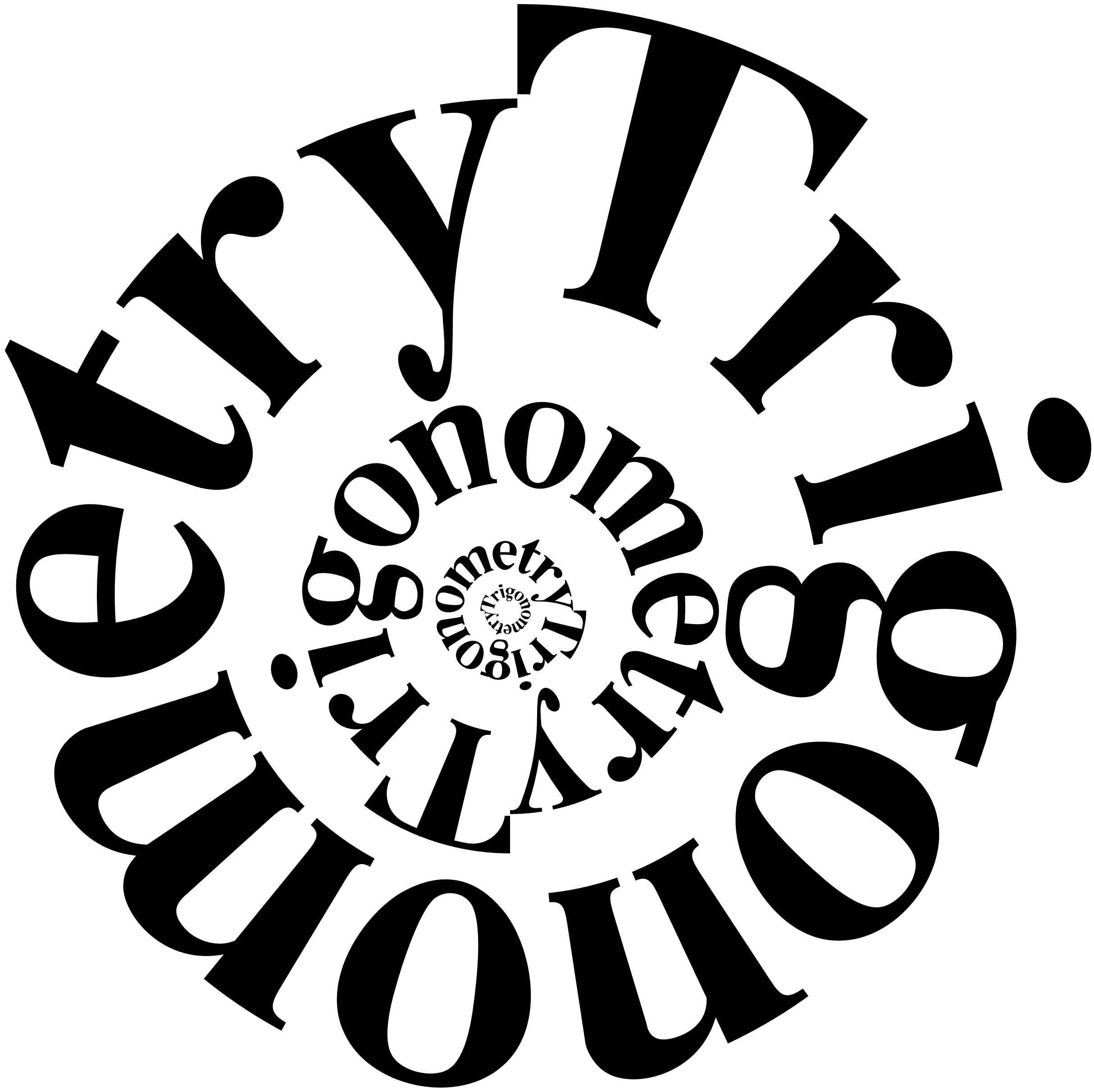
GCSE Mathematics

Non Right Angled

TRIGONOMETRY III



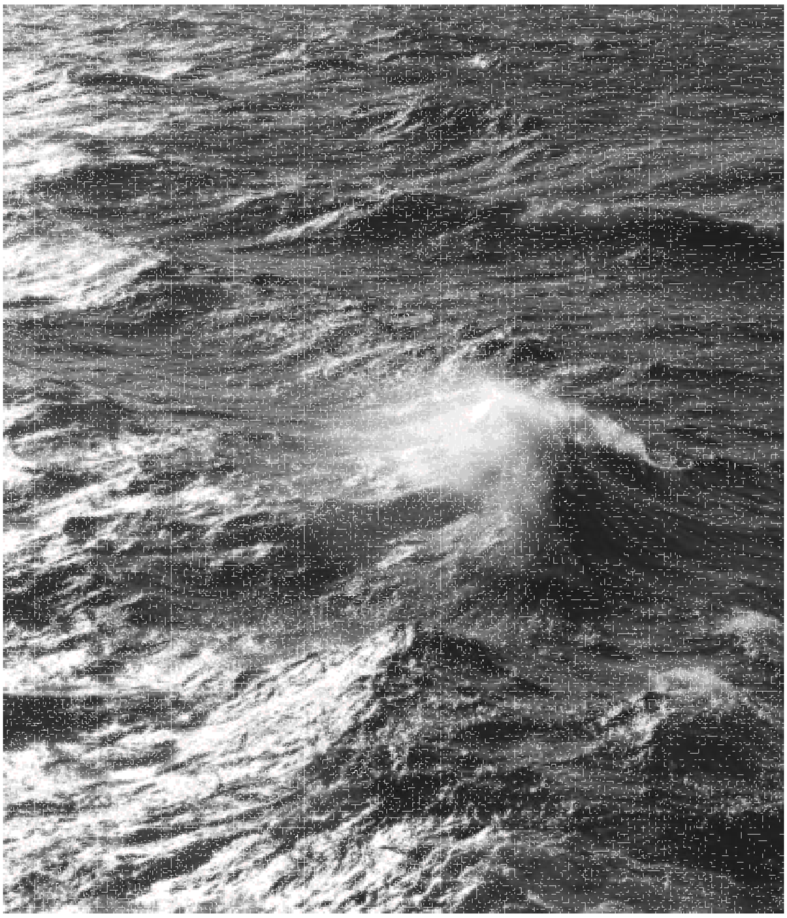
Lesson 1

GCSE Mathematics

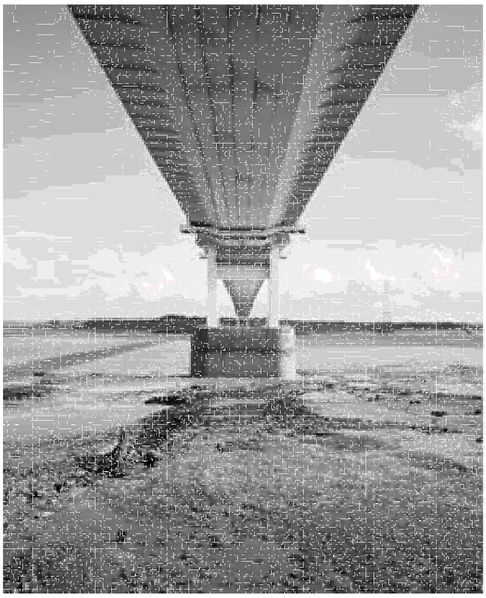
Non-Right Angled Trigonometry

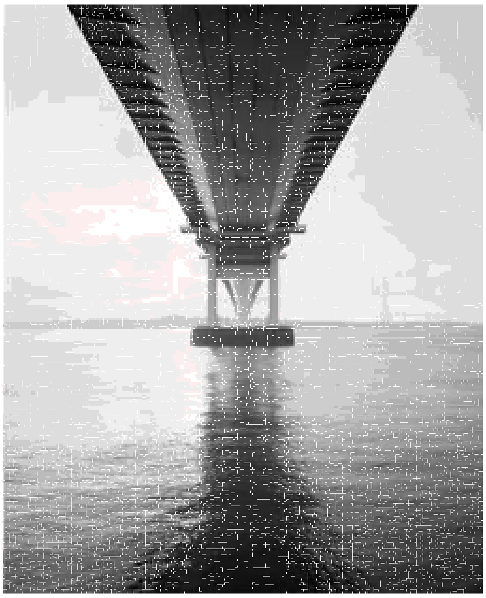
1.1 Waves

If asked to think of the Atlantic Ocean, amongst your thoughts would be 'waves'.

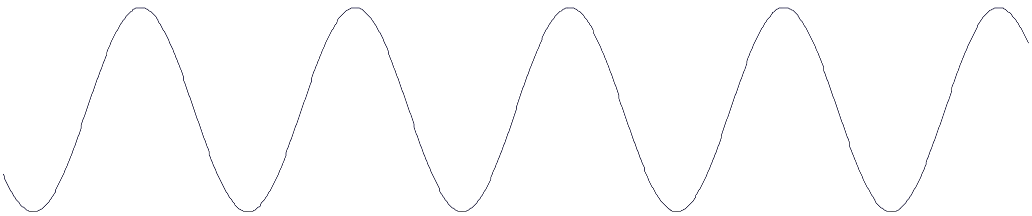


The tide is another 'wave' on the ocean that peaks and troughs about twice a day. Below, the left photo was taken at 8.30am, the right at 2.45pm, both of the same place below the Severn Bridge across the Bristol Channel.





Mathematically, a very pure and simple wave that is very useful at capturing many aspects of real-world waves is the sine wave.



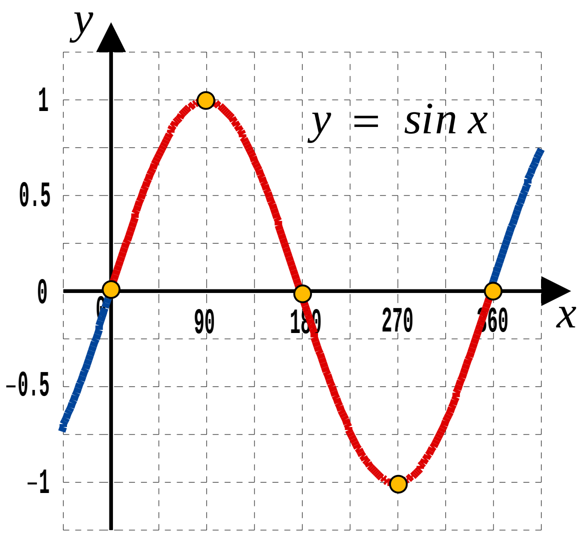
1.2 Mathematics Involving Waves

Example

(i) Use your calculator to find a solution to the equation;

$$y \;=\; sin\,30°$$

(ii) A section of the graph of $y = sin\,x$ is shown below;



Show, by drawing on the graph, where your part **(i)** solution is.

(iii) Use your calculator to find a solution to the equation;

$$y \;=\; sin\,270°$$

(iv) Show, by drawing on the graph, where your part **(iii)** solution is.

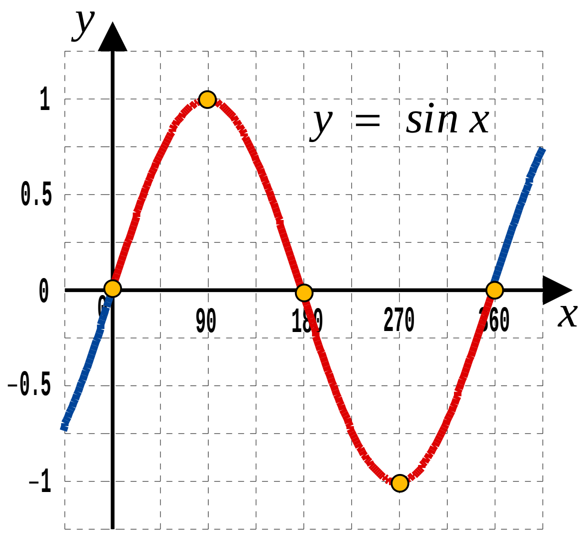
1.3 Exercise

Question 1

(i) Use your calculator to find a solution to the equation;

$$y = \mathit{sin}\,45°$$

(ii) A section of the graph of $y = \mathit{sin}\ x$ is shown below;



Show, by drawing on the graph, where your part **(i)** solution is.

(iii) Use your calculator to find a solution to the equation;

$$y = \mathit{sin}\,90°$$

(iv) Show, by drawing on the graph, where your part **(iii)** solution is.

Question 2

Use your calculator to calculate in one go, the value of the following;

$$(\ \mathit{sin}\,30° + \mathit{sin}\,90°\ )^2$$

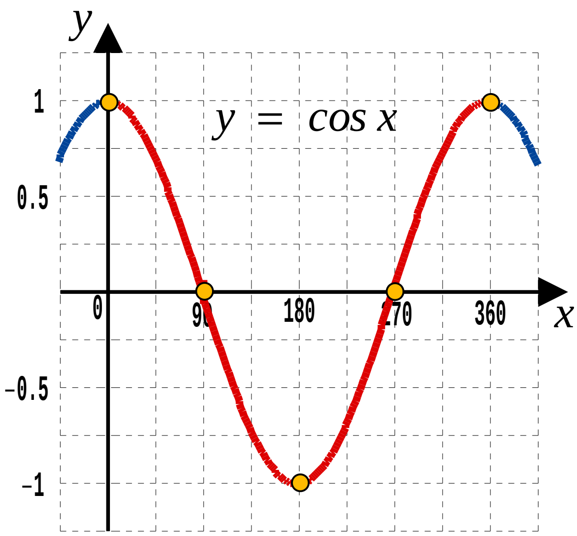
Give the exact answer.

Question 3

(i) Use your calculator to find a solution to the equation;

$$y = \cos 50°$$

(ii) A section of the graph of $y = \cos x$ is shown below;



Show, by drawing on the graph, where your part **(i)** solution is.

(iii) Use your calculator to find a solution to the equation;

$$y = \cos 120°$$

(iv) Show, by drawing on the graph, where your part **(iii)** solution is.

Question 4

Use your calculator to calculate in one go, the value of the following;

$$\sqrt{\cos 50°} + \cos 120°$$

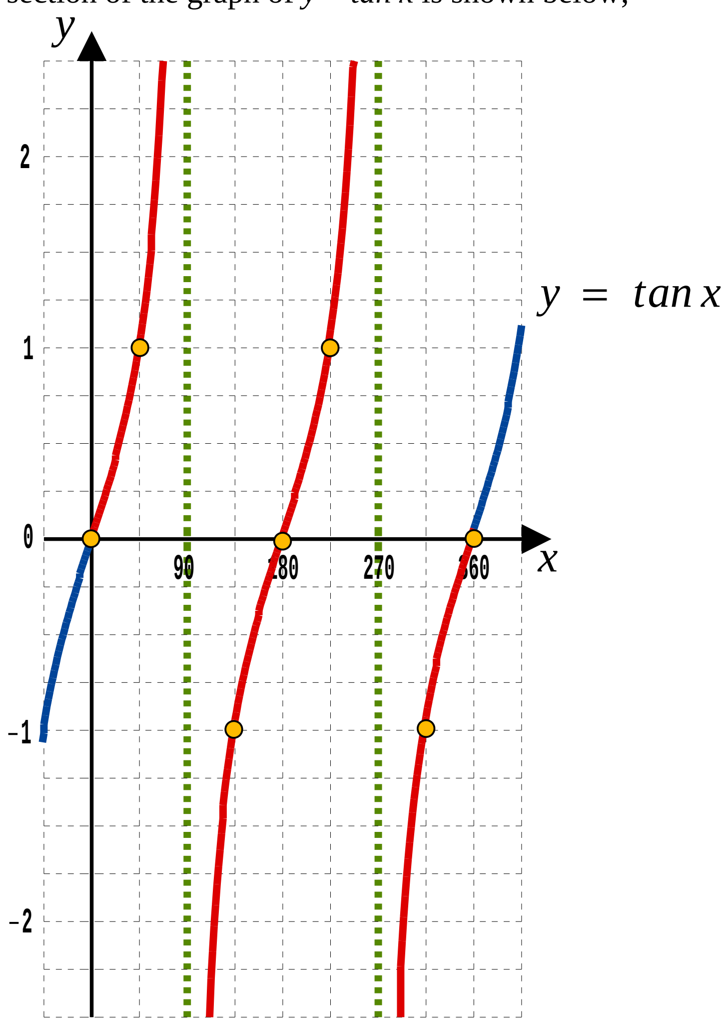
Give your answer accurate to 3 decimal places.

