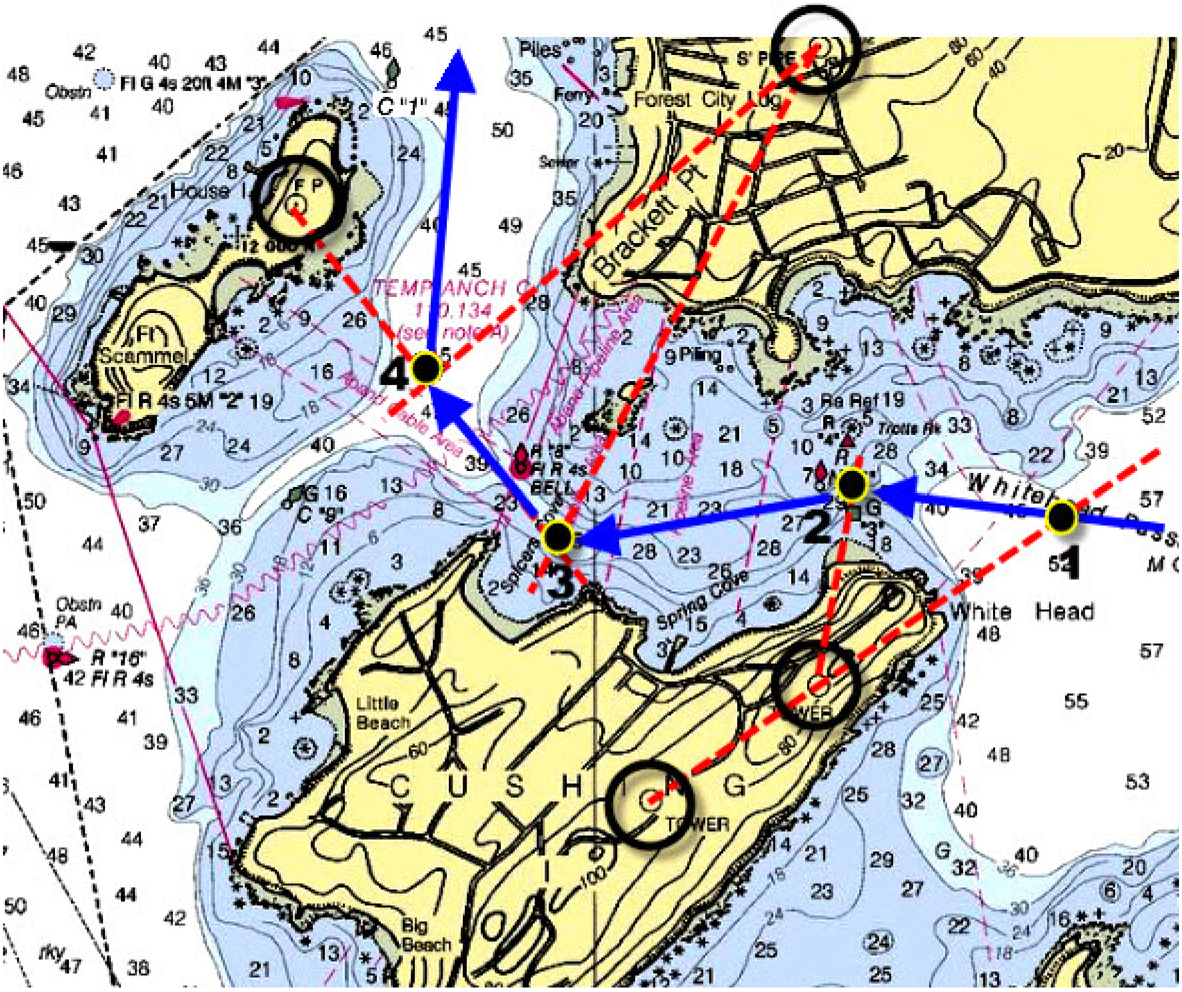
A-Level Pure Mathematics

Year 1 and Year 2

# VECTORS II



# V E C T O R S  I I

**Chapter 1**

**A-Level Pure Mathematics**
**Vectors II : Year 1 and Year 2**

## 1.1 Vectors and Kinematics

**Example 1**
A particle moves with initial velocity $( 7\,\boldsymbol{i} + 6\,\boldsymbol{j} )$ ms$^{-1}$
It is accelerating at $( -3\,\boldsymbol{i} + 5\,\boldsymbol{j} )$ ms$^{-2}$
**( i )** What is its velocity when $t = 4$ seconds ?

**( ii )** What is its speed when $t = 4$ seconds ?

**Example 2**
A particle is moving with initial velocity $( -2\,\boldsymbol{i} + \boldsymbol{j} )$ ms$^{-1}$
A constant acceleration of $( \boldsymbol{i} - 2\,\boldsymbol{j} )$ ms$^{-2}$ acts upon it.
**( i )** What is its displacement vector over the next 5 seconds ?

**( ii )** If it was initially at position $( 3\,\boldsymbol{i} + 4\,\boldsymbol{j} )$, where is it when $t$ is 5 seconds ?

## 1.2 Exercise

**Question 1**

A particle is initially moving with velocity ( 3 ***i*** + ***j*** ) $ms^{-1}$
It is constantly accelerating at ( – ***i*** + 2 ***j*** ) $ms^{-2}$

**( i )** What is its velocity when $t = 7$ seconds ?

**( ii )** What is its speed when $t = 7$ seconds ?

**Question 2**

A particle is moving with initial velocity ( 3 ***i*** + 2 ***j*** ) $ms^{-1}$
A constant acceleration of ( 4 ***i*** – **j** ) $ms^{-2}$ acts upon it.

**( i )** What is its displacement vector over the next 3 seconds ?

**( ii )** If initially at position ( – 20 ***i*** + 2 ***j*** ), what is its position when $t$ is 3 seconds ?

**Question 3**

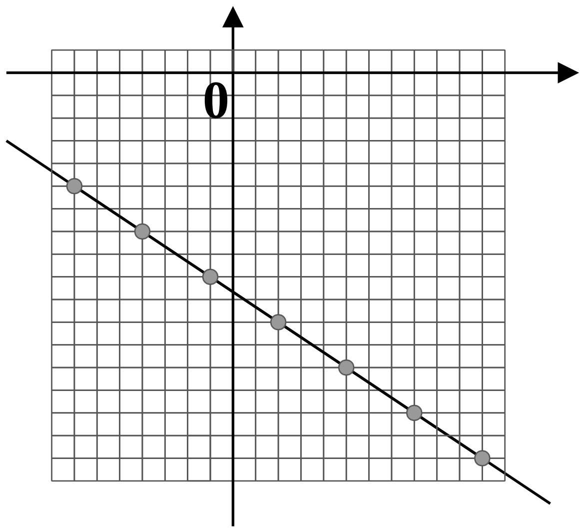
*M1 examination question, May 2010, Q1 with Hint added*

