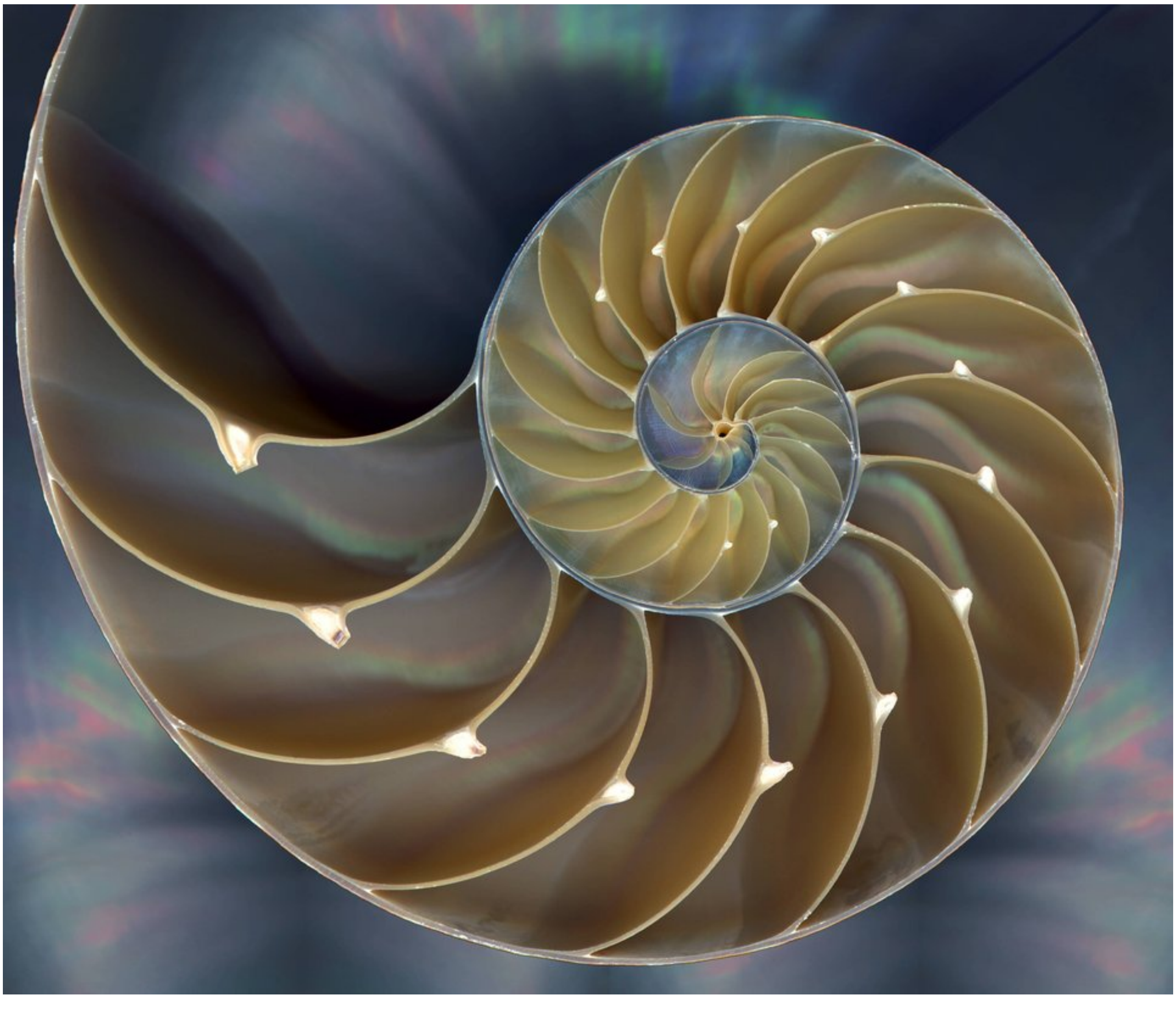
A-Level Pure Mathematics

NUMERICAL METHODS

Solving Equations using Iteration



~ Year 2 ~

N U M E R I C A L M E T H O D S
Solving Equations using Iteration

Lesson 1

Numerical Methods : Pure Year 2

1.1 Solving 'Impossible' Equations

Much of GCSE and A level mathematics is about learning methods for finding the exact solutions to equations, when such solutions exist. However, it is easy to write down an equation for which such methods do not yield solutions, even when it is known that solutions must exist.

For example, consider the following equation;

$$\sin x^c - \ln x = 0$$

It's a job to know where to begin with trying to solve this.
Perhaps it has no solutions, or one, or several, or even an infinite number.

We could make some progress towards deciding how many solutions this equation has (if any) by rearranging it like this;

$$\sin x^c = \ln x$$

and on a graph sketching each of the curves;

$$y = \sin x^c \qquad and \qquad y = \ln x$$

From the sketch, and perhaps calculating $\ln(2\pi)$, you have a reasoned argument that the equation $\sin x^c - \ln x = 0$ has one solution exactly.

Although we would struggle to solve the equation algebraically, the sketch graph suggests that not only is there one solution but that it lies between 1.5 and 3.5.

In fact, to the nearest integer, the solution is 2.
Here is how we numerically prove this:

When $x = 1.5$:

When $x = 2.5$:

As $\sin x^c - \ln x$ is POSITIVE when $x = 1.5$ and NEGATIVE when $x = 2.5$ there must be a value of x between 1.5 and 2.5 where the equation equals zero exactly.

As all values of x between 1.5 and 2.5 round off to 2, the solution to the nearest integer is 2.

1.2 Exercise

Question 1

$$f(x) = \sin x^c - \ln x$$

Calculate

(i) $f(2.15)$

(ii) $f(2.25)$

Explain how this proves the solution of $f(x) = 0$ correct to one decimal place is 2.2
You'll need to write out the POSITIVE / NEGATIVE argument we used above
and the bit about ROUNDING.

Question 2

$$f(x) = \sin x^c - \ln x$$

Calculate

(i) $f(2.215)$

(ii) $f(2.225)$

(iii) $f(2.235)$

What is the solution of $f(x) = 0$ correct to two decimal places ?

Question 3

$$f(x) = \sin x^c - \ln x$$

Calculate

(i) $f(2.2175)$

(ii) $f(2.2185)$

(iii) $f(2.2195)$

What is the solution of $f(x) = 0$ correct to three decimal places ?

Question 4

$$f(x) = \sin x^c - \ln x$$

What is the solution of $f(x) = 0$ correct to four decimal places ?

Question 5

$$g(x) = x^3 - 5x + 1$$

(a) Show that the equation $g(x) = 0$ has a solution, correct to 1 decimal place of 2.1. Do this by first calculating $g(2.05)$ and $g(2.15)$ and then explain how those answers imply the claimed result.

(b) Improve upon the part **(a)** answer by finding a solution correct to 2 decimal places.
HINT : $g(2.105)$, $g(2.115)$, $g(2.125)$, $g(2.135)$ etc...

Question 6

$$h(x) = x^3 - 4x + 2$$

In trying to quickly locate a solution to $h(x) = 0$ a student calculates $h(0.5)$ and $h(3.5)$ and concludes that there is no root in between.

(i) Calculate $h(0.5)$

(ii) Calculate $h(3.5)$

(iii) Explain why the student thinks there is no root between $x = 0.5$ & $x = 3.5$

(iv) Calculate $h(1.5)$

(v) Explain the significance of your part (iv) answer.

(vi) Find one of the roots of $h(x) = 0$, correct to 2 decimal places.
There are three, but you are only asked to find one.
(Your choice which you decide to work on)

1.3 Answers (1.2 Exercise)

Answer 1

(i) $f(2.15) = + 0.071\ 431$

(ii) $f(2.25) = - 0.032\ 857$

As $f(2.15)$ is +ve and $f(2.25)$ is -ve (and the curves are smooth and continuous) there must be a value of x between 2.15 and 2.25 where the equation equals zero exactly.

As all values of x between 2.15 and 2.25 round off to 2.2, the solution to 1 decimal place is 2.2 □

Answer 2

(i) $f(2.215) = + 0.004\ 326$

(ii) $f(2.225) = - 0.006\ 224$

(iii) $f(2.235) = - 0.016\ 833$

$\therefore$ Solution to 2 decimal places is 2.22

Answer 3

(i) $f(2.2175) = + 0.001\ 694$

(ii) $f(2.2185) = + 0.000\ 640$

(iii) $f(2.2195) = - 0.000\ 414$

$\therefore$ Solution to 3 decimal places is 2.219

Answer 4

Here are the likely numbers to be tried...

$f(2.21855) = + 0.000\ 587$

$f(2.21865) = + 0.000\ 482$

$f(2.21875) = + 0.000\ 377$

$f(2.21885) = + 0.000\ 271$

$f(2.21895) = + 0.000\ 166$

$f(2.21905) = + 0.000\ 060$

$f(2.21915) = - 0.000\ 045$

$f(2.21925 = - 0.000\ 151$

$f(2.21935) = - 0.000\ 256$

$f(2.21945) = - 0.000\ 362$

$\therefore$ Solution to 4 decimal places is 2.2191

Out of Interest 2.219 107 158 915 2 is the solution to 13 decimal places

Answer 5

(a) g (2.05) = - 0.634 875

g (2.15) = + 0.188 375

As g (2.05) is -ve and g (2.15) is +ve (and the curves are smooth and continuous) there must be a value of x between 2.05 and 2.15 for which $g(x) = 0$ exactly.

As all values of x between 2.05 and 2.15 round off to 2.1, the solution to 1 decimal place is 2.1 ☐

(b) g (2.125) = - 0.029 297

g (2.135) = + 0.056 810

$\therefore$ Solution to 2 decimal places is 2.13

Answer 6

(i) h (0.5) = + 0.125

(ii) h (3.5) = + 30.875

(iii) There is no change of sign.