Question 5

(i) Use your calculator to find a solution to the equation;

$$y = \tan 60°$$

(ii) A section of the graph of $y = \tan x$ is shown below;



Show, by drawing on the graph, where your part **(i)** solution is.

(iii) Use your calculator to find a solution to the equation;

$$y = \tan 315°$$

(iv) Show, by drawing on the graph, where your part **(iii)** solution is.

Question 6

Use your calculator to calculate in one go, the value of the following;

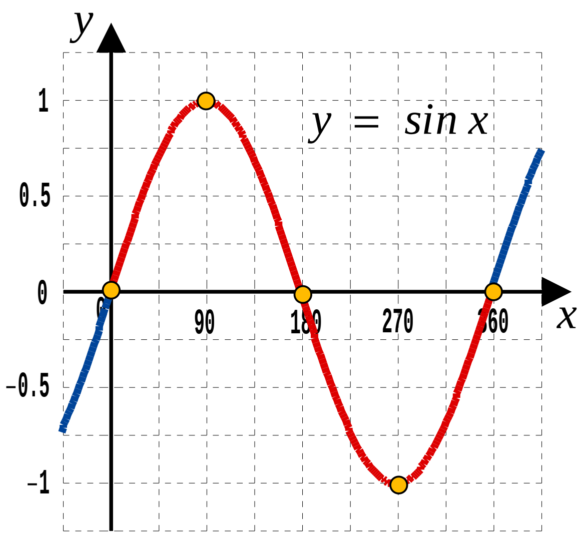
$$\sqrt{(\tan 60)^2 + (\tan 315)^2}$$

Question 7

(i) Use your calculator to find a solution to the equation;

$$y = \arcsin 0.75$$

(ii) A section of the graph of $y = \sin x$ is shown below;



Show, by drawing on the graph, where your part **(i)** solution is.

(iii) Use your calculator to find a solution to the equation;

$$y = \arcsin(-0.6)$$

(iv) Show, by drawing on the graph, where your part **(iii)** solution is.

Question 8

Use your calculator to calculate in one go, the value of the following;

$$\frac{\arcsin 0.75}{\arcsin(-0.6)}$$

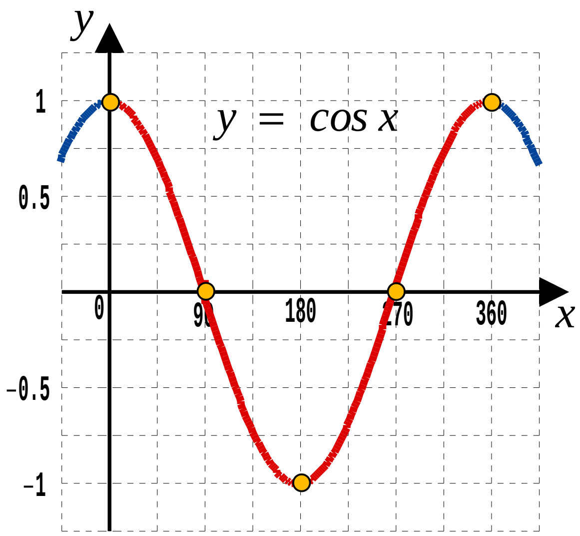
Give your answer correct to 3 decimal places.

Question 9

(i) Use your calculator to find a solution to the equation;

$$y = arccos\, 0.4$$

(ii) A section of the graph of $y = cos\ x$ is shown below;



Show, by drawing on the graph, where your part **(i)** solution is.

(iii) Use your calculator to find a solution to the equation;

$$y = arccos\ (-0.5)$$

(iv) Show, by drawing on the graph, where your part **(iii)** solution is.

Question 10

Use your calculator to calculate in one go, the value of the following;

$$\sqrt{arccos\, 0.4} + \sqrt{arccos\ (-0.5)}$$

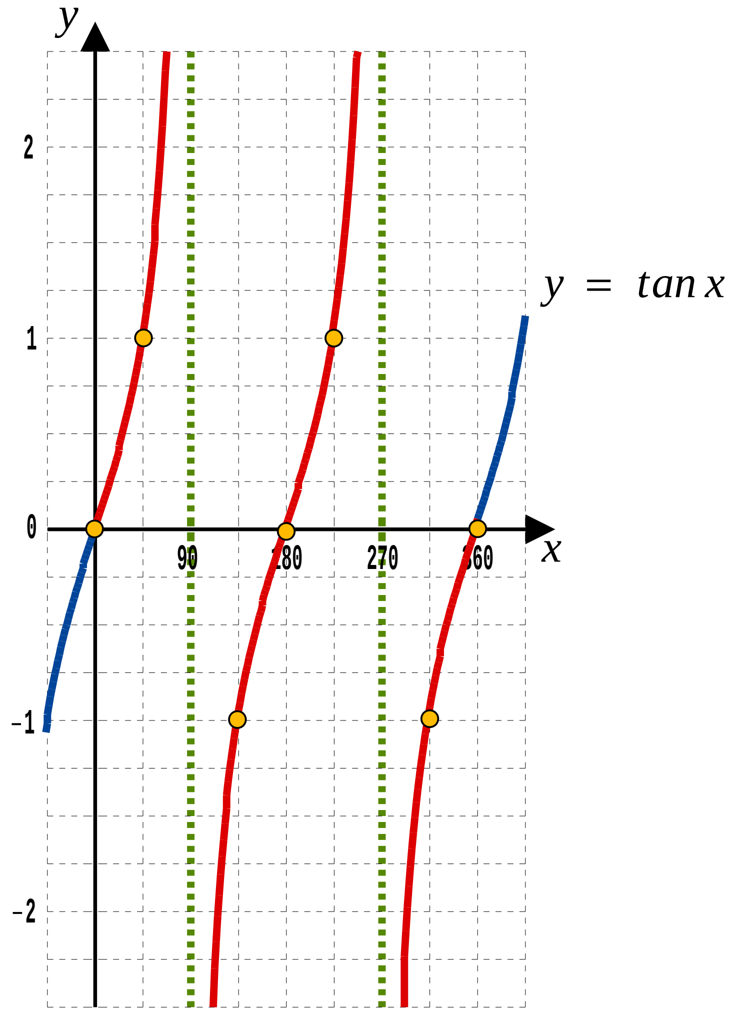
Give your answer accurate to 3 decimal places.

Question 11

(i) Use your calculator to find a solution to the equation;

$$y = arctan\, 2$$

(ii) A section of the graph of $y = tan\; x$ is shown below;



Show, by drawing on the graph, where your part **(i)** solution is.

(iii) Use your calculator to find a solution to the equation;

$$y = arctan\left(\frac{1}{\sqrt{3}}\right)$$

(iv) Show, by drawing on the graph, where your part **(iii)** solution is.

Image : Storm Waves at Drake's Passage by Ralph Lee Hopkins
Image : Sea Change at the Severn Bridge by Michael Marten

1.4 Answers

1.4.1 Solution to 1.2 Example

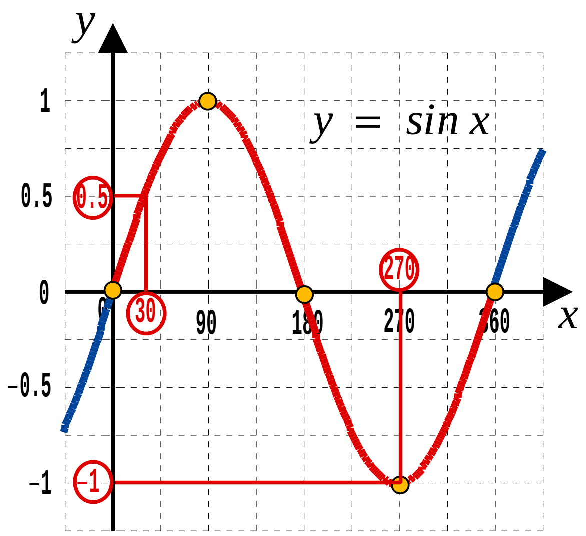
(i) $y = sin\,30°$

$= 0.5$

(iii) $y = sin\,270°$

$= -1$

(ii) and **(iv)**

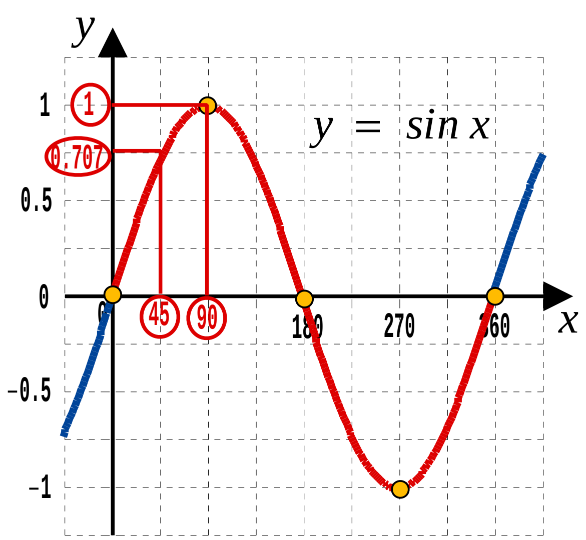


1.4.2 Solutions to 1.3 Exercise

Answer 1

(i) 0.707 **(iii)** 1

(ii) and **(iv)**



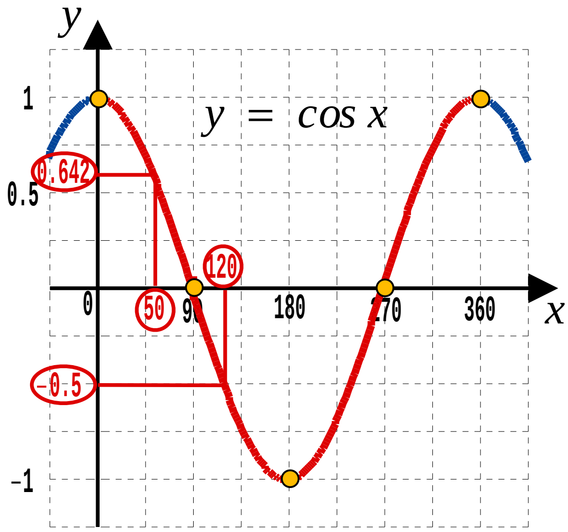
Answer 2

2.25

Answer 3

(i) 0.642 **(iii)** – 0.5

(ii) and **(iv)**



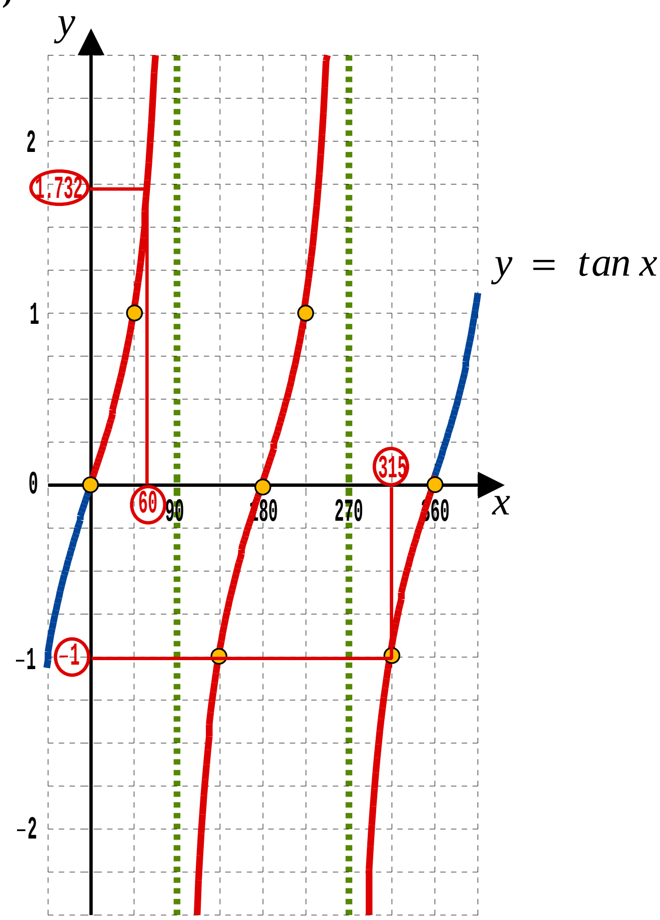
Answer 4

0.302

Answer 5

(i) 1.732 **(iii)** – 1

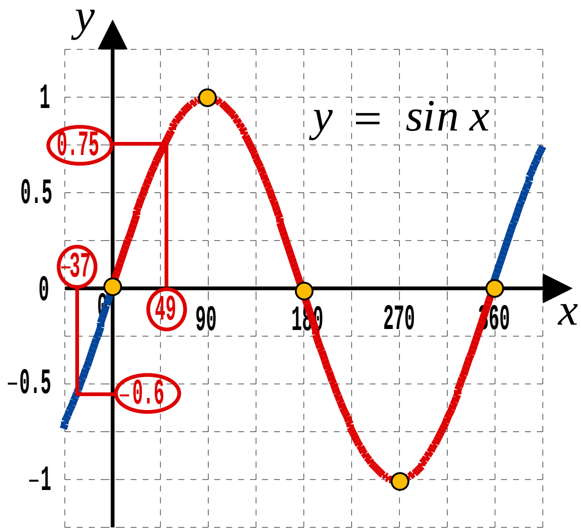
(ii) and **(iv)**



Answer 6
2

Answer 7
(i) 48.6° **(iii)** – 36.8°

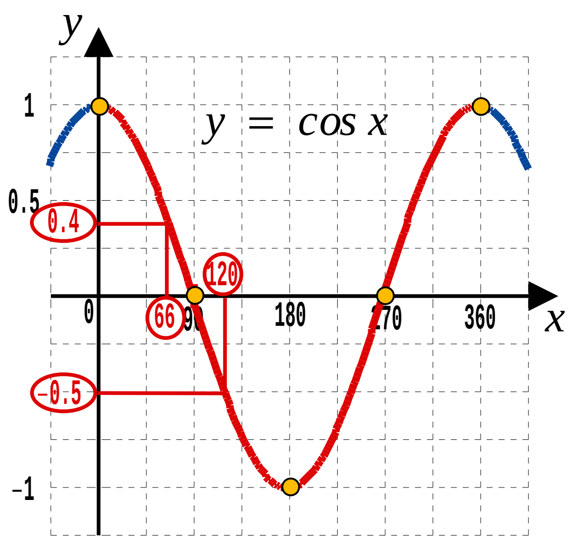
(ii) and **(iv)**



Answer 8
– 1.318

Answer 9
(i) 66.4° **(iii)** 120°

(ii) and **(iv)**



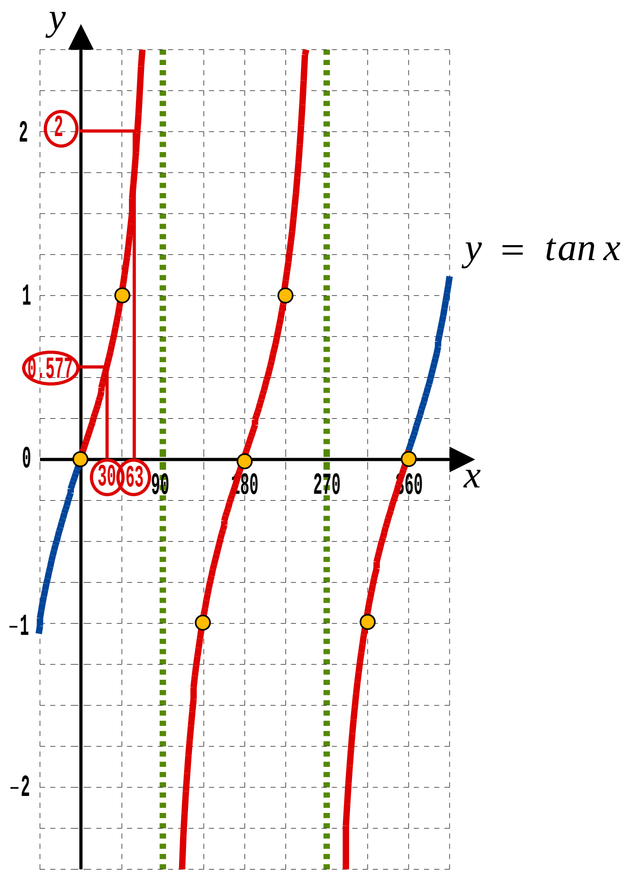
Answer 10
19.104

Answer 11

(i) 63.93° **(iii)** 30°

$$Note \;:\; \frac{1}{\sqrt{3}} = 0.577$$

(ii) and **(iv)**



Lesson 2

GCSE Mathematics

Non-Right Angled Trigonometry

2.1 Relationships

Mathematicians write; $\sin^2 x$ when they mean $(\sin x)^2$

$\cos^2 x$ when they mean $(\cos x)^2$

$\tan^2 x$ when they mean $(\tan x)^2$

Fill in the table below, using your calculator and working to three decimal places.