A particle $P$ is moving with constant velocity $( -3\,\boldsymbol{i} + 2\,\boldsymbol{j} )$ ms$^{-1}$

At time $t = 6$ s $P$ is at the point with position vector $( -4\,\boldsymbol{i} - 7\,\boldsymbol{j} )$ m

Find the distance of $P$ from the origin at time $t = 2$ s

**HINT :** This diagram may help...




**[ 5 marks ]**

**Question 4**

*M1 examination question, January 2009, Q1*

A particle $P$ moves with constant acceleration $( 2\,\boldsymbol{i} - 5\,\boldsymbol{j} )$ ms$^{-2}$

At time $t = 0$ $P$ has speed $u$ ms$^{-1}$

At time $t = 3$ s, $P$ has velocity $( -6\,\boldsymbol{i} + \boldsymbol{j} )$ ms$^{-1}$

Find the value of $u$

**[ 5 marks ]**

**Question 5**

*M1 examination question, January 2008, Q6*

*[ In this question, the unit vectors* **i** *and* **j** *are due east and due north respectively ]*

A particle $P$ is moving with constant velocity $( -5\,\boldsymbol{i} + 8\,\boldsymbol{j} )$ ms$^{-1}$

**( a )** Find the speed of $P$

**[ 2 marks ]**

**( b )** Find the direction of motion of $P$, giving your answer as a bearing

**[ 3 marks ]**

At time $t = 0$ $P$ is at the point $A$ with position vector $( 7\,\boldsymbol{i} - 10\,\boldsymbol{j} )$ m relative to a fixed origin $O$. When $t = 3$ s, the velocity of $P$ changes and it moves with velocity $( u\,\boldsymbol{i} + v\,\boldsymbol{j} )$ ms$^{-1}$, where $u$ and $v$ are constants. After a further 4 s, it passes through $O$ and continues to move with velocity $( u\,\boldsymbol{i} + v\,\boldsymbol{j} )$ ms$^{-1}$

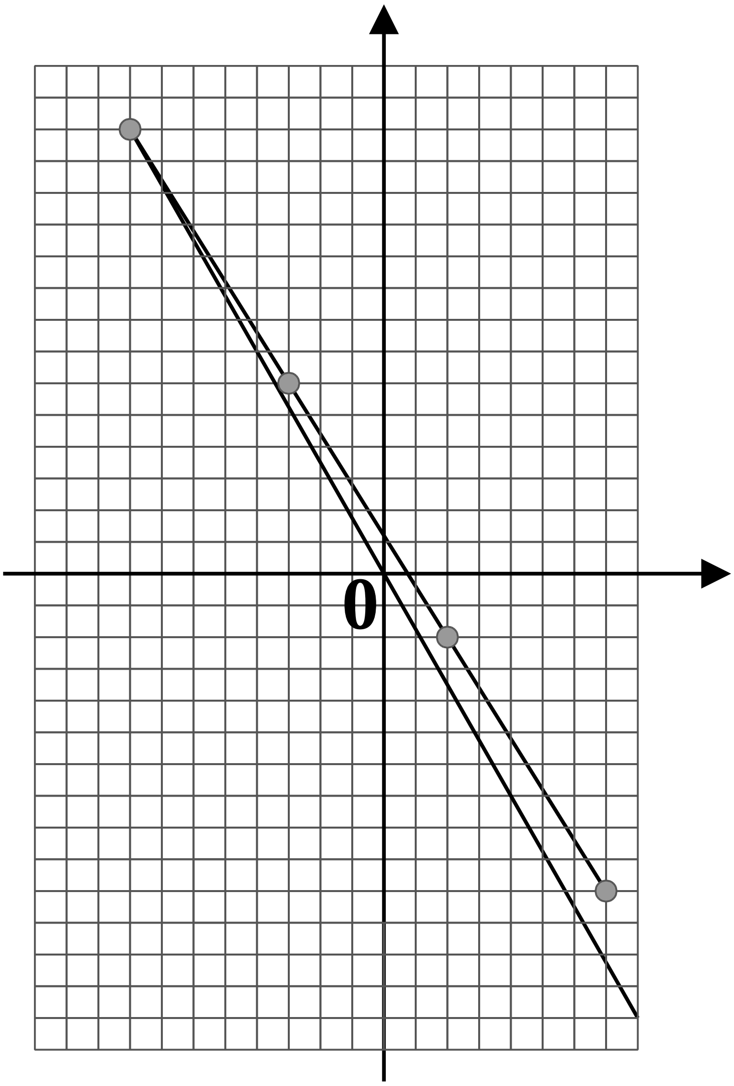
**( c )** Find the values of $u$ and $v$

**[ 5 marks ]**

**( d )** Find the total time taken for $P$ to move from $A$ to a position which is due south of $A$

**[ 3 marks ]**

**HINT :** This diagram may help...



**Question 6**

*M1 examination question, January 2010, Q7*

*[ In this question, the unit vectors* ***i*** *and* ***j*** *are horizontal unit vectors due east and due north respectively and position vectors are given with respect to a fixed origin ]*

A ship $S$ is moving along a straight line with constant velocity.
At time $t$ hours the position vector of $S$ is $\boldsymbol{s}$ km
When $t = 0$, $\boldsymbol{s} = 9\,\boldsymbol{i} - 6\,\boldsymbol{j}$
When $t = 4$, $\boldsymbol{s} = 21\,\boldsymbol{i} + 10\,\boldsymbol{j}$

**( a )** Find the speed of $S$

**[ 4 marks ]**

**( b )** Find the direction in which $S$ is moving, giving your answer as a bearing

**[ 2 marks ]**

**( c )** Show that $\boldsymbol{s} = (3t + 9)\,\boldsymbol{i} + (4t - 6)\,\boldsymbol{j}$

**[ 2 marks ]**

A lighthouse $L$ is located at the point with position vector $(18\,\boldsymbol{i} + 6\,\boldsymbol{j})$ km
When $t = T$, the ship $S$ is 10 km from $L$.

**( d )** Find the possible values of $T$.

**[ 6 marks ]**

**Question 7**

*M1 examination question, June 2007, Q7*

A boat $B$ is moving with constant velocity. At noon, $B$ is at the point with position vector $( 3 \boldsymbol{i} - 4 \boldsymbol{j} )$ km with respect to a fixed origin $O$. At 14:30 on the same day, $B$ is at the point with position vector $( 8 \boldsymbol{i} + 11 \boldsymbol{j} )$ km

**( a )** Find the velocity of $b$, giving your answer in the form $p \boldsymbol{i} + q \boldsymbol{j}$

**[ 3 marks ]**

At time $t$ hours after noon, the position vector of $B$ is $\boldsymbol{b}$ km

**( b )** Find, in terms of $t$, an expression for $\boldsymbol{b}$

**[ 3 marks ]**

Another boat $C$ is also moving with constant velocity. The position vector of $C$, $\boldsymbol{c}$ km, at time $t$ hours after noon, is given by

$$\boldsymbol{c} = ( -9\,\boldsymbol{i} + 20\,\boldsymbol{j} ) + t\,( 6\,\boldsymbol{i} + \lambda\,\boldsymbol{j} )$$

where $\lambda$ is a constant.