(iv) h (1.5) = - 0.625

(v) There are at least two roots in the interval between 0.5 and 3.5

(vi)

h (-2.215) = - 0.007 288

h (-2.205) = + 0.099 235

$\therefore$ Solution to 2 decimal places is - 2.21

h (0.535) = + 0.013 130

h (0.545) = - 0.018 121

$\therefore$ Solution to 2 decimal places is 0.54

h (0.5385) = + 0.000 216

h (0.5395) = - 0.000 973

$\therefore$ Solution to 3 decimal places is 0.538

h (1.675) = - 0.000 578

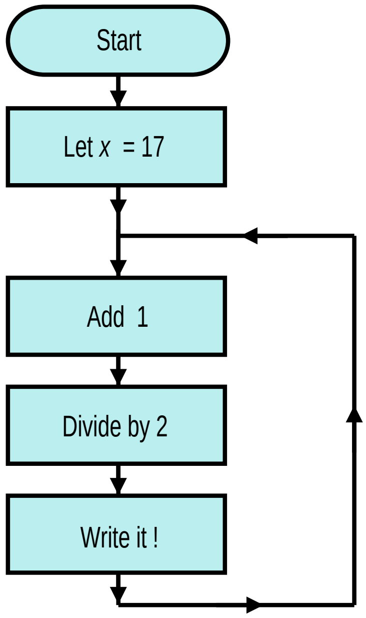
h (1.685) = + 0.044 094

$\therefore$ Solution to 2 decimal places is 1.68

Lesson 2

2.1 A First Iteration

To iterate means to repeatedly carry out the same set of instructions.
Iteration is mathematical feedback.



In the flow chart, starting with 17 we repeatedly Add 1, Divide by 2 and Write it !

Thus:

9
5
3
2
1.5
1.25
1.125
1.0625
1.03125
1.015625
1.0078125
1.00390625
1.001953125

(i) This iteration is heading towards a fixed value.
What do you think the fixed value is ?

(ii) What happens if you start the iteration using the fixed value ?

This method of checking the fixed value only works for rational numbers.

2.2 Solving an equation using iteration.

Question :

$$g(x) = x^3 - 3x - 5$$

Use iteration to find, to 3 decimal places, a solution to $g(x) = 0$.

Solution : First we do some algebraic manipulation
(So algebra is not redundant, after all!)

$$\begin{aligned} g(x) &= 0 \\ x^3 - 3x - 5 &= 0 \\ x^3 &= 3x + 5 \\ x &= \sqrt[3]{3x + 5} \\ x_{n+1} &= \sqrt[3]{3x_n + 5} \end{aligned}$$

Let's start our iteration with $x_0 = 2$

$$\begin{aligned} x_0 &= 2 \\ x_1 &= 2.223980091 \\ x_2 &= 2.268372388 \\ x_3 &= 2.276967161 \\ x_4 &= 2.278623713 \\ x_5 &= 2.278942719 \\ x_6 &= 2.279004141 \end{aligned}$$

The iteration suggests that the solution to 3 decimal places is 2.279.

We confirm this by calculating that

$$\begin{aligned} g(2.2785) &= -0.0065254 \\ g(2.2795) &= 0.0060561 \end{aligned}$$

As $g(x)$ is NEGATIVE when $x = 2.2785$ and POSITIVE when $x = 2.2795$ there must be a value of x between 2.2785 and 2.2795 where the equation equals zero exactly.

As all values of x between 2.2785 and 2.2795 round off to to 2.279, the solution to 3 decimal places is 2.279.

2.3 Exercise

Question 1

The iteration formula for a sequence, *C*, is;

$$C_{n+1} = \frac{C_n}{2} + 3 \qquad with \qquad C_1 = 8$$

Calculate the first six terms of this sequence and then write down a guess of what you think the limit might be. Prove that your guess is correct by putting it into the iteration formula for *C* and observing what comes out.

Question 2

The iterative formula for a sequence, *D*, is;

$$D_{n+1} = \left(\sqrt{D_n} + 1\right)^2 \qquad with \qquad D_1 = 1$$

(i) Calculate D_2, D_3, D_4 and D_5.

(ii) This iteration is *divergent*.
Explain what *divergent* means.

(iii) You should recognise the sequence of numbers being generated.
What special name is given to these numbers ?

Question 3

The iteration formula for a sequence, E, is;

$$E_{n+1} = \frac{E_n}{4} + 6 \qquad with \qquad E_1 = 6$$

Calculate the first six terms of this sequence and then write down a guess of what you think the limit might be. Prove that your guess is correct by putting it into the iteration formula for E and observing what comes out.

Question 4

Here is an interesting iterative rule:

$$F_{n+1} = \frac{2}{F_n} - 1.$$

It is interesting because it has two limits.

(i) Show that - 2 is one limit and try to guess the other.

(ii) Let $F_1 = 2$.

Something goes horribly wrong !

What is it ?

Question 5

A sequence has the iteration formula;

$$G_{n+1} = \frac{1}{G_n} + 5 \qquad with \qquad G_1 = 2$$

(i) Calculate the values of G_2, G_3, G_4 ..... G_9 writing each to at least 8 decimal places.

(ii) What type of number is the limit this time ?

(iii) The perfect answer is actually

$$\frac{5 + \sqrt{a}}{2}$$

for some integer, a.

What is the value of a ?

Question 6

The *jumping tortoise* sequence has the iteration formula;

$$T_{n+1} = \frac{14}{T_n + 1.1} \qquad with \qquad G_1 = 10$$

Calculate sufficient values of $T_2,\ T_3,\ T_4$ ..... until you feel you know the fixed point of the iteration correct to five decimal places.

(i) Write down the fixed point correct to 5 decimal places.

(ii) Why do you think this is called the *jumping tortoise* sequence ?

Question 7

A famous sequence is described iteratively as follows,

$$F_1 = 1, \qquad F_2 = 1, \qquad F_{n+1} = F_n + F_{n-1}$$

(i) Write out the first ten terms of the sequence.

(ii) What is this sequence's famous name ?

Question 8

A sequence is described iteratively as follows,

$$S_1 = 1, \qquad S_2 = 1, \qquad S_{n+1} = S_n - S_{n-1}$$

(i) Write out the first ten terms of the sequence.

(ii) What is S_{33} ?

Question 9

The first five terms of the Fibonacci sequence are;

$F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$

(i) Write out the next five terms.

$F_6 =$, $F_7 =$, $F_8 =$, $F_9 =$, $F_{10} =$

[3 marks]

(ii) Work out each of the following;

(i) $F_1 \times F_3 - 1 =$

(ii) $F_2 \times F_4 + 1 =$

(iii) $F_3 \times F_5 - 1 =$

(iv) $F_4 \times F_6 + 1 =$

(v) $F_5 \times F_7 - 1 =$

(vi) $F_6 \times F_8 + 1 =$

(vii) $F_7 \times F_9 - 1 =$

(viii) $F_8 \times F_{10} + 1 =$

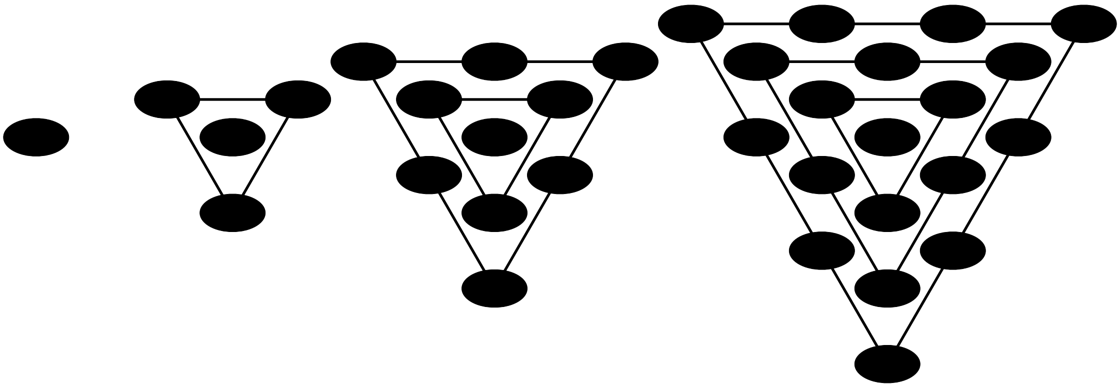
[3 marks]

(iii) Comment on the pattern.

[4 marks]

Question 10

Study the sequence of patterns below;



The sequence can be described iteratively;

$$G_1 = 1, \qquad G_2 = 4, \qquad G_{n+1} = 2G_n - G_{n-1} + 3$$

(i) Show how you would use the information $G_1 = 1$ and $G_2 = 4$ and the iterative relationship to calculate the value of G_3.

(ii) Show how you would use the iterative description to calculate G_4.

(iii) Write out the sequence starting with G_1 and up to G_{10}.

2.4 Answers

2.4.1 Solutions (2.1 A First Iteration)

(i) 1

(ii) Putting precisely 1 in causes precisely 1 to come out.

A neat way of tidying up what is going on is to write out a solution as follows;

To ***prove*** that 1 is the limit.

$$A_{n+1} = \frac{A_n + 1}{2}$$

Let $A_n = 1$, *in which case....*

$$A_{n+1} = \frac{1 + 1}{2}$$

$$= 1$$

So 1 *is indeed the limit*

(*as putting precisely* 1 *in causes precisely* 1 *to come out*) □

2.4.2 Solutions (2.3 Exercise)

Answer 1

$C_1 =$ 8

$C_2 =$ 7

$C_3 =$ 6.5

$C_4 =$ 6.25

$C_5 =$ 6.125

$C_6 =$ 6.0625 The limit <u>seems</u> to be 6

Now to ***prove*** that 6 is the limit.

$$C_{n+1} = \frac{C_n}{2} + 3$$

Let $C_n = 6$, *in which case....*

$$C_{n+1} = \frac{6}{2} + 3$$

$$= 3 + 3$$

$$= 6$$

So 6 *is indeed the limit*

(*as putting* 6 *in precisely causes precisely* 6 *to come out*) □

Answer 2

(i) $D_1 = 1$
$D_2 = 4$
$D_3 = 9$
$D_4 = 16$
$D_5 = 25$

(ii) Divergent means that the iterative series does not converge upon a fixed limit.
In this case it's heading off to + ∞

(iii) The Square Numbers

Answer 3

$E_1 = 6$
$E_2 = 7.5$
$E_3 = 7.875$
$E_4 = 7.96875$
$E_5 = 7.9921875$
$E_6 = 7.998046875$

The limit <u>seems</u> to be 8

Now to ***prove*** that 8 is the limit....

$$E_{n+1} = \frac{E_n}{4} + 6$$

$$\textit{Let } E_n = 8, \textit{ in which case....}$$

$$E_{n+1} = \frac{8}{4} + 6$$

$$= 8$$

$$\textit{So } 8 \textit{ is indeed the limit}$$

$$(\textit{ as putting precisely } 8 \textit{ in causes precisely } 8 \textit{ to come out }) \qquad \square$$

Answer 4

(i) To ***prove*** that - 2 is a limit;

$$F_{n+1} = \frac{2}{F_n} - 1$$

$$\textit{Let } F_n = -2, \textit{ in which case....}$$

$$F_{n+1} = \frac{2}{-2} - 1$$

$$= -2$$

$$\textit{So } -2 \textit{ is indeed a limit}$$

$$(\textit{ as putting precisely } -2 \textit{ in causes precisely } -2 \textit{ to come out }) \qquad \square$$

The other limit is 1

(ii) A division by zero occurs.
A first hint that iteration is fraught with difficulties !