	$\sin x$	$\cos x$	$\tan x$	$\sin^2 x$	$\cos^2 x$	$\frac{\sin x}{\cos x}$	$\cos^2 x + \sin^2 x$
0°							
10°							
20°							
30°							
40°							
50°							
60°							
70°							
80°							
90°							

Looking over your table of results, can you spot two interesting relationships ?
One "always true formula" involves sin x, *cos* x and *tan* x.
Write it down here:

The other involves $cos^2 x$, the number 1, and $sin^2 x$.
Write it down here:

Check that your "always true formulae" (that will be true no matter what the value of x is, work for the example of when $x = 35°$.
Show your working here;

These two "always true formulae" are examples of trigonometric identities. Unlike an equation which is typically only true for one or two values of x (and solving an equation is about finding those special values of x) an identity is true for all (or almost all as division by zero is not allowed, for example) values of x.

2.2 Answers

	$\sin x$	$\cos x$	$\tan x$	$\sin^2 x$	$\cos^2 x$	$\frac{\sin x}{\cos x}$	$\cos^2 x + \sin^2 x$
0°	0.000	1.000	0.000	0.000	1.000	0.000	1.000
10°	0.174	0.985	0.176	0.030	0.970	0.177	1.000
20°	0.342	0.940	0.364	0.117	0.883	0.364	1.000
30°	0.500	0.866	0.577	0.250	0.750	0.577	1.000
40°	0.643	0.766	0.839	0.413	0.587	0.839	1.000
50°	0.766	0.643	1.192	0.587	0.413	1.191	1.000
60°	0.866	0.500	1.732	0.750	0.250	1.732	1.000
70°	0.940	0.342	2.747	0.883	0.117	2.749	1.000
80°	0.985	0.174	5.672	0.970	0.030	5.661	1.000
90°	1.000	0.000	- - -	1.000	0.000	- - -	1.000

The numerical investigation suggests that the following may be true;

$$\frac{\sin x}{\cos x} = \tan x \qquad \text{provided} \qquad \cos x \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

These are, in fact, true. However, no amount of numerical evidence can ***prove*** they are true. For that a watertight argument is needed that gives a reason for ***why*** they must be true. Such as reason is called ***a mathematical proof***.

An accessible proof of the above (for $0° < x < 90°$) involves the definitions of *sin x*, *cos x* and *tan x* in terms of a triangle's *opposite*, *adjacent* and *hypotenuse* and the use of the *theorem of Pythagoras*. An extension activity would be to try to find it (or look it up).

Lesson 3

3.1 Trigonometric Formulae

Many useful formulae include the trigonometric functions *sin x*, *cos x* or *tan x*.
This Lesson is about extracting useful information from such formulae.

3.2 Example

In the UK, domestic mains electricity is supplied as alternating current.
This has maximum value of 240 volts and minimum voltage of – 240 volts.
It comes as a fairly perfect sine wave at a frequency of 50 Hz.

As a mathematical formula; $V = 240 \sin(18000t + \alpha)$

where V is the voltage (volts) at time, t (seconds).
α is one of three phases, either $\alpha = 0°$, or $\alpha = 120°$ or $\alpha = 240°$.
If your house is on $\alpha = 120°$ then your neighbours house on one side will be on $\alpha = 0°$ and your neighbours house on the other side will be on $\alpha = 240°$.
Find the voltage when

(i) $t = 0.0025$ seconds, and $\alpha = 120°$

(ii) $t = 0.005$ seconds, and $\alpha = 240°$

3.3 Exercise

Question 1

In the harbour at Holyhead, Wales, the height of the tide on a certain day is given by;

$$H = 3.1 + 2.9 \sin(28.8t + 40)$$

where H is the height of the tide in metres at time t in hours after midnight.
Find the height of the tide in Holyhead harbour at

(i) $t = 1$ am

(ii) $t = 2$ am

(iii) $t = 3$ am

(iv) One of these is high tide. Was high tide at 1am, 2am or 3am ?

(v) Experiment to find the time, to the nearest hour, of low tide.

Question 2

In the drying harbour at Paignton, Devon, the height of the tide on a certain day is given by;

$$H = 2.4 + 2.7\,cos\,(\,28.8t - 74\,)$$

where H is the height in hours and t is the time in hours after midnight.

Find the height of the tide in Paignton harbour at

(i) t = 17:00

(ii) t = 18:00

(iii) t = 19:00

(iv) If the harbour dries when the tide is below 1.4 m, did the harbour become dry at 17:00, 18:00 or 19:00 ?

(v) Experiment to find the time, to the nearest hour, at which the harbour will start to refill with water.
(i.e. When there is next more than 1.4 m of tide)

Question 3

A boy, bouncing on a pogo stick, has eyes at a height, H, above the ground given by;

$$H = 2.3 + 0.5\,cos\,(120t + 45\,)$$

where t is the time in seconds after he first started bouncing.
How high are his eyes above the ground when

(i) t = 6 seconds,

(ii) t = 6.5 seconds

(iii) t = 7 seconds

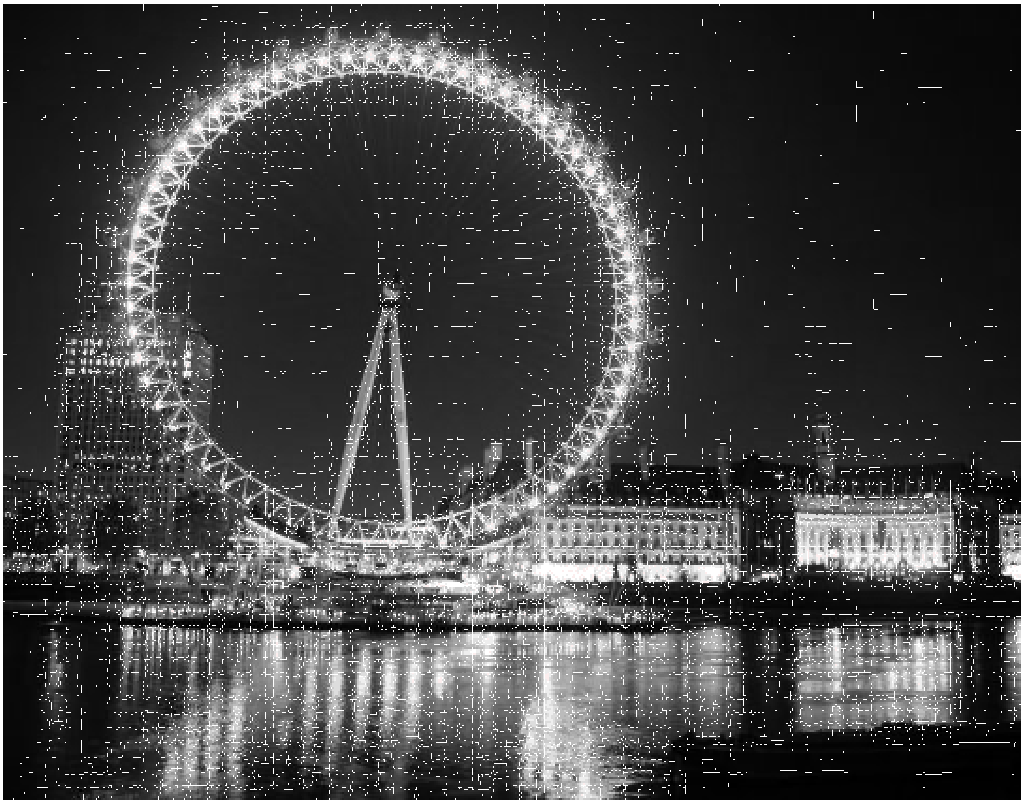
(iv) Experiment to find the first time after 7 seconds, to the nearest half second, when he gets to his maximum height.

Question 4

The photograph below is of the London Eye, a large ferris wheel.

The height above the ground, H, of a passenger, t minutes after stepping aboard is;

$$H = 170 + 130\sin(12t - 90)$$



How high is a passenger at the following times after stepping aboard;

(i) $t = 3$ minutes

(ii) $t = 6$ minutes

(iii) $t = 9$ minutes

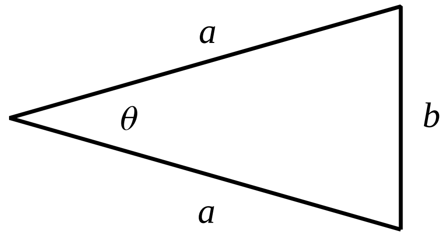
(iv) Experiment to find the time, to the nearest three minutes, for the passenger to get to the maximum height.

(v) How long does it take the wheel to revolve once ?

Question 5

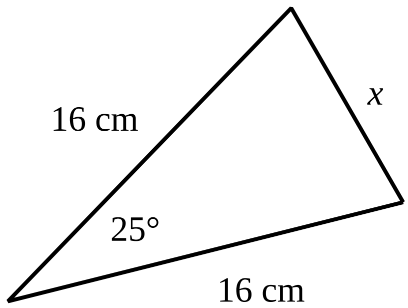
Given an isosceles triangle with two sides of length a, at an angle of θ apart, the length of the third side, b, is given by the formula

$$b^2 = 2a^2(1 - \cos\theta)$$

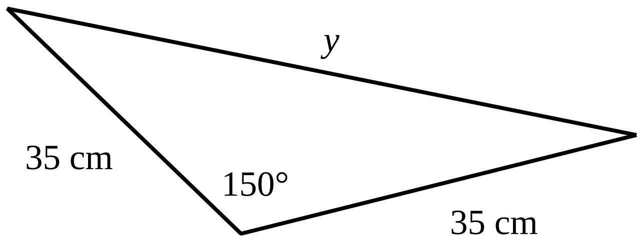


Use the formulae to find the length of the missing side of each of these triangles;
Give your answers to three significant figures.

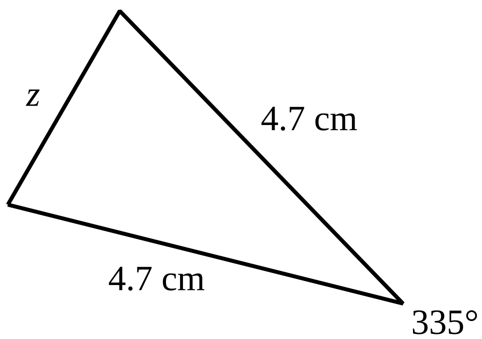
(i)



(ii)



(iii)

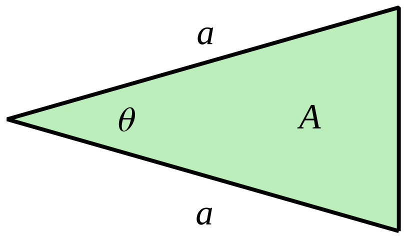


Can the formulae cope with having 335° inserted or do you have to use 360° – 335° instead ?

Question 6

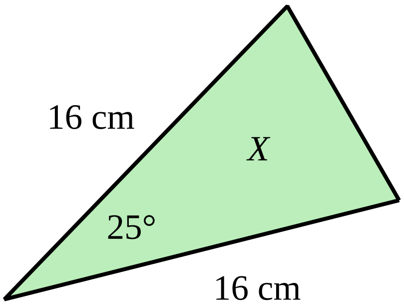
Given an isosceles triangle with two sides of length a, at an angle of θ apart, the area, A, is given by the formula

$$A = \frac{1}{2} a^2 \sin \theta$$

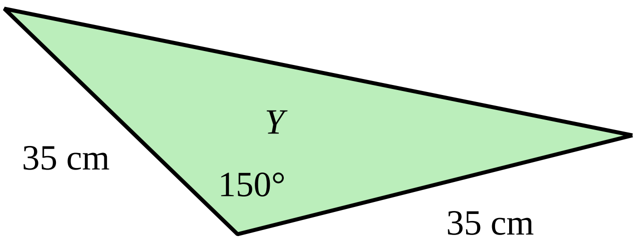


Use the formulae to find the area of each of these triangles;
Give your answers to 3 significant figures.

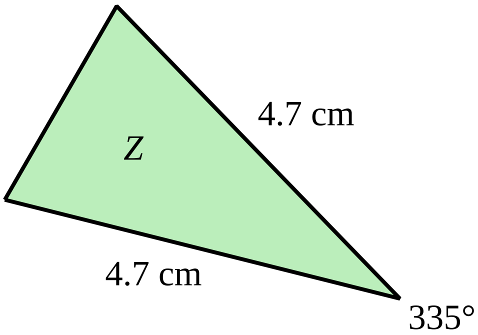
(i)



(ii)



(iii)



Can the formulae cope with having 335° inserted or do you have to use 360° – 335° instead ?

Question 7

The area of any regular polygon is given by the formula;

$$A = \frac{n\,a^2}{4\,tan\left(\frac{180}{n}\right)}$$

where n is the number of sides,
and a is the length of a side.

(i) A square of side 8 cm has an area of 64 cm^2
Show clearly that the formula also gives the area of the square to be 64 cm^2

(ii) Use the formula to find the area of a regular decagon of side length 6.2 cm.

(iii) Use the formula to find the area of a regular hexagon of side length 17.2 cm.

(iv) Find the area of a regular 100 sided polygon of with a perimeter of 80 cm.

Image : Millennium Wheel, London by Dreamer

3.4 Homework

This exercise shows you how to make a sine wave that will predict the tide. It's a crude predictor, as real tidal curves can be (and often are) more complicated. So don't expect perfection, especially in places like (for example) the Solent where the Isle of Wight gives rise to a very interesting currents, and an unusual tidal curve that is more complicated than a simple sine wave.

As a well behaved example we'll look at Conwy on 25th December 2019

From the internet, we can easily get times of the first low and following high tide.

25th December 2019 : Conwy, Wales

Time of LOW tide :	04:35	Height of :	1.4 metres
Time of HIGH tide :	09:59	Height of :	7.7 metres

We're after a formula of the form

$$H = M + A \sin(28.8t + T)$$

Where we have to work out numbers for *M*, *A* and *T*.

$$M = \frac{High\ height + Low\ height}{2}$$

$$M = \frac{7.7 + 1.4}{2} =$$

$$A = \frac{High\ height - Low\ height}{2}$$

$$A = \frac{7.7 - 1.4}{2} =$$

The time 04:35 is $4.\left(\frac{35}{60}\right)$ which is 4.6 hours

$$T = \frac{Decimal\ time\ of\ LOW\ tide \times 360}{28.8} + 90$$

$$T = \frac{4.6 \times 360}{28.8} + 90 =$$

(i) Complete the above calculations to work out *M*, *A* and *T*.

(ii) Write out the formulae for the tide at Conwy on 25th December 2019 with *M*, *A* and *T* replaced with your answers from part (i)

(iii) Insert $t = 4.6$ into your formula, and calculate H.
Did you get what you expected ?

(iv) Use your formula to calculate the height of the tide at hourly intervals starting at 5am.

t	5am	6am	7am	8am	9am	10am	11am	12am
H								

Space for workings...

(v) How close does the formula get to predicting the high tide at 09:59 ?

