 \end{pmatrix}$

[2 marks]

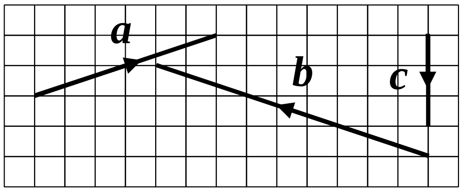


Vectors in use on a weather chart.
Each vector shows the direction of the wind at a point and the magnitude of the vector corresponds to the strength of the wind at that point.
These vectors are tied to a location and are not free to be translated.
Such a collection of fixed vectors is termed a vector field.

1.6 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 50

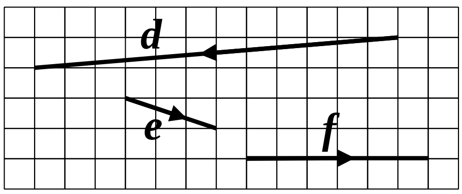
Question 1



Write the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in the form $\begin{pmatrix} p \\ q \end{pmatrix}$ where p and q are integers.

[3 marks]

Question 2



Write the vectors $\boldsymbol{d}$, $\boldsymbol{e}$ and $\boldsymbol{f}$ in the form $p\,\mathbf{i} + q\,\boldsymbol{j}$ where p and q are integers.

[3 marks]

Question 3

On the grid draw the following vectors, labelling each with its letter and an arrow.

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

[2 marks]

Question 4

On the grid draw the following vectors, labelling each with its letter and an arrow.

$$\boldsymbol{c} = 2\boldsymbol{i} - 4\boldsymbol{j} \qquad \boldsymbol{d} = -2\boldsymbol{i} + 5\boldsymbol{j}$$

[2 marks]

Question 5

Factorise completely the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$.

$$\boldsymbol{p} = \begin{pmatrix} 22 \\ 77 \end{pmatrix} \qquad \boldsymbol{q} = \begin{pmatrix} 20 \\ 24 \end{pmatrix}$$

[2 marks]

Question 6

Show that vectors ***y*** and ***z*** are parallel.

$$\boldsymbol{y} = \begin{pmatrix} 15 \\ 12 \end{pmatrix} \qquad \boldsymbol{z} = \begin{pmatrix} 35 \\ 28 \end{pmatrix}$$

[2 marks]

Question 7

Show that the following vectors are parallel,

$$\boldsymbol{c} = 12\,\boldsymbol{i} + 15\,\boldsymbol{j} \qquad \boldsymbol{v} = 28\,\boldsymbol{i} + 35\,j$$

[2 marks]

Question 8

$$\boldsymbol{T} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

Find $|\boldsymbol{T}|$ giving the answer in surd form.

[2 marks]

Question 9

$$\boldsymbol{m} = \begin{pmatrix} -7 \\ 5 \end{pmatrix}$$

Find $|\boldsymbol{m}|$ giving the answer in surd form.

[2 marks]

Question 10

Let vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be :

$$\boldsymbol{A} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \qquad \boldsymbol{C} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Find :

(i) $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}$

[1 mark]

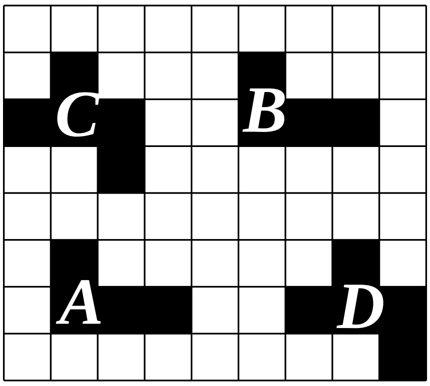
(ii) $2\,\boldsymbol{A} + \boldsymbol{B} + 3\,\boldsymbol{C}$

[2 marks]

(iii) $\boldsymbol{A} + 2\,\boldsymbol{B} + 3\,\boldsymbol{C}$

[2 marks]

Question 11



(i) **(a)** Write down the vector that translates shape *A* onto shape *B*.

[1 mark]

(b) Circle which describes the direction of your part **(i) (a)** answer.

$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

[1 mark]

(c) Circle which describes the direction of the vector that translates shape *B* onto shape *A*.

$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

[1 marks]

(ii) **(a)** Write down the vector that translates shape *C* onto shape *D*.

[1 mark]

(b) Circle which describes the direction of your part **(ii) (a)** answer.

$\begin{pmatrix} -3 \\ -2 \end{pmatrix}$ $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

[1 mark]

(c) Circle which describes the direction of the vector that translates shape *D* onto shape *C*.

$\begin{pmatrix} -3 \\ -2 \end{pmatrix}$ $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

[1 mark]

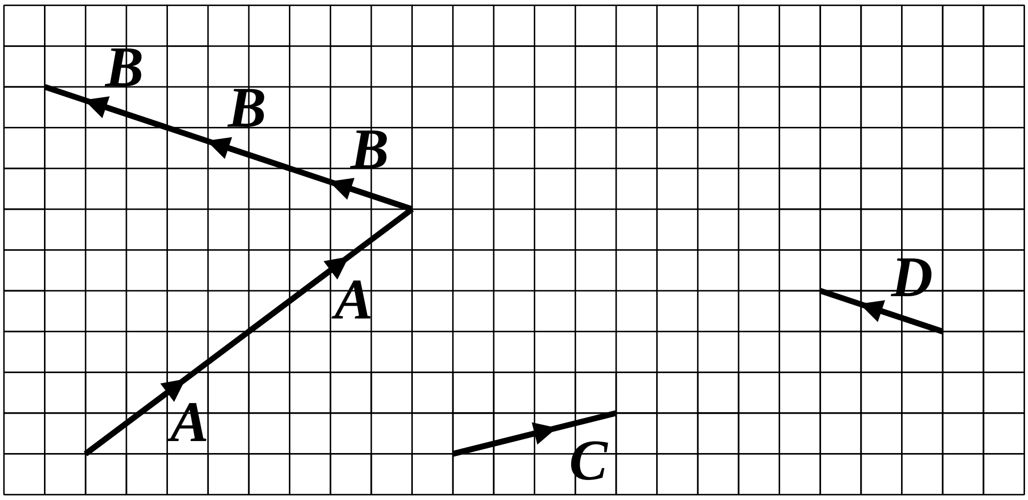
Question 12
Let vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ be :

$$\boldsymbol{A} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

(i) Calculate vector $\boldsymbol{R}$ where $\boldsymbol{R} = 2\,\boldsymbol{A} + 3\,\boldsymbol{B}$

[2 marks]

(ii) On the diagram below, draw vector $\boldsymbol{R}$ in the appropriate place.



[1 mark]

Question 13
Let vectors $\boldsymbol{C}$ and $\boldsymbol{D}$ be :

$$\boldsymbol{C} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \boldsymbol{D} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

(i) Calculate vector $\boldsymbol{S}$ where $\boldsymbol{S} = 3\,\boldsymbol{C} + 2\,\boldsymbol{D}$

[2 marks]

(ii) Complete the diagram above to illustrate the vector sum $\boldsymbol{S} = 3\,\boldsymbol{C} + 2\,\boldsymbol{D}$

[1 mark]

Question 14

Let vectors $\boldsymbol{D}$, $\boldsymbol{E}$ and $\boldsymbol{F}$ be :

$$\boldsymbol{D} = -6\,\boldsymbol{i} - 2\,\boldsymbol{j} \qquad \boldsymbol{E} = 4\,\boldsymbol{i} - 3\,\boldsymbol{j} \qquad \boldsymbol{F} = -5\,\boldsymbol{i} + 3\,\boldsymbol{j}$$

Find :

(i) $2\,\boldsymbol{D} + 2\,\boldsymbol{E} + 3\,\boldsymbol{F}$

[2 marks]

(ii) $\boldsymbol{D} + 5\,\boldsymbol{E} + 2\,\boldsymbol{F}$

[2 marks]

(iii) $3\,(\,\boldsymbol{E} + 2\,\boldsymbol{F}\,) - \boldsymbol{D}$

[2 marks]

Question 15

(a) Two translations are described by the following vectors;

$$F = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \qquad G = \begin{pmatrix} -3 \\ 10 \end{pmatrix}$$

Describe the single transformation which is equivalent to combining the two translations.

[1 mark]

(b) Two translations are described by the following vectors;

$$\boldsymbol{f} = 7\,\boldsymbol{a} - 2\,\boldsymbol{b} \qquad \boldsymbol{g} = -3\,\boldsymbol{a} + 10\,\boldsymbol{b}$$

Describe the single transformation which is equivalent to combining the two translations.
(You will have ***a*** and ***b*** in your answer)

[1 mark]

Question 16

Show that the following vectors are parallel,

$$\boldsymbol{b} = \begin{pmatrix} 45 \\ 15 \end{pmatrix} \qquad \boldsymbol{t} = \begin{pmatrix} 36 \\ 12 \end{pmatrix}$$

[2 marks]

Question 17

Consider the vector,

$$\boldsymbol{R} = \begin{pmatrix} 13 \\ 16 \end{pmatrix}$$

Without using a calculator, show that $|\boldsymbol{R}| = 5\sqrt{17}$.

[1 mark]

Question 18

In general the 3 dimensional vector;

$$\boldsymbol{d} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

has a magnitude given by the formula;

$$|\boldsymbol{d}| = \sqrt{x^2 + y^2 + z^2}$$

Show that the magnitude of the following vector is an integer;

$$\boldsymbol{t} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$$

[1 mark]

Question 19

Try to find some more 3 dimensional vectors of integer components which have magnitudes which are also integers.

[1 marks]

1.7 Answers

1.7.1 Solutions (1.1.2 Exercise)

$\boldsymbol{b} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$ $\qquad$ $\boldsymbol{b} = -6\,\boldsymbol{i} + 2\,\boldsymbol{j}$

$\boldsymbol{c} = \begin{pmatrix} -6 \\ 0 \end{pmatrix}$ $\qquad$ $\boldsymbol{c} = -6\,\boldsymbol{i}$

[4 marks]

1.7.2 Solutions (1.2.2 Exercise)

$\boldsymbol{m} = 5\begin{pmatrix} 4 \\ 11 \end{pmatrix}$ $\qquad$ $\boldsymbol{n} = 4\,(\,3\,\boldsymbol{i} + 2\,\boldsymbol{j}\,)$

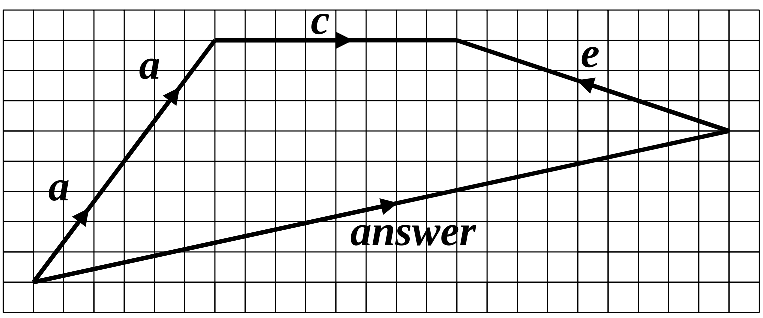
[2 marks]

1.7.3 Solutions (1.3.2 Exercise)

$\boldsymbol{a} = 6\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\qquad$ $\boldsymbol{z} = 8\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\qquad$ Same direction, $\therefore$ parallel ☐

[3 marks]

1.7.4 Solution (1.4.1 Example)



Vectors ***a***, ***a*** and ***c*** are joined "head to tail" then add on ***e*** having reversed it first.
From the start to the finish is now the answer vector.

[1 mark]

1.7.5 Solutions (1.4.2 Exercise)

$$\begin{aligned} \boldsymbol{a} + 2\,\boldsymbol{c} - \boldsymbol{e} &= \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 8 \\ 0 \end{pmatrix} - \begin{pmatrix} -9 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 16 \\ 0 \end{pmatrix} + \begin{pmatrix} 9 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 28 \\ 1 \end{pmatrix} \end{aligned}$$

[3 marks]

1.7.6 Solutions (1.5.2 Exercise)

$$\boldsymbol{p} = \begin{pmatrix} 6 \\ 11 \end{pmatrix} \Rightarrow |\boldsymbol{p}| = \sqrt{6^2 + 11^2} = \sqrt{36 + 121} = \sqrt{157}$$

[2 marks]

1.7.8 Solutions (1.6 Exercise)

Answer 1

$\boldsymbol{a} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} -9 \\ 3 \end{pmatrix} \qquad \boldsymbol{c} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

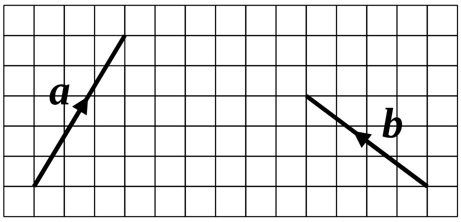
[3 marks]

Answer 2

$\boldsymbol{d} = -12\,\boldsymbol{i} - \boldsymbol{j} \qquad \boldsymbol{e} = 3\,\boldsymbol{i} - \boldsymbol{j} \qquad \boldsymbol{f} = 6\,\boldsymbol{i}$

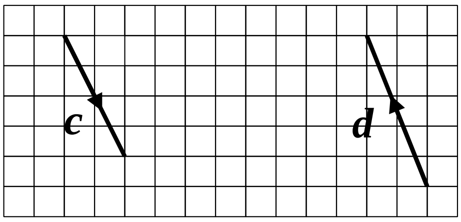
[3 marks]

Answer 3



[2 marks]

Answer 4



[2 marks]

Answer 5

$\boldsymbol{p} = 11 \begin{pmatrix} 2 \\ 7 \end{pmatrix} \qquad \boldsymbol{q} = 4 \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

[2 marks]

Answer 6

$\boldsymbol{y} = 3 \begin{pmatrix} 5 \\ 4 \end{pmatrix} \qquad \boldsymbol{z} = 7 \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ Same direction, $\therefore$ parallel □

[2 marks]

Answer 7

$\boldsymbol{c} = 3\,(4\,\boldsymbol{i} + 5\,\boldsymbol{j}) \qquad \boldsymbol{v} = 7\,(4\,\boldsymbol{i} + 5\,\boldsymbol{j})$ Same direction, $\therefore$ parallel □

[2 marks]

Answer 8

$\sqrt{145}$ (12.04)

Answer 9

$\sqrt{74}$ (8.60)

[2, 2 marks]

Answer 10

(i) $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 18 \\ 24 \end{pmatrix}$ **(iii)** $\begin{pmatrix} 12 \\ 24 \end{pmatrix}$

[1, 2, 2 marks]

Answer 11

(i) **(a)** $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

(ii) **(a)** $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **(c)** $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

[6 marks]

Answer 12

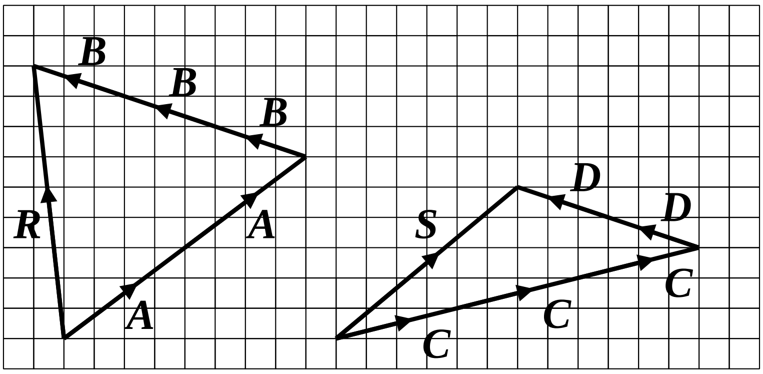
(i) $\begin{pmatrix} -1 \\ 9 \end{pmatrix}$

(ii)

Answer 13

(i) $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$

(ii)



[3, 3 marks]

Answer 14

(i) $\begin{pmatrix} -19 \\ -1 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 4 \\ -11 \end{pmatrix}$ **(iii)** $\begin{pmatrix} -12 \\ 11 \end{pmatrix}$

[2, 2, 2 marks]

Answer 15

(a) $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$ **(b)** $4\boldsymbol{a} + 8\boldsymbol{b}$

[1, 1 mark]

Answer 16

$\boldsymbol{b} = 15\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\boldsymbol{t} = 12\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ Same direction, $\therefore$ parallel ☐

[2 marks]

Answer 17

$$|\boldsymbol{R}| = \sqrt{13^2 + 16^2} = \sqrt{169 + 256} = \sqrt{425} = \sqrt{25 \times 17}$$
$$= \sqrt{25} \times \sqrt{17} = 5\sqrt{17} \quad ☐$$

[1 mark]

Answer 18

6

Answer 19

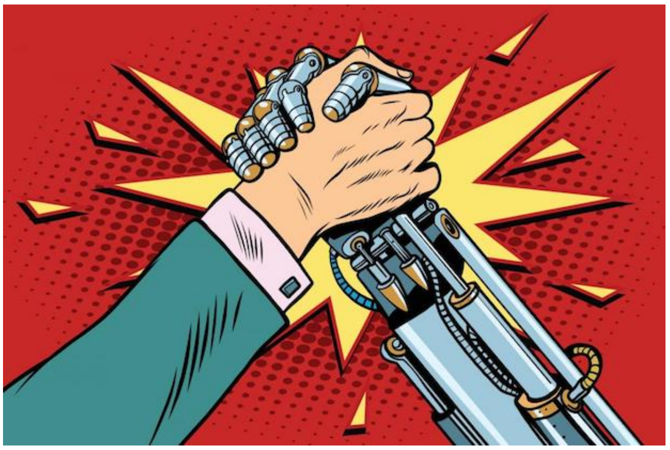
Various !

[1, 1 mark]

Calculator Needed

2.1 Magnitude and Direction

An interesting aspect of today's questions is that humans and computers would answer them using different methods. To a human, the computer method is long and tedious. To a computer, the human method is flawed: it seemingly can't tell the difference between two possible answers, only one of which is correct.



2.2 Example

Find the magnitude and the direction of the vector, ***r***, where

$$\boldsymbol{r} = \begin{pmatrix} 8.4 \\ -3.7 \end{pmatrix}$$

Man v Machine

In answering this question the crucial first step is to draw a diagram. This is because the human solution method uses “short cut” mathematics in which the diagram is used to guide the mathematics.

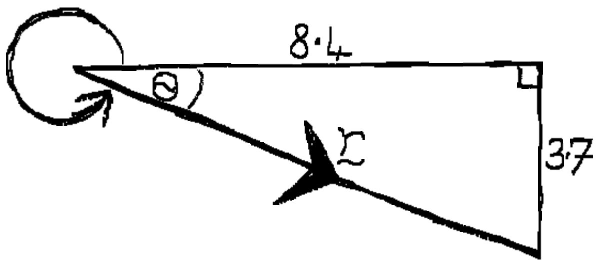
To draw the diagram keep in mind that the horizontal instruction is the upper number, and the vertical number is the lower number.

Sketch the vector ***r*** in the space below;

☞

Now turn the page to see if you got it correct.

Here is my sketch of ***r*.**
How does your sketch compare with mine ?



Finding the magnitude (the length of the vector) is the easy part of the question. It's a straight forward use of the Theorem of Pythagoras,

$$|\boldsymbol{r}| = \sqrt{8.4^2 + (3.7)^2} \qquad \text{Notice I used the } + \text{ 3.7 from the diagram}$$

$$= \sqrt{70.56 + 13.69}$$

$$= \sqrt{84.25}$$

$$= 9.179$$

Now to work out the angle marked θ, again using + 3.7 from the diagram

$$tan\,\theta = \frac{3.7}{8.4}$$

$$\theta = arctan\left(\frac{3.7}{8.4}\right)$$

$$\theta = 23.8°$$

By convention the direction of a vector is given as an angle measured at;

- the tail of the vector,
- from the positive x-axis direction,
- turning anticlockwise.

$\therefore$ In this particular question, $Direction = 360° - 23.8°$

$$= 336.2°$$

This last step is where the diagram totally guided the mathematics.

If one tried to use the – 3.7 with the *tan* function the mathematics cannot tell the difference between

$$tan\left(\frac{-3.7}{8.4}\right) \text{ and } tan\left(\frac{3.7}{-8.4}\right)$$

and yet the vectors $\begin{pmatrix} 8.4 \\ -3.7 \end{pmatrix}$ and $\begin{pmatrix} -8.4 \\ 3.7 \end{pmatrix}$ are in totally different directions !

2.1.2 Example

Let $\boldsymbol{s}$ be the following vector :

$$\boldsymbol{s} = \begin{pmatrix} -2.6 \\ -5.1 \end{pmatrix}$$

Find $|\, \boldsymbol{s} \,|$ and also the direction in which $\boldsymbol{s}$ acts.

Teaching Video : http://www.NumberWonder.co.uk/v9009/2.mp4

After watching the teaching video, write out an answer to the question below.

☞

[4 marks]

2.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable. Make the method used clear.

Marks available : 50

Question 1

Find the magnitude and the direction (with the positive x-axis) of each of the following vectors.

(i) $\boldsymbol{a} = \begin{pmatrix} 11.2 \\ -8.7 \end{pmatrix}$

[4 marks]

(ii) $\boldsymbol{b} = \begin{pmatrix} 7.5 \\ 6 \end{pmatrix}$

[4 marks]

(iii) $\boldsymbol{c} = \begin{pmatrix} -3.5 \\ 8 \end{pmatrix}$

[4 marks]

(iv) $\boldsymbol{d} = \begin{pmatrix} -13 \\ -7.2 \end{pmatrix}$

[4 marks]

(v) $\boldsymbol{e} = \begin{pmatrix} 14.3 \\ -19 \end{pmatrix}$

[4 marks]

Question 2

Find the magnitude and direction (with the positive x-axis) of each of these vectors.

(i) $\boldsymbol{f} = \begin{pmatrix} 7.4 \\ -2.7 \end{pmatrix}$

[4 marks]

(ii) $\boldsymbol{g} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$

[4 marks]

(iii) $\boldsymbol{h} = \begin{pmatrix} -3.5 \\ 3.5 \end{pmatrix}$

[4 marks]

(iv) $\boldsymbol{i} = \begin{pmatrix} -15 \\ -6.2 \end{pmatrix}$

[4 marks]

(v) $\boldsymbol{j} = \begin{pmatrix} 1.3 \\ -16 \end{pmatrix}$

[4 marks]

Question 3

If $\boldsymbol{k} = \boldsymbol{l} + 2\,\boldsymbol{m} + 3\,\boldsymbol{n}$ and

$$\boldsymbol{l} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \qquad \boldsymbol{m} = \begin{pmatrix} 8 \\ 5 \end{pmatrix} \qquad \boldsymbol{n} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

find the magnitude and direction of $\boldsymbol{k}$.

[2 marks]

Question 4

A vector is of magnitude 8.2 cm and acts at an angle of 65°.

(i) Accurately sketch, full size, this vector.

(ii) From the sketch, make measurements, and so write the vector in the form

$$\begin{pmatrix} p \\ q \end{pmatrix}$$

(iii) Use right angled trigonometry (i.e. SOH CAH TOA) to determine the answer to part **(ii)** to an accuracy of four decimal places places.

[4 marks]

Question 5

The vector $\boldsymbol{A}$ is of length 9.3 cm.

$$\boldsymbol{A} = \begin{pmatrix} 6 \\ y \end{pmatrix}$$

Find the two possible values of y and the two possible directions in which this vector is acting.

[2 marks]

Question 6

Write the vector $\boldsymbol{z}$ as a length and a ***bearing.***

$$\boldsymbol{z} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

[2 marks]

2.3 Answers

2.3.1 Solutions (2.1.2 Example)

$\|\boldsymbol{s}\| = 5.72$	$\theta = 63°$	Direction = 243°

[4 marks]

2.3.2 Solutions (2.1.2 Exercise)

Answer 1

$\|\boldsymbol{a}\| = 14.18$	$\theta = 37.8°$	Direction = 322.2°
$\|\boldsymbol{b}\| = 9.60$	$\theta = 38.6°$	Direction = 38.6°
$\|\boldsymbol{c}\| = 8.73$	$\theta = 66.4°$	Direction = 113.6°
$\|\boldsymbol{d}\| = 14.86$	$\theta = 29.0°$	Direction = 209.0°
$\|\boldsymbol{e}\| = 23.78$	$\theta = 53.0°$	Direction = 307.0°

[4, 4, 4, 4, 4 marks]

Answer 2

$\|\boldsymbol{f}\| = 7.88$	$\theta = 20.0°$	Direction = 340.0°
$\|\boldsymbol{g}\| = 13.42$	$\theta = 26.6°$	Direction = 26.6°
$\|\boldsymbol{h}\| = 4.95$	$\theta = 45.0°$	Direction = 135.0°
$\|\boldsymbol{i}\| = 16.23$	$\theta = 22.5°$	Direction = 202.5°
$\|\boldsymbol{e}\| = 16.05$	$\theta = 85.4°$	Direction = 274.6°

[4, 4, 4, 4, 4 marks]

Answer 3

$\|\boldsymbol{k}\| = 29.68$	$\theta = 57.4°$	Direction = 57.4°

[2 marks]

Answer 4 $\begin{pmatrix} 3.4655 \\ 7.4317 \end{pmatrix}$

[4 marks]

Answer 5

$y = \pm 7.1$ 49.8° or 310.2°

[2 marks]

Answer 6

$\|\boldsymbol{z}\| = 6.71$	$\theta = 26.6°$	Bearing = 117°

[2 marks]

Calculator Needed

3.1 Cartesian Form ⇒ Polar Form

Previously, the vector $\boldsymbol{r}$ was studied,

$$\boldsymbol{r} = \begin{pmatrix} 8.4 \\ -3.7 \end{pmatrix}$$

A vector written in this way is in ***Cartesian*** form.