Answer 5

(i) $G_1 =$ 2

$G_2 =$ 5.5

$G_3 =$ 5.181 818 181

$G_4 =$ 5.192 982 456

$G_5 =$ 5.192 567 568

$G_6 =$ 5.192 582 954

$G_7 =$ 5.192 582 383

$G_8 =$ 5.192 582 404

$G_9 =$ 5.192 582 404

(ii) The limit is an *irrational* number.

(iii) a is 29

i.e.

$$\frac{5 + \sqrt{29}}{2}$$

Answer 6

$T_1 =$ 10

$T_2 =$ 1.26126

$T_3 =$ 5.92903

$T_4 =$ 1.99173

$T_5 =$ 4.52820

$T_6 =$ 2.48747

$T_7 =$ 3.90247

$T_8 =$ 2.79862

$T_9 =$ 3.59101

(i) 3.23186

(ii) It jumps repeatedly above and below the fixed value.
It converges relatively slowly compared with previous questions.

Answer 7

(i) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

(ii) The Fibonacci sequence.

Answer 8

(i) 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, ...

(ii) $S_{33} = 0$

Answer 9

(i)

$F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$

$F_6 = \mathbf{8}$, $F_7 = \mathbf{13}$, $F_8 = \mathbf{21}$, $F_9 = \mathbf{34}$, $F_{10} = \mathbf{55}$

[3 marks]

(ii) Work out each of the following;

(i) $F_1 \times F_3 - 1 = 1 \times 2 - 1 = 1$

(ii) $F_2 \times F_4 + 1 = 1 \times 3 + 1 = 4$

(iii) $F_3 \times F_5 - 1 = 2 \times 5 - 1 = 9$

(iv) $F_4 \times F_6 + 1 = 3 \times 8 + 1 = 25$

(v) $F_5 \times F_7 - 1 = 5 \times 13 - 1 = 64$

(vi) $F_6 \times F_8 + 1 = 8 \times 21 + 1 = 169$

(vii) $F_7 \times F_9 - 1 = 13 \times 34 - 1 = 441$

(viii) $F_8 \times F_{10} + 1 = 21 \times 55 + 1 = 1156$

[3 marks]

(iii) The answers are the Fibonacci numbers squared.

$1^2, 2^2, 3^2, 5^2, 8^2, \ldots = (F_2)^2, (F_3)^2, (F_4)^2, (F_5)^2, (F_6)^2, \ldots.$

$$F_n \times F_{n-2} + (-1)^n = (F_{n+1})^2$$

[4 marks]

Answer 10

(i) $G_3 = 10$

(ii) $G_4 = 19$

(iii) 1, 4, 10, 19, 31, 46, 64, 85, 109, 136, ...

Lesson 3

Numerical Methods : Pure Year 2

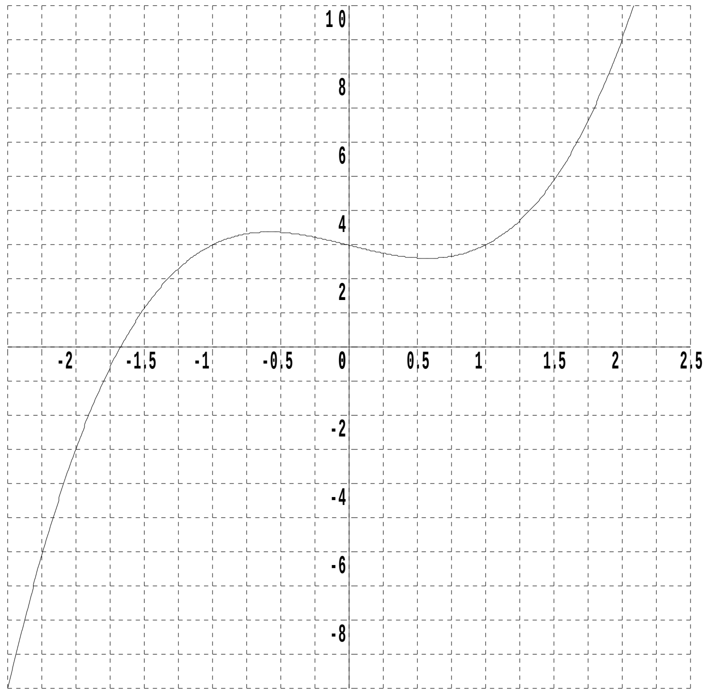
3.1 Numerical Methods Exercise

Question 1

Consider the function

$$f(x) = x^3 - x + 3$$

The graph of this function suggests that it has a root in $-2 \leqslant x \leqslant -1$



(i) Calculate **(a)** $f(-2)$

(b) $f(-1)$

(ii) How do the part **(i)** answers prove that there is a root in $-2 \leqslant x \leqslant -1$?

(iii) Prove that a root is in the interval $-1.8 \leqslant x \leqslant -1.6$.

Question 2

This question is about the function

$$g(x) = \ln x - \frac{1}{x}$$

(i) On one graph sketch the curves

(a) $y = \ln x$

(b) $y = \frac{1}{x}$

to show that $g(x) = 0$ has a root.

(ii) Calculate **(a)** $g(1.7)$

(b) $g(1.8)$

(iii) How do the part **(ii)** answers prove that there is a root in $1.7 \leqslant x \leqslant 1.8$?

(iv) Show that $g(x) = 0$, implies

$$x = e^{\frac{1}{x}}$$

(v) Using the iterative formula

$$x_{n+1} = e^{\frac{1}{x_n}}$$

$$x_0 = 1.4$$

find

(a) $x_1 =$

(b) $x_2 =$

(c) $x_3 =$

(vi) Keep iterating to find a root of $g(x) = 0$ to about eight decimal places.

Question 3

This question is about the function

$$h(x) = x^2 - \sin x$$

where x is measured in **RADIANS**.

(i) On one graph sketch the curves

(a) $y = x^2$

(b) $y = \sin x$

to show that $h(x) = 0$ has a root.

(ii) Use the iteration formula

$$x_{n+1} = \frac{\sin x_n}{x_n}$$

$$x_0 = 1$$

to calculate

$x_1 =$

$x_2 =$

$x_3 =$

$x_4 =$

$x_5 =$

(iii) To prove that a root is 0.8767 to four decimal places, calculate;

(a) $h(0.87665) =$

(b) $h(0.87675) =$

and observe that as one answer is negative and the other positive there is at least one root in the interval $0.87665 \leqslant x < 0.87675$ and that as all numbers in this interval round to 0.8767 the root is indeed 0.8767, correct to four decimal places.

This proof is a technique to remember, including the worded reasoning.

Question 4
Initially the number of fish in a lake is 500 000.

The population is then modelled by the recurrence relation;

$$u_{n+1} = 1.05\,u_n - d$$

$$u_0 = 500\,000$$

In this relation u_n is the number of fish in the lake after n years and d is the number of fish which are caught each year.

Given that $d = 15\,000$,

(a) calculate u_1, u_2 and u_3 and comment briefly on your results.

Given instead that $d = 100\,000$,

(b) Show that the population of fish dies out during the sixth year.

(c) find the value of d which would leave the population each year unchanged.

Question 4 is an Edexcel examination question from January 2002.

Question 5

(i) Use the iteration formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{12}{x_n}\right)$$

$$with\ x_1 = 2$$

to calculate

$x_2 =$

$x_3 =$

$x_4 =$

$x_5 =$

$x_6 =$

(ii) It is thought that this iterative sequence may converges to $\sqrt{n}$ where n is an integer.

If this is so, what is n ?

(iii) At the fixed point, X,

$$x_{n+1} = x_n = X$$

By replacing x_{n+1} and x_n with X in the iterative formula, rearrange the formula and so prove that that it does indeed converge to your part **(ii)** answer.

Question 6

Let

$$f(\theta) = sin\,\theta + 1 - \theta$$

where θ is in **<u>RADIANS</u>**.

(i) Calculate $f(1)$ and $f(2)$ and explain how this proves that

$$f(\theta) = 0$$

has a solution between $\theta = 1$ and $\theta = 2$.

(ii) Use the iterative formula

$$\theta_{n+1} = sin\,\theta_n + 1$$

to find this root correct to 5 decimal places.

(iii) Prove that your part **(ii)** answer is correct using the ***Question 3, part (iii)*** technique, ***including some worded reasoning***.

3.2 Answers

Answer 1

(i) **(a)** $f(-2) = -3$

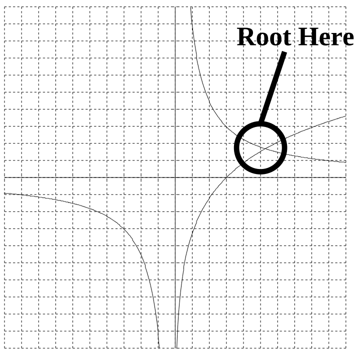
(b) $f(-1) = 3$

(ii) One answer is negative and one is positive.
As polynomials are smooth and continuous functions there must be a value of x between -2 and -1 that causes $f(x) = 0$.

(iii) $f(-1.8) = -1.032$
$f(-1.6) = 0.504$
Again, one answer is negative and one is positive.
As polynomials are smooth and continuous functions there must be a value of x between -1.8 and -1.6 that causes $f(x) = 0$.

Answer 2

(i)



(ii) **(a)** $g(1.7) = -0.057\,6$

(b) $g(1.8) = 0.032\,2$

(iii) One answer is negative and one is positive.
As polynomials are smooth and continuous functions there must be a value of x between 1.7 and 1.8 that causes $g(x) = 0$.