3.5 Answers

3.5.1 Solution to 3.2 Example

(i)

$$\begin{aligned} V &= 240\,sin\,(\,18000\,t\ +\ \alpha) \\ &= 240\,sin\,(\,18000\times 0.0025 + 120\,) \\ &= 62.1\ volts \end{aligned}$$

(ii)

$$\begin{aligned} V &= 240\,sin\,(\,18000\,t\ +\ \alpha) \\ &= 240\,sin\,(\,18000\times 0.005 + 240\,) \\ &= -\,120\ volts \end{aligned}$$

3.5.2 Solutions to 3.3 Exercise

Answer 1

(i) 5.8 m **(ii)** 5.97 m
(iii) 5.43 m **(iv)** 2 am
(v) Here is the A-Level method, but trial and improvement is OK
Low tide occurs when the sine curve has a minimum value

$$\begin{aligned} sin\,(\,28.8t + 40\,) &= -\,1 \\ 28.8t + 40 &= arcsin\,(-\,1) \\ &= 270 \\ 28.8t &= 230 \\ &= 7.986 \end{aligned}$$

That is, low tide is at 8 am

Answer 2

(i) 3.93 m **(ii)** 2.66 m
(iii) 1.33 m **(iv)** Just before 19:00
(v) Here is the A-Level method, but trial and improvement is OK

$$\begin{aligned} 2.4 + 2.7\,cos\,(\,28.8t - 74\,) &= 1.4 \\ 28.8t - 74 &= arccos\left(-\,\frac{10}{27}\right) \\ 28.8t - 74 &= 111.7,\ 248.3,\ 471.7,\ 608.3,\ \ldots \\ 28.8t &= 185.7,\ 322.3,\ 545.7,\ 682.3,\ \ldots \\ &= 6.45,\ 11.2,\ 18.9,\ 23.7,\ \ldots \end{aligned}$$

The harbour starts to refill at 23.7
That is, at 23:47 (a little before midnight).

Answer 3

(i) 2.65 m **(ii)** 2.17 m

(iii) 1.82 m

(iv) when t = 7.5 s H = 1.65 m
when t = 8.0 s H = 2.43 m
when t = 8.5 s H = 2.78 m <=== Highest
when t = 9.0 s H = 2.65 m
when t = 9.5 s H = 2.17 m
To the nearest 5 seconds his maximum height is reached at 8.5 seconds

Answer 4

(i) 65 m **(ii)** 130 m

(iii) 210 m

(iv) when t = 12 minutes H = 275 m
when t = 15 minutes H = 300 m <=== Highest
when t = 18 minutes H = 275 m
To the nearest 3 minutes his maximum height is reached at 15 minutes

(v) As it took 15 minutes (to the nearest 3 minutes) to reach maximum height, it'll take about 15 minutes to get back down.
That is, it takes 30 minutes to revolve once.
(Some further trial and improvement will confirm this to be correct)

Answer 5

(i) 6.93 cm

(ii) 67.6 cm

(iii) 2.03 cm
You can indeed use 335°

Answer 6

(i) 54.1 cm^2

(ii) 306 cm^2

(iii) 4.67 cm^2
You need to remove a minus sign from the answer if using 335°

Answer 7

(i) It does !

(ii) 296 cm^2

(iii) 769 cm^2

(iv) a = 0.8 cm giving an area of 509 cm^2

3.5.3 Solutions to 3.4 Homework

(i)

$$M = \frac{7.7 + 1.4}{2} = 4.55$$

$$A = \frac{7.7 - 1.4}{2} = 3.15$$

$$T = \frac{4.6 \times 360}{28.8} + 90 = 147.5$$

(ii)

$$H = M + A\sin(28.8t + T)$$

$$H_{CONWY} = 4.55 + 3.15\sin(28.8t + 147.5)$$

(iii)

$$H_{CONWY} = 4.55 + 3.15\sin(28.8 \times 4.6 + 147.5)$$

$$= 1.447664831...$$

$$= 1.45$$

This is close to the 1.4 m that was expected (Low tide)

(iv)

t	5am	6am	7am	8am	9am	10am	11am	12am
H	1.61	2.54	3.95	5.52	6.84	7.60	7.60	6.85

(v) It's about half an hour out as the figures suggest the high tide is at 10.30 am
This can be checked by putting 10.5 into H_{conwy}

$$H_{CONWY} = 4.55 + 3.15\sin(28.8 \times 10.5 + 147.5)$$

$$= 7.7\,m$$

So our model gives a high tide 10 cm higher and 30 minutes later than the actual high tide.
However, for such a simple model it's yielded a reasonable prediction.

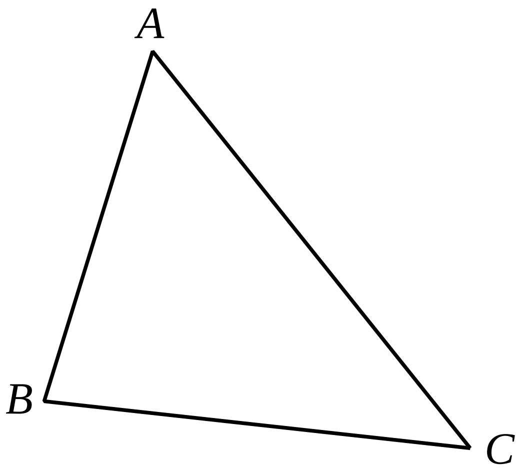
Lesson 4

4.1 The Cosine Rule

This is Pythagoras' Theorem for triangles without a right angle.
Used when two sides and the included angle are known and the third side sought.

4.2 Introductory Question

(i) On $\triangle ABC$, place the letters a, b and c.



(ii) On $\triangle ABC$, mark on that;

- AC is of length 6.3cm.
- AB is of length 7.8cm.
- $\angle A$ is 70°.
- BC is the length x, which is to be found.

(iii) Mark on the *included* angle with a *.

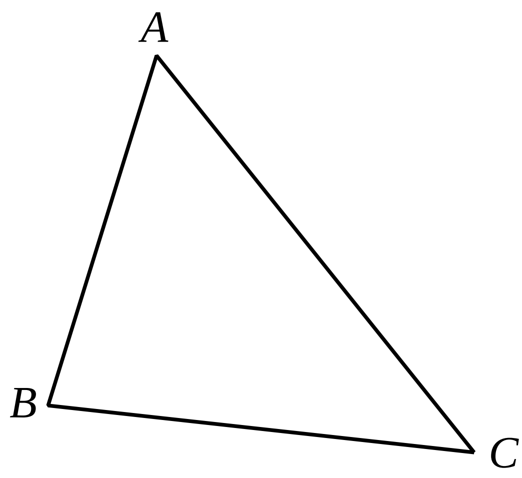
(iv) Use the cosine rule to find the length of x correct to three significant figures.

(v) Find the area of $\triangle ABC$, correct to three significant figures.

4.3 Exercise

Question 1

(i) On $\triangle ABC$, place the letters a, b and c.



(ii) On $\triangle ABC$, mark on that;

- AC is of length 8cm.
- AB is of length 3cm.
- $\angle A$ is 60°.
- BC is the length x, which is to be found.

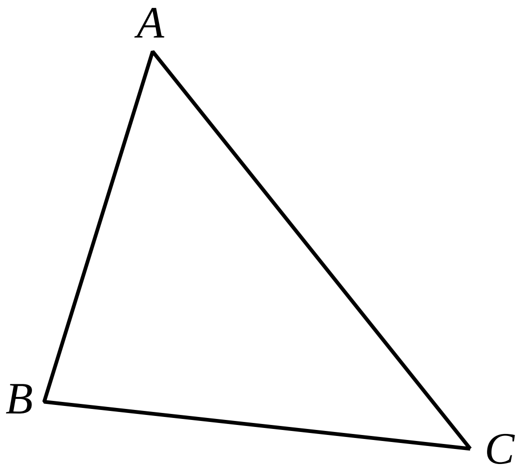
(iii) Mark on the *included* angle with a *.

(iv) Use the cosine rule to find the length of x.

(v) Find the area of $\triangle ABC$.

Question 2

(i) On $\triangle ABC$, place the letters a, b and c.



(ii) On $\triangle ABC$, mark on that;

- AC is of length 17 cm.
- AB is of length 15 cm.
- $\angle A$ is 70°.
- BC is the length x, which is to be found.

(iii) Mark on the *included* angle with a *.

(iv) Use the cosine rule to find the length of x.

(v) Find the area of $\triangle ABC$.

Question 3

$\triangle ABC$ has $AB = 5$ cm, $AC = 2$ cm, and $\angle A = 78.5°$.

(i) Sketch, roughly, $\triangle ABC$ and mark on the known lengths and angle.

(ii) Find the length of side BC.

(iii) Find the area of $\triangle ABC$.

Question 4

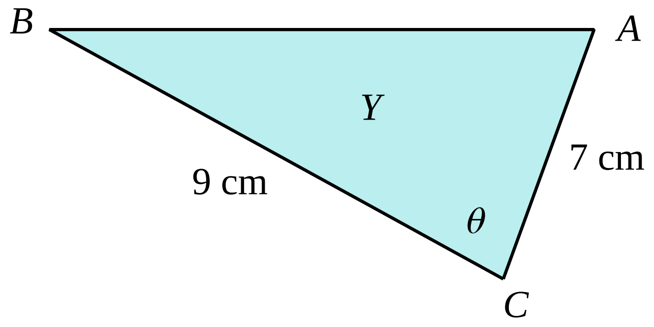
$\triangle ABC$ has $AB = 6$ cm, $AC = 7$ cm, and $\angle A = 125°$.

(i) Sketch, roughly, $\triangle ABC$ and mark on the known lengths and angle.

(ii) Find the length of side BC.

(iii) Find the area of $\triangle ABC$.

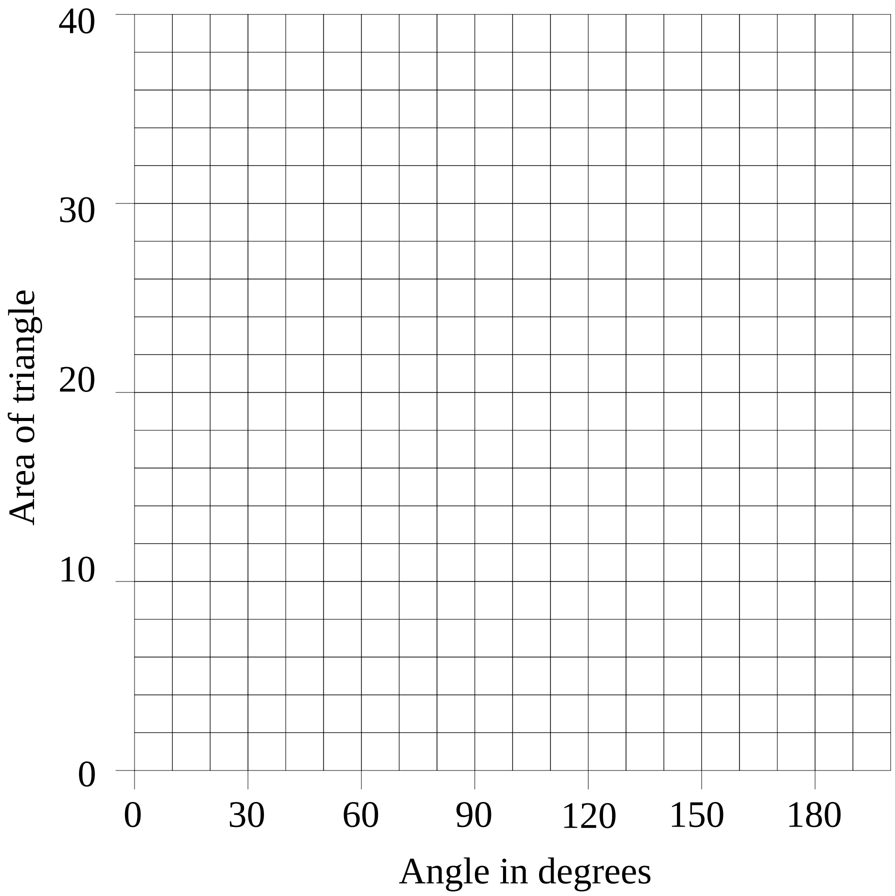
Question 5



(i) Work out the area Y for the values of θ given in the table below;

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
Y										

(ii) Plot a graph of your results with θ in the x-axis and area Y on the y-axis

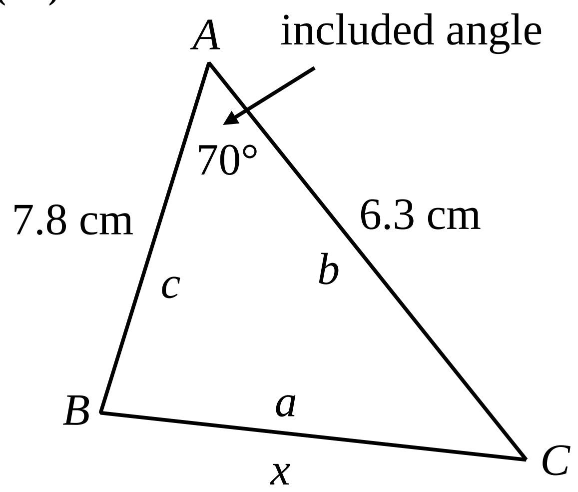


(iii) What happens as θ continues to increase beyond 90° and up to 180° ?

4.4 Answers

4.4.1 Solution to 4.2 Introductory Question

(i) **(ii)** and **(iii)**



(iv)

$a^2 = b^2 + c^2 - 2bc\cos A$ as given in GCSE formulae page

$= 6.3^2 + 7.8^2 - 2 \times 6.3 \times 7.8 \times \cos 70°$

$= 66.9$

$\therefore\ x = 8.18$ cm

(v)

$Area\ \Delta = \frac{1}{2}\, a\, b \sin C$ as given in GCSE formulae page

$= \frac{1}{2}\, b\, c \sin A$ rewritten for an included angle A

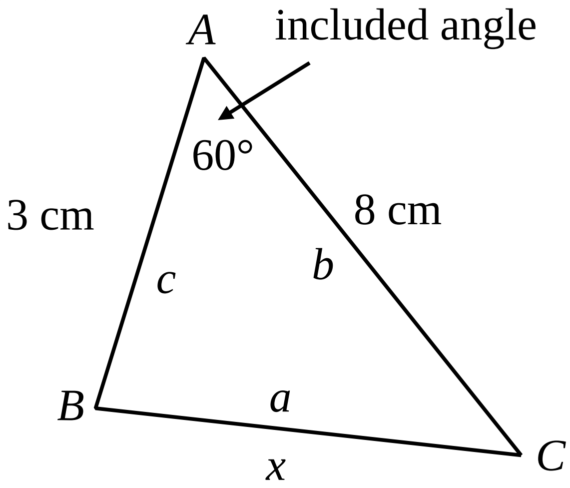
$= \frac{1}{2} \times 6.3 \times 7.8 \times \sin 70$

$= 23.1\ \text{cm}^2$

4.4.2 Solution to 4.3 Exercise

Answer 1

(i) **(ii)** and **(iii)**



(iv)

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \text{as given in GCSE formulae page}$$

$$= 8^2 + 3^2 - 2 \times 8 \times 3 \times \cos 60°$$

$$= 49$$

$$\therefore\ x = 7\ \text{cm}$$

(v)

$$Area\ \triangle = \frac{1}{2}\, a\, b \sin C \qquad \text{as given in GCSE formulae page}$$

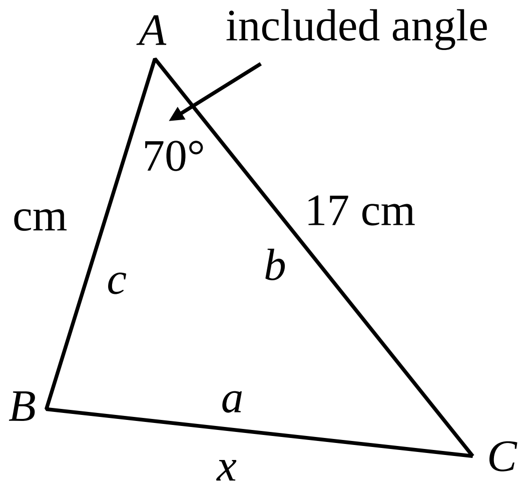
$$= \frac{1}{2}\, b\, c \sin A \qquad \text{rewritten for an included angle } A$$

$$= \frac{1}{2} \times 8 \times 3 \times \sin 60$$

$$= 10.4\ \text{cm}^2$$

Answer 2

(i) **(ii)** and **(iii)**



(iv)