Given that $C$ intercepts $B$,

**( c )** find the value of $\lambda$

**[ 5 marks ]**

**( d )** show that, before $C$ intercepts $B$, the boats are moving with the same speed

**[ 3 marks ]**

## 1.3 Answers



### 1.3.1 Solutions (1.1 Examples)

**Answer 1**

**( i )**

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}\,t$$

$$\boldsymbol{v} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \end{pmatrix} \times 4$$

$$\boldsymbol{v} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} + \begin{pmatrix} -12 \\ 20 \end{pmatrix}$$

$$\boldsymbol{v} = \begin{pmatrix} -5 \\ 26 \end{pmatrix} \text{ ms}^{-1}$$

$$\boldsymbol{v} = -5\,\boldsymbol{i} + 26\,\boldsymbol{j} \quad \text{ms}^{-1}$$

**( ii )**

$$\begin{aligned} speed &= |\,\boldsymbol{v}\,| \\ &= \sqrt{(-5)^2 + (26)^2} \\ &= 26.5 \text{ ms}^{-1} \end{aligned}$$

**Answer 2**

**( i )**

$$\boldsymbol{s} = \boldsymbol{u}t + \frac{1}{2}\boldsymbol{a}\,t^2$$

$$\boldsymbol{s} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \times 5 + \frac{1}{2}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \times 5^2$$

$$\boldsymbol{s} = \begin{pmatrix} -10 \\ 5 \end{pmatrix} + \begin{pmatrix} 12.5 \\ -25 \end{pmatrix}$$

$$\boldsymbol{s} = \begin{pmatrix} 2.5 \\ -20 \end{pmatrix}$$

**( ii )**

$$\begin{aligned} \boldsymbol{r} &= \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2.5 \\ -20 \end{pmatrix} \\ &= \begin{pmatrix} 5.5 \\ -16 \end{pmatrix} \end{aligned}$$

or $5.5\,\boldsymbol{i} - 16\,\boldsymbol{j}$

### 1.3.2 Solutions (1.2 Exercise)

**Answer 1**

**( i )**

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}\, t$$

$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} 7$$

$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -7 \\ 14 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 15 \end{pmatrix}$$

$$\boldsymbol{v} = -4\,\boldsymbol{i} + 15\,\boldsymbol{j} \quad \text{ms}^{-1}$$

**( ii )**

$$speed = |\boldsymbol{v}|$$

$$= \sqrt{(-4)^2 + (15)^2}$$

$$= 15.5\,\text{ms}^{-1}$$

**Answer 2**

**( i )**

$$\boldsymbol{s} = \boldsymbol{u}t + \frac{1}{2}\boldsymbol{a}\,t^2$$

$$\boldsymbol{s} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \times 3 + \frac{1}{2}\begin{pmatrix} 4 \\ -1 \end{pmatrix} \times 3^2$$

$$\boldsymbol{s} = \begin{pmatrix} 9 \\ 6 \end{pmatrix} + \begin{pmatrix} 18 \\ -4.5 \end{pmatrix}$$

$$\boldsymbol{s} = \begin{pmatrix} 27 \\ 1.5 \end{pmatrix}$$

**( ii )**

$$\boldsymbol{r} = \begin{pmatrix} -20 \\ 2 \end{pmatrix} + \begin{pmatrix} 27 \\ 1.5 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 3.5 \end{pmatrix}$$

$$\text{or} \quad 7\,\boldsymbol{i} + 3.5\,\boldsymbol{j}$$

**Answer 3**

To get to the $t$ = 2 location, move for 4 s from the $t$ = 6 s position back down the path using the reversed velocity vector.

Note : The acceleration is zero as the question states the velocity is constant.

i.e.

$$\begin{pmatrix} -4 \\ -7 \end{pmatrix} + 4\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ -15 \end{pmatrix}$$

which is the particle's displacement from the origin when $t$ = 2 s

Question asks for *distance* which is the magnitude of the *displacement* vector

i.e.

$$\begin{aligned} distance &= \sqrt{8^2 + (-15)^2} \\ &= 17\text{ m} \end{aligned}$$

**Answer 4**

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{u} + \boldsymbol{a}t \\ \begin{pmatrix} -6 \\ 1 \end{pmatrix} &= \boldsymbol{u} + \begin{pmatrix} 2 \\ -5 \end{pmatrix} \times 3 \\ \boldsymbol{u} &= \begin{pmatrix} -12 \\ 16 \end{pmatrix} \end{aligned}$$

Question requests the initial speed which is the magnitude of the velocity vector

That is,

$$\begin{aligned} u &= \sqrt{(-12)^2 + 16^2} \\ &= 20\text{ ms}^{-1} \end{aligned}$$

**Answer 5**

**( a )** 9.43 $\text{ms}^{-1}$

**( b )** 328°

**( c )** Work out where particle is at $t$ = 3 seconds

$$\begin{pmatrix} 7 \\ -10 \end{pmatrix} + \begin{pmatrix} -5 \\ 8 \end{pmatrix} \times 3 = \begin{pmatrix} -8 \\ 14 \end{pmatrix}$$

New constant velocity from this point places particle at origin after 4 s

$$\begin{pmatrix} -8 \\ 14 \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} \times 4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ u = 2 \ \text{ and } \ v = -3.5$$