By using

- A quick sketch
- The Theorem of Pythagoras,
- The *arctan* function

vector $\boldsymbol{r}$ rewrote was rewritten as a magnitude and a direction.

It turned out that $\boldsymbol{r}$ has a magnitude of 9.179, at an angle of 336.2°
A vector written in this way is in ***polar*** form.

3.2 Polar Form ⇒ Cartesian Form

Converting from ***polar*** form back into ***Cartesian*** form is the reverse of the problem studied last lesson. A simple, clear diagram avoids "obviously wrong answers".

Example

Convert the vector $\boldsymbol{r}$, of magnitude 9.179 and direction 336.2° into the form

$$\boldsymbol{r} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Teaching Video: http://www.NumberWonder.co.uk/v9009/3.mp4

Watch the video before
writing out your answer.

☞

[4 marks]

3.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 50

Question 1

Write each of the following vectors in the form $\begin{pmatrix} p \\ q \end{pmatrix}$

(i) Vector $\boldsymbol{a}$ of magnitude 32, and direction 160°.

[4 marks]

(ii) Vector $\boldsymbol{b}$ of magnitude 640, and direction $245°$.

[4 marks]

(iii) Vector ***c*** of magnitude 145, and direction 110°.

[4 marks]

(iv) Vector ***d*** of magnitude 2.6, and direction 300°.

[4 marks]

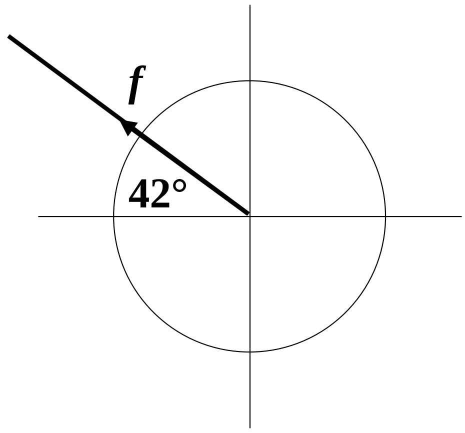
(v) Vector ***e*** of magnitude 13, and direction 72°.

[4 marks]

Question 2

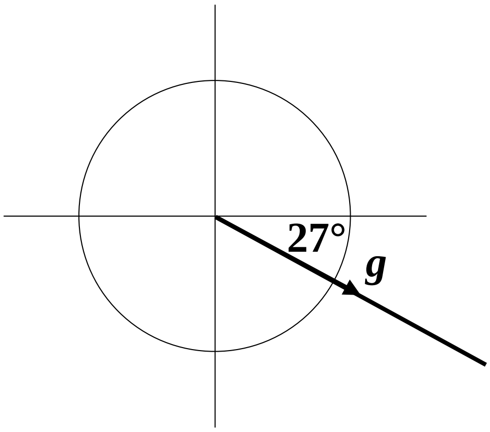
Express each of the vectors shown in the following diagrams in the form $\begin{pmatrix} p \\ q \end{pmatrix}$

(i) $|\, \boldsymbol{f} \,| = 32$



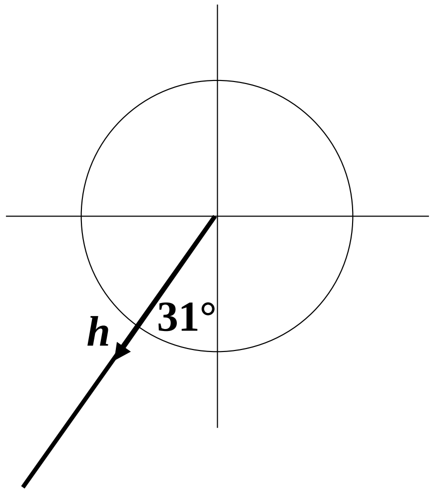
[4 marks]

(ii) $|\, \boldsymbol{g} \,| = 78$



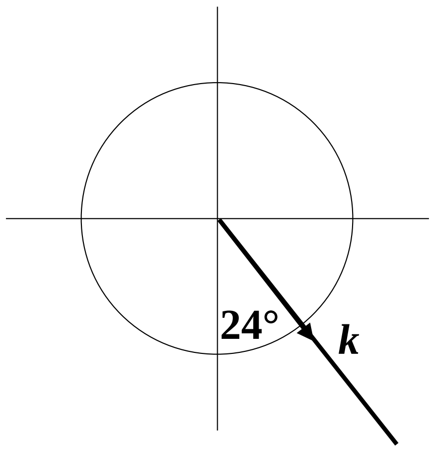
[4 marks]

(iii) | $\boldsymbol{h}$ | = 104



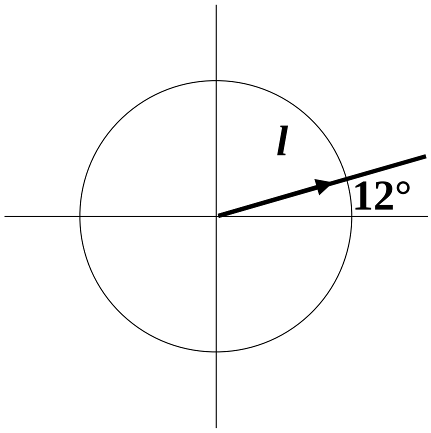
[4 marks]

(iv) | $\boldsymbol{k}$ | = 0.3



[4 marks]

(v) $|\,\boldsymbol{l}\,| = 1700$



[4 marks]

Question 3

Two vectors are $\boldsymbol{a} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

Determine the magnitude and the direction of the vector $\boldsymbol{a} + \boldsymbol{b}$

[4 marks]

Question 4

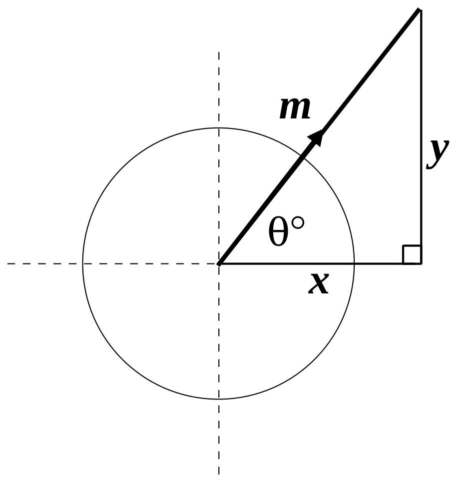
Two vectors are $\boldsymbol{c} = \begin{pmatrix} 17 \\ -3 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} 12 \\ 9 \end{pmatrix}$

Determine the magnitude and the direction of the vector $\boldsymbol{c} - \boldsymbol{d}$

[4 marks]

Question 5

In the right angled triangle, calculate, in terms of $|\boldsymbol{m}|$ and θ, the length of the sides marked x and y.



[2 marks]

3.4 Answers

3.4.1 Solution to Teaching Video Example (3.2 Polar Form ⇒ Cartesian Form)

$$\boldsymbol{r} = \begin{pmatrix} 9.179 \, cos \, 23.8 \\ -9.179 \, sin \, 23.8 \end{pmatrix} = \begin{pmatrix} 8.4 \\ -3.7 \end{pmatrix}$$

[4 marks]

3.4.2 Solutions (3.2 Exercise)

Answer 1

$$\boldsymbol{a} = \begin{pmatrix} -30.1 \\ 10.9 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} -270 \\ -580 \end{pmatrix} \quad \boldsymbol{c} = \begin{pmatrix} -49.6 \\ 136.26 \end{pmatrix} \quad \boldsymbol{d} = \begin{pmatrix} 1.3 \\ -2.3 \end{pmatrix} \quad \boldsymbol{e} = \begin{pmatrix} 4.02 \\ 12.3 \end{pmatrix}$$

[4, 4, 4, 4, 4 marks]

Answer 2

$$\boldsymbol{f} = \begin{pmatrix} -23.8 \\ 21.4 \end{pmatrix} \quad \boldsymbol{g} = \begin{pmatrix} 69.5 \\ -35.4 \end{pmatrix} \quad \boldsymbol{h} = \begin{pmatrix} -53.6 \\ -89.1 \end{pmatrix} \quad \boldsymbol{k} = \begin{pmatrix} 0.12 \\ -0.27 \end{pmatrix} \quad \boldsymbol{l} = \begin{pmatrix} 1662.9 \\ 353.4 \end{pmatrix}$$

[4, 4, 4, 4, 4 marks]

Answer 3

$$\boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} -7 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

This has magnitude 5 and direction 126.9°

[4 marks]

Answer 4

$$\boldsymbol{c} - \boldsymbol{d} = \begin{pmatrix} 17 \\ -3 \end{pmatrix} - \begin{pmatrix} 12 \\ 9 \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$$

This has magnitude 13 and direction 292.6°

[4 marks]

Answer 5

A vector, $\boldsymbol{m}$, of magnitude $|\boldsymbol{m}|$ and direction θ can be written as :

$$\boldsymbol{m} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} |\boldsymbol{m}| \; cos \, \theta \\ |\boldsymbol{m}| \, sin \, \theta \end{pmatrix}$$

[2 marks]

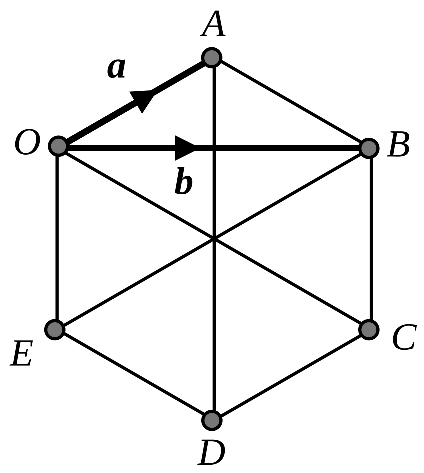
4.1 Vector Geometry

Many GCSE vector questions specify a couple of vectors and ask that they be used to move between points on a simple geometric shape such as a triangle, rectangle, hexagon, trapezium or parallelogram.

4.2 Example

The diagram, which is not drawn to scale, shows a regular hexagon *OABCDE*

$$\overrightarrow{OA} = \boldsymbol{a} \qquad \overrightarrow{OB} = \boldsymbol{b}$$



Teaching Video : http://www.NumberWonder.co.uk/v9009/4a.mp4
http://www.NumberWonder.co.uk/v9009/4b.mp4

<= Part 1 Part 2 =>

Watch the teaching video, then express the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$;

(i) $\overrightarrow{EB} =$

(ii) $\overrightarrow{CD} =$

(iii) $\overrightarrow{AB} =$

(iv) $\overrightarrow{BC} =$

(v) $\overrightarrow{AD} =$

(vi) $\overrightarrow{BD} =$

[6 marks]

4.3 Exercise

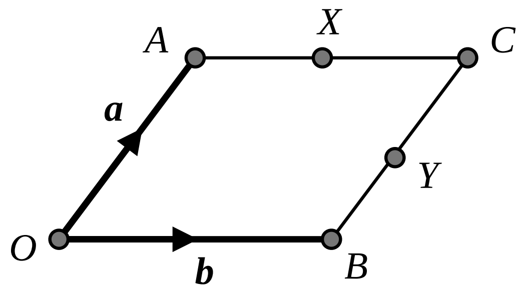
Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 40

Question 1

The diagram, which is not drawn to scale, shows a parallelogram *OACB* with

$$\overrightarrow{OA} = \boldsymbol{a} \qquad \overrightarrow{OB} = \boldsymbol{b}$$

The point *X* is the mid-point of *AC* and the point *Y* is the mid-point of *BC*.



(a) Express the following vectors in terms of ***a*** and ***b***;

(i) $\overrightarrow{BC}$ =   **(ii)** $\overrightarrow{AC}$ =

(iii) $\overrightarrow{BO}$ =   **(iv)** $\overrightarrow{CY}$ =

(v) $\overrightarrow{AB}$ =   **(vi)** $\overrightarrow{OX}$ =

(vii) $\overrightarrow{AX}$ =   **(viii)** $\overrightarrow{XY}$ =

[8 marks]

(b) If $|\boldsymbol{a}| = |\boldsymbol{b}|$ which one of of following best describes shape *OACB* ?

A triangle   **B** trapezium
C rhombus   **D** cube

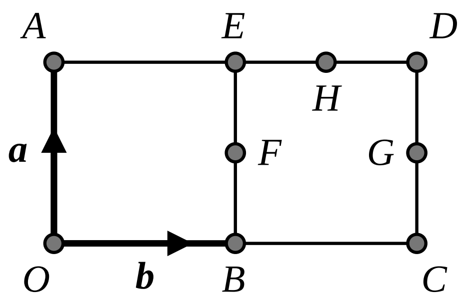
[1 mark]

Question 2

The diagram, which is not drawn to scale, shows two squares *OBEA* and *BCDE*.

$$\overrightarrow{OA} = \boldsymbol{a} \quad \text{and} \quad \overrightarrow{OB} = \boldsymbol{b}$$

The points *F*, *G* and *H* are the mid-points of *BE*, *CD* and *ED* respectively.



(a) Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$;

(i) $\overrightarrow{BG} =$

(ii) $\overrightarrow{GO} =$

(iii) $\overrightarrow{AC} =$

(iv) $\overrightarrow{GE} =$

(v) $\overrightarrow{GA} =$

(vi) $\overrightarrow{HO} =$

[6 marks]

(b) **(i)** Show that $\overrightarrow{AC} = \lambda\,\overrightarrow{GE}$ for an integer value of λ which is to be found.

[1 mark]

(ii) WWhich one of the following is true about $\overrightarrow{AC}$ and $\overrightarrow{GE}$?

A same magnitude

B same direction

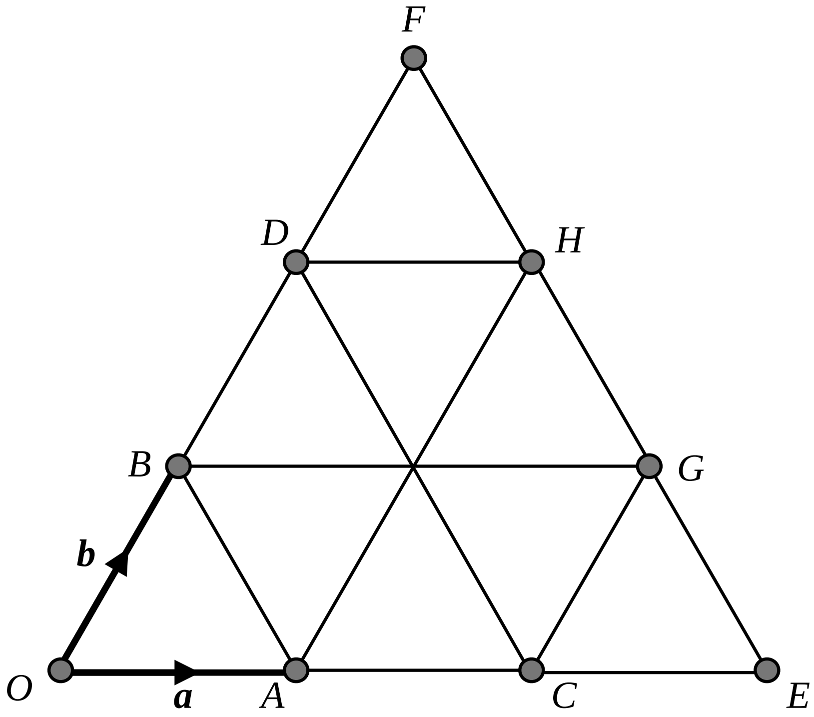
C perpendicular

D parallel

[1 mark]

Question 3

On the lattice of equilateral triangles, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$



(a) Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$;

(i) $\overrightarrow{OH} =$ **(ii)** $\overrightarrow{BE} =$

(iii) $\overrightarrow{ED} =$ **(iv)** $\overrightarrow{DA} =$

[4 marks]

(b) Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors, find the magnitude of $\overrightarrow{OG}$

[2 marks]

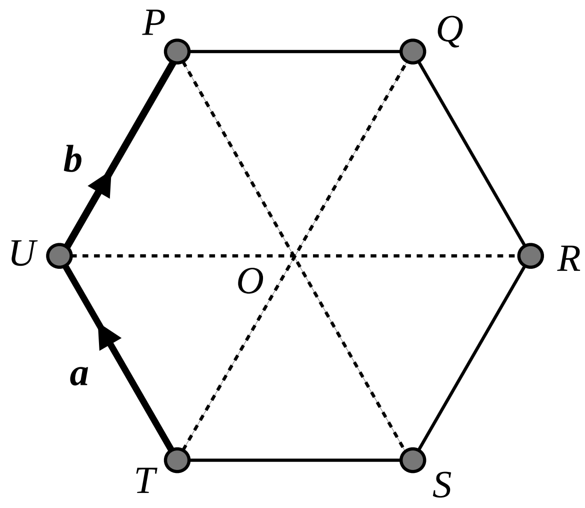
Question 4

GCSE Examination Question, May 2008, 4H, Q21

PQRSTU, which is not drawn to scale, is a regular hexagon, centre *O*

The hexagon is made from six equilateral triangles of side 2.5 cm

$$\overrightarrow{TU} = \boldsymbol{a} \quad \text{and} \quad \overrightarrow{UP} = \boldsymbol{b}$$



(a) Find, in terms of ***a*** and ***b***, the vectors

(i) $\overrightarrow{TP}$ =

[1 mark]

(ii) $\overrightarrow{PO}$ =

[1 mark]

(iii) $\overrightarrow{UO}$ =

[1 mark]

(b) Find the modulus (magnitude) of $\overrightarrow{UR}$

[1 marks]

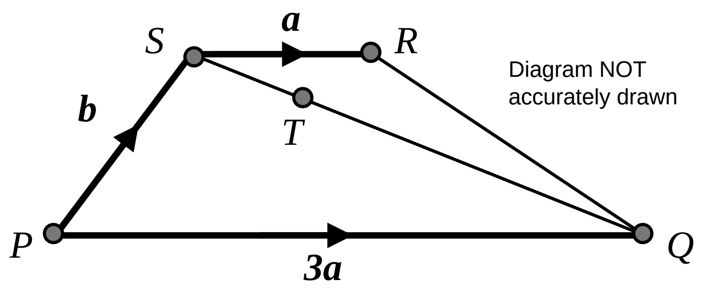
Question 5

GCSE Examination Question, November 2010, 4H, Q21

PQRS, which is not drawn to scale, is a trapezium with *PQ* parallel to *SR*

$$\overrightarrow{SR} = \boldsymbol{a} \qquad \overrightarrow{PQ} = 3\boldsymbol{a} \qquad \overrightarrow{PS} = \boldsymbol{b}$$

T is the point on SQ such that $ST = \dfrac{1}{4} SQ$



(a) Find, in terms of ***a*** and ***b***,

(i) $\overrightarrow{PR}$ =

[1 mark]

(ii) $\overrightarrow{SQ}$ =

[1 mark]

(iii) $\overrightarrow{PT}$ =

[1 mark]

(b) $\overrightarrow{PT} = k\,\overrightarrow{PR}$ where k is a fraction

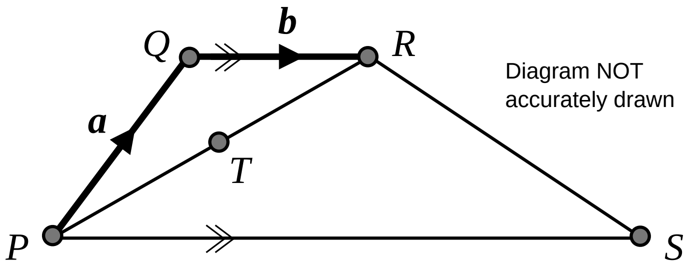
(i) What does this result tell you about the points P, T and R ?

(ii) Find the value of k

[2 marks]

Question 6

GCSE Examination Question, June 2011, 4H, Q24 edited



The diagram shows a trapezium *PQRS*, which is not drawn to scale.

- *T* is the midpoint of *PR*
- *PS* parallel to *QR*
- *PS* = 4 *QR*
- $\overrightarrow{PQ} = \boldsymbol{a}$ and $\overrightarrow{QR} = \boldsymbol{b}$

Find, in terms of ***a*** and ***b*,**

(i) $\overrightarrow{PS} =$

[1 mark]

(ii) $\overrightarrow{PR} =$

[1 mark]

(iii) $\overrightarrow{RS} =$

[1 mark]

(iv) $\overrightarrow{QT} =$

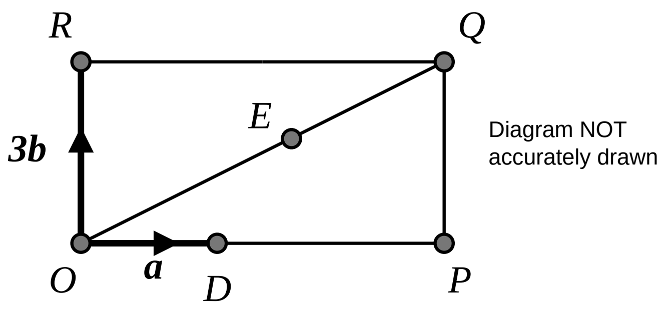
[1 mark]

(v) $\overrightarrow{TS} =$

[1 mark]

Question 7

GCSE Examination Question, January 2012, 4H, Q22



$OPQR$, which is not drawn to scale, is a rectangle

D is the point on OP such that $OD = \frac{1}{3} OP$

E is the point on OQ such that $OE = \frac{2}{3} OQ$

$\overrightarrow{OD} = \boldsymbol{a}$ $\qquad \overrightarrow{OR} = 3\boldsymbol{b}$

Find, in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

(i) $\overrightarrow{OQ} =$

[1 mark]

(ii) $\overrightarrow{OE} =$

[1 mark]

(iii) $\overrightarrow{DE} =$

[1 mark]

4.4 Answers

4.4.1 Solution to 4.2 Example (Teaching Video)

(i) $\overrightarrow{EB} = 2\boldsymbol{a}$ **(ii)** $\overrightarrow{CD} = -\boldsymbol{a}$

(iii) $\overrightarrow{AB} = -\boldsymbol{a} + \boldsymbol{b}$ **(iv)** $\overrightarrow{BC} = -2\boldsymbol{a} + \boldsymbol{b}$

(v) $\overrightarrow{AD} = -4\boldsymbol{a} + 2\boldsymbol{b}$ **(vi)** $\overrightarrow{BD} = -3\boldsymbol{a} + \boldsymbol{b}$

[6 marks]

4.4.2 Solution to 4.3 Exercise

Answer 1

(a) **(i)** $\overrightarrow{BC} = \boldsymbol{a}$ **(ii)** $\overrightarrow{AC} = \boldsymbol{b}$

(iii) $\overrightarrow{BO} = -\boldsymbol{b}$ **(iv)** $\overrightarrow{CY} = -\frac{1}{2}\boldsymbol{a}$

(v) $\overrightarrow{AB} = -\boldsymbol{a} + \boldsymbol{b}$ **(vi)** $\overrightarrow{OX} = \boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

(vii) $\overrightarrow{AX} = \frac{1}{2}\boldsymbol{b}$ **(viii)** $\overrightarrow{XY} = -\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

[8 marks]

(b) **C** rhombus

[1 mark]

Answer 2

(a) **(i)** $\overrightarrow{BG} = \frac{1}{2}\boldsymbol{a} + \boldsymbol{b}$ **(ii)** $\overrightarrow{GO} = -\frac{1}{2}\boldsymbol{a} - 2\boldsymbol{b}$

(iii) $\overrightarrow{AC} = -\boldsymbol{a} + 2\boldsymbol{b}$ **(iv)** $\overrightarrow{GE} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b}$

(v) $\overrightarrow{GA} = \frac{1}{2}\boldsymbol{a} - 2\boldsymbol{b}$ **(vi)** $\overrightarrow{HO} = -\boldsymbol{a} - \frac{3}{2}\boldsymbol{b}$

[6 marks]

(b) **(i)**

$$\overrightarrow{AC} = -\boldsymbol{a} + 2\boldsymbol{b} \quad \text{and} \quad \overrightarrow{GE} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b}$$

$$\therefore \overrightarrow{AC} = -2\,\overrightarrow{GE}$$

$$\text{i.e. } \lambda = -2$$

[1 mark]

(ii) **D** parallel

[1 mark]

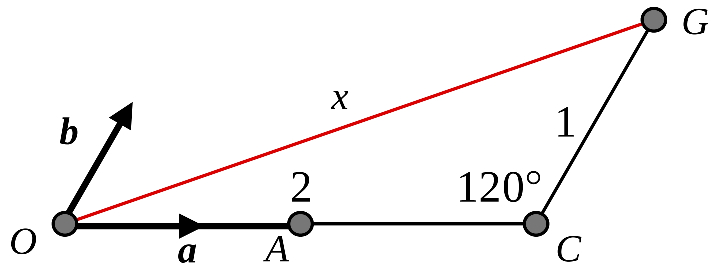
Answer 3

(a) **(i)** $\overrightarrow{OH} = \boldsymbol{a} + 2\boldsymbol{b}$ **(ii)** $\overrightarrow{BE} = 3\boldsymbol{a} - \boldsymbol{b}$

(iii) $\overrightarrow{ED} = -3\boldsymbol{a} + 2\boldsymbol{b}$ **(iv)** $\overrightarrow{DA} = \boldsymbol{a} - 2\boldsymbol{b}$

[4 marks]

(b)



By the cosine rule...

$$x^2 = 2^2 + 1^2 - 2 \times 2 \times 1 \times \cos 120°$$
$$= 4 + 1 + 2$$
$$= 7$$

Thus $\left|\overrightarrow{OG}\right| = \sqrt{7}$ units

[2 marks]

Answer 4

(a) **(i)** $\overrightarrow{TP} = \boldsymbol{a} + \boldsymbol{b}$

(ii) $\overrightarrow{PO} = -\boldsymbol{a}$

(iii) $\overrightarrow{UO} = -\boldsymbol{a} + \boldsymbol{b}$

(b) $\left|\overrightarrow{UR}\right| = 5\,\text{cm}$

[4 marks]

Answer 5

(a) **(i)** $\overrightarrow{PR} = \boldsymbol{a} + \boldsymbol{b}$

(ii) $\overrightarrow{SQ} = 3\boldsymbol{a} - \boldsymbol{b}$

(iii) $\overrightarrow{PT} = \overrightarrow{PS} + \overrightarrow{ST}$

$$= \overrightarrow{PS} + \frac{1}{4}\overrightarrow{SQ}$$

$$= \boldsymbol{b} + \frac{1}{4}\left(3\boldsymbol{a} - \boldsymbol{b}\right)$$

$$= \frac{3}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$$

$$= \frac{3}{4}\left(\boldsymbol{a} + \boldsymbol{b}\right)$$

[3 marks]

(b) **(i)** *P*, *T* and *R* are parallel with *P* in common.
$\therefore$ They are all on a straight line.