$a^2 = b^2 + c^2 - 2bc\cos A$ as given in GCSE formulae page

$= 17^2 + 15^2 - 2 \times 17 \times 15 \times \cos 70°$

$= 339$

$\therefore\ x = 18.4$ cm

(v)

$Area\ \triangle = \frac{1}{2}\, a\, b \sin C$ as given in GCSE formulae page

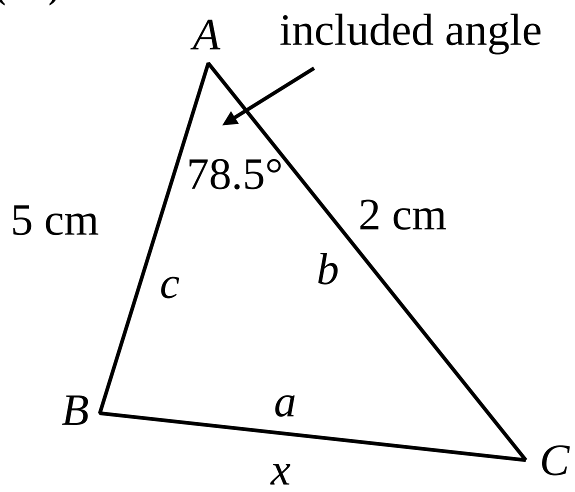
$= \frac{1}{2}\, b\, c \sin A$ rewritten for an included angle A

$= \frac{1}{2} \times 17 \times 15 \times \sin 70$

$= 120$ cm^2

Answer 3

(i) **(ii)** and **(iii)**



(iv)

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \text{as given in GCSE formulae page}$$

$$= 2^2 + 5^2 - 2 \times 2 \times 5 \times \cos 78.5°$$

$$= 25.0126$$

$$\therefore\ x = 5\,\text{cm}$$

(v)

$$Area\ \triangle = \frac{1}{2}\,a\,b\sin C \qquad \text{as given in GCSE formulae page}$$

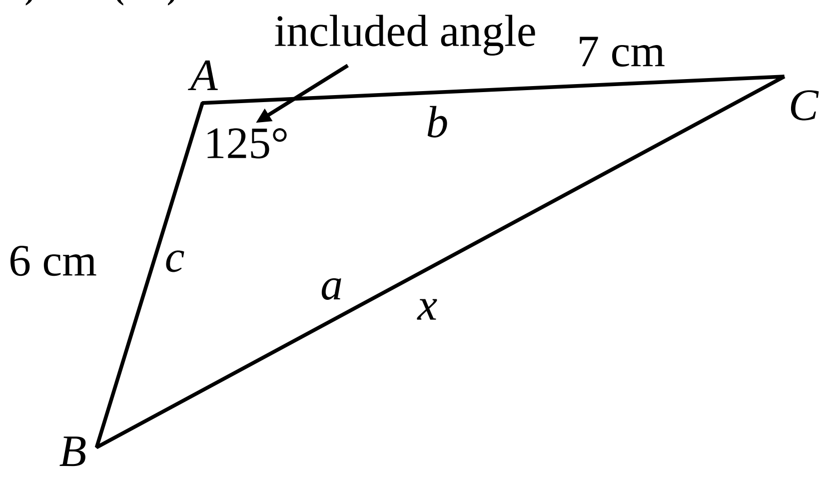
$$= \frac{1}{2}\,b\,c\sin A \qquad \text{rewritten for included angle } A$$

$$= \frac{1}{2} \times 2 \times 5 \times \sin 78.5$$

$$= 4.9\ \text{cm}^2$$

Answer 4

(i) **(ii)** and **(iii)**



(iv)

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \text{as given in GCSE formulae page}$$

$$= 7^2 + 6^2 - 2 \times 7 \times 6 \times \cos 125°$$

$$= 133$$

$$\therefore\ x = 11.5 \text{ cm}$$

(v)

$$Area\ \Delta = \frac{1}{2}\, a\, b \sin C \qquad \text{as given in GCSE formulae page}$$

$$= \frac{1}{2}\, b\, c \sin A \qquad \text{rewritten for an included angle } A$$

$$= \frac{1}{2} \times 7 \times 6 \times \sin 125$$

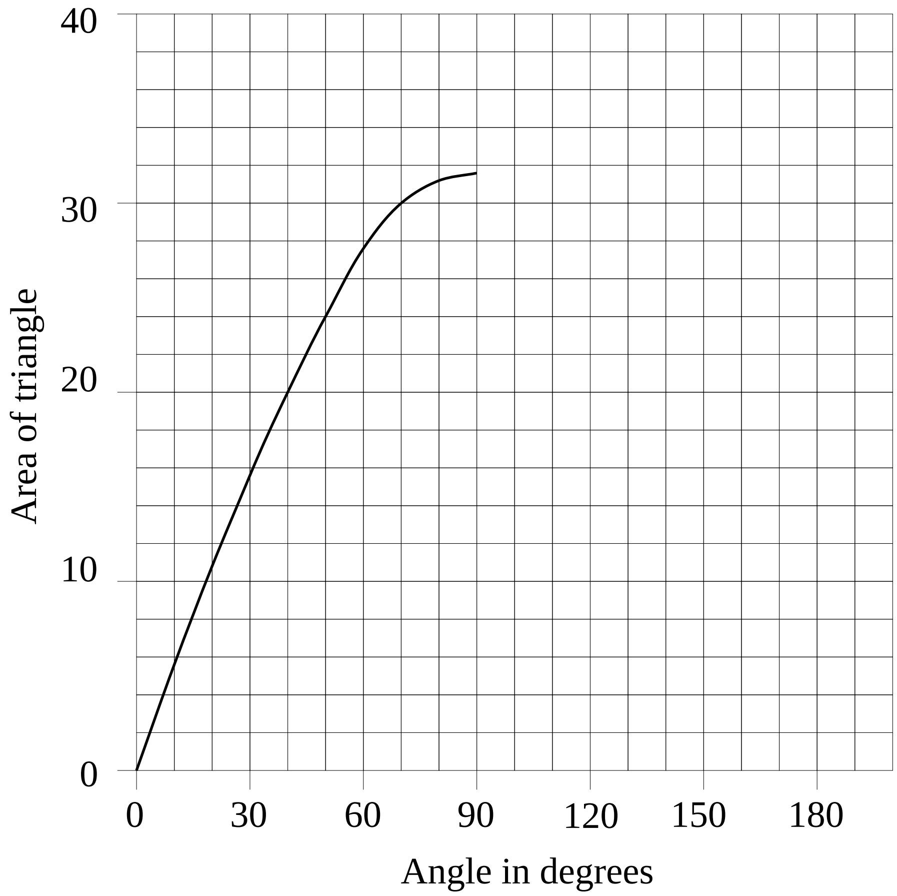
$$= 17.2 \text{ cm}^2$$

Answer 5

(i)

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
Y	0	5.5	10.8	15.8	20.2	24.1	27.3	29.6	31.0	31.5

(ii)



(iii) The curve is symmetrical in the mirror line, $\theta = 90°$

Starter for Lesson 5

GCSE Mathematics

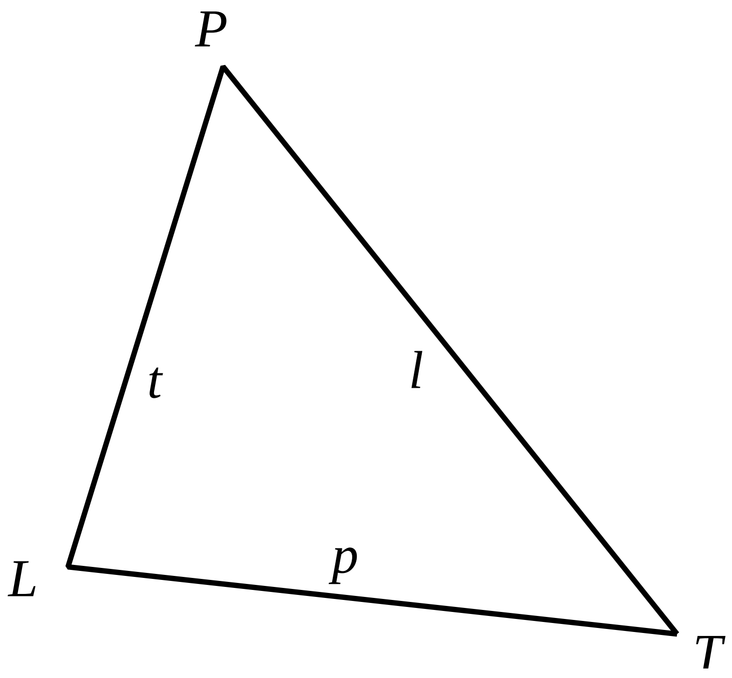
Non-Right Angled Trigonometry

5.start.1 △ *TLP* (Three Little Pigs)

You are reminded that in examinations you will be told the cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Consider this triangle:



Write down the cosine rule formula you would use to find the length of side *p*

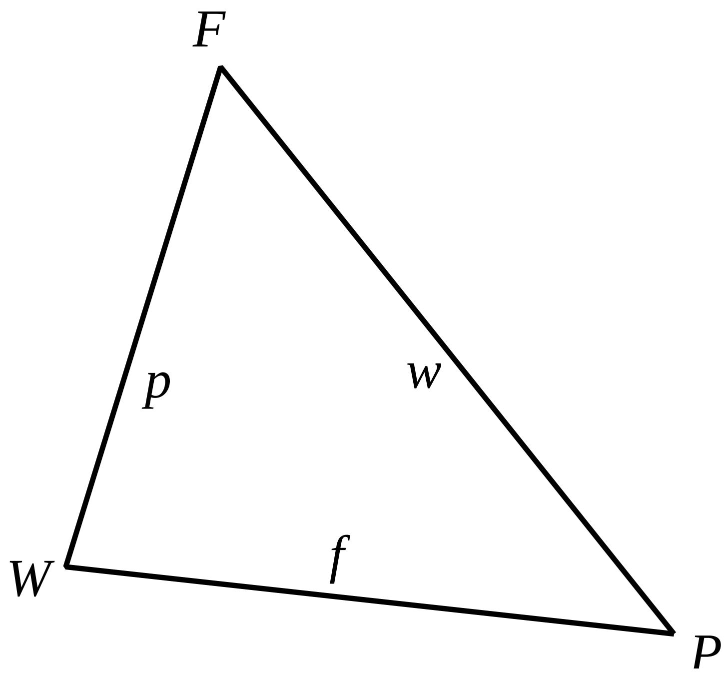
$p^2 =$

5.start.2 $\triangle$ *PWF*   (Pigs Will Fly)

You are reminded that in examinations you will be told the cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Consider this triangle:



Write down the cosine rule formula you would use to find the length of side *w*

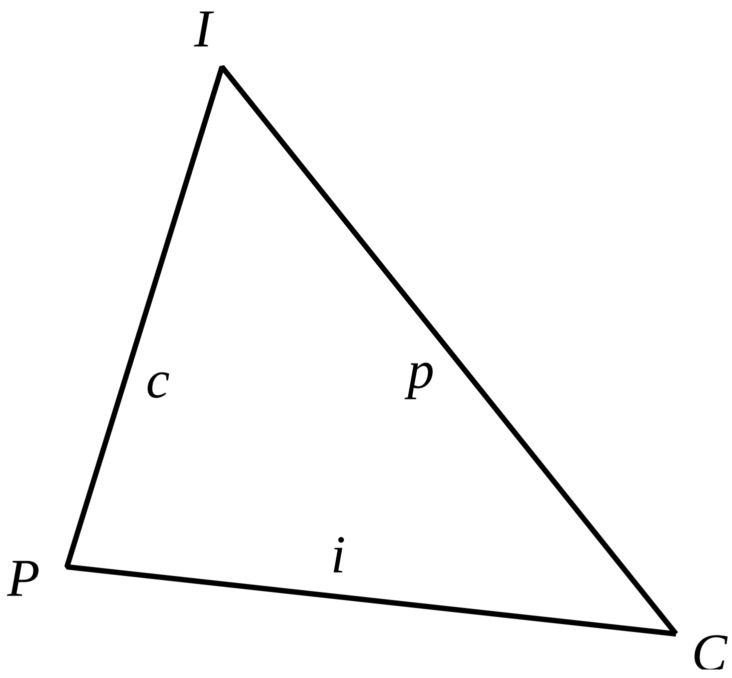
$w^2 =$

5.start.3 $\triangle$ *PIC* (Pigs In Clover)

You are reminded that in examinations you will be told the cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Consider this triangle:



(i) Write down the cosine rule formula you would use to find length p

$p^2 =$

(ii) Write down the cosine rule formula you would use to find length i

$i^2 =$

(iii) Write down the cosine rule formula you would use to find length c

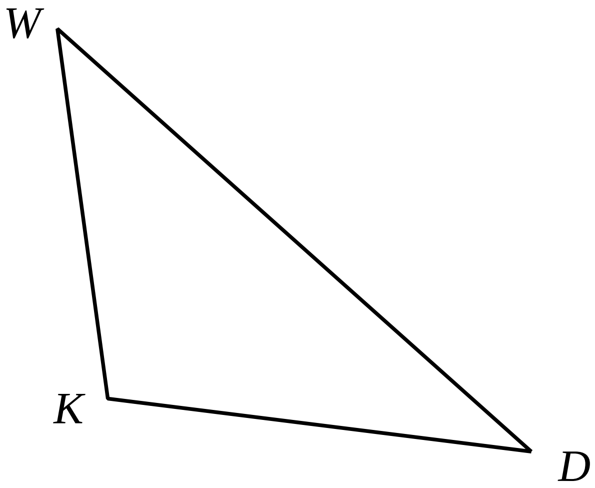
$c^2 =$

5.1 Alphabet Soup Cosine Rule

In the previous Lesson, all our triangles were labelled A, B and C and every problem was arranged so that length c was to be found.
Often different lettering is used, as the next example illustrates.

5.2 Example

(i) On $\triangle WKD$, place the letters w, k and d.



(ii) On $\triangle WKD$, mark on that;

- WD is of length 16.3 cm.
- WK is of length 11.6 cm.
- $\angle W$ is 34°.
- KD is the length x, which is to be found.

(iii) Mark the *included* angle with a star, *.

(iv) For $\triangle WKD$, write down the cosine rule for w^2, in terms of k, d and W.

(v) Use the cosine rule to find the length of x.

(vi) Find the area of $\triangle WKD$.

5.3 Exercise

Question 1

For $\triangle HMS$, write down the cosine rule for h^2, in terms of m, s and H.

Question 2

For $\triangle BAT$, write down the cosine rule for a^2, in terms of b, t and A.

Question 3

For $\triangle MAD$, write down the cosine rule for m^2.

Question 4

For $\triangle ABC$, write down the cosine rule for;

(i) c^2

(ii) b^2

(iii) a^2

Question 5

$\triangle ZEN$ has $EN = 4.3$ cm, $ZE = 6.5$ cm, and $\angle E = 66°$.

(i) Sketch, roughly, $\triangle ZEN$ marking on the two known lengths & included angle.

(ii) Write down the cosine rule for e^2.

(iii) Find the length of side ZN, accurate to 2 decimal places.

(iv) Find the area of $\triangle ZEN$, accurate to 2 decimal places.

Question 6

$\triangle ICU$ has IU = 10.2 cm, IC = 3.8 cm, and $\angle I$ = 41.5°.

(i) Sketch, roughly, $\triangle ICU$ marking on the two known lengths & included angle.

(ii) Write down the cosine rule for i^2.

(iii) Find the length of side CU, clearly stating the units of your answer.

(iv) Find the area of $\triangle ICU$, clearly stating the units of your answer.