**( d )** To be due south of $A$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -3.5 \end{pmatrix} \times t = \begin{pmatrix} 7 \\ \text{don't care} \end{pmatrix}$$

$$t = 3.5$$

$$\therefore \text{ Total time is 10.5 seconds}$$

**Answer 6**

**( a )** As we have no acceleration, just constant velocity...

$$velocity = \frac{displacement}{time} \qquad \left( Vector\ version\ of\ \ speed = \frac{distance}{time} \right)$$

$$= \frac{\begin{pmatrix} 21 \\ 10 \end{pmatrix} - \begin{pmatrix} 9 \\ -6 \end{pmatrix}}{4}$$

$$= \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$speed = \sqrt{3^2 + 4^2}$$

$$= 5 \text{ kmh}^{-1}$$

**( b )** 037°

**( c )**

$$\boldsymbol{s} = (\ start\ location\ ) + (velocity) \times t$$

$$= \begin{pmatrix} 9 \\ -6 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} \times t$$

$$= \begin{pmatrix} 3t + 9 \\ 4t - 6 \end{pmatrix}$$

$$= (3t + 9)\,\boldsymbol{i} + (4t - 6)\,\boldsymbol{j}$$

**( d )**

$$\left| \begin{pmatrix} 3t + 9 \\ 4t - 6 \end{pmatrix} - \begin{pmatrix} 18 \\ 6 \end{pmatrix} \right| = 10$$

$$\left| \begin{pmatrix} 3t - 9 \\ 4t - 12 \end{pmatrix} \right| = 10$$

$$\sqrt{(3t - 9)^2 + (4t - 12)^2} = 10$$

square both sides

$$(3t - 9)^2 + (4t - 12)^2 = 100$$

$$9t^2 - 54t + 81 + 16t^2 - 96t + 144 = 100$$

$$25t^2 - 150t + 125 = 0$$

divide through by 25

$$t^2 - 6t + 5 = 0$$

$$(t - 1)(t - 5) = 0$$

$$t = 1 \quad \text{or} \quad t = 5$$

**Answer 7**

**( a )** As we have no acceleration, just constant velocity...

$$velocity = \frac{displacement}{time} \qquad \left( Vector\ version\ of\ \ speed = \frac{distance}{time} \right)$$

$$= \frac{\begin{pmatrix} 8 \\ 11 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \end{pmatrix}}{2.5}$$

$$= \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

$$= 2\,\boldsymbol{i} + 6\,\boldsymbol{j}$$

**( b )**

$$b = \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} \times t$$

$$= \begin{pmatrix} 2t + 3 \\ 6t - 4 \end{pmatrix}$$

$$= (\,2t + 3\,)\,\boldsymbol{i} + (\,6t - 4\,)\,\boldsymbol{j}$$

**( c )** At intercept...

$$\begin{pmatrix} 2t + 3 \\ 6t - 4 \end{pmatrix} = \begin{pmatrix} 6t - 9 \\ \lambda t + 20 \end{pmatrix}$$

First solve the upper $\boldsymbol{i}$ component line;

$$2t + 3 = 6t - 9$$

$$4t = 12$$

$$t = 3$$

Use this answer in the lower $\boldsymbol{j}$ component line;

$$6 \times 3 - 4 = 3\lambda + 20$$

$$14 - 20 = 3\lambda$$

$$\lambda = -2$$

**( d )** From part (a), $B$ has speed $\sqrt{2^2 + 6^2} = \sqrt{40}$

Knowing that $\lambda = 2$, $C$ has speed $\sqrt{6^2 + (-2)^2} = \sqrt{40}$

$\therefore$ $B$ and $C$ had the same speed.

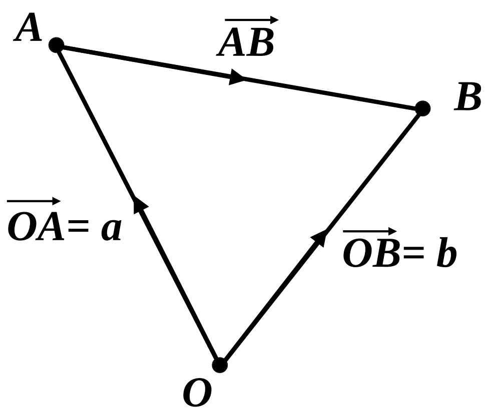
## 2.1 The Vector Between Two points

**Statement :**

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$$

**Proof :**

The result is obvious from a study of the following diagram




A more mathematical proof is to argue as follows;

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} \qquad \square$$

In words we say that $\overrightarrow{AB}$ is **b** relative to **a**
with the words "relative to" being the interpretation of the minus sign.

Interpretation:

If these were displacement vectors $\overrightarrow{AB}$ is the position of ***b*** relative to ***a*** which tells you how to get to ***b*** from ***a***.

## 2.2 Exercise

**Question 1**

$A$ is ( 1, 4 )

$B$ is ( 7, 6 )

Write down $\overrightarrow{AB}$

**Question 2**

$C$ is ( – 3, 4 )

$D$ is ( 2, 3 )

Write down $\overrightarrow{CD}$

**Question 3**

$P$ is ( 7, 3 )

$Q$ is ( 1, 2 )

Write down $\overrightarrow{PQ}$

**Question 4**

$A$ is ( – 1, 4 )

$B$ is ( 5, 9 )

Write down $\overrightarrow{AB}$

**Question 5**

$A$ is ( – 3, – 5 )

$B$ is ( 2, 1 )

Write down $\overrightarrow{BA}$

**Question 6**

$A$ is ( 5, – 2 )

$B$ is ( 3, 0 )

Write down $\overrightarrow{BA}$

**Question 7**

$P$ is ( 4, 1 )

$Q$ is ( 6, 3 )

Write down $\overrightarrow{QP}$

**Question 8**

$A$ is ( – 1, – 3 )

$B$ is ( – 5, – 8 )

Write down $\overrightarrow{AB}$

**Question 9**
$P$ is the point ( 6, 5 ) and $Q$ is the point ( – 3, 3 )

*Determine the vector* $\overrightarrow{PQ}$

Does your vector take you from $P$ to $Q$ or from $Q$ to $P$ ?
( Draw a sketch of the situation to convince yourself that your answer is correct )

**Question 10**
$M$ is the point ( 7, – 4 ) and $N$ is the point ( 11, 8 )

*Determine the vector* $\overrightarrow{MN}$

Have you worked out the position of $M$ relative to $N$ or of $N$ relative to $M$ ?
( Draw a sketch of the situation to convince yourself that your answer is correct )

**Question 11**
$C$ is the point ( – 7, 12 ) and $D$ is the point ( 8, – 3 )
*Determine the position of C relative to D*

**Question 12**

At the start of a walk, I am at the position given by $\boldsymbol{r}_{\mathbf{A}} = 1.3\,\boldsymbol{i} + 0.4\,\boldsymbol{j}$ km

I walk directly, in a straight line, to $\boldsymbol{r}_{\mathbf{B}} = 0.3\,\boldsymbol{i} - 0.7\,\boldsymbol{j}$ km

**( i )** Determine the vector that describes my walk.

**( ii )** By using the theorem of Pythagoras, and your part (i) answer, determine the distance that I have walked.

**Question 13**

Two motor boats, *The Dragon*, and *The Runner*, sit side by side upon the ocean. They then separate, each at a constant velocity.