(iv)

$$\begin{aligned} g(x) &= 0 \\ \ln x - \frac{1}{x} &= 0 \\ \ln x &= \frac{1}{x} \\ x &= e^{\frac{1}{x}} \qquad \square \end{aligned}$$

(v) **(a)** $x_1 = 2.042\,727\,07$

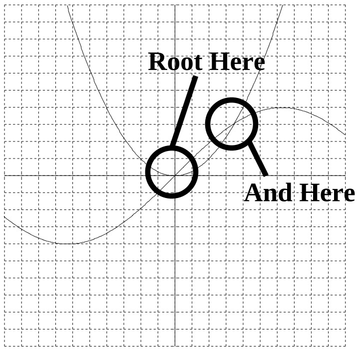
(b) $x_2 = 1.631\,568\,235$

(c) $x_3 = 1.845\,789\,778$

(vi) 1.763 222 83

Answer 3

(i)



(ii) $x_0 = 1$

$x_1 = 0.841\ 470\ 984\ 8$

$x_2 = 0.886\ 096\ 080\ 7$

$x_3 = 0.874\ 181\ 339\ 2$

$x_4 = 0.877\ 413\ 474\ 4$

$x_5 = 0.876\ 540\ 328\ 2$

(iii) h (0.87665) = - 0.000 084 8

h (0.87675) = 0.000 026 49

One answer is negative and one is positive.
As both functions are smooth and continuous there must be a value of x between 0.87665 and 0.87675 that causes $h(x) = 0$.
Furthermore, all numbers in this interval round to 0.8767 and so the root must be 0.8767 to four decimal places.

Answer 4

(a) $u_0 = 500\ 000$

$u_1 = 510\ 000$

$u_2 = 520\ 500$

$u_3 = 531\ 525$

(b) $u_0 = 500\ 000$

$u_1 = 425\ 000$

$u_2 = 346\ 250$

$u_3 = 263\ 562.5$

$u_4 = 176\ 740.625$

$u_5 = 85\ 577.65625$

$u_6 = -\ 10\ 143.46$

(c)

$$\textit{Assume that a fixed value exists of value } x$$

$$\textit{Then } u_n = x \qquad \textit{and} \qquad u_{n+1} = x$$

$$\textit{From} \qquad u_{n+1} = 1.05\,u_n - d$$

$$\textit{it follows that} \qquad x = 1.05x - d$$

$$\therefore \quad d = 0.05x \qquad (\textit{in general})$$

$$d = 0.05 * 500\,000 \qquad (\textit{as we start with half a million fish})$$

$$d = 25\,000$$

Answer 5

(i) $x_1 = 2$
$x_2 = 4$
$x_3 = 3.5$
$x_4 = 3.464\ 285\ 714$
$x_5 = 3.464\ 101\ 62$
$x_6 = 3.464\ 101\ 615$

(ii) $n = 12$

(iii)

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{12}{x_n}\right)$$

$$X = \frac{1}{2}\left(X + \frac{12}{X}\right)$$

$$2X = X + \frac{12}{X}$$

$$X = \frac{12}{X}$$

$$X^2 = 12$$

$$X = \pm\sqrt{12} \qquad \textit{i.e. } \sqrt{12} \textit{ is a root} \qquad \square$$

Answer 6

(i) $f(1) = \ \ 0.841\ 47$
$f(2) = -\ 0.090\ 70$
One answer is positive and one is negative.
As the function is smooth and continuous there must be a value of θ between 1 and 2 that causes $f(\theta) = 0$.

(ii) 1.93456

(iii) $f(1.934\ 5\mathbf{55}) = \ \ 0.000\ 011\ 132\ 08$
$f(1.934\ 5\mathbf{65}) = -\ 0.000\ 002\ 425\ 86$
One answer is positive and one is negative.
As the function is smooth and continuous there must be a value of θ between 1.934 555 and 1.934 565 that causes $f(\theta) = 0$.
Furthermore, all numbers in this interval round to 1.934 56 and so the root must be 1.934 56 to five decimal places.

Lesson 4

Numerical Methods : Pure Year 2

4.1 Past Paper Questions

Question 1

C3 Examination question from January 2013, Q2.

$$g(x) = e^{x-1} + x - 6$$

(a) Show that the equation $g(x) = 0$ can be written as

$$x = ln(6 - x) + 1, \qquad x < 6$$

[2 marks]

The root of $g(x) = 0$ is a

The iterative formula

$$x_{n+1} = ln(6 - x_n) + 1, \qquad x_0 = 2$$

is used to find an approximate value for a

(b) Calculate the values of x_1, x_2 and x_3 to 4 decimal places.

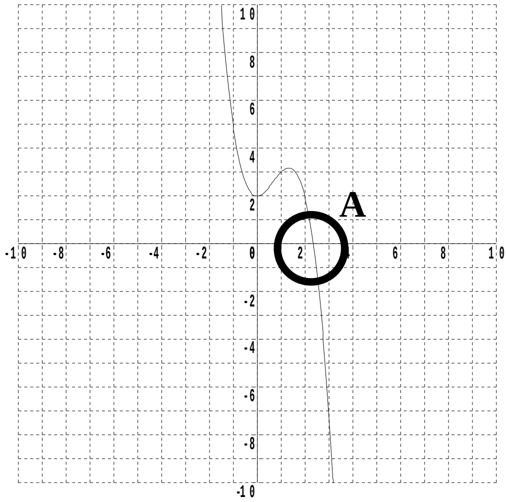
[3 marks]

(c) By choosing a suitable interval, show that $a = 2.307$ correct to 3 decimal places.

[3 marks]

Question 2

C3 Examination question from June 2009, Q1.



The graph shows part of the curve with equation

$$y = -x^3 + 2x^2 + 2$$

which intersects the x-axis at the point A where $x = \alpha$

To find an approximation to α the following iterative formula is used;

$$x_{n+1} = \frac{2}{(x_n)^2} + 2$$

(a) Taking $x_0 = 2.5$, find the values of x_1, x_2, x_3 and x_4
Give your answers to 3 decimal places where appropriate.

[3 marks]

(b) Show that $\alpha = 2.359$ correct to 3 decimal places.

[3 marks]

Question 3

C3 Examination question from January 2010, Q2.

$$f(x) = x^3 + 2x^2 - 3x - 11$$

(a) Show that $f(x) = 0$ can be rearranged as

$$x = \sqrt{\left(\frac{3x + 11}{x + 2}\right)} \qquad x \neq -2$$

[2 marks]

The equation $f(x) = 0$ has one positive root α.

The following iterative formula is used to find an approximation to α.

$$x_{n+1} = \sqrt{\left(\frac{3x_n + 11}{x_n + 2}\right)}$$

(b) Taking $x_1 = 0$, find, to 3 decimal places, the values of x_2, x_3 and x_4

[3 marks]

(c) Show that $\alpha = 2.057$ correct to 3 decimal places

[3 marks]

Question 4

C3 Examination question from June 2010, Q3.

$$f(x) = 4\,cosec\,x - 4x + 1, \qquad \textit{where } x \textit{ is in radians}$$

(a) Show that there is a root α of $f(x) = 0$ in the interval $[1.2, 1.3]$

[2 marks]

(b) Show that the equation $f(x) = 0$ can be written in the form

$$x = \frac{1}{\sin x} + \frac{1}{4}$$

[2 marks]

(c) Use the iterative formula

$$x_{n+1} = \frac{1}{\sin x_n} + \frac{1}{4}, \qquad x_0 = 1.25$$

to calculate the values of x_1, x_2 and x_3 giving answers to 4 decimal places .

[3 marks]

(d) By considering the change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 1.291$ correct to 3 decimal places.

[2 marks]

4.2 Answers

Answer 1

(a)

$$\begin{aligned} g(x) &= 0 \\ e^{x-1} + x - 6 &= 0 \\ e^{x-1} &= 6 - x \\ ln(e^{x-1}) &= ln(6 - x) \\ x - 1 &= ln(6 - x) \\ x &= ln(6 - x) + 1 \qquad \square \end{aligned}$$

(b) $x_0 = 2$
$x_1 = 2.3863$
$x_2 = 2.2847$
$x_3 = 2.3125$

(c) g (2.30**<u>65</u>**) = - 0.000 27
g (2.30**<u>75</u>**) = + 0.004 4
One answer is positive and one is negative.
As the function is smooth and continuous there must be a value of x between 2.3065 and 2.3075 that causes $g(x) = 0$.
Furthermore, all numbers in this interval round to 2.307 and so the root must be 2.307 to three decimal places.

Answer 2

(a) $x_0 = 2.5$
$x_1 = 2.32$
$x_2 = 2.371\ 581\ 451$
$x_3 = 2.355593575$
$x_4 = 2.360436923$

(b) Let $g(x) = -x^3 + 2x^2 + 2$
g (2.35**<u>85</u>**) = + 0.005 835 77
g (2.35**<u>95</u>**) = - 0.001 422 86
One answer is positive and one is negative.
As the function is smooth and continuous there must be a value of x between 2.3585 and 2.3595 that causes $g(x) = 0$.
Furthermore, all numbers in this interval round to 2.359 and so the root must be 2.359 to three decimal places.

Answer 3

(a)

$$f(x) = 0$$

$$x^3 + 2x^2 - 3x - 11 = 0$$

$$x^3 + 2x^2 = 3x + 11$$

$$x^2 (x + 2) = 3x + 11$$

$$x^2 = \frac{3x + 11}{x + 2} \qquad x \neq 2$$

$$x = \sqrt{\left(\frac{3x + 11}{x + 2}\right)} \qquad x \neq 2 \qquad \square$$

(b) $x_1 = 0$

$x_2 = 2.345\ 207\ 88$

$x_3 = 2.037\ 324\ 945$

$x_4 = 2.058\ 748\ 112$

(c) f (2.05**<u>65</u>**) = - 0.013 781 637

f (2.05**<u>75</u>**) = + 0.004 140 109 4

One answer is positive and one is negative.

As the function is smooth and continuous there must be a value of x between 2.0565 and 2.0575 that causes $f(x) = 0$.

Furthermore, all numbers in this interval round to 2.057 and so the root must be 2.057 to three decimal places.