(ii) $\overrightarrow{PT} = \frac{3}{4}\left(\boldsymbol{a} + \boldsymbol{b}\right)$ $\quad\quad$ $\overrightarrow{PR} = \boldsymbol{a} + \boldsymbol{b}$

$$\therefore \overrightarrow{PT} = \frac{3}{4}\overrightarrow{PR}$$

[2 marks]

Answer 6

(i) $\overrightarrow{PS} = 4\boldsymbol{b}$

(ii) $\overrightarrow{PR} = \boldsymbol{a} + \boldsymbol{b}$

(iii) $\overrightarrow{RS} = -\boldsymbol{a} + 3\boldsymbol{b}$

(iv) $\overrightarrow{QT} = -\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

(v) $\overrightarrow{TS} = -\frac{1}{2}\boldsymbol{a} + \frac{5}{2}\boldsymbol{b}$

[5 marks]

Answer 7

(i) $\overrightarrow{OQ} = 3\boldsymbol{a} + 3\boldsymbol{b}$

(ii) $\overrightarrow{OE} = 2\boldsymbol{a} + 2\boldsymbol{b}$

(iii) $\overrightarrow{DE} = \boldsymbol{a} + 2\boldsymbol{b}$

[3 marks]

Chapter 5

Vectors I

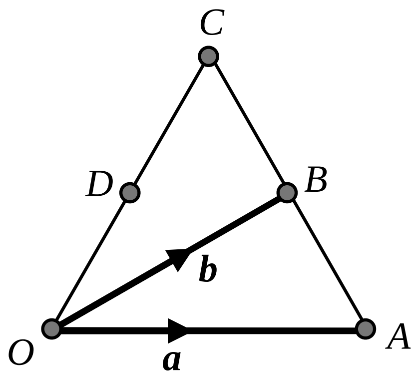
5.1 Vector Algebra

The more difficult GCSE vector questions can't be done directly from the provided diagram. Do use the diagram but also work with and trust your vector algebra.

5.2 Example

The diagram, which is not drawn to scale, shows an equilateral triangle OAC.
The point B is the mid-point of AC and the point D is the mid-point of OC.

Furthermore $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$



Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$;

(i) $\overrightarrow{AB} =$

(ii) $\overrightarrow{BC} =$

(iii) $\overrightarrow{OC} =$

(iv) $\overrightarrow{OD} =$

(v) $\overrightarrow{DB} =$

(vi) Given that $\overrightarrow{DB} = k\,\overrightarrow{OA}$ state the value of k

[6 marks]

Teaching Video : http://www.NumberWonder.co.uk/v9009/5a.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9009/5b.mp4 (Part 2)

<= Part 1 Part 2 =>

Complete the example above after watch the teaching videos.

5.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable. Make the method used clear.

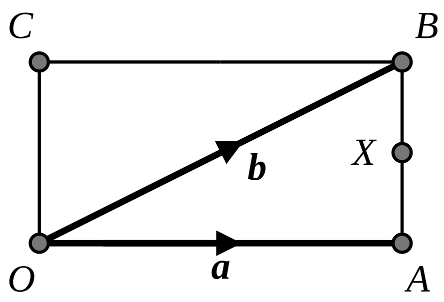
Marks available : 50

Question 1

The diagram, which is not drawn to scale, shows a rectangle *OABC* with

$$\overrightarrow{OA} = \boldsymbol{a} \quad \text{and} \quad \overrightarrow{OB} = \boldsymbol{b}$$

The point X is the mid-point of AB.



(a) Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$;

(i) $\overrightarrow{CB} =$

(ii) $\overrightarrow{AB} =$

(iii) $\overrightarrow{AX} =$

[3 marks]

(b) Work out $\overrightarrow{CX}$ by using the path.

(i) $\overrightarrow{CX} = \overrightarrow{CB} - \frac{1}{2}\overrightarrow{AB}$

[2 marks]

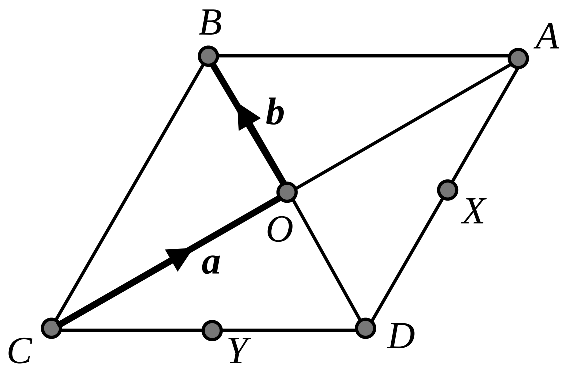
(ii) $\overrightarrow{CX} = \overrightarrow{CB} - \overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{AX}$

[2 marks]

Question 2

The diagram, which is not drawn to scale, shows a rhombus *ABCD*.
The two diagonals of the rhombus intersect at *O*.
The point *X* is the mid-point of *AD* and the point *Y* is the mid-point of *CD*.

Furthermore, $\overrightarrow{CO} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$



(a) Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$;

(i) $\overrightarrow{DA} =$

(ii) $\overrightarrow{DX} =$

(iii) $\overrightarrow{CD} =$

(iv) $\overrightarrow{YX} =$

[1, 1, 1, 1 mark]

(b) Given that $\overrightarrow{YX} = k\,\overrightarrow{CA}$ state the value of k

[1 mark]

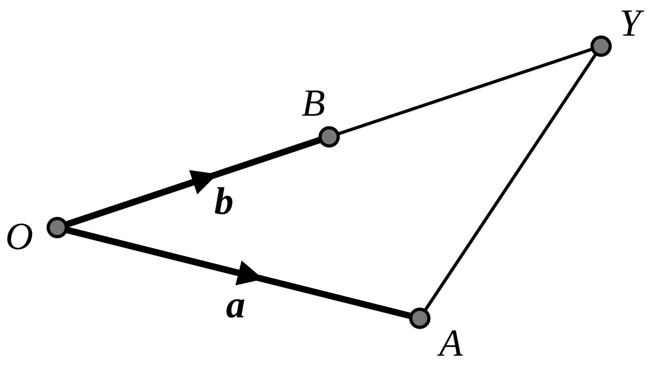
(c) If $|\boldsymbol{a}| = 15$ and $|\boldsymbol{b}| = 7$, determine $|\boldsymbol{a} + \boldsymbol{b}|$ using the fact that the diagonals of a rhombus are mutually perpendicular.

[1 mark]

Question 3

The diagram, which is not drawn to scale, shows a triangle OAY.
The point B is the mid-point of OY.

Furthermore, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$



(a) Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$;

(i) $\overrightarrow{OY} =$ **(ii)** $\overrightarrow{AY} =$

[2 marks]

X is the mid-point of OA

(b) Write down, in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, an expression for $\overrightarrow{XB}$

[1 mark]

(c) Show that $\overrightarrow{XB}$ is parallel to $\overrightarrow{AY}$, by writing a relationship between them of the form $\overrightarrow{XB} = k\,\overrightarrow{AY}$

[1 mark]

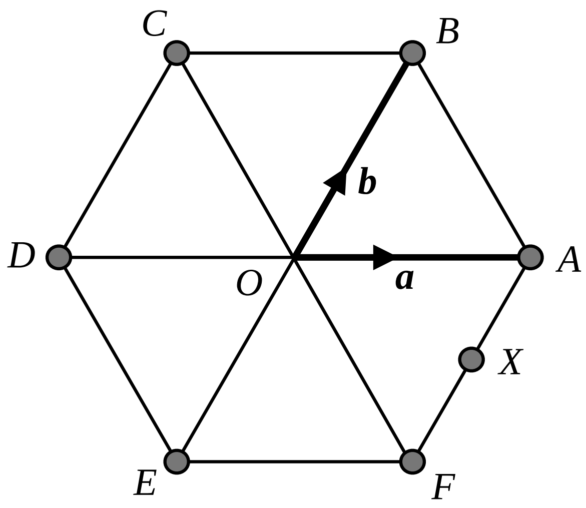
(d) If $\overrightarrow{OA}$ and $\overrightarrow{AY}$ are mutually perpendicular and $|\boldsymbol{a}| = |\boldsymbol{b}| = 3$ cm
what is $|\overrightarrow{AY}|$?

[1 mark]

Question 4

The diagram, which is not drawn to scale, shows a regular hexagon *ABCDEF*.
The spokes of the hexagon intersect at *O*.

Furthermore, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$



(a) Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$;

(i) $\overrightarrow{DB} =$ **(ii)** $\overrightarrow{DC} =$

(iii) $\overrightarrow{FC} =$ **(iv)** $\overrightarrow{FD} =$

[4 marks]

The point *X* is the mid-point of *FA*.

(b) Write down, in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, an expression for $\overrightarrow{DX}$

[1 mark]

(c) If the hexagon has sides of length 4.3 cm, what is $\left|\overrightarrow{DB}\right|$?
HINT : The Cosine Rule

[2 marks]

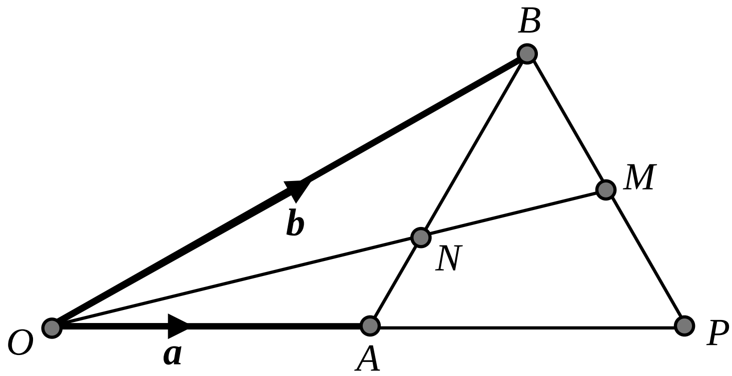
Question 5

The diagram, which is not drawn to scale, shows an equilateral triangle *APB* and an isosceles triangle, *OAB*, where $\left|\overrightarrow{OA}\right| = \left|\overrightarrow{AB}\right|$

The point *M* is the mid-point of *PB*.

$\overrightarrow{AN} = \frac{1}{3}\overrightarrow{AB}$

Furthermore, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$



(a) Express the following vectors in terms of ***a*** and ***b***;

(i) $\overrightarrow{OP} =$

(ii) $\overrightarrow{PB} =$

(iii) $\overrightarrow{OM} =$

(iv) $\overrightarrow{AN} =$

(v) $\overrightarrow{ON} =$

[1, 1, 1, 1, 2 marks]

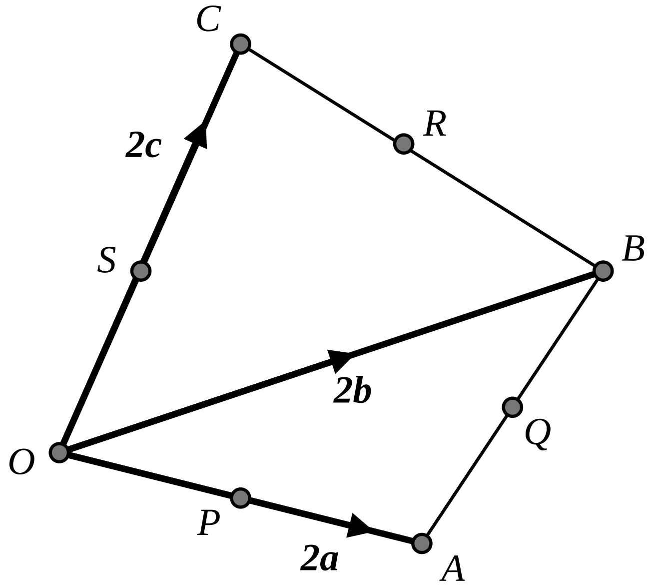
(b) Given that $\overrightarrow{OM} = k\,\overrightarrow{ON}$ find *k*.

[2 marks]

Question 6

The diagram, which is not drawn to scale, shows a quadrilateral *OABC* in which, $\left|\overrightarrow{OA}\right| = 2\,\boldsymbol{a}$, $\left|\overrightarrow{OB}\right| = 2\,\boldsymbol{b}$ and $\left|\overrightarrow{OC}\right| = 2\,\boldsymbol{c}$

Points *P*, *Q*, *R* and *S* are the midpoints of the sides *OA*, *AB*, *BC* and *CO* respectively.



(a) Express the following vectors in terms of ***a***, ***b*** and ***c***;

(i) $\overrightarrow{AB}$ =

(ii) $\overrightarrow{BC}$ =

(iii) $\overrightarrow{PQ}$ =

(iv) $\overrightarrow{QR}$ =

(v) $\overrightarrow{PS}$ =

[1, 1, 1, 2, 1 marks]

(b) Describe the relationship between $\overrightarrow{QR}$ and $\overrightarrow{PS}$

[1 mark]

(c) What sort of quadrilateral is *PQRS* ?

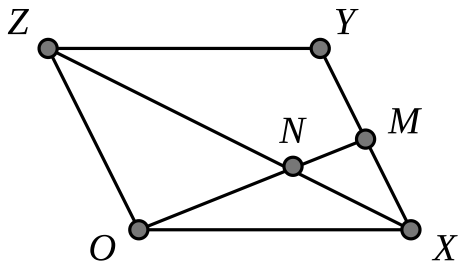
[1 mark]

Question 7

In the diagram, *OXYZ* is a parallelogram.

M is the mid-point of $\overrightarrow{XY}$

Furthermore, $\overrightarrow{OX} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ and $\overrightarrow{OZ} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$



(i) Write down the vectors $\overrightarrow{XM}$ and $\overrightarrow{XZ}$

[2 marks]

(ii) Given that $\overrightarrow{ON} = v\ \overrightarrow{OM}$ write down in terms of *v* the vector $\overrightarrow{ON}$

[2 marks]

(iii) Given that $\overrightarrow{ON} = \overrightarrow{OX} + w\ \overrightarrow{XZ}$ find in terms of *w* the vector $\overrightarrow{ON}$

[1 mark]

(iv) Solve two simultaneous equations to find *v* and *w*

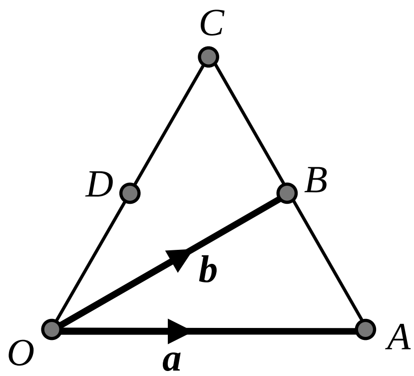
[3 marks]

(v) Explain the significance of your solution.

[1 mark]

5.4 Answers

5.4.1 Solution to 5.2 Example (Teaching Video)



(i) $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AC}$

$= -\boldsymbol{a} + \boldsymbol{b}$

(ii) $\overrightarrow{BC} = \frac{1}{2}\overrightarrow{AC}$

$= -\boldsymbol{a} + \boldsymbol{b}$

(iii) $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$

$= \boldsymbol{b} + (-\boldsymbol{a} + \boldsymbol{b})$

$= -\boldsymbol{a} + 2\boldsymbol{b}$

(iv) $\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OC}$

$= \frac{1}{2}(-\boldsymbol{a} + 2\boldsymbol{b})$

$= -\frac{1}{2}\boldsymbol{a} + \boldsymbol{b}$

(v) $\overrightarrow{DB} = \overrightarrow{DO} + \overrightarrow{OB}$

$= -\overrightarrow{OD} + \overrightarrow{OB}$

$= \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{b}$

$= \frac{1}{2}\boldsymbol{a}$

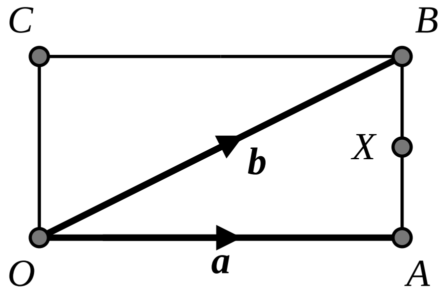
(vi) Told $\overrightarrow{DB} = k\,\overrightarrow{OA}$

$\frac{1}{2}\boldsymbol{a} = k\,\boldsymbol{a}$

$\therefore\ k = \frac{1}{2}$

5.4.2 Solutions (5.3 Exercise)

Answer 1



(a) **(i)** $\overrightarrow{CB} = \boldsymbol{a}$

(ii) $\overrightarrow{AB} = -\boldsymbol{a} + \boldsymbol{b}$

(iii) $\overrightarrow{AX} = \frac{1}{2}\overrightarrow{AB}$

$$= \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{b})$$

$$= -\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$$

[3 marks]

(b) **(i)** $\overrightarrow{CX} = \overrightarrow{CB} - \frac{1}{2}\overrightarrow{AB}$

$$= \boldsymbol{a} - \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{b})$$

$$= \boldsymbol{a} + \frac{1}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}$$

$$= \frac{3}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}$$

[2 marks]

(ii) $\overrightarrow{CX} = \overrightarrow{CB} - \overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{AX}$

$$= \boldsymbol{a} - \boldsymbol{b} + \boldsymbol{a} - \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$$

$$= \frac{3}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}$$

[2 marks]

Answer 2

(a) **(i)** $\overrightarrow{DA} = \boldsymbol{a} + \boldsymbol{b}$

[1 mark]

(ii) $\overrightarrow{DX} = \frac{1}{2}\overrightarrow{DA} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

[1 mark]

(iii) $\overrightarrow{CD} = \boldsymbol{a} - \boldsymbol{b}$

[1 mark]

(iv)
$$\begin{aligned}\overrightarrow{YX} &= \overrightarrow{YD} + \overrightarrow{DX}\\ &= \frac{1}{2}\overrightarrow{CD} + \overrightarrow{DX}\\ &= \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b}) + \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{a}\\ &= \boldsymbol{a}\end{aligned}$$

[1 mark]

(b) $\overrightarrow{YX} = \boldsymbol{a}$ and $\overrightarrow{CA} = 2\boldsymbol{a}$

$\therefore\ \overrightarrow{YX} = \frac{1}{2}\overrightarrow{CA}$ i.e. $\lambda = \frac{1}{2}$

[1 mark]

(c) $\boldsymbol{a}$ and $\boldsymbol{b}$ are mutually perpendicular

Use the Theorem of Pythagoras

$$\Rightarrow\ |\boldsymbol{a} + \boldsymbol{b}| = \sqrt{274} = 16.6 \text{ cm}$$

[1 mark]

Answer 3

(a) **(i)** $\overrightarrow{OY} = 2\boldsymbol{b}$ **(ii)** $\overrightarrow{AY} = -\boldsymbol{a} + 2\boldsymbol{b}$

[2 marks]

(b) $\overrightarrow{XB} = -\frac{1}{2}\boldsymbol{a} + \boldsymbol{b}$

[1 mark]

(c) $\overrightarrow{XB} = -\frac{1}{2}\boldsymbol{a} + \boldsymbol{b}$ and $\overrightarrow{AY} = -\boldsymbol{a} + 2\boldsymbol{b}$

$\therefore\ \overrightarrow{XB} = \frac{1}{2}\overrightarrow{AY}$ so $\overrightarrow{XB}$ is parallel to $\overrightarrow{AY}$

[1 mark]

(d) By Pythagoras, $\left|\overrightarrow{AY}\right| = \sqrt{(3 + 3)^2 - 3^2} = 3\sqrt{3}$

$= 5.20$ cm

[1 mark]

Answer 4

(a) **(i)** $\overrightarrow{DB} = \boldsymbol{a} + \boldsymbol{b}$ **(ii)** $\overrightarrow{DC} = \boldsymbol{b}$

(iii) $\overrightarrow{FC} = -2\,\boldsymbol{a} + 2\,\boldsymbol{b}$ **(iv)** $\overrightarrow{FD} = \boldsymbol{b} - 2\,\boldsymbol{a}$

[4 marks]

(b)

$$\begin{aligned}\overrightarrow{DX} &= \overrightarrow{DA} + \overrightarrow{AX} \\ &= \overrightarrow{DA} - \overrightarrow{XA} \\ &= 2\,\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}\end{aligned}$$

[1 mark]

(c) 7.45 cm

[2 marks]

Answer 5

(a) **(i)** $\overrightarrow{OP} = 2\,\boldsymbol{a}$ **(ii)** $\overrightarrow{PB} = -2\,\boldsymbol{a} + \boldsymbol{b}$

(iii) $\overrightarrow{OM} = \boldsymbol{a} + \frac{1}{2}\,\boldsymbol{b}$ **(iv)** $\overrightarrow{AN} = -\frac{1}{3}\,\boldsymbol{a} + \frac{1}{3}\,\boldsymbol{b}$

(v)

$$\begin{aligned}\overrightarrow{ON} &= \overrightarrow{OA} + \overrightarrow{AN} \\ &= \boldsymbol{a} - \frac{1}{3}\,\boldsymbol{a} + \frac{1}{3}\,\boldsymbol{b} \\ &= \frac{2}{3}\,\boldsymbol{a} + \frac{1}{3}\,\boldsymbol{b}\end{aligned}$$

[1, 1, 1, 1, 2 marks]

(b) $\overrightarrow{OM} = \boldsymbol{a} + \frac{1}{2}\,\boldsymbol{b}$ and $\overrightarrow{ON} = \frac{2}{3}\,\boldsymbol{a} + \frac{1}{3}\,\boldsymbol{b}$

$\overrightarrow{OM} = k\,\overrightarrow{ON}$ with $k = \frac{3}{2}$

[2 marks]

Answer 6

(a) **(i)** $\overrightarrow{AB} = -2\boldsymbol{a} + 2\boldsymbol{b}$ **(ii)** $\overrightarrow{BC} = -2\boldsymbol{b} + 2\boldsymbol{c}$