*The Dragon* has velocity $\boldsymbol{V}_{\mathbf{D}} = 4\,\boldsymbol{i} + 7\,\boldsymbol{j}$ kmh$^{-1}$

*The Runner* has velocity $\boldsymbol{V}_{\mathbf{R}} = 5\,\boldsymbol{i} + 5\,\boldsymbol{j}$ kmh$^{-1}$

**( i )** Which boat is faster and by how much ?

**( ii )** Calculate the velocity of *The Dragon* relative to *The Runner*.

**( iii )** Use your part (ii) answer to determine how long it takes until the two motor boats are 5 km apart.

**Question 14**

The velocities of particles $A$ and $B$ are $( u\,\boldsymbol{i} - 7\,\boldsymbol{j} )\ \text{ms}^{-1}$ and $( 5\,\boldsymbol{i} + v\,\boldsymbol{j} )\ \text{ms}^{-1}$ respectively. The velocity of $A$ relative to $B$ is $( 2\,\boldsymbol{i} - 3\,\boldsymbol{j} )\ \text{ms}^{-1}$

Find the values of $u$ and $v$.

**Question 15**

The velocities of two particles $A$ and $B$ are $( 13\,\boldsymbol{i} - 3\,\boldsymbol{j} )\ \text{ms}^{-1}$ and $( 5\,\boldsymbol{i} + 12\,\boldsymbol{j} )\ \text{ms}^{-1}$ respectively.

Find;

**( i )** the speed of $B$,

**( ii )** the velocity of $B$ relative to $A$,

**( iii )** the angle between this relative velocity and the positive $x$-axis direction, giving your answer to the nearest degree.

**Question 16**
I am at the position $\boldsymbol{r} = 7\,\boldsymbol{i} + 5\,\boldsymbol{j}$ m.
My velocity is given by $\boldsymbol{v} = 2\,\boldsymbol{i} + 4\,\boldsymbol{j}$ $\text{ms}^{-1}$
If I have no acceleration, what is my position 4 seconds later ?

**Question 17**
The position of a particle at time $t$ is given by $\boldsymbol{r} = (\,2t - 9\,)\,\boldsymbol{i} + (\,t - 2\,)\,\boldsymbol{j}$

**( i )** If $d$ is the distance of $\boldsymbol{r}$ from the origin at time $t$, find an expression for $d$ that involves the square root of a quadratic equation in $t$ (Hint: Pythagoras).

**( ii )** Show, by completing the square on the quadratic, that;

$$\frac{1}{5}d^2 = (\,t - 4\,)^2 + 1$$

**( iii )** What value of $t$ makes $\frac{1}{5}d^2$ as small as possible ?
This is the time at which the particle is closest to the origin.

**( iv )** What is this minimum distance ?

## 2.3 Answers



### 2.3.1 Solutions (2.2 Exercise)

**Answer 1**

$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

**Answer 2**

$\overrightarrow{CD} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

**Answer 3**

$\overrightarrow{PQ} = \begin{pmatrix} -6 \\ -1 \end{pmatrix}$

**Answer 4**

$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

**Answer 5**

$\overrightarrow{BA} = \begin{pmatrix} -5 \\ -6 \end{pmatrix}$

**Answer 6**

$\overrightarrow{BA} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

**Answer 7**

$\overrightarrow{QP} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

**Answer 8**

$\overrightarrow{BA} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$

**Answer 9**

$\overrightarrow{PQ} = \begin{pmatrix} -9 \\ -2 \end{pmatrix}$

$P$ to $Q$

**Answer 10**

$\overrightarrow{MN} = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$

$N$ relative to $M$

**Answer 11**

$\overrightarrow{DC} = \begin{pmatrix} -15 \\ 15 \end{pmatrix}$

**Answer 12.**

$\overrightarrow{AB} = \begin{pmatrix} -1 \\ -1.1 \end{pmatrix}$ Distance is 1.49 km

**Answer 13**

**( i )** Dragon : $\sqrt{65}$ = 8.06 kmh$^{-1}$
Runner : $\sqrt{50}$ = 7.07 kmh$^{-1}$

$\therefore$ Dragon is faster by about 1 km/h

**( ii )**

$\overrightarrow{RD} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

**( iii )** Speed of separation is $\sqrt{5}$

$Time = \dfrac{Distance}{Speed} = \sqrt{5}$ hours, which is 2 hours 14 minutes

**Answer 14**

$u = 7$, $v = -4$

**Answer 15**

**( i )** Speed = 13 ms$^{-1}$

**( ii )** $\overrightarrow{AB} = \begin{pmatrix} -8 \\ 15 \end{pmatrix}$

**( iii )** 118°

**Answer 16**

$\boldsymbol{r} = 15\,\boldsymbol{i} + 21\,\boldsymbol{j}$

**Answer 17**

**( i )**

$$d = \sqrt{(2t - 9)^2 + (t - 2)^2}$$

$$d = \sqrt{4t^2 - 36t + 81 + t^2 - 4t + 4}$$

$$d = \sqrt{5t^2 - 40t + 85}$$

**( ii )**

$$d^2 = 5t^2 - 40t + 85$$

$$\frac{1}{5}d^2 = t^2 - 8t + 17$$

$$\frac{1}{5}d^2 = \left(t^2 - 8t + 16\right) + 1$$

$$\frac{1}{5}d^2 = (t - 4)^2 + 1$$

**( iii )** $t = 4$ seconds

**( iv )** $\sqrt{5}$ or 2.24

Chapter 3



## 3.1 Vectors and Motion

Vectors are used to keep track of shipping and aircraft.
Most importantly, they are used by air traffic control centres to calculate minimum distances between aircraft flight paths, so that "near misses" are avoided.

Unless told otherwise, $\boldsymbol{i}$ and $\boldsymbol{j}$ are **unit** vectors due east and due north respectively.
A **unit** vector has a magnitude of 1.

## 3.2 The Near Miss Problem

The position of a particle at time $t$ is given by; $\boldsymbol{r} = ( t - 4 )\,\boldsymbol{i} + ( t - 2 )\,\boldsymbol{j}$

**( i )** If $d$ is the distance of $\boldsymbol{r}$ from the origin at time $t$, find an expression for $d$ that involves the square root of a quadratic equation in $t$.

**( ii )** Show, by completing the square on the quadratic that;

$$\frac{1}{2} d^2 = ( t - 3 )^2 + 1$$

**( iii )** What value of $t$ makes $\frac{1}{2} d^2$ as small as possible ?

This is the time at which the particle is closest to the origin.