Question 7

$\triangle OMG$ has $OM = 5.3$ km, $MG = 7.9$ km, and $\angle M = 127°$.

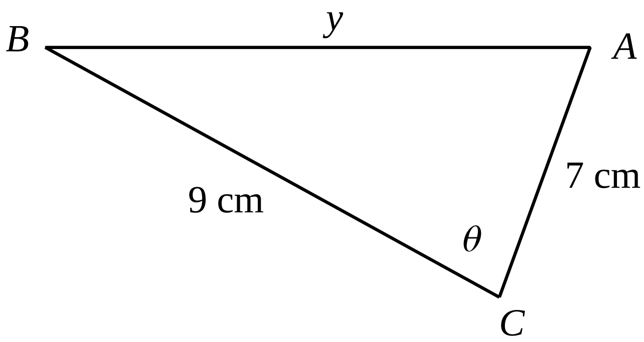
(i) Sketch, roughly, $\triangle OMG$ marking on the two known lengths & included angle.

(ii) Write down the cosine rule for m^2.

(iii) Find the length of side OG, clearly stating the units of your answer.

(iv) Find the area of $\triangle OMG$, clearly stating the units of your answer.

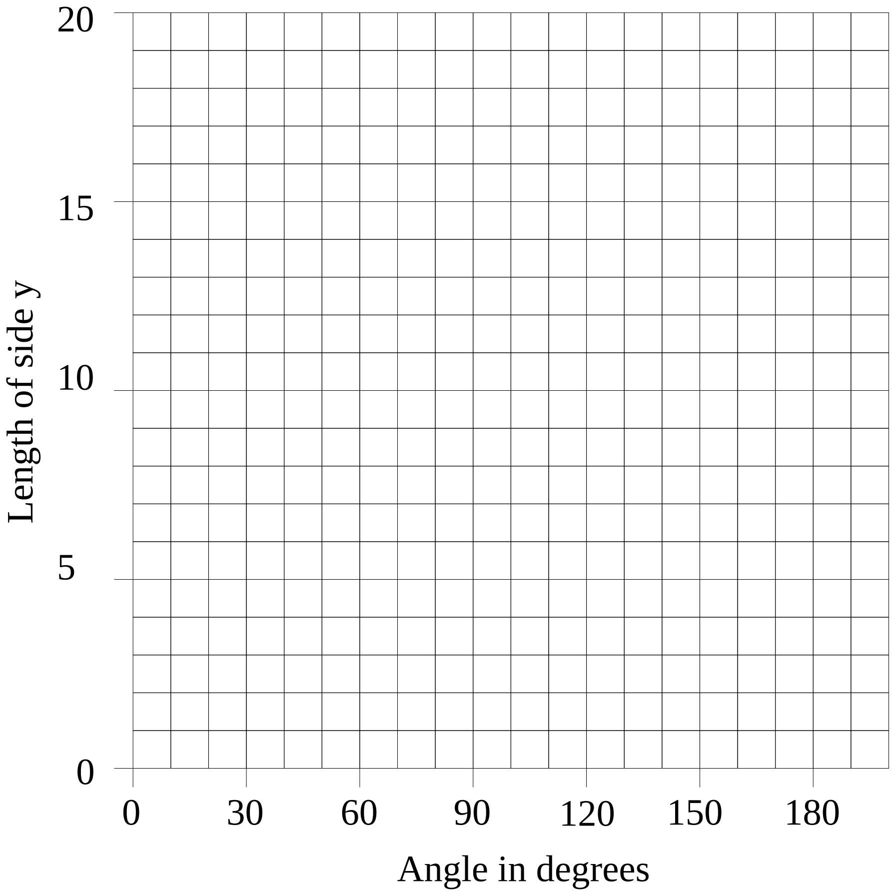
Question 8



(i) Work out the length y for the values of θ given in the table below;

θ	0°	30°	60°	90°	120°	150°	180°
y							

(ii) Plot a graph of your results with θ on the x-axis and length y on the y-axis



(iii) What happens as θ continues to increase beyond 180° and up to 360° ?

5.4 Homework

Question 1

ΔKFC has $FC = 10$ m, $KC = 8$ m, and $\angle C = 37°$.

(i) Sketch, roughly, ΔKFC marking on the two known lengths & included angle.

(ii) Write down the cosine rule for c^2.

(iii) Find the length of side KF, giving your answer accurate to 2 decimal places.

(iv) Find the area of ΔKFC, giving your answer accurate to 2 decimal places.

Question 2

$\triangle FBI$ has $FI = 14$ m, $FB = 5$ m, and $\angle F = 24.5°$.

(i) Sketch, roughly, $\triangle FBI$ marking on the two known lengths & included angle.

(ii) Write down the cosine rule for f^2.

(iii) Find the length of side BI, clearly stating the units of your answer.

(iv) Find the area of $\triangle FBI$, clearly stating the units of your answer.

Question 3

In $\triangle CIA$, $\angle I = 137°$, $CI = 84$ m, $IA = 123$ m.

Find length CA and the area of the triangle, both accurate to 3 significant figures.

5.5 Answers

5.5.1 Solution to Starter for Lesson 5

Answer 5.start.1

$$p^2 = t^2 + l^2 - 2tl \cos P$$

Answer 5.start.2

$$w^2 = p^2 + f^2 - 2pf \cos W$$

Answer 5.start.3

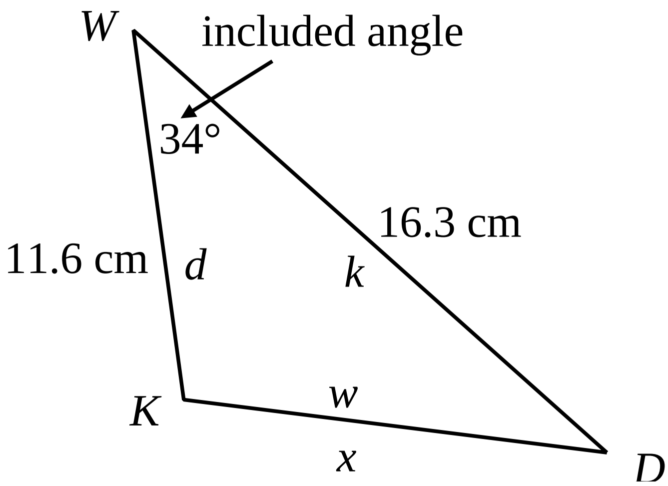
(i) $p^2 = i^2 + c^2 - 2ic \cos P$

(ii) $i^2 = p^2 + c^2 - 2pc \cos I$

(iii) $c^2 = i^2 + p^2 - 2ip \cos C$

5.5.2 Solution to 5.2 Example

(i) (ii) and **(iii)**



(iv)

$$w^2 = d^2 + k^2 - 2dk \cos W$$
$$= 11.6^2 + 16.3^2 - 2 \times 11.6 \times 16.3 \times \cos 34°$$
$$= 86.74$$
$$\therefore\ x = 9.31 \text{ cm}$$

(v)

$$Area\ \Delta = \frac{1}{2}\, d\, k \sin W$$
$$= \frac{1}{2} \times 11.6 \times 16.3 \times \sin 34$$
$$= 52.9 \text{ cm}^2$$

5.5.3 Solutions to 5.3 Exercise

Answer 1

$$h^2 = m^2 + s^2 - 2ms\cos H$$

Answer 2

$$a^2 = b^2 + t^2 - 2bt\cos A$$

Answer 3

$$m^2 = a^2 + d^2 - 2ad\cos M$$

Answer 4

(i)

$$c^2 = a^2 + b^2 - 2ab\cos C$$

(ii)

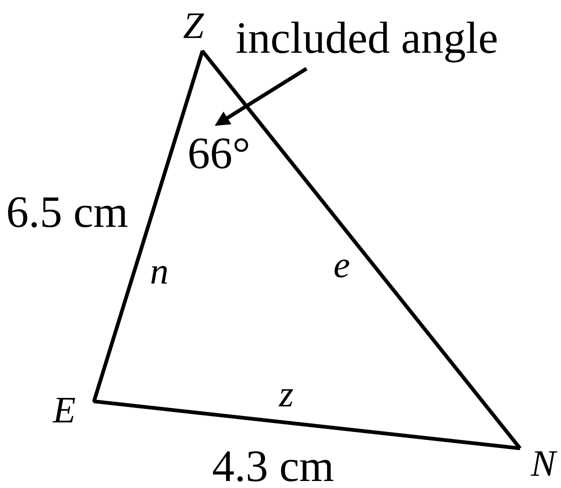
$$b^2 = a^2 + c^2 - 2ac\cos B$$

(iii)

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Answer 5

(i)



(ii) and **(iii)**

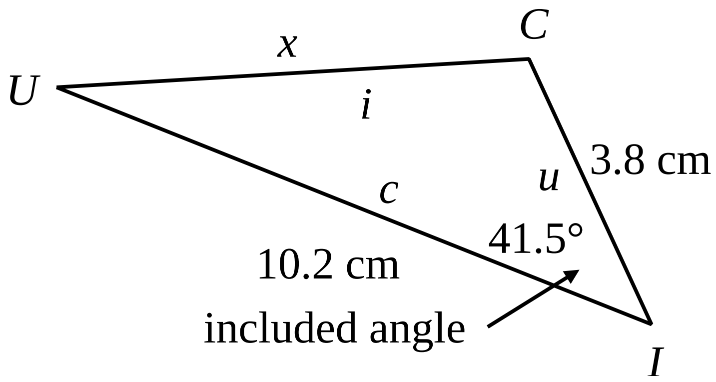
$$\begin{aligned} e^2 &= z^2 + n^2 - 2zn\cos E \\ &= 4.3^2 + 6.5^2 - 2 \times 4.3 \times 6.5 \times \cos 66 \\ &= 38 \\ e &= 6.16 \text{ cm (2 decimal places asked for)} \end{aligned}$$

(iv)

$$\begin{aligned} Area\,\Delta &= \frac{1}{2} zn\sin E \\ &= \frac{1}{2} \times 4.3 \times 6.5 \times \sin 66 \\ &= 12.77 \text{ cm}^2 \text{ (2 decimal places asked for)} \end{aligned}$$

Answer 6

(i)



(ii) and **(iii)**

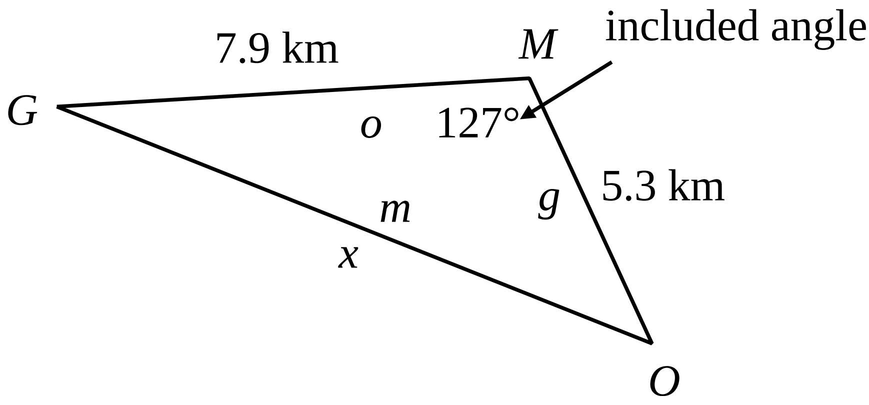
$$
\begin{aligned}
i^2 &= c^2 + u^2 - 2cu\cos I \\
&= 10.2^2 + 3.8^2 - 2 \times 10.2 \times 3.8 \times \cos 41.5 \\
&= 60.42 \\
i &= 7.77 \text{ cm}
\end{aligned}
$$

(iv)

$$
\begin{aligned}
Area\ \triangle &= \frac{1}{2} cu \sin I \\
&= \frac{1}{2} \times 10.2 \times 3.8 \times \sin 41.5 \\
&= 12.84 \text{ cm}^2
\end{aligned}
$$

Answer 7

(i)



(ii) and **(iii)**

$$\begin{aligned} m^2 &= o^2 + g^2 - 2og\cos M \\ &= 7.9^2 + 5.3^2 - 2 \times 7.9 \times 5.3 \times \cos 127 \\ &= 140.9 \\ m &= 11.87 \text{ km} \end{aligned}$$

(iv)

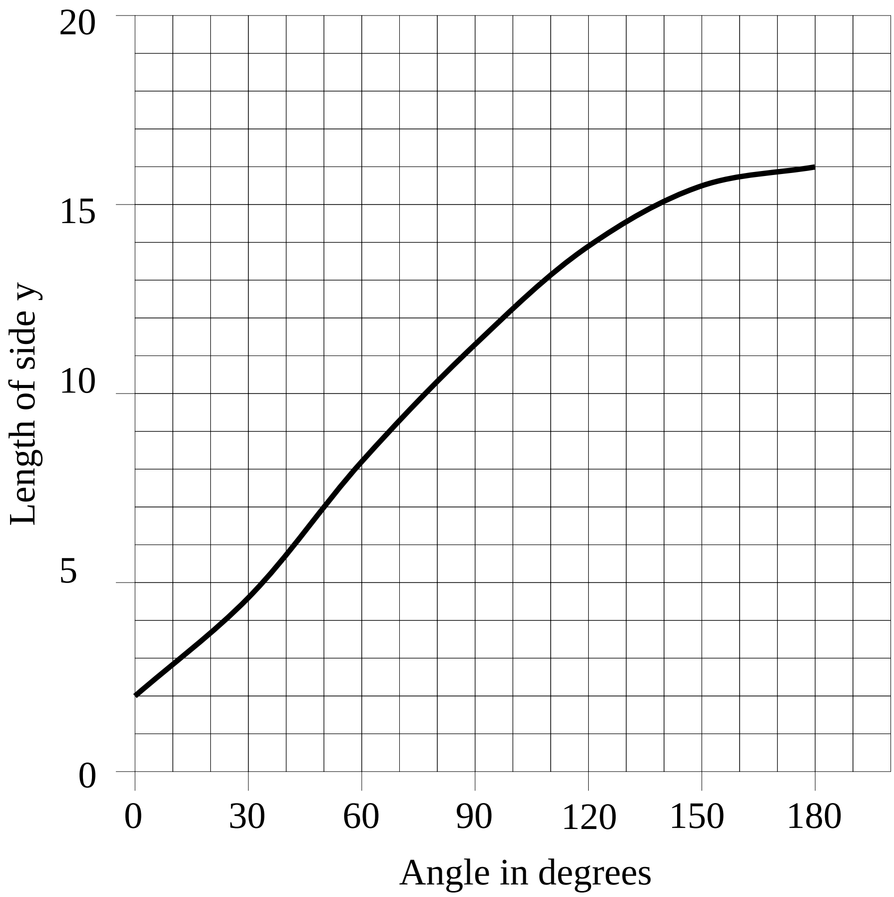
$$\begin{aligned} Area\ \triangle &= \frac{1}{2} og \sin M \\ &= \frac{1}{2} \times 7.9 \times 5.3 \times \sin 127 \\ &= 16.72 \text{ km}^2 \end{aligned}$$

Answer 8

(i)

θ	0°	30°	60°	90°	120°	150°	180°
y	2.00	4.57	8.19	11.4	13.9	15.5	16

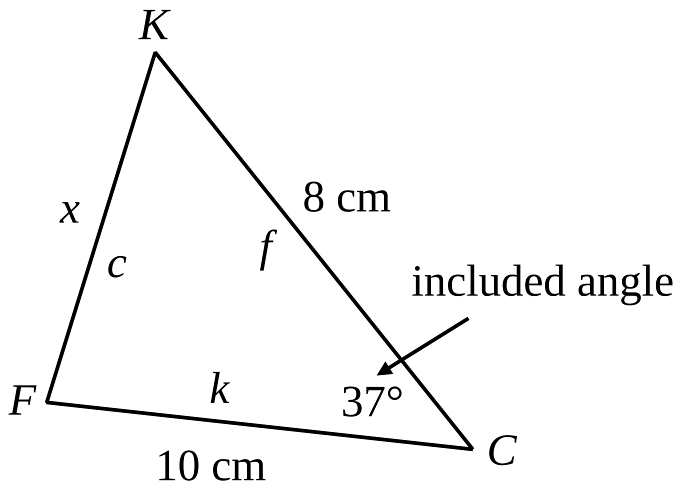
(ii)