Answer 4

(a) $f(1.2) = +0.491\,665\,51$

$f(1.3) = -0.048\,719\,817$

One answer is positive and one is negative.

As the function is smooth and continuous there must be a value of x between 1.2 and 1.3 that causes $f(x) = 0$.

(b)

$$
\begin{aligned}
f(x) &= 0 \\
4\,cosec\,x - 4x + 1 &= 0 \\
\frac{4}{\sin x} - 4x + 1 &= 0 \\
\frac{1}{\sin x} - x + \frac{1}{4} &= 0 \\
x &= \frac{1}{\sin x} + \frac{1}{4} \qquad \sin x \neq 0 \qquad \square
\end{aligned}
$$

(c) $x_0 = 1.25$

$x_1 = 1.3038$

$x_2 = 1.2867$

$x_3 = 1.2917$

(d) $f(1.29\underline{\mathbf{05}}) = +0.000\,445\,666\,95$

$f(1.29\underline{\mathbf{15}}) = -0.004\,750\,172\,79$

One answer is positive and one is negative.

As the function is smooth and continuous there must be a value of x between 1.2905 and 1.2915 that causes $f(x) = 0$.

Furthermore, all numbers in this interval round to 1.291 and so the root must be 1.291 to three decimal places.

5.1 Iteration Theory

In general, iterations are of the form;

$$x_{n+1} = f(x_n)$$

Once sufficiently near the fixed point of an iteration, whether the iteration will convergence or diverge depends upon the gradient of

$$y = f(x)$$

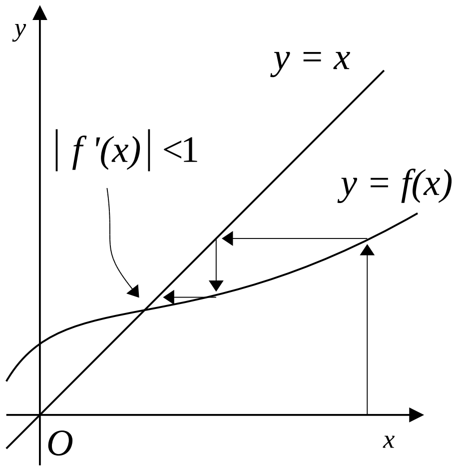
at the intersection with

$$y = x$$

The condition for convergence is that at that point of intersection,

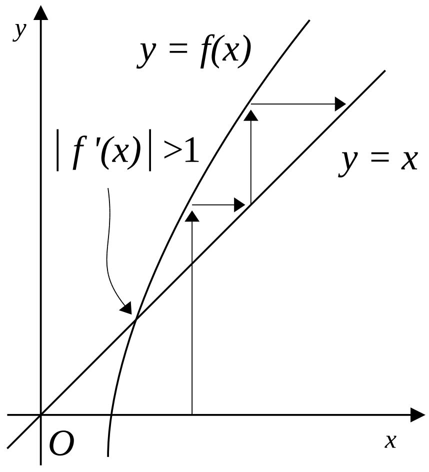
$$|f'(x)| < 1$$

Diagrams of iterations will be investigated shortly - look out for these situations;



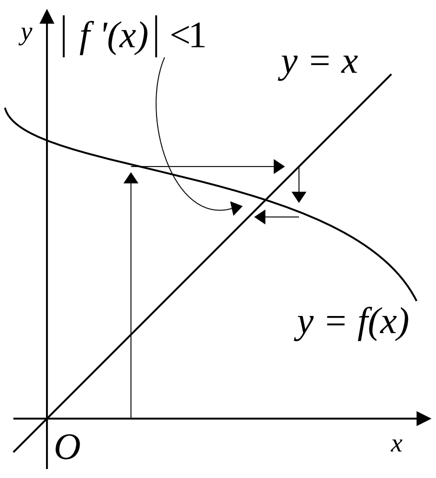
$|f'(x)| < 1$

staircase converges



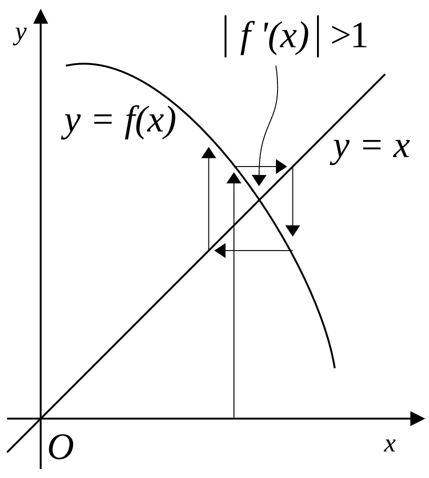
$|f'(x)| > 1$

staircase diverges



$|f'(x)| < 1$

cobweb converges



$|f'(x)| > 1$

cobweb diverges

5.2 Picturing Iteration : Cobweb & Staircase Diagrams

An Investigation

Consider the iterative relation;

$$x_{n+1} = \frac{x_n^2 - 2x_n + 5}{4}$$

which you are going to iterate from four different starting points.

x_0	1	3	5	5.5
x_1				
x_2				
x_3				
x_4				
x_5				
x_6				

(i) Complete the table, each column being from a different initial value, x_0

(ii) Given that

$$g(x) = \frac{x_n^2 - 2x_n + 5}{4}$$

differentiate to find $g'(x)$

(iii) Find $g'(1)$ and $g'(5)$

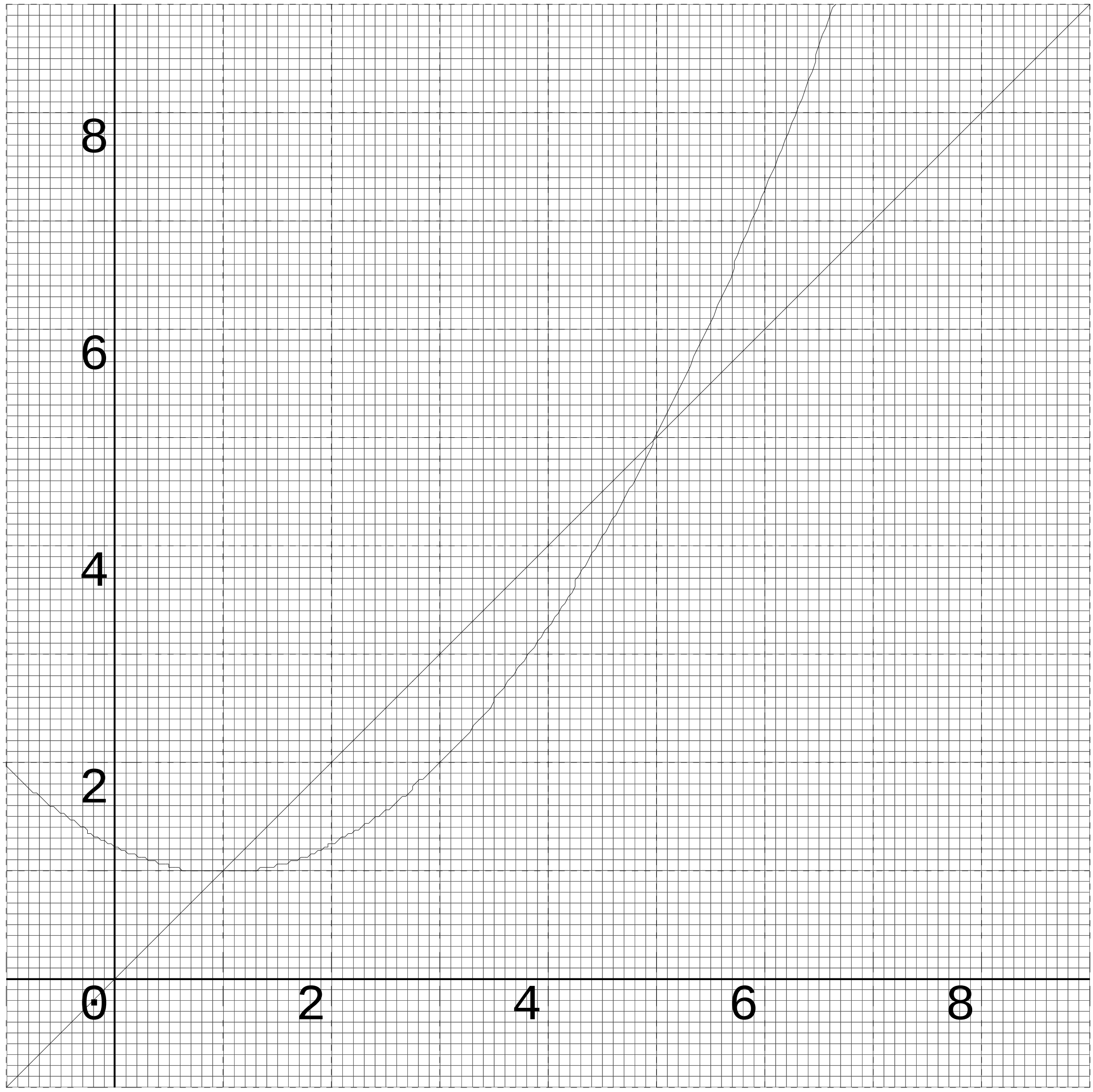
Plotted below is the curve;

$$y = \frac{x^2 - 2x + 5}{4}$$

and the line;

$$y = x$$

(iv) Plot your iterations onto this graph.



5.3 Exercise

Question 1

Consider the iterative relation;

$$x_{n+1} = \frac{12}{x_n} + 1$$

$$x_0 = 6$$

(a) Calculate to four decimal places;

(i) $x_1 =$ **(iv)** $x_4 =$

(ii) $x_2 =$ **(vi)** $x_5 =$

(iii) $x_3 =$ **(vii)** $x_6 =$

(b) Plot a cobweb diagram of the iterative process for x_0 , x_1 , x_2 , ... , x_6. Use the graph on the following page.

(c) Prove that 4 is a fixed point of the iteration by showing that if $x_n = 4$ ***exactly***, then $x_{n+1} = 4$ ***exactly***.