**( iv )** What is this minimum distance ?

### 3.3 Exercise

**Question 1**

Two motor boats $P$ and $Q$ sit side by side upon the ocean.
At 12.15 pm they separate, each at a constant velocity.
$P$ has velocity $\boldsymbol{V}\mathbf{p} = 8\,\boldsymbol{i} + 7\,\boldsymbol{j}$ $\text{ms}^{-1}$ and $Q$ has velocity $\boldsymbol{V}\mathbf{q} = 3\,\boldsymbol{i} - 5\,\boldsymbol{j}$ $\text{ms}^{-1}$

**( i )** Find the velocity of $P$ relative to $Q$.

**( ii )** Use the theorem of Pythagoras, and your part (i) answer to calculate the speed of separation.

**( iii )** After how many seconds are they 520 metres apart ?

**Question 2**

I am at the position $\boldsymbol{r} = 16\,\boldsymbol{i} - 7\,\boldsymbol{j}$ m.
My initial velocity is given by $\boldsymbol{u} = 3\,\boldsymbol{i} + \boldsymbol{j}$ $\text{m.s}^{-1}$
I am under constant a acceleration of $\boldsymbol{a} = -\,\boldsymbol{i} + \boldsymbol{j}$

**( i )** Use the formula $\boldsymbol{s} = \boldsymbol{u}t + \dfrac{1}{2}\boldsymbol{a}\,t^2$ to find my position 3 seconds later.

**( ii )** Find an expression (in terms of $t$) for the velocity at time $t$.
HINT: $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}\,t$

**( iii )** By using your part (ii) answer, or otherwise, find the time at which I am moving in the direction $\boldsymbol{i} + \boldsymbol{j}$

*This next question a "near miss" problem, like that at the start of the lesson.*

**Question 3**

The position of Cedric, my cyber-dog, at time $t$ is given by;

$$\boldsymbol{r} = (t - 5)\,\boldsymbol{i} + (t + 1)\,\boldsymbol{j} \quad \text{metres.}$$

**( i )** What is the distance of Cedric from the origin when $t = 5$ ?

**( ii )** Find an expression involving $t$ for the distance that Cedric is from the origin at time $t$.

If Cedric moves further than 30 metres from the origin I can no longer receive his video signal transmission.

**( iii )** By using your part (ii) answer, or otherwise, find the time at which I lose video signal contact with Cedric.

**Question 4**

At noon the position of a ship relative to a radar station *O* is ( 100 $\boldsymbol{i}$ +60 $\boldsymbol{j}$ ) km
$\boldsymbol{i}$ and $\boldsymbol{j}$ being vectors due east and due north respectively.
The velocity of the ship is constant at $-10\,\boldsymbol{i} + 8\,\boldsymbol{j}$ kmh$^{-1}$

**( i )** Find the time when the ship is due north of *O*

**( ii )** How far the ship then is from *O*

**( iii )** How far the ship is from *O* at 1 pm

**Question 5**

Two ice skaters pass close to each other at constant velocities at an ice rink.

Skater $A$ has velocity $\boldsymbol{V}_A = \boldsymbol{i} + 3\,\boldsymbol{j}$ and skater $B$ has velocity $\boldsymbol{V}_B = 2\,\boldsymbol{i} + \boldsymbol{j}$

**( i )** Find the velocity of $B$ relative to $A$.

At $t = 0$ the positions of the skaters are $\boldsymbol{S}_A = \mathbf{i} - 3\,\mathbf{j}$ and $\boldsymbol{S}_B = -3\,\boldsymbol{i} + 5\,\boldsymbol{j}$

**( ii )** Show that, at time $t$, $A$ has position

$$\boldsymbol{R}_A = (1 + t)\,\boldsymbol{i} + (-3 + 3\,t)\,\boldsymbol{j}$$

and find a similar expression for the position of $B$ at time $t$.

**( iii )** Find an expression for the position of $B$ relative to $A$.

**( iv )** By considering your part (iii) expression find the value of $t$ for which $A$ and $B$ were closest together.

**( v )** What is this closest distance ?

## 3.4 Answers



### 3.4.1 Solutions (3.2 The Near Miss problem)

**( i )**

$$d = \sqrt{(t-4)^2 + (t-2)^2}$$

$$d = \sqrt{t^2 - 8t + 16 + t^2 - 4t + 4}$$

$$d = \sqrt{2t^2 - 12t + 20}$$

**( ii )**

$$d^2 = 2t^2 - 12t + 20$$

$$\frac{1}{2}d^2 = \left[t^2 - 6t\right] + 10$$

$$\frac{1}{2}d^2 = \left[(t-3)^2 - 9\right] + 10$$

$$\frac{1}{2}d^2 = (t-3)^2 + 1$$

**( iii )** The smallest value of $\frac{1}{2}d^2$ occurs when $(t-3)^2 = 0$

i.e. When $t = 3$ seconds

**( iv )** $\boldsymbol{r} = (t-4)\,\boldsymbol{i} + (t-2)\,\boldsymbol{j}$

when $t = 3$ becomes $\boldsymbol{r} = -\boldsymbol{i} + \boldsymbol{j}$

Distance $= \sqrt{2}$

$= 1.414$

### 3.4.2 Solutions (3.3 Exercise)

**Answer 1**

**( i )** $\overrightarrow{PQ} = 5\,\boldsymbol{i} + 12\,\boldsymbol{j}$ **( ii )** 13 ms$^{-1}$ **( iii )** 40 seconds

**Answer 2**

**( i )**

$$\boldsymbol{s} = \boldsymbol{u}\,t + \frac{1}{2}\boldsymbol{a}\,t^2$$

$$= \begin{pmatrix}3\\1\end{pmatrix} \times 3 + \frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix} \times 9$$

$$= \begin{pmatrix}9\\3\end{pmatrix} + \begin{pmatrix}-4.5\\4.5\end{pmatrix}$$

$$= \begin{pmatrix}4.5\\7.5\end{pmatrix}$$

So position is $20.5\,\boldsymbol{i} + 0.5\,\boldsymbol{j}$

**( ii )**

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t$$
$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} t$$
$$= \begin{pmatrix} 3 - t \\ 1 + t \end{pmatrix}$$
$$\boldsymbol{v} = (3 - t)\,\boldsymbol{i} + (1 + t)\,\boldsymbol{j}$$

**( iii )** We move in the direction $\boldsymbol{i} + \boldsymbol{j}$ when $3 - t = 1 + t$
Solving this gives $t = 1$ second

**Answer 3**

**( i )** When $t = 5$, $\boldsymbol{r} = 0\,\boldsymbol{i} + 6\,\boldsymbol{j}$ $\therefore$ Distance from origin is 6 m.