(iii) The curve is symmetrical in the mirror line, θ = 180°

Answer 1

(i)



(ii) and **(iii)**

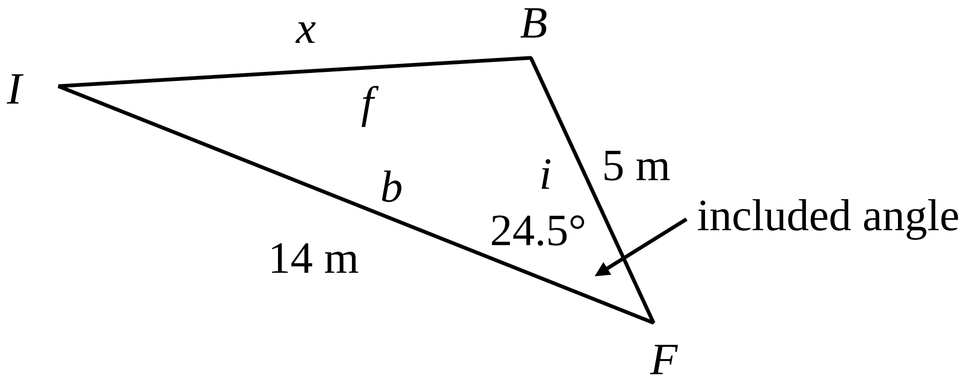
$$\begin{aligned} c^2 &= k^2 + f^2 - 2kf\, \cos C \\ &= 10^2 + 8^2 - 2 \times 10 \times 8 \times \cos 37 \\ &= 36.22 \\ c &= 6.02 \text{ m} \quad \text{(2 decimal places asked for)} \end{aligned}$$

(iv)

$$\begin{aligned} Area\ \triangle &= \frac{1}{2} fk \sin C \\ &= \frac{1}{2} \times 10 \times 8 \times \sin 37 \\ &= 24.07 \text{ m}^2 \quad \text{(2 decimal places asked for)} \end{aligned}$$

Answer 2

(i)



(ii) and **(iii)**

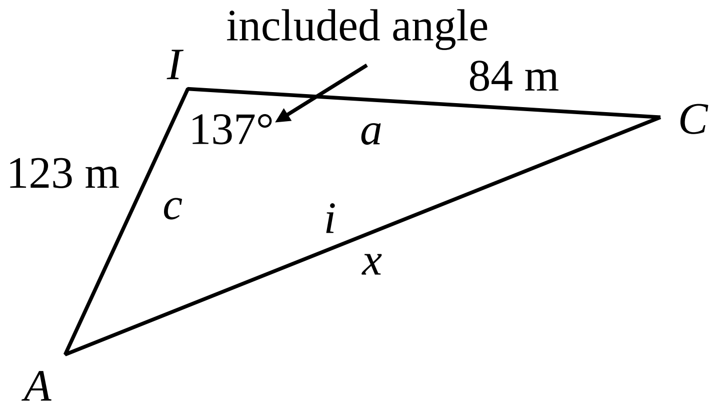
$$f^2 = b^2 + i^2 - 2bi \cos F$$
$$= 14^2 + 5^2 - 2 \times 14 \times 5 \times \cos 24.5$$
$$= 93.6$$
$$f = 9.67 \text{ m}$$

(iv)

$$Area\ \triangle = \frac{1}{2}\, bi \sin F$$
$$= \frac{1}{2} \times 14 \times 5 \times \sin 24.5$$
$$= 14.51 \text{ m}^2$$

Answer 3

(i)



(ii) and **(iii)**

$$
\begin{aligned}
i^2 &= a^2 + c^2 - 2ac\cos I \\
&= 84^2 + 123^2 - 2 \times 84 \times 123 \times \cos 137 \\
&= 37297 \\
i &= 9193\text{ m} \text{ (3 significant figures asked for)}
\end{aligned}
$$

(iv)

$$
\begin{aligned}
Area\ \Delta &= \frac{1}{2} ac \sin I \\
&= \frac{1}{2} \times 84 \times 123 \times \sin 137 \\
&= 3520\text{ m}^2
\end{aligned}
$$

Lesson Starter for Lesson 6

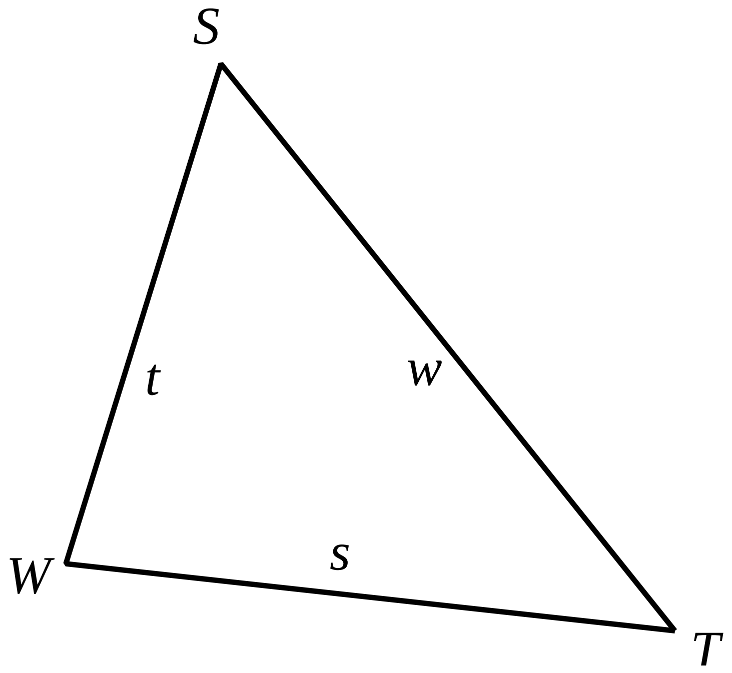
GCSE Mathematics
Non-Right Angled Trigonometry

6.start.1 $\triangle$ ***STW*** **(Save The Whale)**

The reversed cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Consider this triangle:



Write down the reversed cosine rule to find $\cos W$

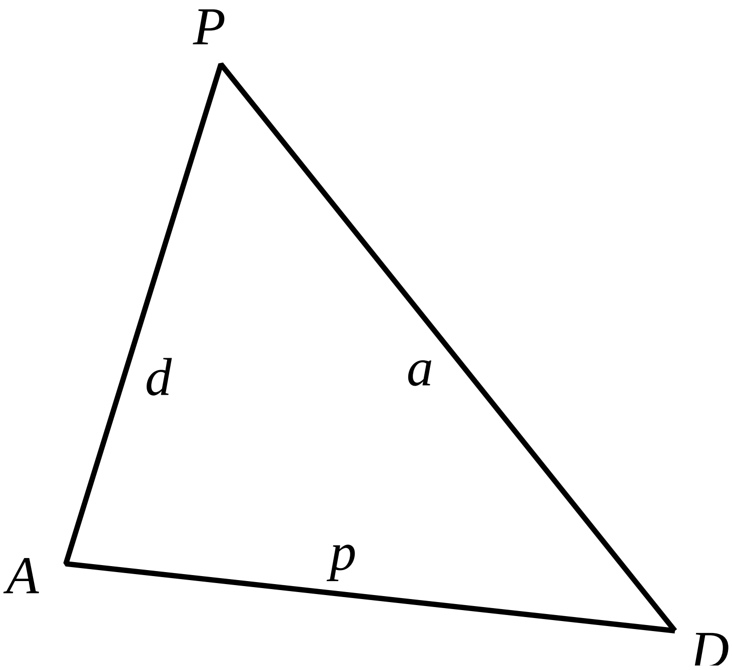
$\cos W =$

6.start.2 $\triangle$ *PAD* (Pandas Are Dying)

The reversed cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Consider this triangle:

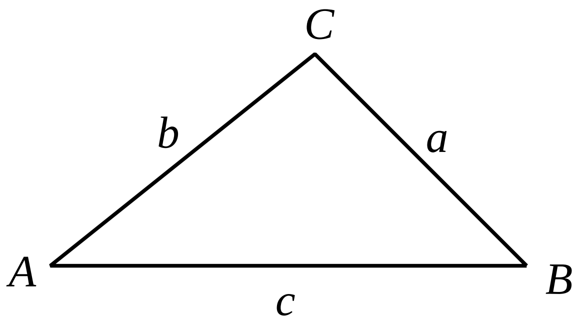


Write down the reversed cosine rule to find $\cos D$

$\cos D =$

6.1 Elur Enisoc : The Cosine Rule Reversed

Here is The Cosine Rule, as given on the GCSE examination formula page;



$$a^2 = b^2 + c^2 - 2bc\cos A$$

Here is how to make $\cos A$ the subject of this formula.

Step 1 : Add $2bc\cos A$ to both sides...

$$2bc\cos A + a^2 = b^2 + c^2$$

Step 2 : Subtract a^2 from both sides...

$$2bc\cos A = b^2 + c^2 - a^2$$

Step 3 : Divide both sides by $2bc$...

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

We shall refer to this result as The Cosine Rule Reversed : "Elur Enisoc".
It is not given on the GCSE examination page so you need to
Either

- Remember it, or rather the form (look at where A and a are).

Or

- Derive it from the version given in the examination, as shown above.

TOP TIP : When entering this into your calculator, use the fraction button.
If your calculator does not have a fraction button put brackets in, like this...

$$\cos A = \frac{(b^2 + c^2 - a^2)}{(2bc)}$$

Use The Cosine Rule Reversed when you know all three sides and want to find an angle

6.2 Exercise

Question 1

For $\triangle FUN$, write down the reversed cosine rule for *cos F*, in terms of *f*, *u* and *n*.

Question 2

For $\triangle HAT$, write down the reversed cosine rule for *cos T*, in terms of *h*, *a* and *t*.

Question 3

For $\triangle DAZ$, write down the reversed cosine rule for *cos Z*.

Question 4

For $\triangle ABC$, write down the reversed cosine rule for;

(i) *cos C*

(ii) *cos B*

(iii) *cos A*

Question 5

(i) Solve the equation $X = arccos\ 0.822$

Give your answer to 1 decimal place.

(ii) Solve the equation $Y = arccos\ (-0.109)$

Give your answer to 1 decimal place.

(iii) If $cos\ Z = 0.571$ what is Z ?

Give your answer in degrees, to 1 decimal place.

(iv) If $cos\ E = 0.311$ what is E ?

Give your answer in degrees, to 1 decimal place.

(v) If $cos\ W = -0.287$ what is W ?

Give your answer in degrees, to 1 decimal place.

Question 6

A triangle has sides of length 3 cm, 7 cm and 9 cm.

(i) Roughly sketch the triangle, with the sides roughly the correct lengths.

(ii) Is the largest angle

- Opposite the 3 cm side ?
- Opposite the 7 cm side ?

Or
- Opposite the 9 cm side ?

Question 7

In $\triangle ABC$, $BC = 13$ m, $CA = 8$ m and $AB = 14$ m.

(i) Sketch, roughly, $\triangle ABC$ marking on the three known lengths.

(ii) Write down the formula for the reversed cosine rule for *cos* A.

(iii) Find the size of angle A.

Question 8

In ΔABC, $BC = 8$ cm, $CA = 12$ cm and $AB = 9$ cm.

(i) Sketch, roughly, ΔABC marking on the three known lengths.

(ii) Write down the reversed cosine rule for *cos* A.

(iii) Find the size of angle A.

Question 9

In $\triangle YXU$, $XU = 280$ m, $UY = 240$ m and $YX = 200$ m.

(i) Sketch, roughly, $\triangle YXU$ marking on the three known lengths.

(ii) Write down the reversed cosine rule for $\cos Y$.

(iii) Find the size of angle Y, giving your answer in degrees to one decimal place.

Question 10

In $\triangle QPR$, $PR = 51\ 400$ km, $RQ = 72\ 300$ km and $QP = 84\ 400$ km.

(i) Sketch, roughly, $\triangle QPR$ marking on the three known lengths.

(ii) Write down the reversed cosine rule for $\cos R$.

(iii) Find the size of angle R, giving your answer in degrees to one decimal place.

Question 11

The AREA of a triangle is given on the GCSE examination formula page as

$$Area\ of\ triangle \ = \ \frac{1}{2}\, ab \sin C$$

(i) Make *sin C* the subject of this formula.

(ii) A triangle has an area of 52 cm^2
It has one side of length 18 cm, and another side of length 11 cm.
What is the size of the acute angle between these two sides ?
Give your answer in degrees to one decimal place.
HINT : Your **part (i)** formula should help !

(iii) If you knew that the angle was obtuse, what would it be ?

Question 12

In $\triangle TVC$, $VC = 11$ cm, $CT = 8$ cm and $TV = 10$ cm.

(i) Sketch, roughly, $\triangle TVC$ marking on the three known lengths.

(ii) Write down the reversed cosine rule for $\cos T$.

(iii) Find the size of angle T.

(iv) Write down the reversed cosine rule for $\cos V$.

(v) Find the size of angle V.

(vi) Using a well known fact about the sum of the angles in a triangle, deduce the size of the remaining angle, angle C.

6.3 Answers

6.3.1 Solution to Starter for Lesson 6

Answer 6.start.1

$$\cos W = \frac{s^2 + t^2 - w^2}{2st}$$

Answer 6.start.2

$$\cos P = \frac{a^2 + d^2 - p^2}{2ad}$$

6.3.2 Solution to 6.2 Exercise

Answer 1

$$\cos F = \frac{u^2 + n^2 - f^2}{2un}$$

Answer 2

$$\cos T = \frac{h^2 + a^2 - t^2}{2ha}$$

Answer 3

$$\cos Z = \frac{d^2 + a^2 - z^2}{2da}$$

Answer 4

(i) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

(ii) $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

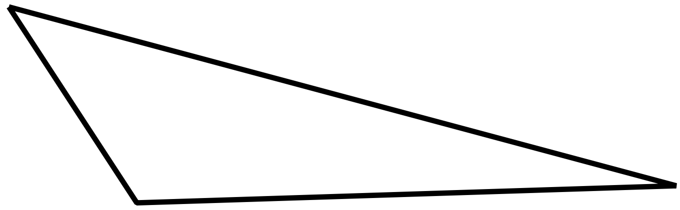
(iii) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Answer 5

(i) 34.7° (1 decimal place asked for)
(ii) 96.3° (1 decimal place asked for)
(iii) 55.2° (1 decimal place asked for)
(iv) 71.9° (1 decimal place asked for)
(v) 106.7° (1 decimal place asked for)

Answer 6

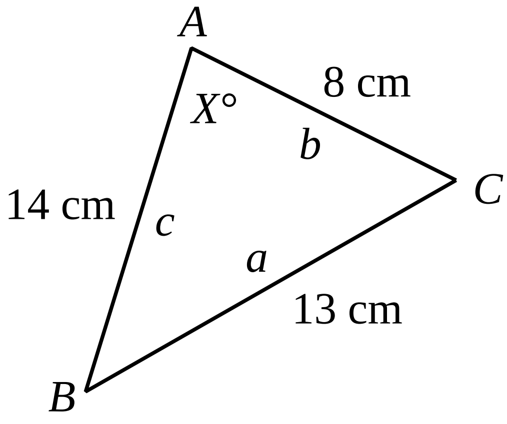
(i)



(ii) The largest angle is opposite the longest side, the 9 cm side

Answer 7

(i)



(ii)

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

(iii)

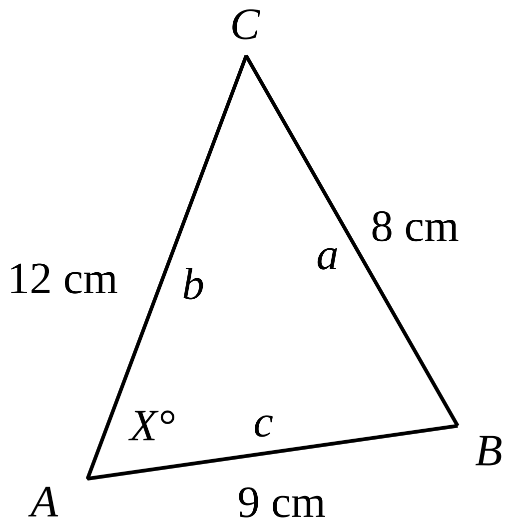
$$= \frac{8^2 + 14^2 - 13^2}{2 \times 8 \times 14}$$

$$= 0.40625$$

$$A = 66°$$

Answer 8

(i)