(iii)
$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PA} + \overrightarrow{AQ}\\ &= \boldsymbol{a} + \frac{1}{2}\overrightarrow{AB}\\ &= \boldsymbol{a} - \boldsymbol{a} + \boldsymbol{b}\\ &= \boldsymbol{b}\end{aligned}$$

(iv)
$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QB} + \overrightarrow{BR}\\ &= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}\\ &= -\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{b} + \boldsymbol{c}\\ &= -\boldsymbol{a} + \boldsymbol{c}\end{aligned}$$

(v)
$$\begin{aligned}\overrightarrow{PS} &= \overrightarrow{PO} + \overrightarrow{OS}\\ &= -\boldsymbol{a} + \boldsymbol{c}\end{aligned}$$

[1, 1, 1, 2, 1 marks]

(b) Same magnitude and direction

[1 mark]

(c) Parallelogram

[1 mark]

Answer 7

(i)

$$\overrightarrow{XM} = \frac{1}{2}\overrightarrow{XY} = \frac{1}{2}\overrightarrow{OZ} = \frac{1}{2}\begin{pmatrix} -2 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{XZ} = \overrightarrow{XO} + \overrightarrow{OZ} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 6 \end{pmatrix} = \begin{pmatrix} -10 \\ 6 \end{pmatrix}$$

[2 marks]

(ii)

$$\overrightarrow{ON} = v\,\overrightarrow{OM} = v\left(\overrightarrow{OX} + \overrightarrow{XM}\right) = v\left(\begin{pmatrix} 8 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) = v\begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

[2 marks]

(iii)

$$\overrightarrow{ON} = \overrightarrow{OX} + w\,\overrightarrow{XZ} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} + w\begin{pmatrix} -10 \\ 6 \end{pmatrix}$$

[1 mark]

(iv)

$$v\begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} + w\begin{pmatrix} -10 \\ 6 \end{pmatrix}$$

$$\left.\begin{aligned} 7v + 10w &= 8 \\ 3v - 6w &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} 21v + 30w &= 24 \\ 21v - 42w &= 0 \end{aligned}\right\} \qquad 72w = 24$$

$$w = \frac{1}{3}$$

and so $v = \dfrac{2}{3}$

[3 marks]

(v) N is a third of the way along XZ

N is two-thirds of the way along OM

[1 mark]

6.1 Vector Ratios

When a point M is described as being the midpoint of the line AB the length of the line is dived in a ratio of,

$$AM \ : \ MB$$

$$\frac{1}{2} \ : \ \frac{1}{2}$$

or, as integers are often preferred, by multiplying through by 2,

$$AM \ : \ MB$$

$$1 \ : \ 1$$

Of course, there is no reason why other ratios could not be used such as, for example, a point X that divides the line AB in the ratio,

$$AX \ : \ XB$$

$$3 \ : \ 2$$

which could also be written, upon dividing through by 5,

$$AX \ : \ XB$$

$$\frac{3}{5} \ : \ \frac{2}{5}$$

This tells the reader that X is $\frac{3}{5}$ of the way along line AB from X

If, for example, $\overrightarrow{AB} = 15\,\boldsymbol{s} + 10\,\boldsymbol{t}$ then $\overrightarrow{AX} = \frac{3}{5}\overrightarrow{AB}$

$$= \frac{3}{5}\,(\,15\,\boldsymbol{s} + 10\,\boldsymbol{t}\,)$$

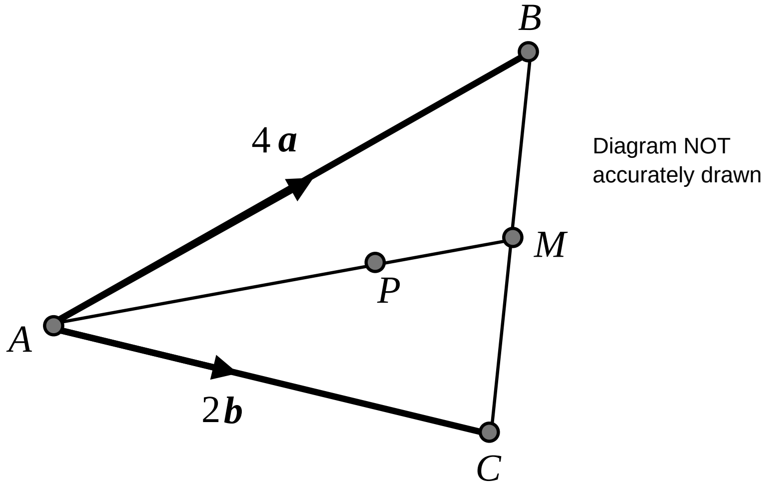
$$= 9\,\boldsymbol{s} + 6\,\boldsymbol{t}$$

□□□□□□□□□□□□□□□△△△△△△△△△△	‖ $15\,\boldsymbol{s} + 10\,\boldsymbol{t}$
□□□△△□□□△△□□□△△□□□△△□□□△△	‖ $5\,(\,3\,\boldsymbol{s} + 2\,\boldsymbol{t}\,)$
□□□△△□□□△△□□□△△ + □□□△△□□□△△	‖ $3\,(\,3\,\boldsymbol{s} + 2\,\boldsymbol{t}\,) + 2\,(\,3\,\boldsymbol{s} + 2\,\boldsymbol{t}\,)$
□□□△△□□□△△□□□△△ : □□□△△□□□△△	‖ $3\,(\,3\,\boldsymbol{s} + 2\,\boldsymbol{t}\,) \ : \ 2\,(\,3\,\boldsymbol{s} + 2\,\boldsymbol{t}\,)$
3 : 2	‖ $3 \ : \ 2$
□□□□□□□□□△△△△△△ : □□□□□□△△△△	‖ $9\,\boldsymbol{s} + 6\,\boldsymbol{t} \ : \ 6\,\boldsymbol{s} + 4\,t$

An interesting similarity between ratios of squares and triangles on the one hand and ratios of vectors $\boldsymbol{s}$ and $\boldsymbol{t}$ in the other

6.2 A Recent Examination Question

GCSE Examination Question from January 2020, Paper 2H, Q23



ABC is a triangle in which the midpoint of *BC* is *M* and *P* is a point on *AM*.

$$\overrightarrow{AB} = 4\boldsymbol{a} \qquad \overrightarrow{AC} = 2\boldsymbol{b} \qquad \overrightarrow{AP} = \frac{3}{2}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$$

Find the ratio $AP : PM$

Teaching Video : http://www.NumberWonder.co.uk/v9009/6.mp4

Watch the teaching video then write
out a solution to the question.

☞

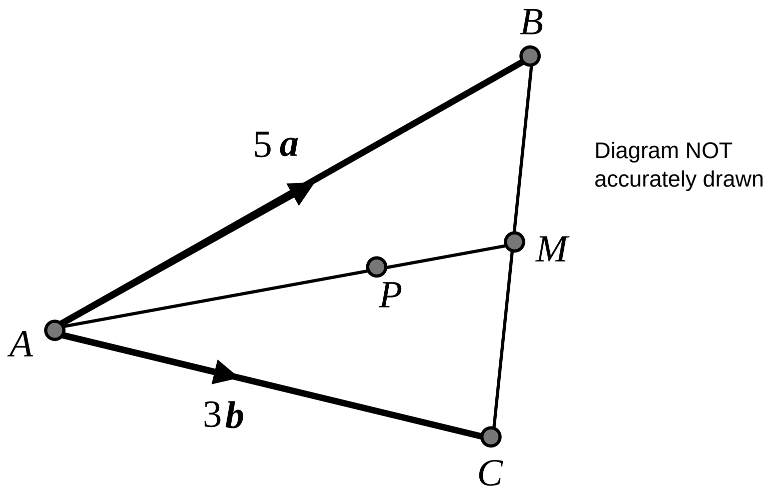
[3 marks]

6.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable. Make the method used clear.

Marks available : 30

Question 1



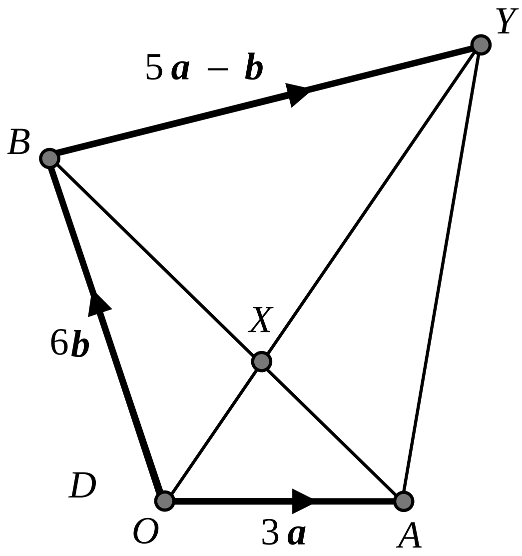
ABC is a triangle in which the midpoint of *BC* is *M* and *P* is a point on *AM*.

$$\overrightarrow{AB} = 5\boldsymbol{a} \qquad \overrightarrow{AC} = 3\boldsymbol{b} \qquad \overrightarrow{AP} = \frac{5}{6}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$$

Find the ratio $AP : PM$

[3 marks]

Question 2



OAYB is a quadrilateral, with the diagonals *AB* and *OY* intersecting at point *X*.
The ratio $AX : XB = 1 : 2$

$$\overrightarrow{OA} = 3\boldsymbol{a} \qquad \overrightarrow{OB} = 6\boldsymbol{b} \qquad \overrightarrow{BY} = 5\boldsymbol{a} - \boldsymbol{b}$$

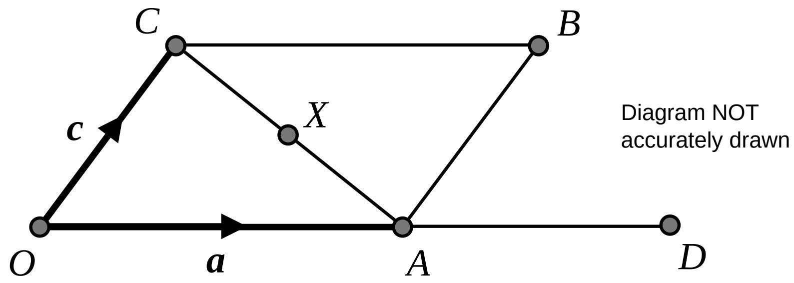
(i) Use vector algebra to show that $\overrightarrow{AX} = -\boldsymbol{a} + 2\boldsymbol{b}$

[2 marks]

(ii) Find the ratio $OX : XY$

[4 marks]

Question 3



$OABC$ is a parallelogram with $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$

X is the midpoint of the line CA.

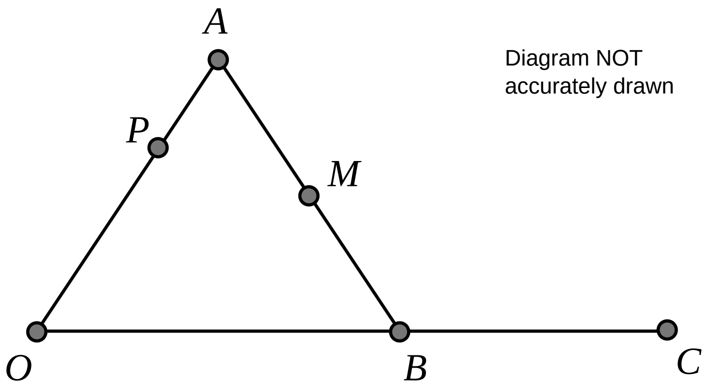
OAD is a straight line.

Given that $\overrightarrow{XD} = 3\boldsymbol{a} - \frac{1}{2}\boldsymbol{c}$ find the ratio $OA : AD$

[4 marks]

Question 4

GCSE Examination Question from January 2016, Paper 3H, Q23



OAB is a triangle

P is the point on *OA* such that *OP* : *PA* = 2 : 1

C is the point such that *B* is the midpoint of *OC*

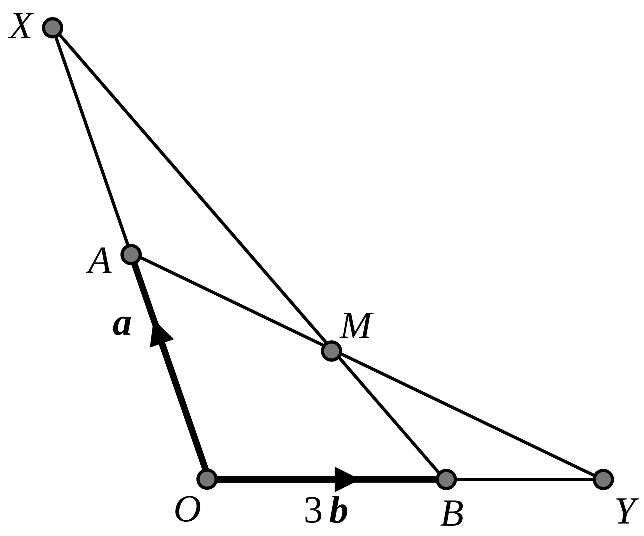
M is the midpoint of *AB*

$$\overrightarrow{OA} = 6\,\boldsymbol{a} \qquad \overrightarrow{OB} = 4\,\boldsymbol{b}$$

Show that *PMC* is a straight line

[5 marks]

Question 5



In the diagram $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = 3\boldsymbol{b}$

$OA : AX = 1 : 1$, $OB : BY = 3 : 1$ and $BM : MX = 1 : 4$

(i) Show that $\overrightarrow{BM} = \dfrac{2}{5}\boldsymbol{a} - \dfrac{3}{5}\boldsymbol{b}$

[2 marks]

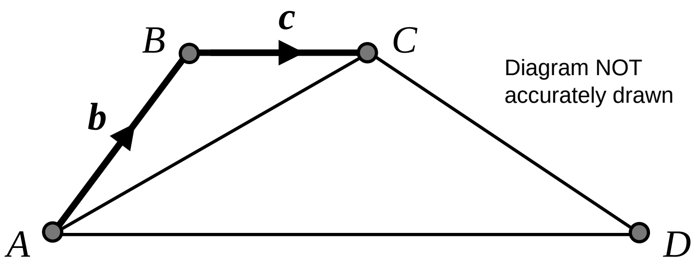
(ii) Use a vector method to find $YM : MA$

Show your working clearly.

[4 marks]

Question 6

GCSE Examination Question from January 2017, Paper 3HR, Q23



The diagram shows trapezium *ABCD*

BC is parallel to *AD*, $AD = 3\,BC$, $\overrightarrow{AB} = \boldsymbol{b}$ and $\overrightarrow{BC} = \boldsymbol{c}$

(a) Find, in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$, the vector $\overrightarrow{CD}$

Give your answer in its simplest form.

[2 marks]

The point *P* lies on the line *AC* such that $AP : PC = 2 : 1$

(b) Is *BPD* a straight line ?

Show your working clearly.

[4 marks]

6.4 Answers

6.4.1 Solutions to 6.2 A Recent Examination Question (Teaching Video)

$$\begin{aligned}
\overrightarrow{AM} &= \overrightarrow{AP} + \overrightarrow{PM} \\
\therefore\ \overrightarrow{PM} &= \overrightarrow{AM} - \overrightarrow{AP} \\
&= \overrightarrow{AB} + \overrightarrow{BM} - \overrightarrow{AP} \\
&= \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} - \overrightarrow{AP} \\
&= 4\boldsymbol{a} + \frac{1}{2}(-4\boldsymbol{a} + 2\boldsymbol{b}) - \left(\frac{3}{2}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}\right) \\
&= 4\boldsymbol{a} - 2\boldsymbol{a} + \boldsymbol{b} - \frac{3}{2}\boldsymbol{a} - \frac{3}{4}\boldsymbol{b} \\
&= \frac{1}{2}\boldsymbol{a} + \frac{1}{4}\boldsymbol{b}
\end{aligned}$$

$$\begin{aligned}
AP &: PM \\
\frac{3}{2}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b} &: \frac{1}{2}\boldsymbol{a} + \frac{1}{4}\boldsymbol{b} \\
3\left(\frac{1}{2}\boldsymbol{a} + \frac{1}{4}\boldsymbol{b}\right) &: \left(\frac{1}{2}\boldsymbol{a} + \frac{1}{4}\boldsymbol{b}\right) \\
3 &: 1
\end{aligned}$$

[3 marks]

6.4.2 Solutions to 6.3 Exercise

Answer 1

$$
\begin{aligned}
\overrightarrow{AM} &= \overrightarrow{AP} + \overrightarrow{PM} \\
\therefore\ \overrightarrow{PM} &= \overrightarrow{AM} - \overrightarrow{AP} \\
&= \overrightarrow{AB} + \overrightarrow{BM} - \overrightarrow{AP} \\
&= \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} - \overrightarrow{AP} \\
&= 5\boldsymbol{a} + \frac{1}{2}(-5\boldsymbol{a} + 3\boldsymbol{b}) - \left(\frac{5}{6}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right) \\
&= 5\boldsymbol{a} - \frac{5}{2}\boldsymbol{a} + \frac{3}{2}\boldsymbol{b} - \frac{5}{6}\boldsymbol{a} - \frac{1}{2}\boldsymbol{b} \\
&= \frac{5}{2}\boldsymbol{a} + \frac{3}{2}\boldsymbol{b} - \frac{5}{6}\boldsymbol{a} - \frac{1}{2}\boldsymbol{b} \\
&= \frac{15}{6}\boldsymbol{a} + \frac{3}{2}\boldsymbol{b} - \frac{5}{6}\boldsymbol{a} - \frac{1}{2}\boldsymbol{b} \\
&= \frac{10}{6}\boldsymbol{a} + \boldsymbol{b}
\end{aligned}
$$

$$
\begin{aligned}
AP &: PM \\
\frac{5}{6}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} &: \frac{10}{6}\boldsymbol{a} + \boldsymbol{b} \\
\left(\frac{5}{6}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right) &: 2\left(\frac{5}{6}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right) \\
1 &: 2
\end{aligned}
$$

[3 marks]

Answer 2

(i) $AX : XB = 1 : 2$

$$\Rightarrow \overrightarrow{XB} = 2\,\overrightarrow{AX}$$

$$\overrightarrow{AB} = \overrightarrow{AX} + \overrightarrow{XB}$$

$$= \overrightarrow{AX} + 2\,\overrightarrow{AX}$$

$$-3\boldsymbol{a} + 6\boldsymbol{b} = 3\,\overrightarrow{AX}$$

$$\overrightarrow{AX} = -\boldsymbol{a} + 2\boldsymbol{b}$$

(ii) $\overrightarrow{XB} = 2\,\overrightarrow{AX}$

$$= -2\boldsymbol{a} + 4\boldsymbol{b}$$

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$$

$$= 3\boldsymbol{a} - \boldsymbol{a} + 2\boldsymbol{b}$$

$$= 2\boldsymbol{a} + 2\boldsymbol{b}$$

$$\overrightarrow{XY} = \overrightarrow{XB} + \overrightarrow{BY}$$

$$= -2\boldsymbol{a} + 4\boldsymbol{b} + 5\boldsymbol{a} - \boldsymbol{b}$$

$$= 3\boldsymbol{a} + 3\boldsymbol{b}$$

$$OX : XY$$

$$2\boldsymbol{a} + 2\boldsymbol{b} : 3\boldsymbol{a} + 3\boldsymbol{b}$$

$$2 : 3$$

[4 marks]

Answer 3

$$\overrightarrow{XD} = \overrightarrow{XA} + \overrightarrow{AD} \quad \Rightarrow \quad \overrightarrow{AD} = \overrightarrow{XD} - \overrightarrow{XA}$$

$$= \overrightarrow{XD} - \frac{1}{2}\overrightarrow{CA}$$

$$= 3\boldsymbol{a} - \frac{1}{2}\boldsymbol{c} - \frac{1}{2}(-\boldsymbol{c} + \boldsymbol{a})$$

$$= 3\boldsymbol{a} - \frac{1}{2}\boldsymbol{c} + \frac{1}{2}\boldsymbol{c} - \frac{1}{2}\boldsymbol{a}$$

$$= \frac{5}{2}\boldsymbol{a}$$

$$OA : AD$$

$$\boldsymbol{a} : \frac{5}{2}\boldsymbol{a}$$

$$2 : 5$$

[4 marks]

Answer 4

$$\begin{aligned}\overrightarrow{PM} &= \overrightarrow{PA} + \overrightarrow{AM}\\ &= \frac{1}{3}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}\\ &= \frac{1}{3} \times 6\boldsymbol{a} + \frac{1}{2}(-6\boldsymbol{a} + 4\boldsymbol{b})\\ &= 2\boldsymbol{a} - 3\boldsymbol{a} + 2\boldsymbol{b}\\ &= -\boldsymbol{a} + 2\boldsymbol{b}\end{aligned}$$

$$\begin{aligned}\overrightarrow{MC} &= \overrightarrow{MB} + \overrightarrow{BC}\\ &= \frac{1}{2}\overrightarrow{AB} + 4\boldsymbol{b}\\ &= \frac{1}{2}(-6\boldsymbol{a} + 4\boldsymbol{b}) + 4\boldsymbol{b}\\ &= -3\boldsymbol{a} + 2\boldsymbol{b} + 4\boldsymbol{b}\\ &= -3\boldsymbol{a} + 6\boldsymbol{b}\\ &= 3(-\boldsymbol{a} + 2\boldsymbol{b})\end{aligned}$$

Thus,

$$\overrightarrow{PM} = \frac{1}{3}\overrightarrow{MC}$$

$\therefore$ $\overrightarrow{PM}$ and $\overrightarrow{MC}$ are parallel and, as they have the point M is common,

PMC is a straight line ☐

[5 marks]

Answer 5

(i) $\overrightarrow{BX} = \overrightarrow{BO} + \overrightarrow{OX}$

$= -3\boldsymbol{b} + 2\boldsymbol{a}$ as $\overrightarrow{AX} = \boldsymbol{a}$

$$BM : MX = 1 : 4$$

$$\Rightarrow \overrightarrow{MX} = 4\overrightarrow{BM}$$

$$\begin{aligned} \overrightarrow{BX} &= \overrightarrow{BM} + \overrightarrow{MX} \\ &= \overrightarrow{BM} + 4\overrightarrow{BM} \\ -3\boldsymbol{b} + 2\boldsymbol{a} &= 5\overrightarrow{BM} \\ \overrightarrow{BM} &= \frac{2}{5}\boldsymbol{a} - \frac{3}{5}\boldsymbol{b} \qquad \square \end{aligned}$$

[2 marks]

(ii)

$$\begin{aligned} \overrightarrow{YM} &= \overrightarrow{YB} + \overrightarrow{BM} \\ &= -\boldsymbol{b} + \frac{2}{5}\boldsymbol{a} - \frac{3}{5}\boldsymbol{b} \\ &= \frac{2}{5}\boldsymbol{a} - \frac{8}{5}\boldsymbol{b} \end{aligned}$$

$$\begin{aligned} \overrightarrow{MA} &= \overrightarrow{MX} + \overrightarrow{XA} \\ &= 4\overrightarrow{BM} + \overrightarrow{XA} \\ &= \frac{8}{5}\boldsymbol{a} - \frac{12}{5}\boldsymbol{b} - \boldsymbol{a} \\ &= \frac{3}{5}\boldsymbol{a} - \frac{12}{5}\boldsymbol{b} \end{aligned}$$

$$YM : MA$$

$$\frac{2}{5}\boldsymbol{a} - \frac{8}{5}\boldsymbol{b} : \frac{3}{5}\boldsymbol{a} - \frac{12}{5}\boldsymbol{b}$$

$$2\left(\frac{1}{5}\boldsymbol{a} - \frac{4}{5}\boldsymbol{b}\right) : 3\left(\frac{1}{5}\boldsymbol{a} - \frac{4}{5}\boldsymbol{b}\right)$$

$$2 : 3$$

[4 marks]

Answer 6

(a)

$$\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AD}$$
$$= -\boldsymbol{c} - \boldsymbol{b} + 3\,\boldsymbol{c}$$
$$= -\boldsymbol{b} + 2\,\boldsymbol{c}$$

[2 marks]

(b)

$$AP : PC = 2 : 1$$
$$\Rightarrow \overrightarrow{AP} = 2\,\overrightarrow{PC}$$

$$\overrightarrow{AC} = \overrightarrow{AP} + \overrightarrow{PC}$$
$$\boldsymbol{b} + \boldsymbol{c} = 2\,\overrightarrow{PC} + \overrightarrow{PC}$$
$$\overrightarrow{PC} = \frac{1}{3}\,\boldsymbol{b} + \frac{1}{3}\,\boldsymbol{c} \qquad \therefore\ \overrightarrow{AP} = \frac{2}{3}\,\boldsymbol{b} + \frac{2}{3}\,\boldsymbol{c}$$

$$\overrightarrow{BP} = \overrightarrow{BA} + \overrightarrow{AP}$$
$$= -\boldsymbol{b} + \frac{2}{3}\,\boldsymbol{b} + \frac{2}{3}\,\boldsymbol{c}$$
$$= -\frac{1}{3}\,\boldsymbol{b} + \frac{2}{3}\,\boldsymbol{c}$$

$$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$$
$$= -\boldsymbol{b} + 3\,\boldsymbol{c}$$

As $\overrightarrow{BP}$ and $\overrightarrow{BD}$ are not multiples of each other BPD is not a straight line

[4 marks]

Vectors I

7.1 Change of Basis

When a point is specified, such as $P(3, 4)$, it can be thought of as a vector description of P's location from the origin. Such vectors that are tied to a location, the origin in this case, are called position vectors, rather than free vectors.

$$\boldsymbol{p} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{or} \quad \boldsymbol{p} = 3\,\boldsymbol{i} + 4\,\boldsymbol{j}$$

The description of $\boldsymbol{p}$ in the style $\boldsymbol{p} = 3\,\boldsymbol{i} + 4\,\boldsymbol{j}$ emphasises that the vector $\boldsymbol{p}$ is expressed in terms of two other vectors; the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ in the x and y-axis directions respectively. The vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are said to form a basis for the Cartesian coordinate system. Any other location on the XY plane can be specified using some combination of $\boldsymbol{i}$ and $\boldsymbol{j}$.