(d) At a fixed point, X, of the iteration

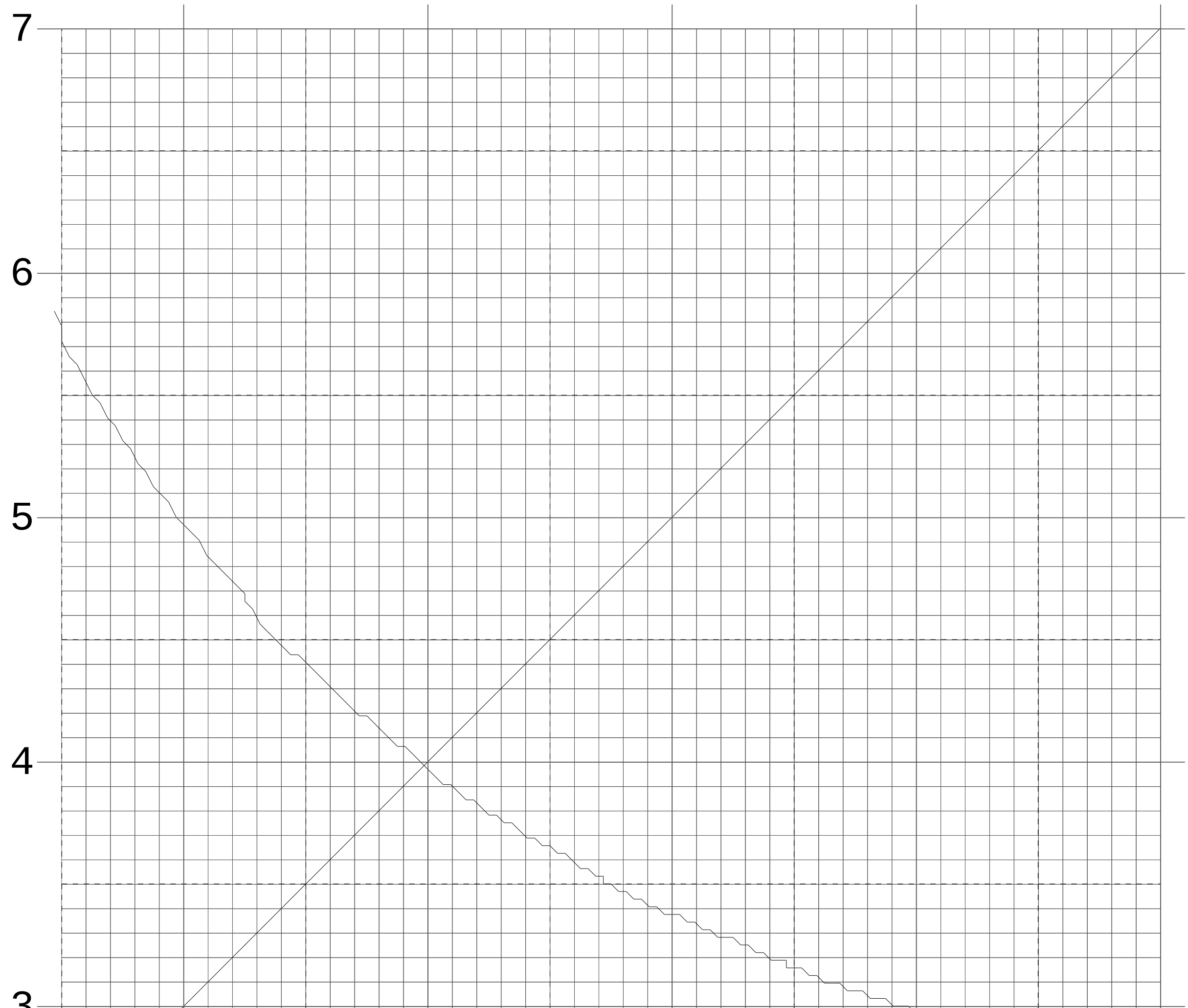
$$x_{n+1} = x_n = X$$

By replacing x_{n+1} and x_n with X in the iterative formula, rearrange the formula and so prove that, in addition to the fixed point at 4 there is another fixed point of this iteration. State the value of this other fixed point.

(e) With

$$f(x) = \frac{12}{x} + 1$$

calculate the value of the gradient at each of the two fixed points. Explain what this tells you about each fixed point.


7
6
5
4
3
3
4
5
6
7

Question 2

Consider the iterative relation;

$$x_{n+1} = ln(100\, x_n)$$

$$x_0 = 0.5$$

(a) Calculate to four decimal places;

(i) $x_1 =$ **(ii)** $x_2 =$ **(iii)** $x_3 =$

(b) Plot a cobweb diagram of the iterative process for x_0 , x_1 , x_2 , x_3.
Use the graph on the following page.

(c) Given that

$$f(x) = ln\,(100\, x) - x$$

prove that the fixed point of the iteration is, to two decimal places 6.47.

(d) Let

$$g(x) = ln\,(100\, x)$$

Use the rule of logs which states;

$$ln\,(a\, b) = ln\, a + ln\, b$$

to help determine $g'(x)$ and hence show that the fixed point is an attractor.

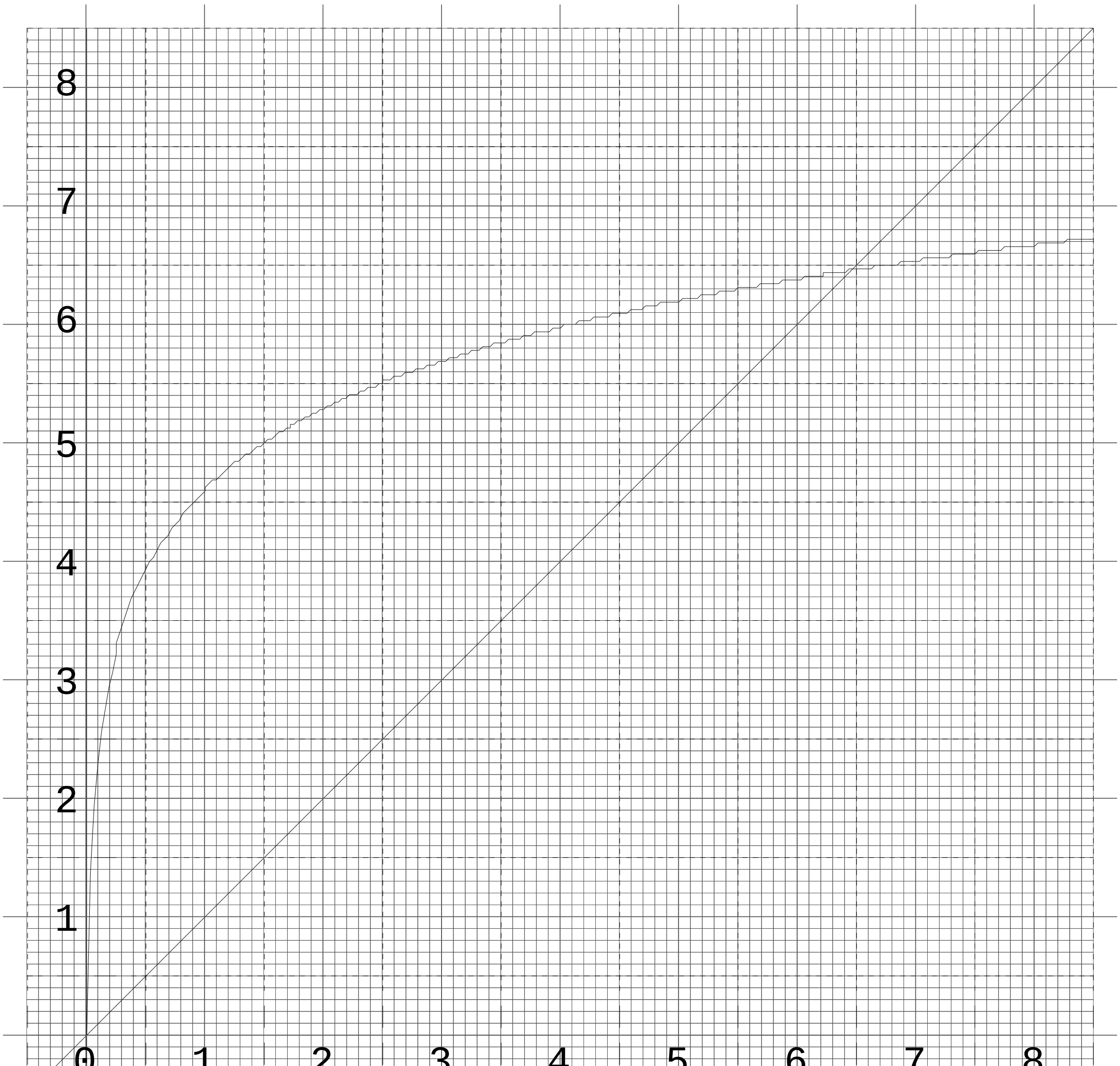
(e) Predict what will happen if the iteration starts with

(i) $x_0 = 8$

(ii) $x_0 = 0.05$

(iii) $x_0 = 0.01$

Test your predictions and comment.



8
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4
5
6
7
8

Question 3

(a) Show that

$$f(x) = x \ln x - 3$$

has a root between 2 and 3.