**( ii )**

$$d = \sqrt{(t - 5)^2 + (t + 1)^2}$$
$$d = \sqrt{t^2 - 10t + 25 + t^2 + 2t + 1}$$
$$d = \sqrt{2t^2 - 8t + 26}$$

**( iii )**

$$30 = \sqrt{2t^2 - 8t + 26}$$
$$900 = 2t^2 - 8t + 26$$
$$2t^2 - 8t - 874 = 0$$
$$t^2 - 4t - 437 = 0$$
$$(t - 23)(t + 19) = 0$$
$$\therefore t = 23 \text{ seconds}$$

**Answer 4**

**( i )** $\boldsymbol{r} = 100\,\boldsymbol{i} + 60\,\boldsymbol{j} + (-10\,\boldsymbol{i} + 8\,\boldsymbol{j})\,t$
$\boldsymbol{r} = (100 - 10t)\,\boldsymbol{i} + (60 + 8t)\,\boldsymbol{j}$
Due north requires $100 - 10t = 0$ in which case $t = 10$ hours
$\therefore$ Time is 10 pm

**( ii )** Distance $= 60 + 8 \times 10$
$= 140$ km

**( iii )** At 1 pm, $t = 1$, $\boldsymbol{r} = (100 - 10)\,\boldsymbol{i} + (60 + 8)\,\boldsymbol{j}$
Distance $= \sqrt{90^2 + 68^2}$
$= 112.8$ km

**Answer 5**

**( i )** $\overrightarrow{AB} = \boldsymbol{i} - 2\,\boldsymbol{j}$

**( ii )** $\boldsymbol{R_A} = (\,\boldsymbol{i} - 3\,\boldsymbol{j}\,) + (\,\boldsymbol{i} + 3\,\boldsymbol{j}\,)\,t$
$= \boldsymbol{i} - 3\,\boldsymbol{j} + t\,\boldsymbol{i} + 3t\,\boldsymbol{j}$
$= (\,1 + t\,)\,\boldsymbol{i} + (\,-3 + 3t\,)\,\boldsymbol{j}$

$\boldsymbol{R_B} = (\,-3\,\boldsymbol{i} + 5\,\boldsymbol{j}\,) + (\,2\,\boldsymbol{i} + \boldsymbol{j}\,)\,t$
$= -3\,\boldsymbol{i} + 5\,\boldsymbol{j} + 2t\,\boldsymbol{i} + t\,\boldsymbol{j}$
$= (\,-3 + 2t\,)\,\boldsymbol{i} + (\,5 + t\,)\,\boldsymbol{j}$

**( iii )** $\boldsymbol{R_{AB}} = \boldsymbol{R_B} - \boldsymbol{R_A}$
$= (\,-4 + t\,)\,\boldsymbol{i} + (\,8 - 2t\,)\,\boldsymbol{j}$

**( iv )**

$$d = \sqrt{(\,4 - t\,)^2 + (\,8 - 2t\,)^2}$$
$$d = \sqrt{t^2 - 8t + 16 + 4\,t^2 - 32t + 64}$$
$$d = \sqrt{5\,t^2 - 40t + 80}$$
$$d^2 = 5\,t^2 - 40t + 80$$
$$\frac{1}{5}\,d^2 = t^2 - 8t + 16$$
$$\frac{1}{5}\,d^2 = (\,t - 4\,)^2$$

This is a minimum for when $t = 4$ seconds

**( v )** When $t = 4$, $d = 0$ i.e. The skaters collide !

## 4.1 Vectors: Topic Summary

**Question 1**

In a desert exercise a tank travels 12 km on a bearing of 070° from an Oasis, *O*, then 14 km on a bearing of 160° to a Bunker *B*.

**( i )** Provide a sketch of the tank's manoeuvres marking on the following;
12 km, 14 km, *O*, *B*, 70°, 90°

**[ 2 marks ]**

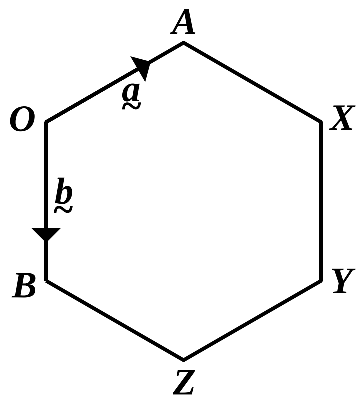
**( ii )** Determine the bearing of the tank's bunker location from the Oasis.

**[ 2 marks ]**

**Question 2**

A regular hexagon has its six vertices marked $O$, $A$, $X$, $Y$, $Z$, and $B$ as shown.

$$\overrightarrow{OA} = \boldsymbol{a} \quad \text{and} \quad \overrightarrow{OB} = \boldsymbol{b}$$




Find, in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

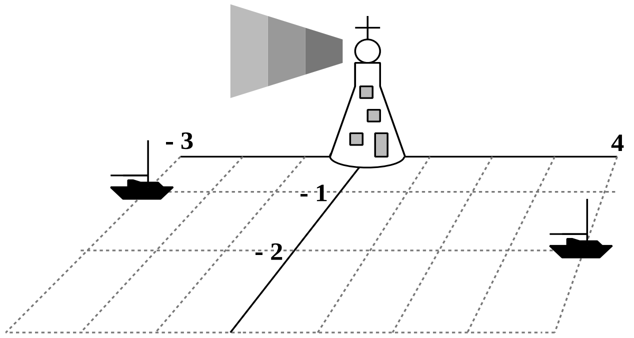
**( i )** $\overrightarrow{YX}$ **( ii )** $\overrightarrow{BX}$ **( iii )** $\overrightarrow{OZ}$

**[ 3 marks ]**

**Question 3**

A yacht is initially at the position, $\boldsymbol{Y_A} = -3\,\boldsymbol{i} - \boldsymbol{j}$ km.

Some time later it is at position, $\boldsymbol{Y_B} = 4\,\boldsymbol{i} - 2\,\boldsymbol{j}$ km.




**( i )** Determine the vector that describes the change in position of the yacht.

**[ 2 marks ]**

**( ii )** By using the theorem of Pythagoras, and your part (i) answer, determine the distance across the sea-bed that the yacht has covered.

**[ 2 marks ]**

**Question 4**

Two motor boats, *The Chunter* and *The Rapid* sit side by side upon the ocean.
They then separate, each at a constant velocity.
*The Chunter* has velocity $\boldsymbol{V_C} = 3\,\boldsymbol{i} + 5\,\boldsymbol{j}$ kmh$^{-1}$
*The Rapid* has velocity $\boldsymbol{V_R} = 8\,\boldsymbol{i} + 4\,\boldsymbol{j}$ kmh$^{-1}$

**( i )** Calculate the speed of *The Chunter*.

**[ 2 marks ]**

**( ii )** How far will *The Chunter* travel in 2 hours 15 minutes ?

**[ 2 marks ]**

**( iii )** Calculate the velocity of *The Chunter* relative to *The Rapid.*

**[ 2 marks ]**

**( iv )** Use your part (iii) answer to calculate, in hours and minutes, how long it will take until the two motor boats are 8 km apart.

**[ 2 marks ]**

**Question 5**

A particle $P$ has velocity ( $3\,\boldsymbol{i} + 2\,\boldsymbol{j}$ ) ms$^{-1}$ when $t = 0$ seconds
and velocity ( $7\,\boldsymbol{i} + 4\,\boldsymbol{j}$ ) ms$^{-1}$ at time $t = 2$ seconds
Find the acceleration of $P$ assuming that it is constant.