Other vectors can be used as a basis for a different coordinate system.

7.2 Example

A coordinate system has basis vectors $\boldsymbol{A} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

(i) Write $\boldsymbol{s} = 6\,\boldsymbol{A} + 4\,\boldsymbol{B}$ in the form $\boldsymbol{s} = k\begin{pmatrix} 20 \\ 21 \end{pmatrix}$ for some constant k.

[1 mark]

(ii) Write $\boldsymbol{t} = 9\,\boldsymbol{A} + 6\,\boldsymbol{B}$ in the form $\boldsymbol{t} = K\begin{pmatrix} 20 \\ 21 \end{pmatrix}$ for some constant K.

[1 mark]

(iii) What do your answers to part (i) and (ii) show ?

[1 mark]

(iv) Show that $\boldsymbol{s}$ and $\boldsymbol{t}$ are parallel for all other vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.
(In other words, $\boldsymbol{s}$ and $\boldsymbol{t}$ are parallel in any linear coordinate system)

[2 marks]

Teaching Video : http://www.NumberWonder.co.uk/v9009/7.mp4

<= Watch the video, complete the above example.

7.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.

Marks available : 40

Question 1

Two vectors, $\boldsymbol{E}$ and $\boldsymbol{Z}$, are $\boldsymbol{E} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\boldsymbol{Z} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

(i) Write $\boldsymbol{a} = 9\,\boldsymbol{E} + 3\,\boldsymbol{Z}$ in the form $\boldsymbol{a} = k\begin{pmatrix} 9 \\ 19 \end{pmatrix}$ for some constant k.

[1 mark]

(ii) Write $\boldsymbol{b} = 15\,\boldsymbol{E} + 5\,\boldsymbol{Z}$ in the form $\boldsymbol{b} = K\begin{pmatrix} 9 \\ 19 \end{pmatrix}$ for some constant K.

[1 mark]

(iii) What do your answers to part (i) and (ii) show ?

[1 mark]

(iv) Show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel for all other vectors $\boldsymbol{E}$ and $\boldsymbol{Z}$.

[2 marks]

Question 2

Two vectors, $\boldsymbol{E}$ and $\boldsymbol{T}$, are given by $\boldsymbol{E} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$ and $\boldsymbol{T} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(i) Write $\boldsymbol{p} = 8\,\boldsymbol{E} - 20\,\boldsymbol{T}$ in the form $\boldsymbol{p} = k\begin{pmatrix} 4 \\ -5 \end{pmatrix}$ for some constant k.

[1 mark]

(ii) Write $\boldsymbol{h} = 6\,\boldsymbol{E} - 15\,\boldsymbol{T}$ in the form $\boldsymbol{h} = K\begin{pmatrix} 4 \\ -5 \end{pmatrix}$ for some constant K.

[1 mark]

(iii) Show that $\boldsymbol{p}$ and $\boldsymbol{h}$ are parallel for all vectors $\boldsymbol{E}$ and $\boldsymbol{T}$.

[2 marks]

Question 3

Two vectors, ***X*** and ***Y*** are parallel if one can be written as a multiple of the other.
In each question decide if the two vectors given are parallel or not.
For those that are parallel, write in the form ***X*** = *k* ***Y*** for some constant *k*.

(i) $\boldsymbol{X} = \begin{pmatrix} -15 \\ 9 \end{pmatrix}$ $\boldsymbol{Y} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$

[1 mark]

(ii) $\boldsymbol{X} = 4\boldsymbol{i} - 2\boldsymbol{j}$ $\boldsymbol{Y} = 4\boldsymbol{i} + 2\boldsymbol{j}$

[1 mark]

(iii) $\boldsymbol{X} = \begin{pmatrix} 8 \\ -4 \\ 6 \end{pmatrix}$ $\boldsymbol{Y} = \begin{pmatrix} 12 \\ -6 \\ 9 \end{pmatrix}$

[1 mark]

(iv) $\boldsymbol{X} = \begin{pmatrix} 16 \\ 9 \end{pmatrix}$ $\boldsymbol{Y} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

[1 mark]

(v) $\boldsymbol{X} = \begin{pmatrix} 9 \\ -6 \end{pmatrix}$ $\boldsymbol{Y} = \begin{pmatrix} 12 \\ -8 \end{pmatrix}$

[1 mark]

(vi) $\boldsymbol{X} = 14\boldsymbol{i} - 21\boldsymbol{j}$ $\boldsymbol{Y} = 21\boldsymbol{i} - 14\boldsymbol{j}$

[1 mark]

(vii) $\boldsymbol{X} = 3\,\boldsymbol{i} - \boldsymbol{j}$ $\boldsymbol{Y} = -6\,\boldsymbol{i} + 2\,\boldsymbol{j}$

[1 mark]

(viii) $\boldsymbol{X} = 4\,\boldsymbol{a} - 12\,\boldsymbol{b}$ $\boldsymbol{Y} = 6\,\boldsymbol{a} - 18\,\boldsymbol{b}$

[1 mark]

(ix) $\boldsymbol{X} = 15\,\boldsymbol{a}$ $\boldsymbol{Y} = 16\,\boldsymbol{a}$

[1 mark]

(x) $\boldsymbol{X} = \boldsymbol{i} - \boldsymbol{j}$ $\boldsymbol{Y} = -\boldsymbol{i} + \boldsymbol{j}$

[1 mark]

(xi) $\boldsymbol{X} = \begin{pmatrix} 1 \\ -7 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ $\boldsymbol{Y} = \begin{pmatrix} -2 \\ 14 \\ 6 \\ 0 \\ -22 \end{pmatrix}$ Five dimensions is blowing my mind !

[1 mark]

(xii) $\boldsymbol{X} = \dfrac{1}{2}\,a + 2\,b$ $\boldsymbol{Y} = \dfrac{1}{3}\,a + 3\,b$

[1 mark]

Question 4

#VectorsFascinatingFact

Here is a fascinating fact about vectors !

The vectors $\boldsymbol{X} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\boldsymbol{Y} = \begin{pmatrix} c \\ d \end{pmatrix}$

are mutually perpendicular (each at 90° to the other) if and only if $ac + bd = 0$

(a) For each of the following pairs of vectors state if they are mutually perpendicular or not.

(i) $\boldsymbol{X} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ and $\boldsymbol{Y} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$

[1 mark]

(ii) $\boldsymbol{X} = \begin{pmatrix} 0 \\ 17 \end{pmatrix}$ and $\boldsymbol{Y} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

[1 mark]

(iii) $\boldsymbol{X} = \begin{pmatrix} 0.5 \\ 16 \end{pmatrix}$ and $\boldsymbol{Y} = \begin{pmatrix} 8 \\ 0.25 \end{pmatrix}$

[1 mark]

(b) Given that the following two vectors are mutually perpendicular.

$\boldsymbol{X} = \begin{pmatrix} -6 \\ 11 \end{pmatrix}$ $\boldsymbol{Y} = \begin{pmatrix} w \\ 9 \end{pmatrix}$

Find the value of w.

[2 marks]

Question 5

Consider the vector $\boldsymbol{X} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

(i) Show that $|\boldsymbol{X}| = 5$

[1 mark]

(ii) Write down a vector of magnitude 5 which is perpendicular to $\boldsymbol{X}$

[2 marks]

(iii) Write down another vector of magnitude 5 which is perpendicular to $\boldsymbol{X}$

[2 marks]

Question 6

Let $\boldsymbol{a} = 5\,\boldsymbol{p} + 4\,\boldsymbol{q}$ $\quad \boldsymbol{b} = -\,2\,\boldsymbol{p} + 2\,\boldsymbol{q}$ $\quad$ and $\quad \boldsymbol{c} = \boldsymbol{p} + 6\,\boldsymbol{q}$

(i) Find an expression for vector, $\boldsymbol{v}$, in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$, if $\boldsymbol{v} = 5\,\boldsymbol{a} - \boldsymbol{b}$
Your expression should not contain any brackets.

HINT : $\boldsymbol{v} = 5 \quad \boldsymbol{a} \quad - \quad \boldsymbol{b}$
$\boldsymbol{v} = 5\,(\,5\,\boldsymbol{p} + 4\,\boldsymbol{q}\,) - (\,-\,2\,\boldsymbol{p} + 2\,\boldsymbol{q}\,)$ Be careful with the double minus.

[1 marks]

(ii) Find an expression for vector, $\boldsymbol{w}$, in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$, if $\boldsymbol{w} = 8\boldsymbol{a} - \boldsymbol{c}$.
Your expression should not contain any brackets.

[1 marks]

(iii) Prove that the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel.

[2 marks]

Question 7

Let $\boldsymbol{d} = \boldsymbol{p} + 3\,\boldsymbol{q}$ $\boldsymbol{e} = \boldsymbol{p} - 3\,\boldsymbol{q}$ and $\boldsymbol{f} = 2\,\boldsymbol{p} - \boldsymbol{q}$

(i) Find an expression for vector, $\boldsymbol{r}$, in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ only, if $\boldsymbol{r} = 3\,\boldsymbol{d} + 6\,\boldsymbol{e}$.
Your expression should not contain any brackets.

[1 marks]

(ii) Find an expression for vector, $\boldsymbol{s}$, in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ only, if $\boldsymbol{s} = -\,2\,\boldsymbol{d} + 8\,\boldsymbol{f}$.
Your expression should not contain any brackets.

[1 marks]

(iii) Find an expression for vector, $\boldsymbol{t}$, in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ only, if $\boldsymbol{t} = 4\,\boldsymbol{e} - 3\,\boldsymbol{f}$.
Your expression should not contain any brackets.

[1 mark]

(iv) Which of the vectors $\boldsymbol{r}$, $\boldsymbol{s}$ and $\boldsymbol{t}$ are parallel ?
Prove your answer.

[2 marks]

7.4 Answers

7.4.1 Solution to 7.2 Example (Teaching Video)

(i)

$$\boldsymbol{s} = 6\boldsymbol{A} + 4\boldsymbol{B} = 6\begin{pmatrix}2\\5\end{pmatrix} + 4\begin{pmatrix}7\\3\end{pmatrix}$$
$$= \begin{pmatrix}12\\30\end{pmatrix} + \begin{pmatrix}28\\12\end{pmatrix}$$
$$= \begin{pmatrix}40\\42\end{pmatrix}$$
$$= 2\begin{pmatrix}20\\21\end{pmatrix}$$

[1 mark]

(ii)

$$\boldsymbol{t} = 9\boldsymbol{A} + 6\boldsymbol{B} = 9\begin{pmatrix}2\\5\end{pmatrix} + 6\begin{pmatrix}7\\3\end{pmatrix}$$
$$= \begin{pmatrix}18\\45\end{pmatrix} + \begin{pmatrix}42\\18\end{pmatrix}$$
$$= \begin{pmatrix}60\\63\end{pmatrix}$$
$$= 3\begin{pmatrix}20\\21\end{pmatrix}$$

[1 mark]

(iii) Parts (i) and (ii) show that $\boldsymbol{s}$ and $\boldsymbol{t}$ are parallel.

[1 mark]

(iv) $\boldsymbol{s} = 6\,\boldsymbol{A} + 4\,\boldsymbol{B}$
$= 2\,(\,3\,\boldsymbol{A} + 2\,\boldsymbol{B}\,)$

$\boldsymbol{t} = 9\,\boldsymbol{A} + 6\,\boldsymbol{B}$
$= 3\,(\,3\,\boldsymbol{A} + 2\,\boldsymbol{B}\,)$

The $3\,\boldsymbol{A} + 2\,\boldsymbol{B}$ is both expressions shows that $\boldsymbol{s}$ and $\boldsymbol{t}$ are parallel regardless of the values of $\boldsymbol{A}$ and $\boldsymbol{B}$.

[2 marks]

7.4.2 Solutions to 7.3 Exercise

Answer 1

(i) $3\begin{pmatrix}9\\19\end{pmatrix}$ **(ii)** $5\begin{pmatrix}9\\19\end{pmatrix}$

[1, 1 mark]

(iii) Parts (i) and (ii) show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

[1 mark]

(iv) $\boldsymbol{a} = 9\,\boldsymbol{E} + 3\,\boldsymbol{Z}$
$= 3\,(\,3\,\boldsymbol{E} + \boldsymbol{Z}\,)$

$\boldsymbol{b} = 15\,\boldsymbol{E} + 5\,\boldsymbol{Z}$
$= 5\,(\,3\,\boldsymbol{E} + \boldsymbol{Z}\,)$

The $3\boldsymbol{E} + \boldsymbol{Z}$ in both expressions shows that $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel regardless of the values of $\boldsymbol{E}$ and $\boldsymbol{Z}$.

[2 marks]

Answer 2

(i) $4\begin{pmatrix} 4 \\ -5 \end{pmatrix}$

(ii) $3\begin{pmatrix} 4 \\ -5 \end{pmatrix}$

[1, 1 mark]

(iii) $\boldsymbol{p} = 8\,\boldsymbol{E} - 20\,\boldsymbol{T}$ $\quad$ $\boldsymbol{h} = 6\,\boldsymbol{E} - 15\,\boldsymbol{T}$

$= 4\,(2\,\boldsymbol{E} - 5\,\boldsymbol{T})$ $\quad$ $= 3\,(2\,\boldsymbol{E} - 5\,\boldsymbol{T})$

The $2\,\boldsymbol{E} - 5\,\boldsymbol{T}$ in both expressions shows that $\boldsymbol{p}$ and $\boldsymbol{h}$ are parallel regardless of the values of $\boldsymbol{E}$ and $\boldsymbol{T}$.

[2 marks]

Answer 3

(i) $\begin{pmatrix} -15 \\ 9 \end{pmatrix} = 3\begin{pmatrix} -5 \\ 3 \end{pmatrix}$ $\quad \therefore\ \boldsymbol{X} = 3\,\boldsymbol{Y} \quad \therefore$ parallel

[1 mark]

(ii) Not parallel

[1 mark]

(iii) $\begin{pmatrix} 8 \\ -4 \\ 6 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 12 \\ -6 \\ 9 \end{pmatrix}$ $\quad \therefore\ \boldsymbol{X} = \frac{2}{3}\,\boldsymbol{Y} \quad \therefore$ parallel

[1 mark]

(iv) Not parallel

[1 mark]

(v) $\begin{pmatrix} 9 \\ -6 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 12 \\ -8 \end{pmatrix}$ $\quad \therefore\ \boldsymbol{X} = \frac{3}{4}\,\boldsymbol{Y} \quad \therefore$ parallel

[1 mark]

(vi) Not parallel

[1 mark]

(vii) $(3\boldsymbol{i} - \boldsymbol{j}) = -\frac{1}{2}\,(-6\boldsymbol{i} + 2\boldsymbol{j})$ $\quad \therefore\ \boldsymbol{X} = -\frac{1}{2}\,\boldsymbol{Y} \quad \therefore$ parallel

[1 mark]

(viii) $(4\,\boldsymbol{a} - 12\,\boldsymbol{b}) = \frac{2}{3}\,(6\,\boldsymbol{a} - 18\,\boldsymbol{b})$ $\quad \therefore\ \boldsymbol{X} = \frac{2}{3}\,\boldsymbol{Y} \quad \therefore$ parallel

[1 mark]

(ix) $15\,\boldsymbol{a} = \frac{15}{16}\,16\,\boldsymbol{a}$ $\quad \therefore\ \boldsymbol{X} = \frac{15}{16}\,\boldsymbol{Y} \quad \therefore$ parallel

[1 mark]

(x) $\boldsymbol{i} - \boldsymbol{j} = -1\,(-\boldsymbol{i} + \boldsymbol{j})$ $\quad \therefore\ \boldsymbol{X} = -\boldsymbol{Y} \quad \therefore$ parallel

[1 mark]

(xi) Not parallel

[1 mark]

(xii) Not parallel

[1 mark]

Answer 4

(i) Yes. They are perpendicular

(ii) Yes. They are perpendicular

(iii) No. They are NOT perpendicular

(iv) $w = 16.5$

[1, 1, 1, 2 marks]

Answer 5

(i) By the Theorem of Pythagoras

(ii) $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ **(iii)** $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

[1, 2, 2 marks]

Answer 6

(i) $\boldsymbol{v} = 9\,(\,3\,\boldsymbol{p} + 2\,\boldsymbol{q}\,)$ **(ii)** $\boldsymbol{w} = 13\,(\,3\,\boldsymbol{p} + 2\,\boldsymbol{q}\,)$

(iii) The $3\,\boldsymbol{p} + 2\,\boldsymbol{q}$ in both expressions shows that $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel regardless of the values of $\boldsymbol{p}$ and $\boldsymbol{q}$.

[1, 1, 2 marks]

Answer 7

(i) $\boldsymbol{r} = 9\,(\,\boldsymbol{p} - \boldsymbol{q}\,)$

(ii) $\boldsymbol{s} = 14\,(\,\boldsymbol{p} - \boldsymbol{q}\,)$

(iii) $\boldsymbol{t} = -\,2\,\boldsymbol{p} - 9\,\boldsymbol{q}$

[1, 1, 1 mark]

(iv) The $\boldsymbol{p} - \boldsymbol{q}$ in the expressions shows that $\boldsymbol{r}$ and $\boldsymbol{s}$ are parallel regardless of the values of $\boldsymbol{p}$ and $\boldsymbol{q}$.

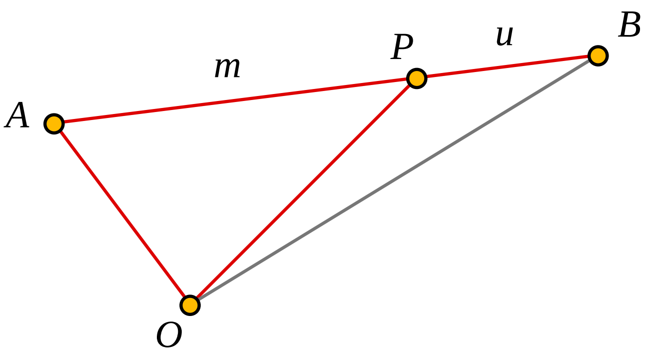
[2 marks]

8.1 The Ratio Theorem

A situation that repeatedly occurs in vectors problems is that of a line *AB* being divided in a given ratio $m : u$. The Ratio Theorem generalises the resulting algebraic manipulations and gives a formula that can be used to skip through this recurring situation at a brisk pace.

The theorem makes use of the simple yet clever idea that for any line *AB* there must be an **O**ther point, *O*, “somewhere”. This other point is often the origin, but it does not have to be and, indeed, part of the skill of using the ratio theorem quickly is to pick a good “other point”. The other point is often on an associated diagram of which the line *AB* is a part but, again, it does not have to be.

The Ratio Theorem



If the point *P* divides the line segment *AB* in the ratio $m : u$ then,

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{m}{u+m}\overrightarrow{AB}$$

Proof

$$\overrightarrow{AP} : \overrightarrow{PB}$$

$$m : u$$

$$u\,\overrightarrow{AP} = m\,\overrightarrow{PB}$$

$$u\left(\overrightarrow{AO} + \overrightarrow{OP}\right) = m\left(\overrightarrow{PO} + \overrightarrow{OB}\right)$$

$$u\,\overrightarrow{AO} + u\,\overrightarrow{OP} = m\,\overrightarrow{PO} + m\,\overrightarrow{OB}$$

$$u\,\overrightarrow{OP} - m\,\overrightarrow{PO} = -u\,\overrightarrow{AO} + m\,\overrightarrow{OB}$$

$$u\,\overrightarrow{OP} + m\,\overrightarrow{OP} = u\,\overrightarrow{OA} + m\left(\overrightarrow{OA} + \overrightarrow{AB}\right)$$

$$(u+m)\,\overrightarrow{OP} = (u+m)\,\overrightarrow{OA} + m\,\overrightarrow{AB}$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{m}{u+m}\overrightarrow{AB} \qquad \square$$

8.2 Example
A line segment AB has endpoints $A(1, 4)$ and $B(11, 19)$
A point P on the line segment AB is such that $AP : PB = 3 : 2$.
Find the coordinates of P

Teaching Video : http://www.NumberWonder.co.uk/v9009/8.mp4

After watching the video
write out your solution.

☞

[3 marks]

To help remember the ratio theorem notice that *mu* is the noise made by a cat

and the fraction in the theorem $\dfrac{m}{u + m}$ is a "sort of" spelling of the word *mum*

8.3 Exercise

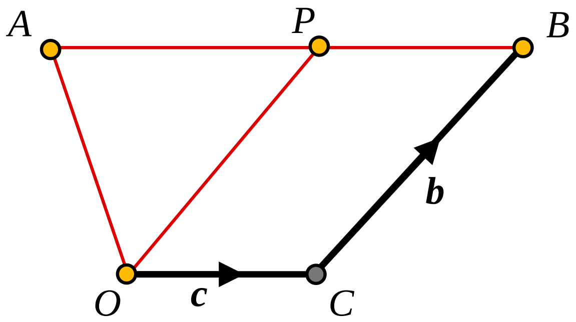
Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 40

Question 1

A line segment AB has endpoints $A(7, 3)$ and $B(42, 59)$
A point P on the line segment AB is such that $AP : PB = 4 : 3$
Find the coordinates of P

[3 marks]

Question 2



$OABC$ is a trapezium with AB parallel to OC and $AB = 5\,OC$.
P divides AB such that $AP : PB = 3 : 2$

$$\overrightarrow{OC} = \boldsymbol{c} \quad \text{and} \quad \overrightarrow{CB} = \boldsymbol{b}$$

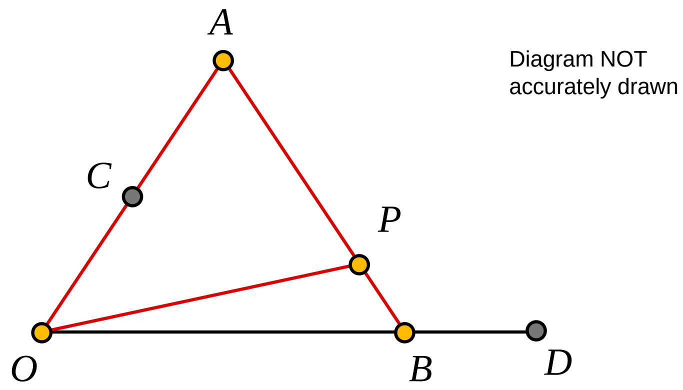
(i) Find $\overrightarrow{OA}$ in terms of $\boldsymbol{c}$ and $\boldsymbol{b}$

[1 mark]

(ii) By using the Ratio Theorem, find $\overrightarrow{OP}$ in terms of $\boldsymbol{c}$ and $\boldsymbol{b}$

[3 marks]

Question 3



OAB is a triangle with $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

C is the midpoint of OA and P is the point on AB such that $AP : PB = 3 : 1$

D is the point such that $\overrightarrow{OB} = 2\,\overrightarrow{BD}$

(i) Use the Ratio Theorem to find $\overrightarrow{OP}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[2 marks]

(ii) Use $\overrightarrow{CP} = \overrightarrow{CO} + \overrightarrow{OP}$ to find $\overrightarrow{CP}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

(iii) Use $\overrightarrow{PD} = \overrightarrow{PO} + \overrightarrow{OD}$ to find $\overrightarrow{PD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

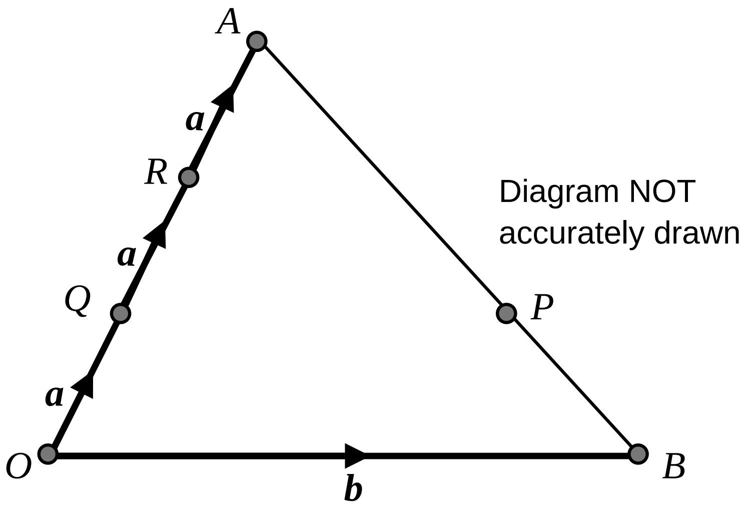
(iv) Prove that the points C, P and D lie on the same straight line

[1 mark]

(v) Determine the ratio $CP : PD$

[1 mark]

Question 4



OAB is a triangle in which $\overrightarrow{OA} = 3\boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

Q is the point on *OA* such that $OA = 3\,OQ$

P is the point on *AB* such that $AB = 3\,PB$

(i) Show how to use the Ratio Theorem to express $\overrightarrow{OP}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[3 marks]

(ii) Show that $\overrightarrow{QP} = k\,\overrightarrow{OB}$ where k is an integer

[2 marks]

(iii) State two things that your answer to part (ii) tells you about the relationship between the line segments *QP* and *OB*.