(b) Show that

$$x \ln x - 3 = 0$$

can be rearranged to give the iterative relationship

$$x_{n+1} = \frac{3}{\ln x_n}$$

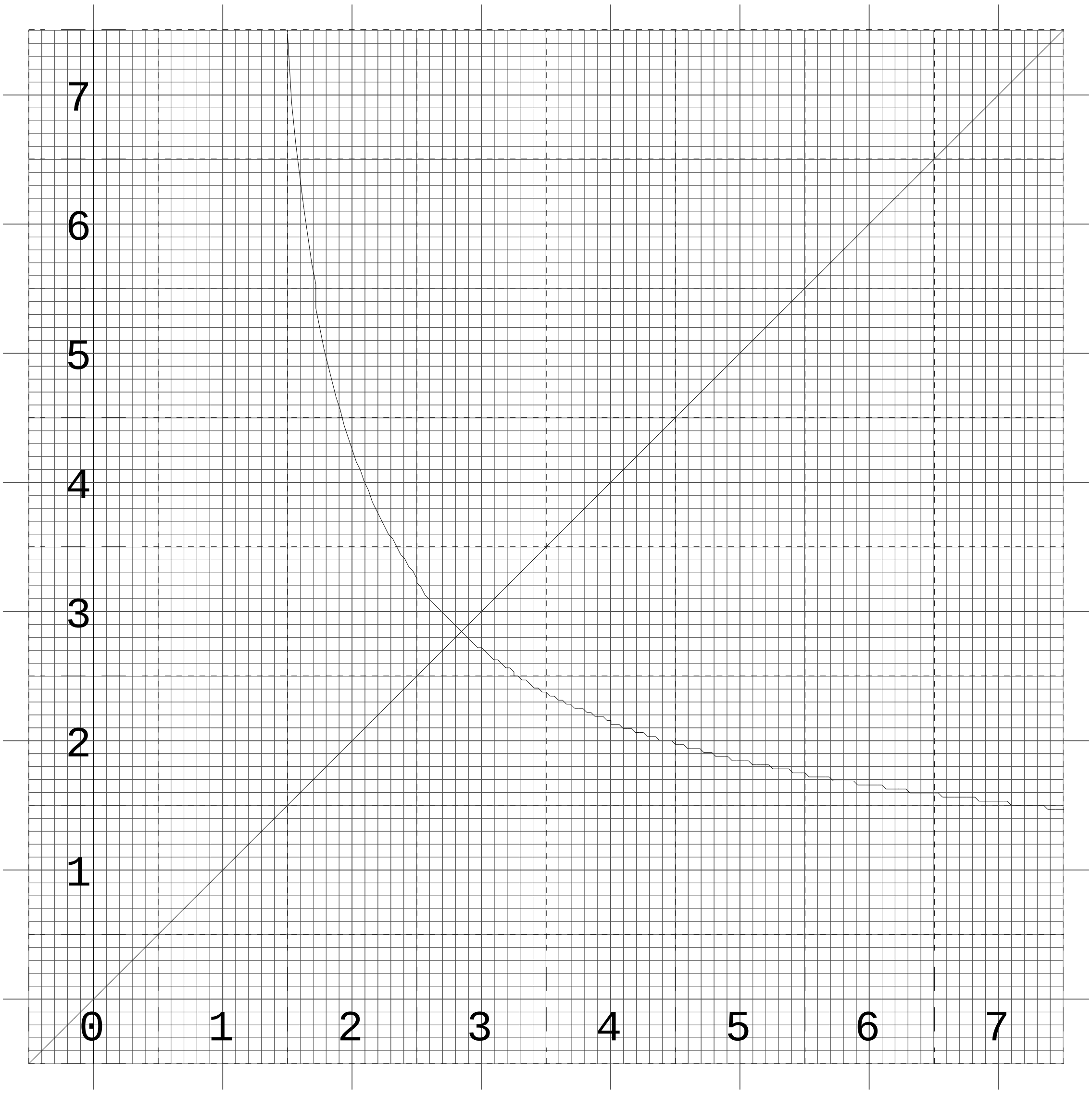
(c) With $x_0 = 2$, calculate x_1 and $x_{2.}$

(d) Plot the iterative process x_0, x_1 and x_2 on a cobweb diagram.
Use the graph on the following page.

(e) What you have done so far, indicates that this iteration may be problematical.
Explain what the nature of the problem is.

(f) Iterate sufficiently to locate the root of $f(x)$ correct to 3 decimal places.
Write down the your answer.

(g) Prove that your **part (f)** answer is indeed the root correct
to 3 decimal places.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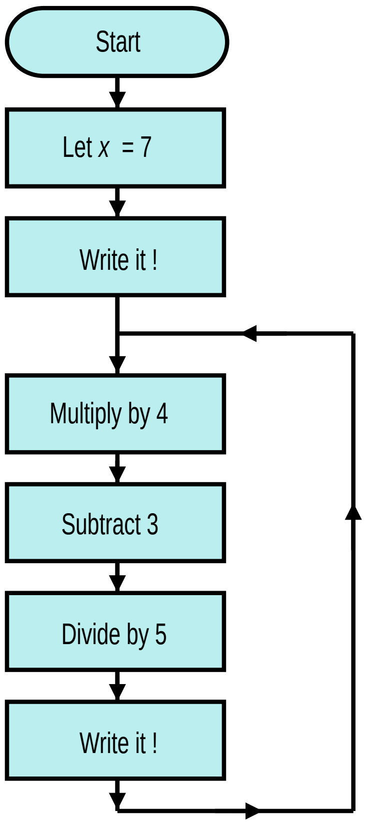
Lesson 6

Numerical Methods : Year 2

6.1 TEST

Question 1



(i) List the first ten numbers produced by the iteration described in the flow chart.

[3 marks]

(ii) This iteration is heading towards a fixed value.
What do you think the fixed value is ?

[1 mark]

(iii) What happens if you start the iteration using the fixed value ?

[1 mark]

(iv) The part **(iii)** method of verifying the fixed value only works for an iteration with a fixed value which is a rational number.
What is a rational number ?

[1 mark]

Question 2

A sequence, G, has the iteration formula;

$$G_{n+1} = \sqrt{\frac{13}{G_n}} \qquad with \qquad G_1 = 13$$

(i) Calculate the values of G_2, G_3, G_4 ..... G_{12} writing each to 3 decimal places.

[3 marks]

(ii) The fixed value is $(13)^{\frac{1}{3}}$.

Prove this by algebraically manipulating the equation

$$x = \sqrt{\frac{13}{x}}$$

into the form

$$x = \sqrt[3]{13}$$

[3 marks]

Question 3

This question is about the function

$$f(x) = e^x - \frac{1}{x}$$

(i) On one graph sketch the curves $y = e^x$ and $y = \frac{1}{x}$

Indicate where your sketch shows that $f(x) = 0$ has a root.

[3 marks]

(ii) Calculate **(a)** $f(0.5)$

(b) $f(0.6)$

[2 marks]

(iii) How do the part **(ii)** answers prove that there is a root to the equation $f(x) = 0$ in the interval $0.5 \leqslant x \leqslant 0.6$?

[1 mark]

(iv) Show that $f(x) = 0$, implies $x = e^{-x}$

[2 marks]

(v) Using the iterative formula

$$x_{n+1} = e^{-x_n} \qquad \textit{with} \qquad x_0 = 0.55$$

to find x_1, x_2 and x_3

[2 marks]

(vi) Keep iterating, finally stating the root of $f(x) = 0$ to about 8 decimal places.

[1 mark]

Question 4

C3 Examination question from January 2006, Q5.

$$f(x) = 2x^3 - x - 4$$

(a) Show that the equation $f(x) = 0$ can be written as

$$x = \sqrt{\left(\frac{2}{x} + \frac{1}{2}\right)}$$

[3 marks]

The equation $2x^3 - x - 4 = 0$ has a root between 1.35 and 1.4.

(b) Use the iteration formula

$$x_{n+1} = \sqrt{\left(\frac{2}{x_n} + \frac{1}{2}\right)}$$

with $x_o = 1.35$, to find, to 2 decimal places, the value of x_1, x_2 and x_3.

[3 marks]

The only real root of $f(x) = 0$ is α.

(c) By choosing a suitable interval, prove that $\alpha = 1.392$, to 3 decimal places.

[3 marks]

Question 5

C3 Examination question from June 2011, Q2.

$$f(x) = 2\sin(x^2) + x - 2. \qquad 0 \leqslant x < 2\pi$$

(a) Show that $f(x) = 0$ has a root α between $x = 0.75$ and $x = 0.85$.

[2 marks]

The equation $f(x) = 0$ can be written as $x = [\,arcsin\,(1 - 0.5x)\,]^{\frac{1}{2}}$.

(b) Use the iterative formula

$$x_{n+1} = [\,arcsin\,(1 - 0.5x)\,]^{\frac{1}{2}}, \qquad x_0 = 0.8$$

to find the values of x_1, x_2 and x_3, giving your answers to 5 decimal places.

[3 marks]

(c) Show that $\alpha = 0.80157$ is correct to 5 decimal places.

[3 marks]

Question 6

C3 Examination question from January 2012, Q6.

$$f(x) = x^2 - 3x + 2\cos\left(\frac{1}{2}x\right). \qquad 0 \leqslant x < \pi$$

(a) Show that the equation $f(x) = 0$ has a solution in the interval $0.8 < x < 0.9$

[2 marks]

The curve with equation $y = f(x)$ has a minimum point P.

(b) Show that the x-coordinate of P is the solution of the equation

$$x = \frac{3 + \sin\left(\frac{1}{2}x\right)}{2}$$

[4 marks]

(c) Using the iteration formula

$$x_{n+1} = \frac{3 + \sin\left(\frac{1}{2}x_n\right)}{2}, \qquad x_0 = 2$$

find the values of x_1, x_2 and x_3, giving your answers to 3 decimal places.

[3 marks]

(d) By choosing a suitable interval, show that the x-coordinate of P is 1.9078 correct to 4 decimal places.

[3 marks]

Lesson 7

7.1 The Newton-Raphson Iteration Formula

Given an equation of the form

$$f(x) = 0$$

the Newton-Raphson iteration formula to find numerical solutions is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In many cases this formula will result in swift convergence upon a root, especially if the starting value is chosen carefully.

However, in common with all iterative methods it can fail for a variety of reasons. For example,

- It may diverge rather than converge
- Convergence may be very slow
- The iteration may get stuck in a periodic loop
- A division by zero may crash the iteration
- The taking of a negative square root may crash the iteration
- The taking of a zero or negative logarithm may crash the iteration

Furthermore, because the Newton-Raphson formula works by 'firing tangents' at the *x*-axis it can fail if $f(x)$ has any points at which the derivative does not exist or if a tangent is 'fired' parallel to the *x*-axis, which makes turning points a source of iteration failure.

7.2 Example

(i) Sketch the graph of $y = \sin x$ and the graph of $y = 0.25x$ over $0 \leqslant x \leqslant 2\pi$

Consider the function

$$f(x) = \sin x - 0.25x$$

(ii) Write down the derivative of this function, $f'(x)$

A root of the equation $f(x) = 0$ is to be found by use of the Newton-Raphson iteration formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad x_0 = 3$$

(iii) Determine the values of the first 6 terms of the iterative sequence, writing your answers to 8 decimal places.

(iv) By considering the change of sign in $f(x)$ over an appropriate interval, show that a root is 2.475, correct to 3 decimal places.