**[ 4 marks ]**

**Question 6**

The position of a particle at time $t$ is given by; $r$ **= ( 3*t* – 7 )** $i$ **+ ( 6*t* + 1 )** $j$

**( i )** If $d$ is the distance in metres of $\boldsymbol{r}$ from the origin at time $t$, find an expression for $d$ involving the square root of a quadratic equation in $t$ (Hint: Pythagoras).

**[ 2 marks ]**

**( ii )** Show, by completing the square on the quadratic, that;

$$\frac{1}{5}d^2 = 9\left(t - \frac{1}{3}\right)^2 + 9$$

**[ 2 marks ]**

**( iii )** What value of $t$ makes $\frac{1}{5}d^2$ as small as possible ?

This is the time at which the particle is closest to the origin.

**[ 1 mark ]**

**( iv )** What is this minimum distance ?

**[ 1 mark ]**

**Question 7**

At 11:00 hour the position vector of an aircraft relative to an airport $O$ is;

$$\boldsymbol{r_A} = ( 200\,\boldsymbol{i} + 30\,\boldsymbol{j} )\ \text{km}$$

Note that $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors due east and due north respectively.

The constant velocity of the aircraft is;

$$\boldsymbol{V_A} = ( 180\,\boldsymbol{i} - 120\,\boldsymbol{j} )\ \text{kmh}^{-1}$$

Find;

**( i )** the time when the aircraft is due east of the airport $O$

**[ 2 marks ]**

**( ii )** how far it then is from $O$

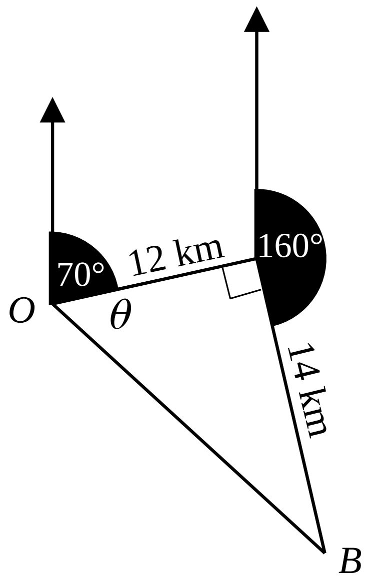
**[ 2 marks ]**

**( iii )** how far it is from $O$ at 12:00

**[ 2 marks ]**

**Answer 1**

**( i )**




**[ 2 marks ]**

**( ii )** $\tan\theta = \dfrac{14}{12} \Rightarrow \theta = 49°$

$\therefore$ Bearing of the bunker from the oasis is 119°

**[ 2 marks ]**

**Answer 2**

**( i )** $\overrightarrow{YX} = -\boldsymbol{b}$

**( ii )** $\overrightarrow{BX} = 2\boldsymbol{a}$

**( iii )** $\overrightarrow{OZ} = \boldsymbol{a} + 2\boldsymbol{b}$

**[ 3 marks ]**

**Answer 3**

**( i )** $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$$= \begin{pmatrix} 4 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ -1 \end{pmatrix} \quad \text{or} \quad 7\boldsymbol{i} - \boldsymbol{j}$$

**[ 2 marks ]**

**( ii )** $distance = \sqrt{7^2 + 1^2} = \sqrt{50} = 5\sqrt{2}$

or 7.07 km

**[ 2 marks ]**

**Answer 4**

**( i )** $\left| V_C \right| = \sqrt{3^2 + 5^2} = \sqrt{34}$ or $5.83$ kmh$^{-1}$

**[ 2 marks ]**

**( ii )** Distance = Speed $\times$ Time

$= 5.83 \times 2.25$

$= 13.12$ km

**[ 2 marks ]**

**( iii )** $\boldsymbol{V_{RC}} = V_C - \boldsymbol{V_R}$

$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 1 \end{pmatrix} \text{ kmh}^{-1} \quad \text{or} \quad \boldsymbol{V_{RC}} = -5\,\boldsymbol{i} + \boldsymbol{j} \text{ kmh}^{-1}$$

**[ 2 marks ]**

**( iv )** $\left| \boldsymbol{V_{RC}} \right| = \sqrt{(-5)^2 + 1^2} = 5.1$ kmh$^{-1}$ (Speed of separation)

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}} = \frac{8}{5.1} = 1.57 \text{ hours} \quad \text{i.e. 1 hour and 34 minutes}$$

**[ 2 marks ]**

**Answer 5**

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t$$

$$\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} a_x \\ a_y \end{pmatrix} \times 2$$

$$2\begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ ms}^{-2} \quad \text{or} \quad \boldsymbol{a} = 2\,\boldsymbol{i} + \boldsymbol{j} \text{ ms}^{-2}$$

**[ 4 marks ]**

**Answer 6**

**( i )**

$$d = \sqrt{(3t - 7)^2 + (6t + 1)^2}$$

$$d = \sqrt{9t^2 - 42t + 49 + 36t^2 + 12t + 1}$$

$$d = \sqrt{45t^2 - 30t + 50}$$

**[ 2 marks ]**

**( ii )**

$$d^2 = 45t^2 - 30t + 50$$

$$\frac{1}{5}d^2 = 9t^2 - 6t + 10$$

$$\frac{1}{5}d^2 = \left(9t^2 - 6t + 1\right) + 9$$

$$\frac{1}{5}d^2 = (3t - 1)^2 + 9$$

$$\frac{1}{5}d^2 = 9\left(t - \frac{1}{3}\right)^2 + 9 \qquad \square$$

**[ 2 marks ]**

**( iii )** This is a minimum when $t = \dfrac{1}{3}$ seconds

**[ 1 mark ]**

**( iv )** $\dfrac{1}{5}d^2 = 9 \Leftrightarrow d^2 = 45 \Rightarrow d = \sqrt{45} = 6.71$ metres

**[ 1 mark ]**

**Answer 7**

**( i )** At time $t$, the aircraft is at $\boldsymbol{r}_A = \begin{pmatrix} 200 \\ 30 \end{pmatrix} + \begin{pmatrix} 180 \\ -120 \end{pmatrix} t$

It's due east when $30 - 120t = 0 \Rightarrow t = 0.25$ hours

i.e. When the time is 11:15

**[ 2 marks ]**

**( ii )** Distance $= 200 + 180 \times 0.25 = 245$ km

**[ 2 marks ]**

**( iii )** When $t = 1$ hour, $\boldsymbol{r}_A = \begin{pmatrix} 380 \\ -90 \end{pmatrix}$

$\therefore$ Distance $= \sqrt{380^2 + (-90)^2} = 390.5$ km

**[ 2 marks ]**