[2 marks]

Question 5

Specimen GCSE Examination Question for the 2018 Examinations

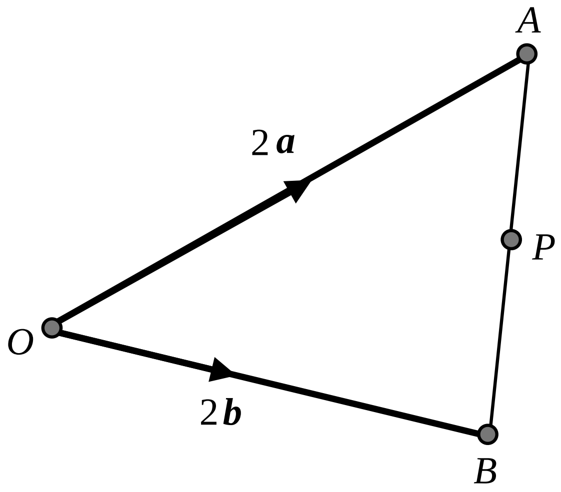


Diagram NOT accurately drawn

OAB is a triangle with $\overrightarrow{OA} = 2\boldsymbol{a}$ and $\overrightarrow{OB} = 2\boldsymbol{b}$

P is the point on *AB* such that $AP : PB = 5 : 3$

$\overrightarrow{OP} = k\ (3\boldsymbol{a} + 5\boldsymbol{b})$ where *k* is a scalar quantity

Find the value of *k*

[3 marks]

Question 6

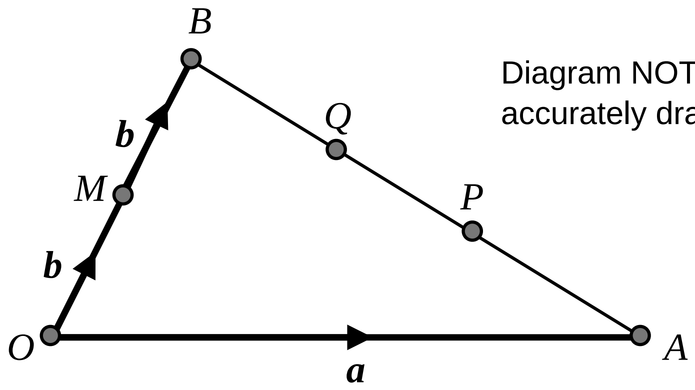


Diagram NOT accurately drawn

OAB is a triangle where *M* is the mid-point of *OB*
P and *Q* are points on *AB* such that $AP = PQ = QB$
$\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = 2\boldsymbol{b}$

(a) Find, in terms of ***a*** and ***b***, expressions for

(i) $\overrightarrow{BA}$

[1 mark]

(ii) $\overrightarrow{MQ}$

[2 marks]

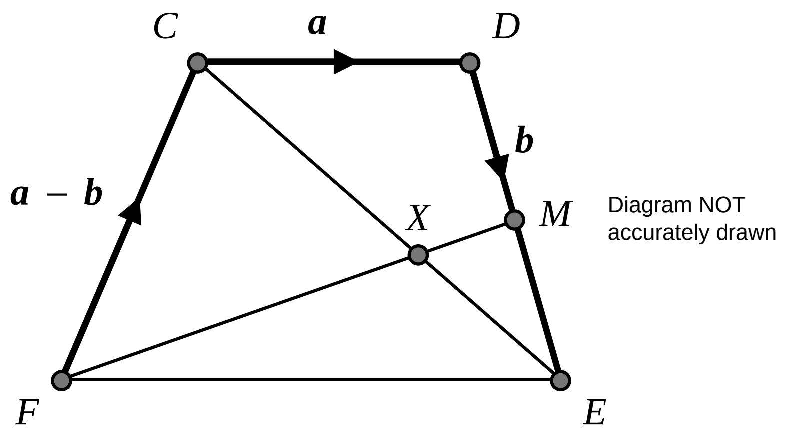
(iii) $\overrightarrow{OP}$

[2 marks]

(b) What can you deduce about quadrilateral *OMQP* ?
Give a reason for your answer

[2 marks]

Question 7



$CDEF$ is a quadrilateral with $\overrightarrow{CD} = \boldsymbol{a}$ $\overrightarrow{DE} = \boldsymbol{b}$ and $\overrightarrow{FC} = \boldsymbol{a} - \boldsymbol{b}$

(i) Express $\overrightarrow{CE}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

(ii) Prove that $\overrightarrow{FE}$ is parallel to $\overrightarrow{CD}$

[1 mark]

M is the midpoint of *DE*

(iii) Express $\overrightarrow{FM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

X is the point on *FM* such that $FX : XM = 4 : 1$

(iv) Prove that *C*, *X* and *E* lie on the same straight line

[3 marks]

Question 8

GCSE Examination Question from May 2014, Paper 3HR, Q21

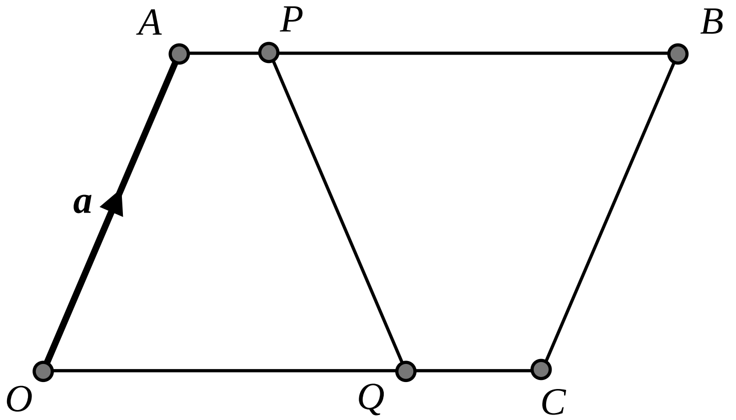


Diagram NOT accurately drawn

$OABC$ is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$

P is the point on AB such that $AP : PB = 1 : 3$

Q is the point on OC such that $OQ : QC = 2 : 1$

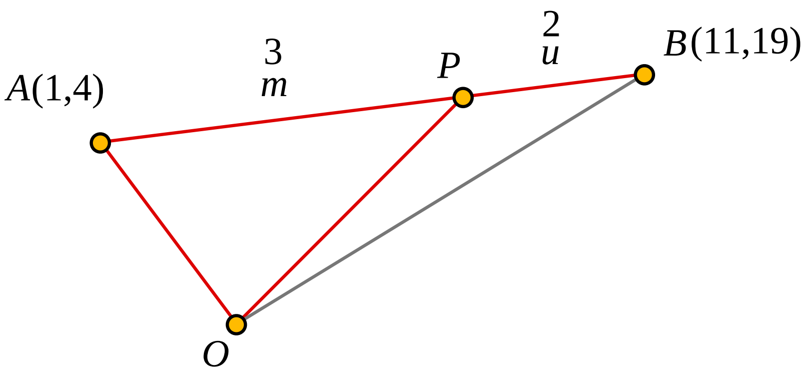
Find, in terms of $\mathbf{a}$ and $\mathbf{c}$, $\overrightarrow{PQ}$

Give your answer in its simplest form.

[4 marks]

8.4 Answers

8.4.1 Solution to 8.2 Example (Teaching Video)



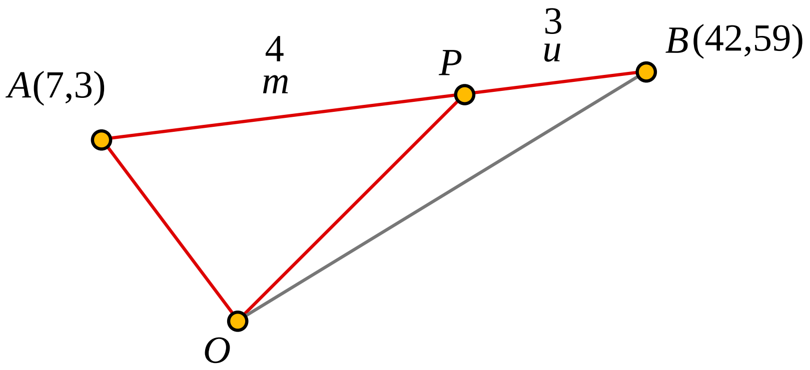
$$
\begin{aligned}
\overrightarrow{OP} &= \overrightarrow{OA} + \frac{m}{u+m}\overrightarrow{AB} \\
&= \overrightarrow{OA} + \frac{m}{u+m}\left(-\overrightarrow{OA} + \overrightarrow{OB}\right) \\
&= \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \frac{3}{2+3}\left(-\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 11 \\ 19 \end{pmatrix}\right) \\
&= \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 10 \\ 15 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 9 \end{pmatrix} \\
&= \begin{pmatrix} 7 \\ 13 \end{pmatrix}
\end{aligned}
$$

$\therefore\ P\ (7,\ 13)$

[3 marks]

8.4.2 Solutions to 8.3 Exercise

Answer 1



$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{m}{u+m}\overrightarrow{AB}$$

$$= \overrightarrow{OA} + \frac{m}{u+m}\left(-\overrightarrow{OA} + \overrightarrow{OB}\right)$$

$$= \begin{pmatrix} 7 \\ 3 \end{pmatrix} + \frac{4}{3+4}\left(-\begin{pmatrix} 7 \\ 3 \end{pmatrix} + \begin{pmatrix} 42 \\ 59 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 7 \\ 3 \end{pmatrix} + \frac{4}{7}\begin{pmatrix} 35 \\ 56 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 3 \end{pmatrix} + \begin{pmatrix} 20 \\ 32 \end{pmatrix}$$

$$= \begin{pmatrix} 27 \\ 35 \end{pmatrix}$$

$\therefore\ P\ (27,\ 35)$

[3 marks]

Answer 2

(i) $\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{CB} + \overrightarrow{BA}$

$= \boldsymbol{c} + \boldsymbol{b} - 5\boldsymbol{c}$

$= \boldsymbol{b} - 4\boldsymbol{c}$

[1 mark]

(ii) $\overrightarrow{OP} = \overrightarrow{OA} + \dfrac{m}{u+m}\overrightarrow{AB}$

$= \boldsymbol{b} - 4\boldsymbol{c} + \dfrac{3}{2+3} \times 5\boldsymbol{c}$

$= \boldsymbol{b} - 4\boldsymbol{c} + \dfrac{3}{5} \times 5\boldsymbol{c}$

$= \boldsymbol{b} - 4\boldsymbol{c} + 3\boldsymbol{c}$

$= \boldsymbol{b} - \boldsymbol{c}$

[3 marks]

Answer 3

(i) $\overrightarrow{OP} = \overrightarrow{OA} + \dfrac{m}{u + m}\overrightarrow{AB}$

$= \boldsymbol{a} + \dfrac{3}{1 + 3}(-\boldsymbol{a} + \boldsymbol{b})$

$= \boldsymbol{a} + \dfrac{3}{4}(-\boldsymbol{a} + \boldsymbol{b})$

$= \boldsymbol{a} - \dfrac{3}{4}\boldsymbol{a} + \dfrac{3}{4}\boldsymbol{b}$

$= \dfrac{1}{4}\boldsymbol{a} + \dfrac{3}{4}\boldsymbol{b}$

[2 marks]

(ii) $\overrightarrow{CP} = \overrightarrow{CO} + \overrightarrow{OP}$

$= \dfrac{1}{2}\overrightarrow{AO} + \overrightarrow{OP}$

$= -\dfrac{1}{2}\overrightarrow{OA} + \overrightarrow{OP}$

$= -\dfrac{1}{2}\boldsymbol{a} + \dfrac{1}{4}\boldsymbol{a} + \dfrac{3}{4}\boldsymbol{b}$

$= -\dfrac{1}{4}\boldsymbol{a} + \dfrac{3}{4}\boldsymbol{b}$

[1 mark]

(iii) $\overrightarrow{PD} = \overrightarrow{PO} + \overrightarrow{OD}$

$= -\overrightarrow{OP} + \overrightarrow{OB} + \overrightarrow{BD}$

$= -\overrightarrow{OP} + \dfrac{3}{2}\overrightarrow{OB}$

$= -\left(\dfrac{1}{4}\boldsymbol{a} + \dfrac{3}{4}\boldsymbol{b}\right) + \dfrac{3}{2}\boldsymbol{b}$

$= -\dfrac{1}{4}\boldsymbol{a} + \dfrac{3}{4}\boldsymbol{b}$

(iv) As $\overrightarrow{CP} = \overrightarrow{PD}$, *CP* and *PD* are parallel.
They also have a common point, *P*, and so *C*, *P* and *D*
lie on the same straight line □

[1 mark]

(v) *CP* : *PD* = 1 : 1
P is the midpoint of *CD*

[1 mark]

Answer 4

(i) $\overrightarrow{OP} = \overrightarrow{OA} + \dfrac{m}{u + m}\overrightarrow{AB}$

$= 3\boldsymbol{a} + \dfrac{2}{1 + 2}\left(-3\boldsymbol{a} + \boldsymbol{b}\right)$

$= 3\boldsymbol{a} + \dfrac{2}{3}\left(-3\boldsymbol{a} + \boldsymbol{b}\right)$

$= 3\boldsymbol{a} - 2\boldsymbol{a} + \dfrac{2}{3}\boldsymbol{b}$

$= \boldsymbol{a} + \dfrac{2}{3}\boldsymbol{b}$

[3 marks]

(ii) $\overrightarrow{QP} = \overrightarrow{QO} + \overrightarrow{OP}$

$= -\boldsymbol{a} + \boldsymbol{a} + \dfrac{2}{3}\boldsymbol{b}$

$= \dfrac{2}{3}\boldsymbol{b}$

$= \dfrac{2}{3}\overrightarrow{OB} \qquad \square$

[2 marks]

(iii) QP is parallel to OB

QP is $\dfrac{2}{3}$ of the length of OB

[2 marks]

Answer 5

By the Ratio Theorem, $\overrightarrow{OP} = \overrightarrow{OA} + \dfrac{m}{u + m}\overrightarrow{AB}$

$= 2\boldsymbol{a} + \dfrac{5}{3 + 5}\left(-2\boldsymbol{a} + 2\boldsymbol{b}\right)$

$= 2\boldsymbol{a} + \dfrac{5}{8}\left(-2\boldsymbol{a} + 2\boldsymbol{b}\right)$

$= 2\boldsymbol{a} - \dfrac{5}{4}\boldsymbol{a} + \dfrac{5}{4}\boldsymbol{b}$

$= \dfrac{3}{4}\boldsymbol{a} + \dfrac{5}{4}\boldsymbol{b}$

$= \dfrac{1}{4}(3\boldsymbol{a} + 5\boldsymbol{b})$

$\therefore\ k = \dfrac{1}{4}$

[3 marks]

Answer 6

(a) **(i)** $\overrightarrow{BA} = \boldsymbol{a} - 2\boldsymbol{b}$

[1 mark]

(ii) By the Ratio Theorem,

$$\overrightarrow{MQ} = \overrightarrow{MB} + \frac{m}{u + m}\overrightarrow{BA}$$
$$= \boldsymbol{b} + \frac{1}{3}\left(-2\boldsymbol{b} + \boldsymbol{a}\right)$$
$$= \boldsymbol{b} - \frac{2}{3}\boldsymbol{b} + \frac{1}{3}\boldsymbol{a}$$
$$= \frac{1}{3}\boldsymbol{b} + \frac{1}{3}\boldsymbol{a}$$
$$= \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$$

[2 marks]

(iii) By the Ratio Theorem,

$$\overrightarrow{OP} = \overrightarrow{OB} + \frac{m}{u + m}\overrightarrow{BA}$$
$$= 2\boldsymbol{b} + \frac{2}{3}(-2\boldsymbol{b} + \boldsymbol{a})$$
$$= 2\boldsymbol{b} - \frac{4}{3}\boldsymbol{b} + \frac{2}{3}\boldsymbol{a}$$
$$= \frac{2}{3}\boldsymbol{b} + \frac{2}{3}\boldsymbol{a}$$
$$= \frac{2}{3}(\boldsymbol{a} + \boldsymbol{b})$$

[2 marks]

(b) As $\overrightarrow{OP} = 2\overrightarrow{MQ}$, OP is parallel to MQ

$\therefore$ $OMQP$ is a trapezium

[2 marks]

Answer 7

(i) $\overrightarrow{CE} = \boldsymbol{a} + \boldsymbol{b}$

[1 mark]

(ii) $\overrightarrow{FE} = \overrightarrow{FC} + \overrightarrow{CE}$

$= \boldsymbol{a} - \boldsymbol{b} + \boldsymbol{a} + \boldsymbol{b}$ using part (i)

$= 2\,\boldsymbol{a}$

$\therefore\ \overrightarrow{FE} = 2\,\overrightarrow{CD}$ and so $\overrightarrow{FE}$ and $\overrightarrow{CD}$ are parallel ☐

[1 mark]

(iii) $\overrightarrow{FM} = \overrightarrow{FC} + \overrightarrow{CD} + \overrightarrow{DM}$

$$= \overrightarrow{FC} + \overrightarrow{CD} + \frac{1}{2}\overrightarrow{DE}$$

$$= \boldsymbol{a} - \boldsymbol{b} + \boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$$

$$= 2\,\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}$$

[1 mark]

(iv) By the Ratio Theorem, using $FX : XM = 4 : 1$

$$\overrightarrow{EX} = \overrightarrow{EF} + \frac{m}{u + m}\overrightarrow{FM}$$

$$= -2\,\boldsymbol{a} + \frac{4}{1 + 4}\left(2\,\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}\right)$$

$$= -2\,\boldsymbol{a} + \frac{4}{5}\left(2\,\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}\right)$$

$$= -2\,\boldsymbol{a} + \frac{8}{5}\boldsymbol{a} - \frac{2}{5}\boldsymbol{b}$$

$$= -\frac{2}{5}\boldsymbol{a} - \frac{2}{5}\boldsymbol{b}$$

$$= -\frac{2}{5}\left(\boldsymbol{a} + \boldsymbol{b}\right)$$

$$= -\frac{2}{5}\overrightarrow{CE}$$

$\therefore$ EX is parallel with CE with a common point E

$\therefore$ C, X and E lie on the same straight line ☐

[3 marks]

Answer 8

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \frac{m}{u + m}\overrightarrow{AB} \\ &= \boldsymbol{a} + \frac{1}{3 + 1}\boldsymbol{c} \\ &= \boldsymbol{a} + \frac{1}{4}\boldsymbol{c}\end{aligned}$$

$$\begin{aligned}\overrightarrow{OQ} &= \frac{2}{3}\overrightarrow{OC} \\ &= \frac{2}{3}\boldsymbol{c}\end{aligned}$$

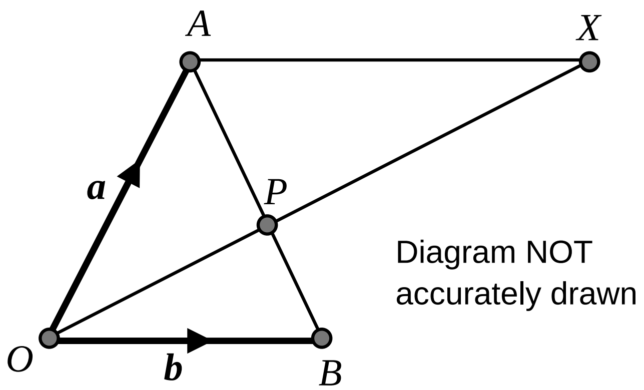
$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\boldsymbol{a} - \frac{1}{4}\boldsymbol{c} + \frac{2}{3}\boldsymbol{c} \\ &= -\boldsymbol{a} + \frac{5}{12}\boldsymbol{c}\end{aligned}$$

[4 marks]

9.1 Line Intersections

Some of the most difficult GCSE vector questions involve finding the position where two lines on a geometric figure intersect. They are hard, partly because they are being done without knowing the mathematical theory of vectors which is not studied until either the Further A-Level mathematics course or at University.

9.2 Example



OAB is a triangle with $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

P is a point on AB such that $AP : PB = 4 : 3$

Given that AX is parallel with OB and that OPX is a straight line, find $\overrightarrow{OX}$

Teaching Video : http://www.NumberWonder.co.uk/v9009/9a.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9009/9b.mp4 (Part 2)

<= Part 1 Part 2 =>

Watch the videos and then answer the question in the space below.

☞

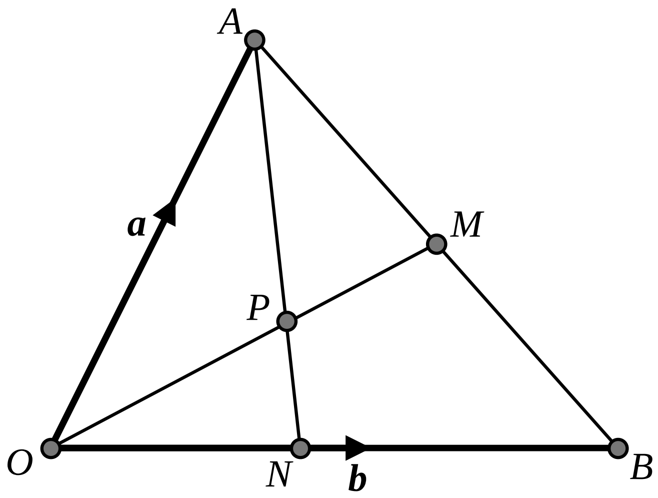
[5 marks]

9.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable. Make the method used clear.