7.3 Exercise

Question 1

(i) Sketch the graph of $y = e^x$ and the graph of $y = \frac{1}{x}$

Consider the function

$$f(x) = e^x - \frac{1}{x}$$

(ii) Write down the derivative of this function, $f'(x)$

A root of the equation $f(x) = 0$ is to be found by use of the Newton-Raphson iteration formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad x_0 = 1$$

(iii) Determine the values of the first 6 terms of the iterative sequence, writing your answers to 8 decimal places.

(iv) By considering the change of sign in $f(x)$ over an appropriate interval, show that a root is 0.567, correct to 3 decimal places.

Question 2

(i) Sketch the graph of $y = \sqrt{x}$ and the graph of $y = e^{-x}$

Consider the function

$$f(x) = \sqrt{x} - e^{-x}$$

(ii) Write down the derivative of this function, $f'(x)$

A root of the equation $f(x) = 0$ is to be found by use of the Newton-Raphson iteration formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad x_0 = 1$$

(iii) Determine the values of the first 6 terms of the iterative sequence, writing your answers to 8 decimal places.

(iv) By considering the change of sign in $f(x)$ over an appropriate interval, show that a root is 0.426, correct to 3 decimal places.

Question 3

(i) Sketch the graph of $y = x \sin x$ and the graph of $y = \cos x$, $0 \leqslant x \leqslant 2\pi$

Consider the function

$$f(x) = x \sin x - \cos x$$

(ii) Write down the derivative of this function, $f'(x)$

A root of the equation $f(x) = 0$ is to be found by use of the Newton-Raphson iteration formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad x_0 = 1$$

(iii) Determine the values of the first 6 terms of the iterative sequence, writing your answers to 8 decimal places.

(iv) By considering the change of sign in $f(x)$ over an appropriate interval, show that a root is 0.860, correct to 3 decimal places.

Question 4

(i) By writing $y = x^x$ in the form $ln\, y = x\, ln\, x$, show that

$$\frac{dy}{dx} = x^x (ln\, x + 1)$$

[4 marks]

(ii) Show that the function $f (x) = x^x - 2$ has a root, β, in the interval [1.4, 1.6]

[2 marks]

(iii) Taking $x_0 = 1.5$ as a first approximation to β, apply the Newton-Raphson procedure once to obtain a second approximation to β, giving your answer to 4 decimal places

[4 marks]

(iv) By considering a change of sign of $f (x)$ over a suitable interval, show that $\beta = 1.5596$ correct to 4 decimal places

[3 marks]

7.4 Solutions

7.4.1 The Example, 7.2

(i) Graph of $y = \sin x$ and $y = 0.25x$

(ii) Emphasise need to use <u>**RADIANS**</u>

$$f(x) = \sin x - 0.25x$$
$$f'(x) = \cos x - 0.25$$

(iii)

$$x_{n+1} = x_n - \frac{\sin x_n - 0.25 x_n}{\cos x_n - 0.25}$$

$$x_0 = 3$$
$$x_1 = 2.508\,964\,777$$
$$x_2 = 2.474\,912\,836$$
$$x_3 = 2.474\,576\,821$$
$$x_4 = 2.474\,576\,787$$

$$\textit{which is now fixed to } 9 \textit{ decimal places}$$

(iv)

$$f(2.4745) = 0.000\,079\,52$$
$$f(2.4755) = -0.0009564$$

Change of sign, along with the part **(i)** graph indicate that there is one root in the interval [2.4745, 2.4755[
As all numbers in this interval round to 2.475, that is the root, correct to 3 decimal places □

7.4.2 The Exercise, 7.3

Answer 1

(i) Graph of $y = e^{x}$ and $y = \frac{1}{x}$

(ii)

$$f(x) = e^{x} - \frac{1}{x}$$

$$f'(x) = e^{x} + \frac{1}{x^{2}}$$

(iii)

$$x_{n+1} = x_n - \frac{e^{x_n} - \frac{1}{x_n}}{e^{x_n} + \frac{1}{(x_n)^2}}$$

$$x_0 = 1$$

$$x_1 = 0.537\,882\,842\,7$$

$$x_2 = 0.566\,277\,007\,7$$

$$x_3 = 0.567\,142\,580\,4$$

$$x_4 = 0.567\,143\,290\,4$$

$$\textit{which is now fixed to } 9 \textit{ decimal places}$$

(iv)

$$f(0.5665) = -0.003136131431$$

$$f(0.5675) = 0.001\,737\,367\,593$$

Change of sign, along with the part **(i)** graph indicate that there is one root in the interval [0.5665, 0.5675[
As all numbers in this interval round to 0.567, that is the root, correct to 3 decimal places ☐

Answer 2

(i) Graph of $y = \sqrt{x}$ and $y = e^{-x}$

(ii)

$$f(x) = \sqrt{x} - e^{-x}$$
$$f'(x) = \frac{1}{2\sqrt{x}} + e^{-x}$$

(iii)

$$x_{n+1} = x_n - \frac{\sqrt{x_n} - e^{-x_n}}{\frac{1}{2\sqrt{x_n}} + e^{-x_n}}$$

$$x_0 = 1$$

$$x_1 = 0.271\,649\,345\,7$$

$$x_2 = 0.411\,602\,332\,8$$

$$x_3 = 0.426\,183\,639\,2$$

$$x_4 = 0.426\,302\,743\,3$$

$$x_5 = 0.426\,302\,751$$

$$\textit{which is now fixed to } 9 \textit{ decimal places}$$

(iv)

$$f(0.4255) = -0.001\,139\,37$$

$$f(0.4265) = 0.000\,279\,809$$

Change of sign, along with the part **(i)** graph indicate that there is one root in the interval [0.4255, 0.4265[
As all numbers in this interval round to 0.426, that is the root, correct to 3 decimal places ☐

Answer 3

(i) Graph of $y = x \sin x$ and $y = \cos x$

(ii)

$$f(x) = x \sin x - \cos x$$

$$f'(x) = \sin x + x \cos x + \sin x$$

$$= 2 \sin x + x \cos x$$

(iii)

$$x_{n+1} = x_n - \frac{x_n \sin x_n - \cos x_n}{2 \sin x_n + x_n \cos x_n}$$

$$x_0 = 1$$

$$x_1 = 0.864\,536\,397\,4$$

$$x_2 = 0.860\,339\,077\,6$$

$$x_3 = 0.860\,333\,589$$

$$\textit{which is now fixed to } 9 \textit{ decimal places}$$

(iv)

$$f(0.8595) = -0.001\,731\,091\,478$$

$$f(0.8605) = 0.000\,345\,689\,721\,2$$

Change of sign, along with the part **(i)** graph indicate that there is one root in the interval [0.8595, 0.8605[

As all numbers in this interval round to 0.860, that is the root, correct to 3 decimal places ☐

Answer 4
(i)

$$y = x^x$$

$$\textit{take the ln of both sides}$$

$$ln\,y = ln\,x^x$$

$$ln\,y = x\,ln\,x$$

$$\textit{exponentiate both sides}$$

$$y = e^{x\,ln\,x}$$

$$\textit{differentiate, using the chain and product rules}$$

$$\frac{dy}{dx} = e^{x\,ln\,x}\,(ln\,x + x\,.\,\frac{1}{x})$$

$$\frac{dy}{dx} = e^{ln\,x^x}\,(\,ln\,x + 1\,)$$

$$\frac{dy}{dx} = x^x\,(\,ln\,x + 1\,) \qquad \square$$

(ii)

$$f\,(x) = x^x - 2$$

$$f\,(1.4) = -0.398\,307\,101\,8$$

$$f\,(1.6) = 0.121\,250\,571\,1$$

The change of sign indicates that there is a root in the interval [1.4, 1.6]

(iii)

$$x_{n+1} = x_n - \frac{(\,x_n\,)^{x_n} - 2}{(\,x_n\,)^{x_n}\,(\,ln\,x_n + 1\,)}$$

$$x_0 = 1.5$$

$$x_1 = 1.563\,083\,82 \qquad i.e\ \ 1.563$$

(iv)

$$f\,(1.55955) = -0.000\,174\,678\,574\,6$$

$$f\,(1.55965) = 0.000\,114\,202\,932\,9$$

Change of sign indicates there is a root in the interval [1.55955, 1.55965[
As all numbers in this interval round to 1.5596, that is the root, correct to 4 decimal places ☐

8.1 Examination Questions On The Newton-Raphson Iteration Formula

Given an equation of the form

$$f(x) = 0$$

the Newton-Raphson iteration formula to find numerical solutions is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

8.2 Exercise

Question 1

FP1 Examination Question, June 2011, Q4

$$f(x) = x^2 + \frac{5}{2x} - 3x - 1 \qquad x \neq 0$$

(a) Use differentiation to find $f'(x)$

[2 marks]

The root α of the equation $f(x) = 0$ lies in the interval [0.7, 0.9]

(b) Taking 0.8 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α.
Give your answer to 3 decimal places

[4 marks]

Question 2

FP1 Examination Question, June 2008, Q2

$$f(x) = 4\cos x + e^{-x}$$

(a) Show that the equation $f(x) = 0$ has a root α between 1.6 and 1.7

[2 marks]

(b) Taking 1.6 as your first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to obtain a second approximation to α.
Give your answer to 3 significant figures

[4 marks]

Question 3
FP1 Examination Question, June 2016, Q2

$$f(x) = 3x^{\frac{3}{2}} - 25x^{-\frac{1}{2}} - 125 \qquad x > 0$$

(a) Find $f'(x)$

[2 marks]

The equation $f(x) = 0$ has a root α in the interval $[12, 13]$

(b) Using $x_0 = 12.5$ as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α.
Give your answer to 3 decimal places

[4 marks]

Question 4
FP1 Examination Question, June 2017, Q1

$$f(x) = \frac{1}{3}x^2 + \frac{4}{x^2} - 2x - 1 \qquad x > 0$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval [6,7]

[2 marks]

(b) Taking 6 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α.
Give your answer to 2 decimal places

[5 marks]

Question 5
FP1 Examination Question, January 2010, Q2

$$f(x) = x\cos x - 2x + 5$$

(a) Show that $f(x) = 0$ has a root α in the interval [2,2.1]

[2 marks]

(b) Taking 2 as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to obtain a second approximation to α.
Give your answer to 2 decimal places

[5 marks]

(c) Show that your answer to part **(b)** gives α **correct** to 2 decimal places

[2 marks]