Marks available : 30

Question 1



OAB is a triangle in which $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

M is the midpoint of AB and OPM and APN are straight lines with $OP : PM = 4 : 3$

(i) Work out $\overrightarrow{OM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

(ii) Work out $\overrightarrow{OP}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

(iii) Work out $\overrightarrow{AP}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

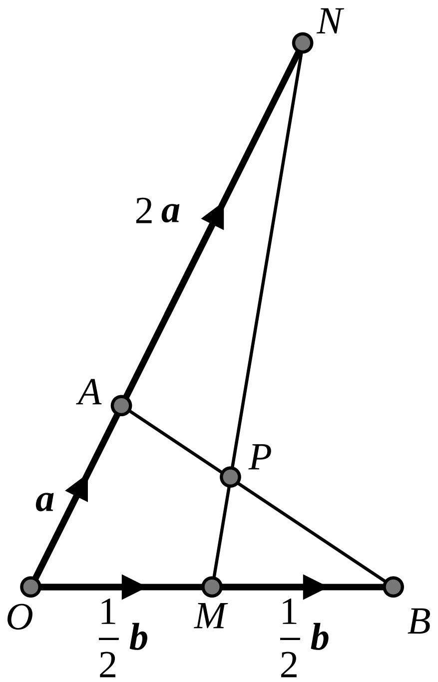
(iv) From the equation $\overrightarrow{AN} = \overrightarrow{AO} + \overrightarrow{ON}$ can be written that,

$s\,\overrightarrow{AP} = -\overrightarrow{OA} + t\,\overrightarrow{OB}$ for some constants s and t

Use this fact to work out the ratio $ON : NB$

[4 marks]

Question 2



In triangle OAB $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

OAN and MPN are straight lines

$OA : AN = 1 : 2$ and $OM : MB = 1 : 1$

(i) Work out $\overrightarrow{AB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

(ii) Work out $\overrightarrow{NM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

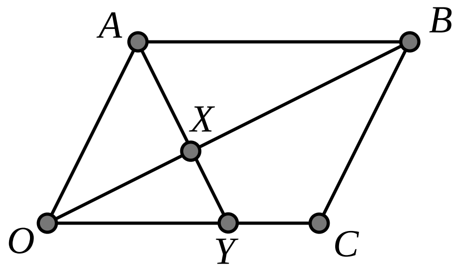
(iii) From the equation $\overrightarrow{AP} = \overrightarrow{AN} + \overrightarrow{NP}$ can be written that,

$s\,\overrightarrow{AB} = \overrightarrow{AN} + t\,\overrightarrow{NM}$ for some constants s and t

Use this fact to work out the ratio $AP : PB$

[3 marks]

Question 3



In parallelogram *OABC*, *Y* is the point on *OC* such that *OY* : *YC* = 2 : 1

$$\overrightarrow{OA} = \boldsymbol{a} \text{ and } \overrightarrow{OC} = \boldsymbol{c}$$

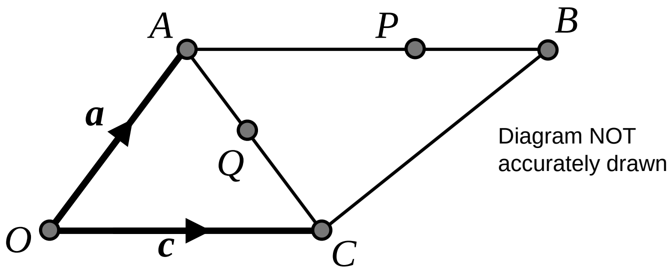
The diagonal *OB* intersects *AY* at *X*.

Calculate the ratio *AX* : *XY*

[6 marks]

Question 4

GCSE Examination Question from May 2019, Paper 1HR, Q24



$$\overrightarrow{OA} = \boldsymbol{a} \qquad \overrightarrow{OC} = \boldsymbol{c} \qquad \overrightarrow{AB} = 2\boldsymbol{c}$$

P is the point on *AB* such that *AP* : *PB* = 3 : 1
Q is the point on *AC* such that *OQP* is a straight line.

Use a vector method to find *AQ* : *QC*
Show your working clearly.

[5 marks]

Question 5

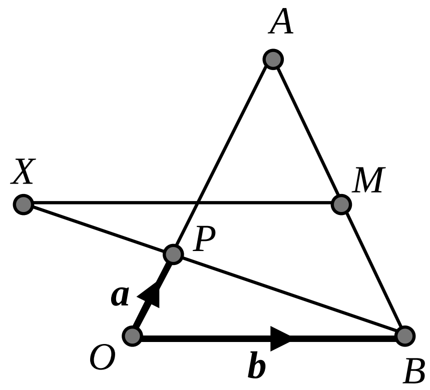


Diagram NOT accurately drawn

In triangle OAB, $\overrightarrow{OP} = \boldsymbol{a}$, $\overrightarrow{OA} = 3\boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

M is the midpoint of AB

(i) Express $\overrightarrow{BP}$ and $\overrightarrow{AB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

(ii) Express $\overrightarrow{MB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

(iii) If X lies on BP produced so that $\overrightarrow{BX} = k\,\overrightarrow{BP}$, express $\overrightarrow{MX}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and k

[3 marks]

(iv) Find the value of k if MX is parallel to BO

[2 marks]

9.4 Answers

9.4.1 Solution to 9.2 Example (Teaching Video)

By the Ratio Theorem,
$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \frac{m}{u+m}\overrightarrow{AB} \\ &= \boldsymbol{a} + \frac{4}{7}(-\boldsymbol{a} + b) \\ &= \frac{3}{7}\boldsymbol{a} + \frac{4}{7}\boldsymbol{b}\end{aligned}$$

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OA} + \overrightarrow{AX} \\ s\left(\frac{3}{7}\boldsymbol{a} + \frac{4}{7}\boldsymbol{b}\right) &= \boldsymbol{a} + t\,\boldsymbol{b} \\ \frac{3s}{7}\boldsymbol{a} + \frac{4s}{7}\boldsymbol{b} &= \boldsymbol{a} + t\,\boldsymbol{b}\end{aligned}$$

$$\therefore\ \frac{3s}{7}\boldsymbol{a} = 1 \Rightarrow s = \frac{7}{3}$$

Also, $\dfrac{4s}{7} = t \Rightarrow t = \dfrac{4}{7} \times \dfrac{7}{3}$

$$t = \frac{4}{3}$$

$$\begin{aligned}\overrightarrow{OX} &= s\left(\frac{3}{7}\boldsymbol{a} + \frac{4}{7}\boldsymbol{b}\right) \\ &= \frac{7}{3}\left(\frac{3}{7}\boldsymbol{a} + \frac{4}{7}\boldsymbol{b}\right) \\ &= \boldsymbol{a} + \frac{4}{3}\boldsymbol{b}\end{aligned}$$

[5 marks]

9.4.2 Solutions to 9.3 Exercise

Answer 1

(i) By the Ratio Theorem, $\overrightarrow{OM} = \overrightarrow{OA} + \dfrac{m}{u + m}\overrightarrow{AB}$

$$= \boldsymbol{a} + \frac{1}{2}\left(-\boldsymbol{a} + \boldsymbol{b}\right)$$

$$= \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$$

[1 mark]

(ii) $\overrightarrow{OP} = \dfrac{4}{7}\overrightarrow{OM} = \dfrac{4}{7}\left(\dfrac{1}{2}\boldsymbol{a} + \dfrac{1}{2}\boldsymbol{b}\right)$

$$= \frac{2}{7}\boldsymbol{a} + \frac{2}{7}\boldsymbol{b}$$

[1 mark]

(iii) $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\boldsymbol{a} + \dfrac{2}{7}\boldsymbol{a} + \dfrac{2}{7}\boldsymbol{b}$

$$= -\frac{5}{7}\boldsymbol{a} + \frac{2}{7}\boldsymbol{b}$$

[1 mark]

(iv)

$$\overrightarrow{AN} = \overrightarrow{AO} + \overrightarrow{ON}$$

$$s\,\overrightarrow{AP} = -\overrightarrow{OA} + t\,\overrightarrow{OB}$$

$$s\left(-\frac{5}{7}\boldsymbol{a} + \frac{2}{7}\boldsymbol{b}\right) = -\boldsymbol{a} + t\,\boldsymbol{b}$$

$$-\frac{5s}{7}\boldsymbol{a} + \frac{2s}{7}\boldsymbol{b} = -\boldsymbol{a} + t\,\boldsymbol{b}$$

$$\therefore\ -\frac{5s}{7}\boldsymbol{a} = -1 \Rightarrow s = \frac{7}{5}$$

$$\text{Also, } \frac{2s}{7} = t \Rightarrow t = \frac{2}{7} \times \frac{7}{5}$$

$$t = \frac{2}{5}$$

$$\overrightarrow{ON} = t\,\boldsymbol{b}$$

$$= \frac{2}{5}\boldsymbol{b}$$

$$\therefore\ ON : NB = 2 : 3$$

[4 marks]

Answer 2

(i) $\overrightarrow{AB} = -\boldsymbol{a} + \boldsymbol{b}$

(ii) $\overrightarrow{NM} = -3\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

(iii)

$$\overrightarrow{AP} = \overrightarrow{AN} + \overrightarrow{NP}$$

$$s(-\boldsymbol{a} + \boldsymbol{b}) = 2\boldsymbol{a} + t\left(-3\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)$$

$$-s\boldsymbol{a} + s\boldsymbol{b} = 2\boldsymbol{a} - 3t\boldsymbol{a} + \frac{t}{2}\boldsymbol{b}$$

$$\therefore \quad -s = 2 - 3t \quad \text{and} \quad s = \frac{t}{2}$$

$$-\frac{t}{2} = 2 - 3t$$

$$-t = 4 - 6t$$

$$5t = 4$$

$$t = \frac{4}{5} \Rightarrow s = \frac{2}{5}$$

$$\therefore AP : PB = 2 : 3$$

[5 marks]

Answer 3

$$\overrightarrow{AY} = -\boldsymbol{a} + \frac{2}{3}\boldsymbol{c} \text{ and } \overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{c}$$

$$\overrightarrow{AX} = \overrightarrow{AO} + \overrightarrow{OX}$$

$$s\overrightarrow{AY} = -\overrightarrow{OA} + t\overrightarrow{OB}$$

$$s\left(-\boldsymbol{a} + \frac{2}{3}\boldsymbol{c}\right) = -\boldsymbol{a} + t(\boldsymbol{a} + \boldsymbol{c})$$

$$-s\boldsymbol{a} + \frac{2s}{3}\boldsymbol{c} = -\boldsymbol{a} + t\boldsymbol{a} + t\boldsymbol{c}$$

$$\therefore \quad -s = -1 + t \quad \text{and} \quad \frac{2s}{3} = t$$

$$-s = -1 + \frac{2s}{3}$$

$$-3s = -3 + 2s$$

$$5s = 3$$

$$s = \frac{3}{5} \Rightarrow t = \frac{2}{5}$$

$$\therefore AX : XY = 3 : 2$$

[6 marks]

Answer 4

$$\overrightarrow{OP} = \boldsymbol{a} + \frac{3}{4} \times 2\,\boldsymbol{c}$$
$$= \boldsymbol{a} + \frac{3}{2}\,\boldsymbol{c}$$

$$\overrightarrow{AC} = -\boldsymbol{a} + \boldsymbol{c}$$

$$\overrightarrow{AQ} = \overrightarrow{AO} + \overrightarrow{OQ}$$
$$s\,\overrightarrow{AC} = -\overrightarrow{OA} + t\,\overrightarrow{OP}$$
$$s(-\boldsymbol{a} + \boldsymbol{c}) = -\boldsymbol{a} + t\left(\boldsymbol{a} + \frac{3}{2}\,\boldsymbol{c}\right)$$
$$-s\,\boldsymbol{a} + s\,\boldsymbol{c} = -\boldsymbol{a} + t\,\boldsymbol{a} + \frac{3t}{2}\,\boldsymbol{c}$$
$$\therefore \quad -s = -1 + t \qquad \text{and} \qquad s = \frac{3t}{2}$$
$$-\frac{3t}{2} = -1 + t$$
$$-3t = -2 + 2t$$
$$5t = 2$$
$$t = \frac{2}{5} \Rightarrow s = \frac{3}{5}$$

$$\overrightarrow{AQ} = s\,\overrightarrow{AC}$$
$$= \frac{3}{5}\,\overrightarrow{AC}$$
$$\therefore\ AQ : QC = 3 : 2$$

[5 marks]

Answer 5

(i) $\overrightarrow{BP} = \boldsymbol{a} - \boldsymbol{b}$ and $\overrightarrow{AB} = -3\boldsymbol{a} + \boldsymbol{b}$

[1 mark]

(ii) $\overrightarrow{MB} = \frac{1}{2}\overrightarrow{AB} = -\frac{3}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

[1 mark]

(iii) $\overrightarrow{MX} = \overrightarrow{MB} + \overrightarrow{BX}$

$= \overrightarrow{MB} + k\,\overrightarrow{BP}$

$= -\frac{3}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + k\,(\boldsymbol{a} - \boldsymbol{b})$

[3 marks]

(iv) $\overrightarrow{MX} = -\frac{3}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + k\,(\boldsymbol{a} - \boldsymbol{b})$

$= \left(-\frac{3}{2} + k\right)\boldsymbol{a} + \left(\frac{1}{2} - k\right)\boldsymbol{b}$

If MX is parallel to BO the $\boldsymbol{a}$ component of $\overrightarrow{MX}$ must be zero.

$$\therefore\ -\frac{3}{2} + k = 0$$

$$k = \frac{3}{2}$$

[2 marks]

Chapter 10

GCSE and **A-Level Pure Mathematics**

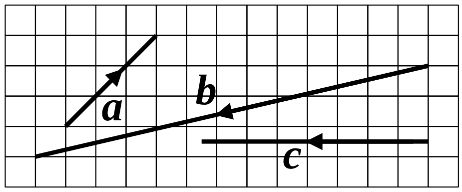
Vectors I

10.1 Revision

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.

Marks available : 50

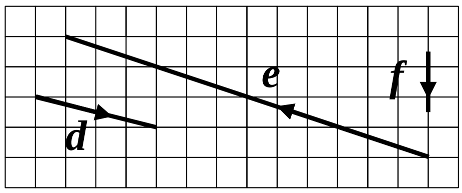
Question 1



Write the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in the form $\begin{pmatrix} p \\ q \end{pmatrix}$ where p and q are integers.

[3 marks]

Question 2



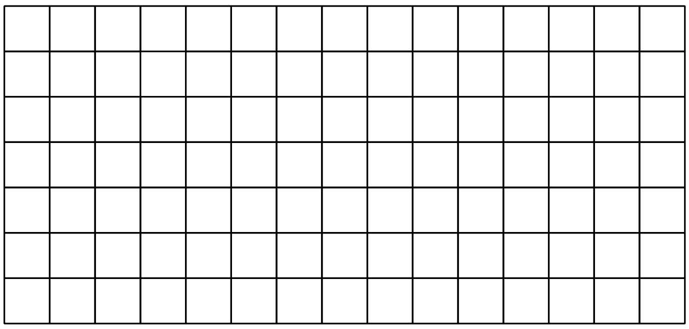
Write the vectors $\boldsymbol{d}$, $\boldsymbol{e}$ and $\boldsymbol{f}$ in the form $p\,\boldsymbol{i} + q\,\boldsymbol{j}$ where p and q are integers.

[3 marks]

Question 3

On the grid draw the following vectors, labelling each with its letter and an arrow.

$$\boldsymbol{a} = \begin{pmatrix} 8 \\ 3 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$$

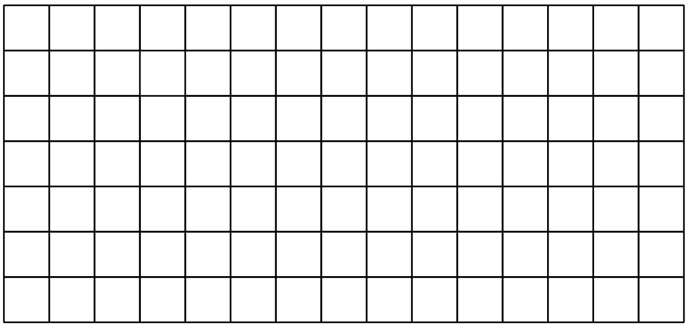


[2 marks]

Question 4

On the grid draw the following vectors, labelling each with its letter and an arrow.

$\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j}$ $\qquad\qquad$ $\boldsymbol{d} = -5\boldsymbol{i} + \boldsymbol{j}$



[2 marks]

Question 5

(i) Find $|\boldsymbol{p}|$ giving the answer in surd form.

$$\boldsymbol{p} = \begin{pmatrix} -7 \\ 10 \end{pmatrix}$$

[2 marks]

(ii) Determine the direction in which the vector $\boldsymbol{p}$ acts.

[2 marks]

Question 6

Circle the two vectors that are parallel;

$$\begin{pmatrix} 18 \\ 75 \end{pmatrix} \qquad \begin{pmatrix} 36 \\ 60 \end{pmatrix} \qquad \begin{pmatrix} -24 \\ -100 \end{pmatrix} \qquad \begin{pmatrix} 20 \\ -80 \end{pmatrix} \qquad \begin{pmatrix} -30 \\ -45 \end{pmatrix}$$

[1 mark]

Question 7

Circle the two vectors that are perpendicular;

$$\begin{pmatrix} 5 \\ 12 \end{pmatrix} \qquad \begin{pmatrix} 12 \\ 5 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ -18 \end{pmatrix} \qquad \begin{pmatrix} 36 \\ -15 \end{pmatrix} \qquad \begin{pmatrix} -24 \\ -10 \end{pmatrix}$$

[1 mark]

Question 8

In polar form, a vector, $\boldsymbol{v}$, has a magnitude of 7.3, and direction 310°

Express $\boldsymbol{v}$ is rectangular form, that is, in the form

$$\boldsymbol{v} = \begin{pmatrix} p \\ q \end{pmatrix}$$

for some values of p and q which you should determine.

[3 marks]

Question 9

Given that;

$$\boldsymbol{f} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} \qquad \boldsymbol{g} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} \qquad \textit{and} \qquad \boldsymbol{h} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

Calculate;

(i) $4\boldsymbol{f} + 3\boldsymbol{g}$

(ii) $\boldsymbol{h} - 5\boldsymbol{g} + \boldsymbol{f}$

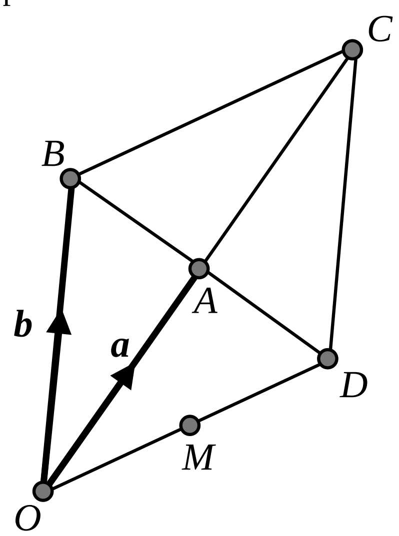
[2, 2 marks]

Question 10

The diagram, which is not drawn to scale, shows a parallelogram *OBCD* with

$$\overrightarrow{OA} = \boldsymbol{a} \quad \text{and} \quad \overrightarrow{OB} = \boldsymbol{b}$$

The point *M* is the mid-point of *OD*.



(a) Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$;

(i) $\overrightarrow{AB} =$ **(ii)** $\overrightarrow{BD} =$

(iii) $\overrightarrow{OD} =$ **(iv)** $\overrightarrow{MA} =$

[4 marks]

(b) **(i)** Given that,

$$\boldsymbol{a} = \begin{pmatrix} 6 \\ 9 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 13 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AB} = \begin{pmatrix} p \\ q \end{pmatrix}$$

determine the values of p and q.

[2 marks]

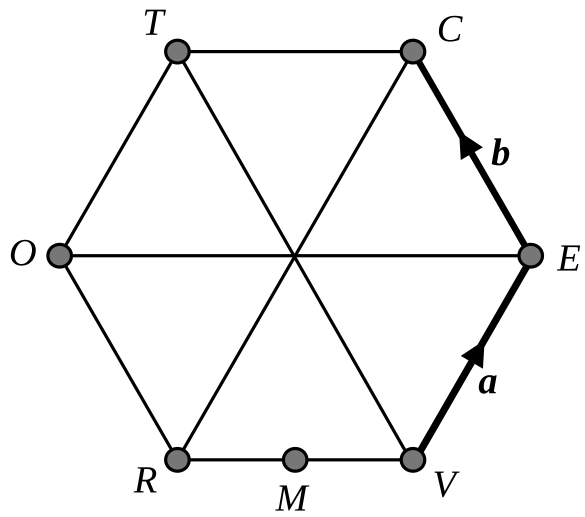
(ii) Hence, or otherwise, show that $\angle\, OAB$ is a right angle.

[2 marks]

Question 11

The diagram, which is not drawn to scale, shows a regular hexagon *VECTOR*.
Each side of the hexagon is of length 4.6 cm.
The point *M* is the mid-point of *RV*.

Furthermore, $\overrightarrow{VE} = \boldsymbol{a}$ and $\overrightarrow{EC} = \boldsymbol{b}$



(a) Express the following vectors in terms of ***a*** and ***b***;

(i) $\overrightarrow{VC} =$

(ii) $\overrightarrow{VO} =$

(iii) $\overrightarrow{VR} =$

(iv) $\overrightarrow{MT} =$

[4 marks]

(b) What is $\left|\overrightarrow{MT}\right|$?

HINT : The Cosine Rule

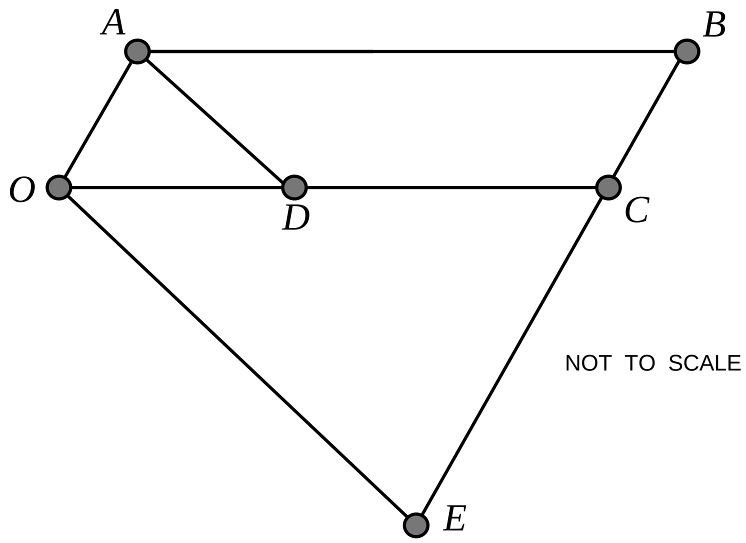
[2 marks]

Question 12

OABC is a parallelogram in which $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$

BCE is a straight line and $\overrightarrow{BE} = 3\overrightarrow{BC}$

D is the midpoint of *OC*.



(a) Write in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$

(i) $\overrightarrow{AD}$

(ii) $\overrightarrow{OE}$

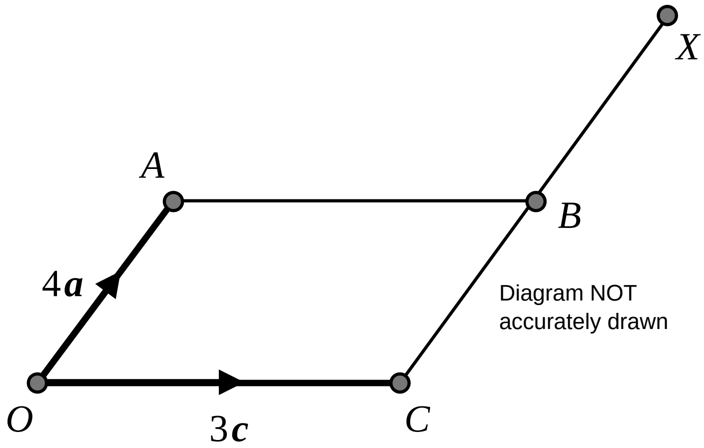
[2 marks]

(b) Deduce the ratio of the lengths of *AD* : *OE*

[2 marks]

Question 13

GCSE Examination Question from January 2018, Paper 3H, Q24



OABC is a parallelogram.

OABC is a parallelogram with $\overrightarrow{OA} = 4\boldsymbol{a}$ and $\overrightarrow{OC} = 3\boldsymbol{c}$

The point *X* is such that *CBX* is a straight line and $CB : BX = 2 : 3$

The point *Y* is such that $\overrightarrow{CY} = 2\overrightarrow{AX}$

Find, in terms of ***a*** and ***c***, the vector $\overrightarrow{OY}$

Give your answer in its simplest form.

[3 marks]

Question 14
#VectorsFascinatingFact
Here is another fascinating fact about vectors !

A vector of magnitude 1 is called a *unit vector*.

(i) Show that the following vector, $\boldsymbol{X}$, is a unit vector;

$$\boldsymbol{X} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$$

[2 marks]

(ii) By first working out the magnitude of the vector, $\boldsymbol{Y}$, write down a *unit vector* that is parallel to $\boldsymbol{Y}$.

$$Y = \begin{pmatrix} 14 \\ 48 \end{pmatrix}$$

[2 marks]

(iii) Find a formulae that will take any vector, $\boldsymbol{Z}$, and convert it into a *unit vector*, where;

$$\boldsymbol{Z} = \begin{pmatrix} p \\ q \end{pmatrix}$$

[2 marks]

10.2 Solutions

Answer 1

$\boldsymbol{a} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ $\boldsymbol{b} = \begin{pmatrix} -13 \\ -3 \end{pmatrix}$ $\boldsymbol{c} = \begin{pmatrix} -7.5 \\ 0 \end{pmatrix}$

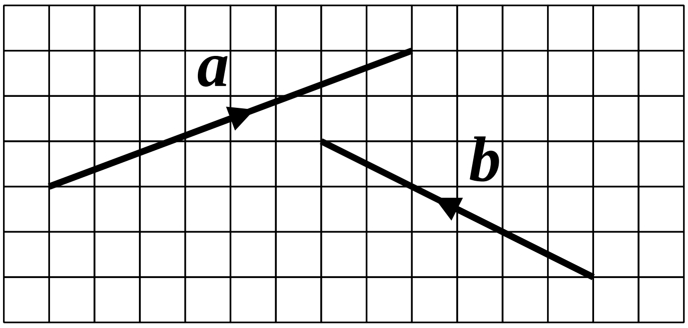
[3 marks]

Answer 2

$\boldsymbol{d} = 4\,\boldsymbol{i} - \boldsymbol{j}$ $\boldsymbol{e} = -12\,\boldsymbol{i} + 4\,\boldsymbol{j}$ $\boldsymbol{f} = -2\,\boldsymbol{j}$

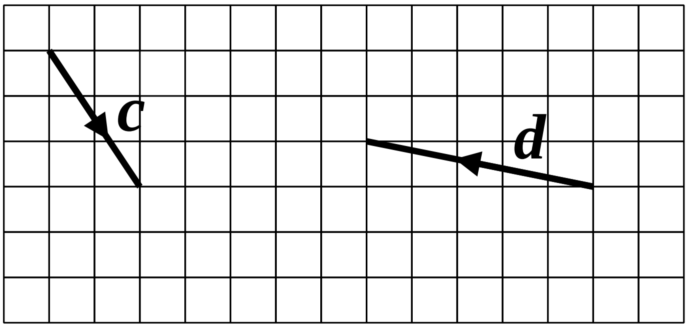
[3 marks]

Answer 3



[2 marks]

Answer 4



[2 marks]

Answer 5

(i) $|\boldsymbol{p}| = \sqrt{149}$ (12.2 but question requests the answer in surd form)

(ii) $180 - 55 = 125°$

[2, 2 marks]

Answer 6

$3\begin{pmatrix} 6 \\ 25 \end{pmatrix}$ $12\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ $-4\begin{pmatrix} 6 \\ 25 \end{pmatrix}$ $20\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ $-15\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

The leftmost and the middle are parallel

[1 mark]

Answer 7

$$\begin{pmatrix} 5 \\ 12 \end{pmatrix} \qquad \begin{pmatrix} 12 \\ 5 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ -18 \end{pmatrix} \qquad \begin{pmatrix} 36 \\ -15 \end{pmatrix} \qquad \begin{pmatrix} -24 \\ -10 \end{pmatrix}$$

$$5 \times 36 + 12 \times (-15) = 0$$

[1 mark]

Answer 8

$$\begin{pmatrix} 4.69 \\ 5.59 \end{pmatrix}$$

[3 marks]

Answer 9

(i) $\begin{pmatrix} 40 \\ 6 \end{pmatrix}$ **(ii)** $\begin{pmatrix} -16 \\ 28 \end{pmatrix}$

[2, 2 marks]

Answer 10

(a) **(i)** $\overrightarrow{AB} = -\boldsymbol{a} + \boldsymbol{b}$ **(ii)** $\overrightarrow{BD} = 2\boldsymbol{a} - 2\boldsymbol{b}$

(iii) $\overrightarrow{OD} = 2\boldsymbol{a} - \boldsymbol{b}$ **(iv)** $\overrightarrow{MA} = \frac{1}{2}\boldsymbol{b}$

[4 marks]

(b) **(i)**

$$\begin{pmatrix} p \\ q \end{pmatrix} = -\begin{pmatrix} 6 \\ 9 \end{pmatrix} + \begin{pmatrix} 0 \\ 13 \end{pmatrix}$$

$$p = -6 \quad \text{and} \quad q = 4$$

[2 marks]

(ii) Consider

$$\overrightarrow{OA} = \begin{pmatrix} 6 \\ 9 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AB} = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$$

And observe that

$$6 \times (-6) + 9 \times 4 = 0$$

$$\therefore \angle OAB \text{ is a right angle} \qquad \square$$

[2 marks]

Answer 11

(a) **(i)** $\overrightarrow{VC} = \boldsymbol{a} + \boldsymbol{b}$ **(ii)** $\overrightarrow{VO} = -\boldsymbol{a} + 2\boldsymbol{b}$

(iii) $\overrightarrow{VR} = -\boldsymbol{a} + \boldsymbol{b}$ **(iv)** $\overrightarrow{MT} = \frac{1}{2}\boldsymbol{a} + \frac{3}{2}\boldsymbol{b}$

[4 marks]

(b) $\left|\overrightarrow{MT}\right|^2 = 2.3^2 + 9.2^2 - 2 \times 2.3 \times 9.2 \times \cos 60°$

$= 8.29 \text{ cm}$

[2 marks]

Answer 12

(a) **(i)** $\overrightarrow{AD} = -\boldsymbol{a} + \frac{1}{2}\boldsymbol{c}$

(ii) $\overrightarrow{OE} = -2\boldsymbol{a} + \boldsymbol{c}$

[2 marks]

(b) 1 : 2

[2 marks]

Answer 13

$CB : BX$

$2 : 3$

$4\boldsymbol{a} : 6\boldsymbol{a} \Rightarrow \overrightarrow{BX} = 6\boldsymbol{a} \Rightarrow \overrightarrow{AX} = 3\boldsymbol{c} + 6\boldsymbol{a} \Rightarrow \overrightarrow{CY} = 6\boldsymbol{c} + 12\boldsymbol{a}$

$\therefore \overrightarrow{OY} = 9\boldsymbol{c} + 12\boldsymbol{a}$

[3 marks]

Answer 14

(i) $\left|\boldsymbol{X}\right| = \sqrt{0.6^2 + 0.8^2}$

$= 1$

$\therefore$ X is a unit vector

[2 marks]

(ii) $\left|Y\right| = \sqrt{14^2 + 48^2}$

$= 50$

$\therefore$ Required unit vector is $\begin{pmatrix} 0.28 \\ 0.96 \end{pmatrix}$

[2 marks]

(iii) $$\begin{pmatrix} \dfrac{p}{\sqrt{p^2 + q^2}} \\[2ex] \dfrac{q}{\sqrt{p^2 + q^2}} \end{pmatrix}$$

[2 marks]

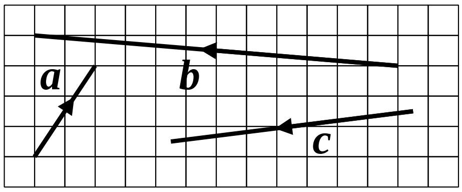
11.1 TEST

Any solution based entirely on graphical or numerical methods is not acceptable.

Make the method used clear.

Marks available : 50

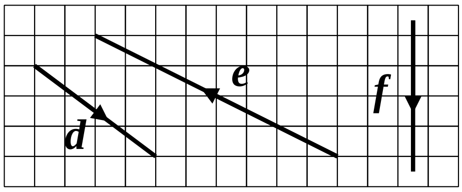
Question 1



Write the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in the form $\begin{pmatrix} p \\ q \end{pmatrix}$ where p and q are integers.

[3 marks]

Question 2



Write the vectors $\boldsymbol{d}$, $\boldsymbol{e}$ and $\boldsymbol{f}$ in the form :

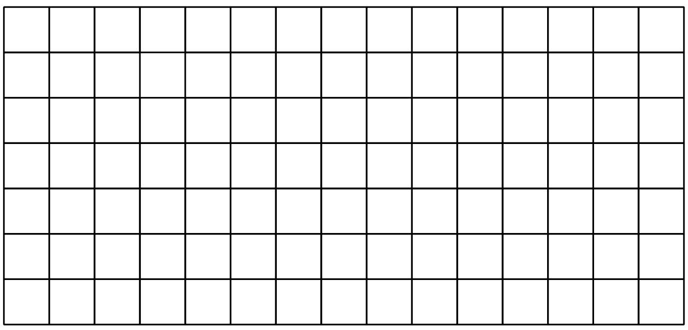
$$p\,\boldsymbol{i} + q\,\boldsymbol{j}$$

[3 marks]

Question 3

On the grid draw the following vectors, labelling each with its letter and an arrow.

$\boldsymbol{a} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ $\qquad$ $\boldsymbol{b} = \begin{pmatrix} 6 \\ -0.5 \end{pmatrix}$

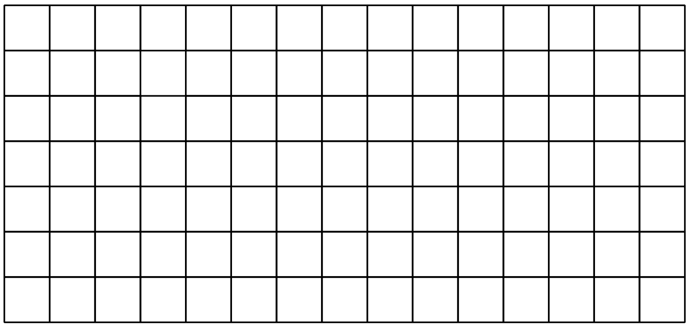


[2 marks]

Question 4

On the grid draw the following vectors, labelling each with its letter and an arrow.

$\boldsymbol{c} = 5\boldsymbol{i} - 2\boldsymbol{j}$ $\qquad\qquad$ $\boldsymbol{d} = -4\boldsymbol{i} + \boldsymbol{j}$



[2 marks]

Question 5

(i) Find $|\boldsymbol{p}|$ giving the answer in surd form.

$$\boldsymbol{p} = \begin{pmatrix} 8 \\ -11 \end{pmatrix}$$

[2 marks]

(ii) Determine the direction in which the vector $\boldsymbol{p}$ acts.

[2 marks]

Question 6

Circle the two vectors that are parallel;

$$\begin{pmatrix} -60 \\ -45 \end{pmatrix} \qquad \begin{pmatrix} 150 \\ 42 \end{pmatrix} \qquad \begin{pmatrix} -80 \\ -24 \end{pmatrix} \qquad \begin{pmatrix} -50 \\ 12 \end{pmatrix} \qquad \begin{pmatrix} 60 \\ 18 \end{pmatrix}$$

[1 mark]

Question 7
Circle the two vectors that are perpendicular;

$$\begin{pmatrix} -21 \\ 36 \end{pmatrix} \qquad \begin{pmatrix} 7 \\ 12 \end{pmatrix} \qquad \begin{pmatrix} -18 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 12 \\ 7 \end{pmatrix} \qquad \begin{pmatrix} -14 \\ -24 \end{pmatrix}$$

[1 mark]

Question 8
In polar form, a vector, ***v***, has a magnitude of 6.4, and direction 250°
Express ***v*** is rectangular form, that is, in the form

$$\boldsymbol{v} = \begin{pmatrix} p \\ q \end{pmatrix}$$

for some values of p and q which you should determine.

[3 marks]

Question 9
Given that;

$$\boldsymbol{f} = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \qquad \boldsymbol{g} = \begin{pmatrix} -7 \\ 4 \end{pmatrix} \qquad \textit{and} \qquad \boldsymbol{h} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

Calculate;
(i) $5\boldsymbol{f} + 2\boldsymbol{g}$ **(ii)** $\boldsymbol{h} - 3\boldsymbol{g} + \boldsymbol{f}$

[2, 2 marks]

Question 10

GCSE Examination Question from May 2007, Paper 3H, Q16

PQR is a triangle.

E is the point on PR such that $PR = 3\,PE$

F is the point on QR such that $QR = 3\,QF$

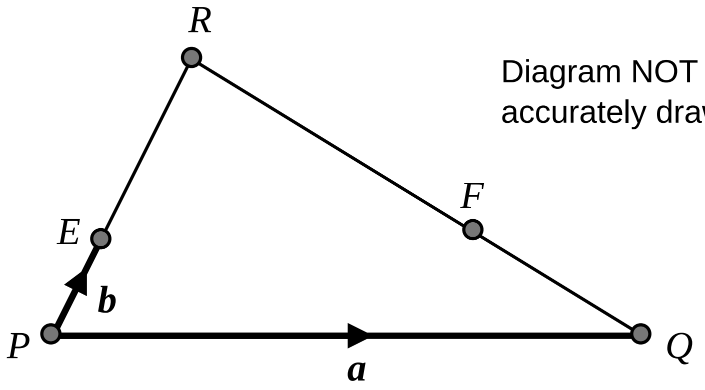


Diagram NOT accurately drawn

$\overrightarrow{PQ} = \boldsymbol{a}$ $\qquad\qquad$ $\overrightarrow{PE} = \boldsymbol{b}$

(a) Find, in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$;

(i) $\overrightarrow{PR} =$

[1 mark]

(ii) $\overrightarrow{QR} =$

[1 mark]

(iii) $\overrightarrow{PF} =$

[1 mark]

(b) Show that $\overrightarrow{EF} = k\,\overrightarrow{PQ}$

where k is a rational number which you must determine

[2 marks]

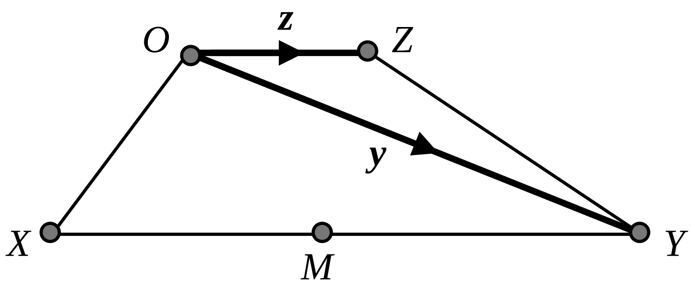
(c) What does the part **(b)** result tell you about vectors $\overrightarrow{EF}$ and $\overrightarrow{PQ}$?

[2 marks]

Question 11

The diagram, which is not drawn to scale, shows a trapezium *OXYZ*
The point *M* is the mid-point of *XY*

Furthermore $\overrightarrow{OY} = \boldsymbol{y}$ $\quad\overrightarrow{OZ} = \boldsymbol{z}$ $\quad\overrightarrow{XY} = 3\,\overrightarrow{OZ}$



(a) Express the following vectors in terms of $\boldsymbol{y}$ and $\boldsymbol{z}$;

(i) $\overrightarrow{XY} =$ **(ii)** $\overrightarrow{OX} =$

(iii) $\overrightarrow{XZ} =$ **(iv)** $\overrightarrow{MZ} =$

[4 marks]

(b) Given that,

$$\boldsymbol{y} = \begin{pmatrix} 17 \\ -8 \end{pmatrix} \quad \text{and} \quad \boldsymbol{z} = \begin{pmatrix} 11 \\ 0 \end{pmatrix}$$

What is $\left|\overrightarrow{YZ}\right|$?

[2 marks]

(c) Find the angle between the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$

HINT : The Reversed Cosine Rule

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

[2 marks]

Question 12

GCSE Examination Question from June 2009, Paper 4H, Q18

The diagram shows a parallelogram, *ABCD*

M is the midpoint of *BC*

N is the midpoint of *AD*

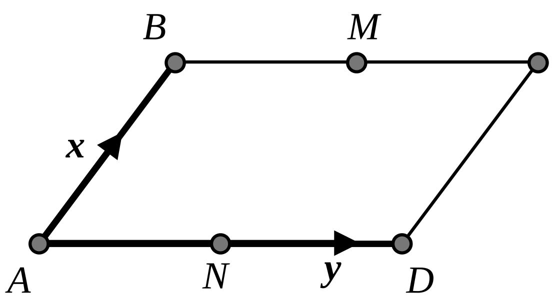


Diagram NOT accurately drawn

$\overrightarrow{AB} = \mathbf{x}$

$\overrightarrow{AD} = \mathbf{y}$

Find, in terms of $\mathbf{x}$ and $\mathbf{y}$, the vectors

(a) $\overrightarrow{MN}$

(b) $\overrightarrow{AC}$

[2 marks]

P is the point such that $\overrightarrow{CP} = \mathbf{y} - \frac{1}{2}\mathbf{x}$

(c) Find, in terms of $\mathbf{x}$ and $\mathbf{y}$, the vector $\overrightarrow{PA}$

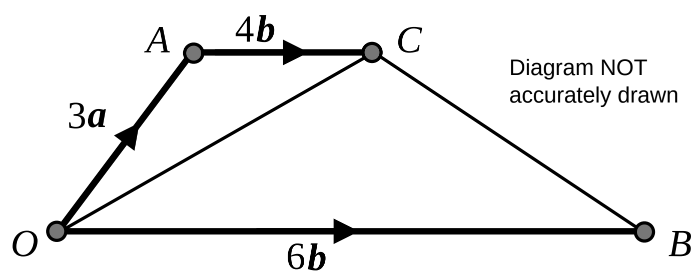
Simplify your answer as much as possible

[2 marks]

Question 13

GCSE Examination Question from January 2020, Paper 2HR, Q26

The diagram shows trapezium *OACB*



$\overrightarrow{OA} = 3\boldsymbol{a}$ $\overrightarrow{OB} = 6\boldsymbol{b}$ $\overrightarrow{AC} = 4\boldsymbol{b}$

N is the point on *OC* such that *ANB* is a straight line

Find $\overrightarrow{ON}$ as a simplified expression in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[5 marks]

Question 14
#VectorFasinatingFact
Here is another fascinating fact about vectors !

Given two vectors, $\boldsymbol{X}$ and $\boldsymbol{Y}$, such that

$$\boldsymbol{X} = \begin{pmatrix} a \\ b \end{pmatrix} \quad and \quad \boldsymbol{Y} = \begin{pmatrix} c \\ d \end{pmatrix}$$

Then the angle, θ, between the vectors is given by;

$$cos\,\theta = \frac{ac + bd}{\sqrt{a^2 + b^2}\,\sqrt{c^2 + d^2}}$$

Demonstrate how to use this result to find the angle between the following two vectors;

$$\boldsymbol{P} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{Q} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

[3 marks]

11.2 Solutions

Answer 1

$\boldsymbol{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\boldsymbol{b} = \begin{pmatrix} -12 \\ 1 \end{pmatrix}$ $\boldsymbol{c} = \begin{pmatrix} -8 \\ -1 \end{pmatrix}$

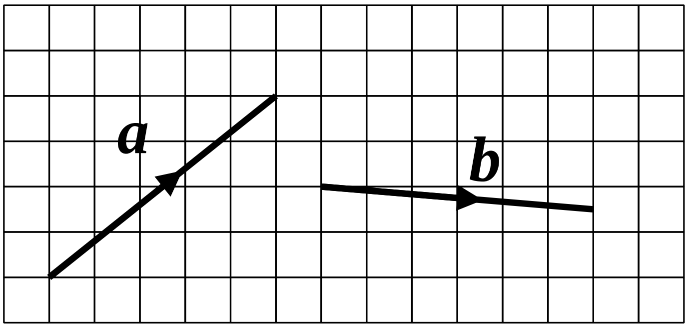
[3 marks]

Answer 2

$\boldsymbol{d} = 4\,\boldsymbol{i} - 3\,\boldsymbol{j}$ $\boldsymbol{e} = -8\,\boldsymbol{i} + 4\,\boldsymbol{j}$ $\boldsymbol{f} = -5\,\boldsymbol{j}$

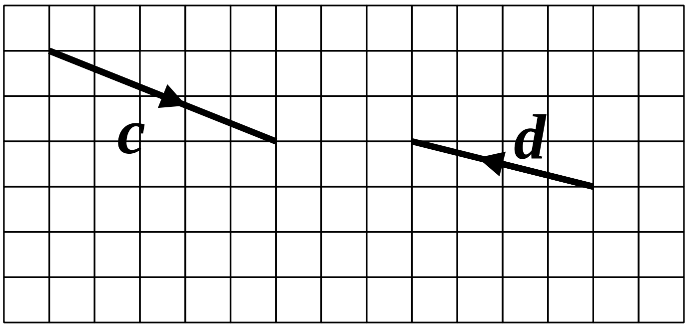
[3 marks]

Answer 3



[2 marks]

Answer 4



[2 marks]

Answer 5

(i) $|\boldsymbol{p}| = \sqrt{185}$ (13.6 but asked for in surd form)

[2 marks]

(ii) $360 - 54 = 306°$

[2 marks]

Answer 6

$-15\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $6\begin{pmatrix} 25 \\ 7 \end{pmatrix}$ $-8\begin{pmatrix} 10 \\ 3 \end{pmatrix}$ $2\begin{pmatrix} -25 \\ 6 \end{pmatrix}$ $6\begin{pmatrix} 10 \\ 3 \end{pmatrix}$

The middle and the rightmost are parallel

[1 mark]

Answer 7

$\begin{pmatrix} -21 \\ 36 \end{pmatrix}$ $\begin{pmatrix} 7 \\ 12 \end{pmatrix}$ $\begin{pmatrix} -18 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 12 \\ 7 \end{pmatrix}$ $\begin{pmatrix} -14 \\ -24 \end{pmatrix}$

$$-21 \times 12 + 36 \times 7 = 0$$

[1 mark]

Answer 8

$\begin{pmatrix} -2.19 \\ -6.01 \end{pmatrix}$

[3 marks]

Answer 9

(i) $\begin{pmatrix} 1 \\ 48 \end{pmatrix}$

(ii) $\begin{pmatrix} 24 \\ -8 \end{pmatrix}$

[2, 2 marks]

Answer 10

(a) **(i)** $\overrightarrow{PR} = 3\boldsymbol{b}$

(ii) $\overrightarrow{QR} = -\boldsymbol{a} + 3\boldsymbol{b}$

(iii) $\overrightarrow{PF} = \overrightarrow{PQ} + \frac{1}{3}\overrightarrow{QR} = \frac{2}{3}\boldsymbol{a} + \boldsymbol{b}$

[3 marks]

(b) $\overrightarrow{EF} = \overrightarrow{EP} + \overrightarrow{PF}$

$= -b + \frac{2}{3}\boldsymbol{a} + \boldsymbol{b}$

$= \frac{2}{3}\boldsymbol{a}$

$= \frac{2}{3}\overrightarrow{PQ}$ i.e. as required with $k = \frac{2}{3}$

[2 marks]

(c) They are parallel.

EF is two thirds the length of *PQ*

[2 marks]

Answer 11

(a) **(i)** $\overrightarrow{XY} = 3\boldsymbol{z}$ **(ii)** $\overrightarrow{OX} = \boldsymbol{y} - 3\boldsymbol{z}$

(iii) $\overrightarrow{XZ} = 4\boldsymbol{z} - \boldsymbol{y}$ **(iv)** $\overrightarrow{MZ} = 2.5\boldsymbol{z} - \boldsymbol{y}$

[4 marks]

(b) $\overrightarrow{YZ} = -y + z = \begin{pmatrix} -17 \\ 8 \end{pmatrix} + \begin{pmatrix} 11 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$

$\therefore \left|\overrightarrow{YZ}\right| = 10$

[2 marks]

(c) $\cos\theta = \dfrac{11^2 + 353 - 10^2}{2 \times 11 \times \sqrt{353}} = 0.905$

$\therefore \theta = 25.2°$

[2 marks]

Answer 12

(a) $\overrightarrow{MN} = -\boldsymbol{x}$

(b) $\overrightarrow{AC} = \boldsymbol{x} + \boldsymbol{y}$

[2 marks]

(c)
$$\begin{aligned}\overrightarrow{PA} &= \overrightarrow{PC} + \overrightarrow{CA} \\ &= \tfrac{1}{2}\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{x} - \boldsymbol{y} \\ &= -\tfrac{1}{2}\boldsymbol{x} - 2\boldsymbol{y}\end{aligned}$$

[2 marks]

Answer 13

$$\overrightarrow{AB} = -3\boldsymbol{a} + 6\boldsymbol{b} \quad \text{and} \quad \overrightarrow{OC} = 3\boldsymbol{a} + 4\boldsymbol{b}$$

$$\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$$

$$s\,\overrightarrow{OC} = \overrightarrow{OA} + t\,\overrightarrow{AB}$$

$$s\,(3\boldsymbol{a} + 4\boldsymbol{b}) = 3\boldsymbol{a} + t\,(-3\boldsymbol{a} + 6\boldsymbol{b})$$

$$3s\,\boldsymbol{a} + 4s\,\boldsymbol{b} = 3\boldsymbol{a} - 3t\,\boldsymbol{a} + 6t\,\boldsymbol{b}$$

$$\therefore\ 3s = 3 - 3t \quad \text{and} \quad 4s = 6t$$

$$3s = 3 - 2s$$

$$5s = 3$$

$$s = \frac{3}{5} \Rightarrow t = \frac{2}{5}$$

$$\overrightarrow{ON} = s\,\overrightarrow{OC} = \frac{3}{5}(3\boldsymbol{a} + 4\boldsymbol{b})$$

$$\overrightarrow{ON} = \frac{9}{5}\boldsymbol{a} + \frac{12}{5}\boldsymbol{b}$$

[5 marks]

Answer 14

$$\begin{aligned}\cos\theta &= \frac{3 \times 12 + 4 \times 5}{\sqrt{3^2 + 4^2} \times \sqrt{12^2 + 5^2}} \\ &= \frac{36 + 20}{5 \times 13} \\ &= \frac{56}{65} \\ &= 0.862\end{aligned}$$

$$\therefore\ \theta = 30.5°$$

[3 marks]