(ii) and **(iii)**

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

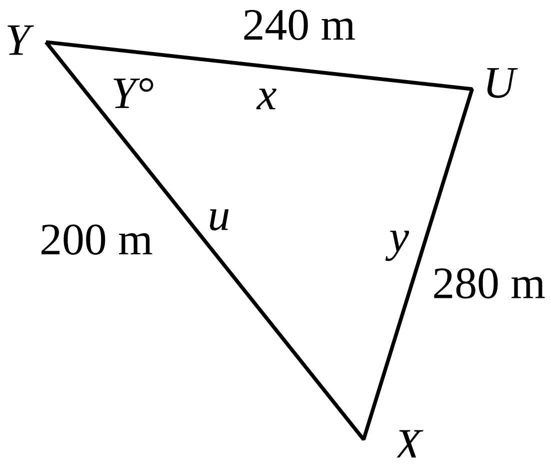
$$= \frac{12^2 + 9^2 - 8^2}{2 \times 12 \times 9}$$

$$= 0.745$$

$$A = 41.8°$$

Answer 9

(i)



(ii) and **(iii)**

$$\cos Y = \frac{x^2 + u^2 - y^2}{2xu}$$

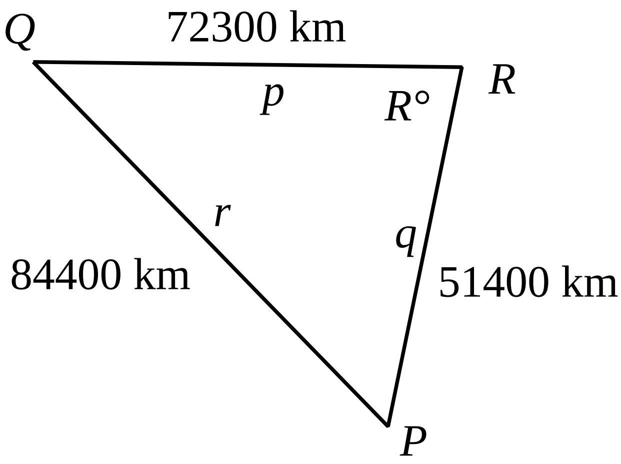
$$= \frac{240^2 + 200^2 - 280^2}{2 \times 240 \times 200}$$

$$= 0.2$$

$Y = 78.5°$ (1 decimal place asked for)

Answer 10

(i)



(ii) and **(iii)**

$$\cos R = \frac{p^2 + q^2 - r^2}{2pq}$$

$$= \frac{72300^2 + 51400^2 - 84400^2}{2 \times 72300 \times 51400}$$

$$= 0.1$$

$$R = 84.3° \text{ (1 decimal place asked for)}$$

Answer 11

(i)

$$\sin C = \frac{2 \times Area}{ab}$$

(ii)

$$\sin C = \frac{2 \times 52}{18 \times 11}$$

$$= 0.525$$

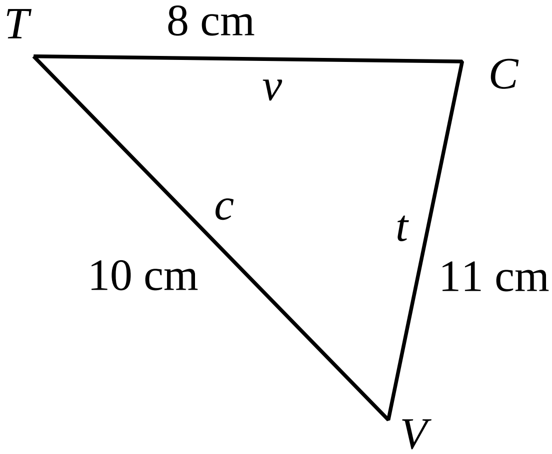
$$C = 31.7° \text{ (acute(}$$

(iii)

$$C = 148.3° \text{ (obtuse)}$$

Answer 12

(i)



(ii) and **(iii)**

$$\cos T = \frac{v^2 + c^2 - t^2}{2vc}$$

$$= \frac{8^2 + 10^2 - 11^2}{2 \times 8 \times 10}$$

$$= 0.26875$$

$$T = 74.4°$$

(iv) and **(v)**

$$\cos V = \frac{c^2 + t^2 - v^2}{2ct}$$

$$= \frac{10^2 + 11^2 - 8^2}{2 \times 10 \times 11}$$

$$= 0.7136$$

$$V = 44.45°$$

(vi)

$$\sum \textit{angles in a } \Delta = 180°$$

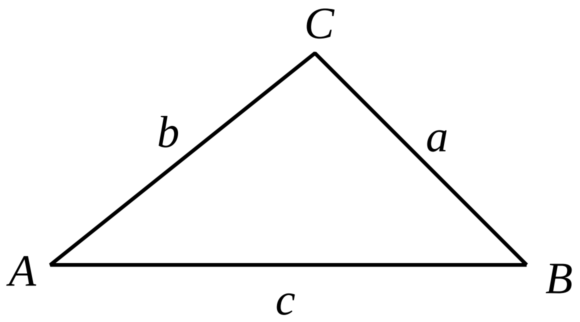
$$\therefore C = 61.1°$$

Lesson 7

7.1 The Sine Rule

On the GCSE examination formulae page...

In any triangle ABC



$$\text{Sine Rule} : \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\text{Cosine Rule} : a^2 = b^2 + c^2 - 2bc \cos A$$

$$\text{Area of triangle} = \frac{1}{2} ab \sin C$$

The lower two of these rules we recognise; it is the upper one we now consider.
(Notice that The Cosine Rule Reversed is not given)

The Sine Rule may look somewhat odd at first glance as it contains two equals signs.
It's a "three in one" formula !

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Tells us that:

(i)

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

(ii)

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

(iii)

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Use it when you know one side and two (and ∴ three) angles and want to find a side

7.2 Example

The Question

$\triangle XTC$ has $\angle X = 54°$ and $\angle T = 80°$.

Side XT is of length 8.3 cm.

(i) What is the size of angle C, the third angle ?

(ii) Sketch the triangle, not to scale, and mark on all known lengths and angles.

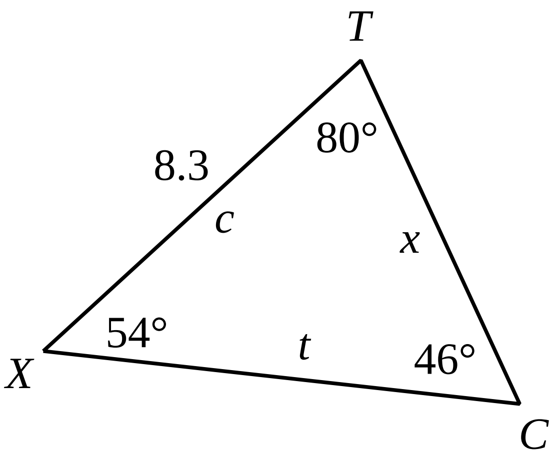
(iii) Find the length of side TC correct to three significant figures.

The Solution

(i)

$$\angle C = 180 - (54 + 80)$$
$$= 46°$$

(ii)



(iii) To find TC, or x;

☐ Write out The Sine Rule in full for $\triangle XTC$

☐ Circle which to find

☐ Box known pair

$$\frac{x}{\sin X} = \frac{t}{\sin T} = \boxed{\frac{c}{\sin C}}$$

$$\frac{x}{\sin X} = \frac{c}{\sin C}$$

$$\frac{x}{\sin 54} = \frac{8.3}{\sin 46}$$

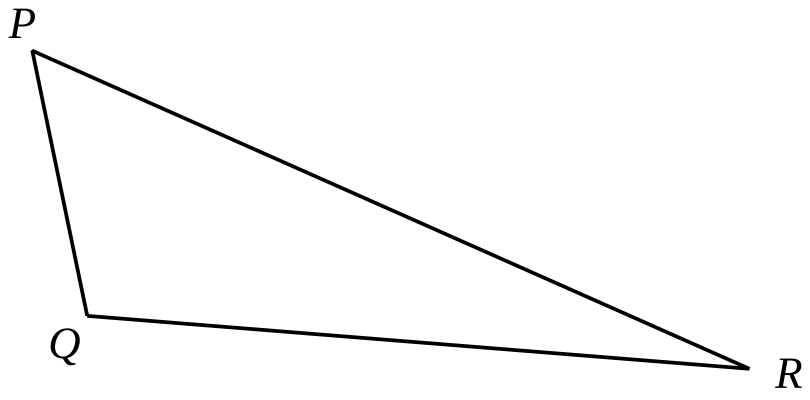
$$x = \frac{8.3 \times \sin 54}{\sin 46}$$

$$x = 9.33 \text{ cm} \qquad \text{to 3 significant figures}$$

7.3 Exercise

Question 1

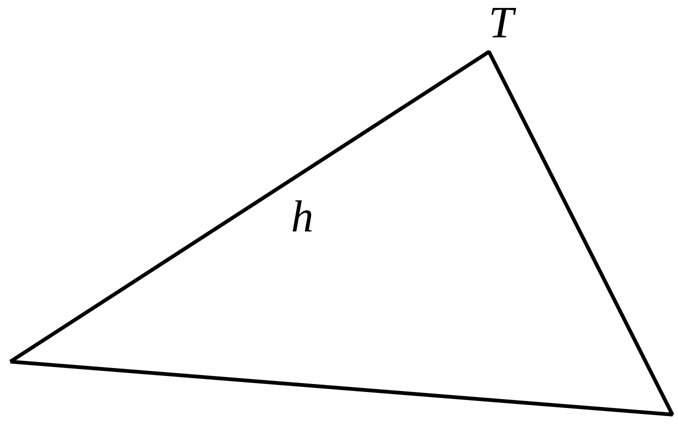
(i) On the following diagram place the letters *p*, *q* and *r* on the sides opposite the similarly labelled angles, *P*, *Q* and *R*.



(ii) Write down the sine rule in terms of *P*, *p*, *Q*, *q*, *R* and *r*.

Question 2

(i) On the following diagram place the letters *H*, *t*, *V* and *v*.



(ii) Write down the sine rule in terms of *H*, *h*, *T*, *t*, *V* and *v*.

Question 3

In each case, calculate the value of *x* correct to three significant figures;

(i)

$$x = \frac{246.4 \times 0.5643}{0.8025}$$

(ii)

$$x = \frac{5.8 \times \sin 66^\circ}{\sin 23^\circ}$$

(iii)

$$x = \frac{0.71 \times \sin 122^\circ}{\sin 31^\circ}$$

(iv)

$$x = \frac{2461 \times \sin 62^\circ}{\sin 34^\circ}$$

Question 4

Solve the following equations to find the values of *x* correct to three significant figures.

(i)

$$\frac{x}{8} = \frac{5}{7}$$

(ii)

$$\frac{x}{\sin 68^\circ} = \frac{74}{\sin 18^\circ}$$

(iii)

$$\frac{x}{\sin 106^\circ} = \frac{2375}{\sin 76^\circ}$$

(iv)

$$\frac{x}{\sin 60} = \frac{63}{\sin 90^\circ}$$

Question 5

$\triangle ABC$ has $\angle A = 28°$, and $\angle B = 74°$.

Side AC is of length 51 cm.

(i) What is the size of angle C, the third angle ?

(ii) Sketch the triangle, not to scale, and mark on all known lengths and angles.

(iii) Find the length of side BC correct to three significant figures.

HINT ☐ Write out The Sine Rule in full for $\triangle ABC$

☐ Circle which to find

☐ Box known pair

(iv) Find the area of $\triangle ABC$ by using the following formula;

$$Area\ \triangle = \frac{1}{2}\, a\, b\, \sin C$$

Question 6

$\triangle DEF$ has $\angle E = 18°$ and $\angle F = 110°$.

Side EF is of length 6.4 cm.

(i) What is the size of angle D, the third angle ?

(ii) Sketch the triangle, not to scale, and mark on all known lengths and angles.

(iii) Find the length of side DF correct to three significant figures.

HINT ☐ Write out The Sine Rule in full for $\triangle DEF$

☐ Circle which to find

☐ Box known pair

(iv) Dr Goodsum has worked out that the triangle's area is given by;

$$Area\,\triangle = \frac{1}{2}\,d\,e\,\sin F$$

Calculate the area of $\triangle DEF$.

Question 7

$\triangle PDQ$ has $\angle P = 44°$ and $\angle D = 69°$.

Side PD is of length 18 cm.

(i) What is the size of angle Q, the third angle ?

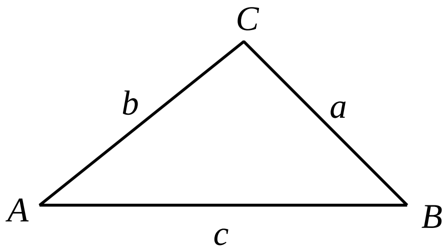
(ii) Sketch the triangle, not to scale, and mark on all known lengths and angles.

(iii) Find the length of side DQ correct to three significant figures.

(iv) Find the length of side PQ correct to three significant figures.

(v) The following formula gives the area of a triangle when you don't know the triangle's perpendicular height;

$$Area\ \triangle \ = \ \frac{1}{2}\, a\, b\, \sin C$$



Use an appropriate version of the area formula to determine area of $\triangle PDQ$.

7.4 Homework

Question 1

ΔABC has $\angle A = 38°$, and $\angle B = 64°$.

Side AB is of length 30 cm.

(i) What is the size of angle C, the third angle ?

(ii) Sketch the triangle, not to scale, and mark on all known lengths and angles.

(iii) Find the length of side BC correct to three significant figures.

HINT ☐ Write out The Sine Rule in full for ΔABC

☐ Circle which to find

☐ Box known pair

Question 2

$\triangle FRT$ has $\angle R = 30°$ and $\angle T = 100°$.

Side RT is of length 16 km.

(i) What is the size of angle F, the third angle ?

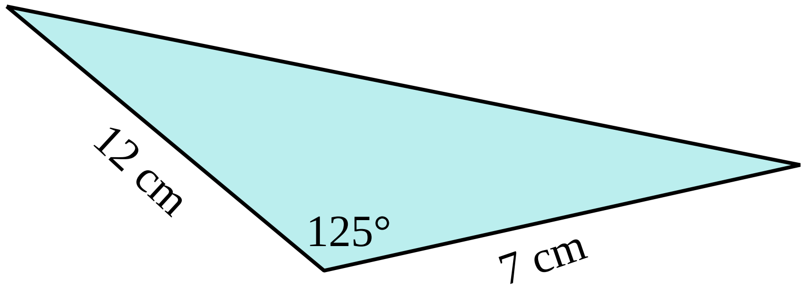
(ii) Sketch the triangle, not to scale, and mark on all known lengths and angles.

(iii) Find the length of side FT clearly stating the units of your answer.

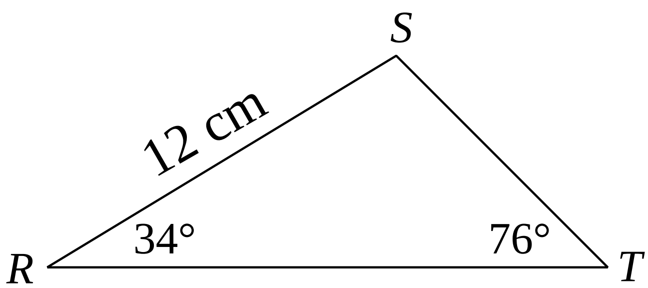
HINT
- ☐ Write out The Sine Rule in full for $\triangle FRT$
- ☐ Circle which to find
- ☐ Box known pair

Question 3

Find the area of the following triangle:



Question 4



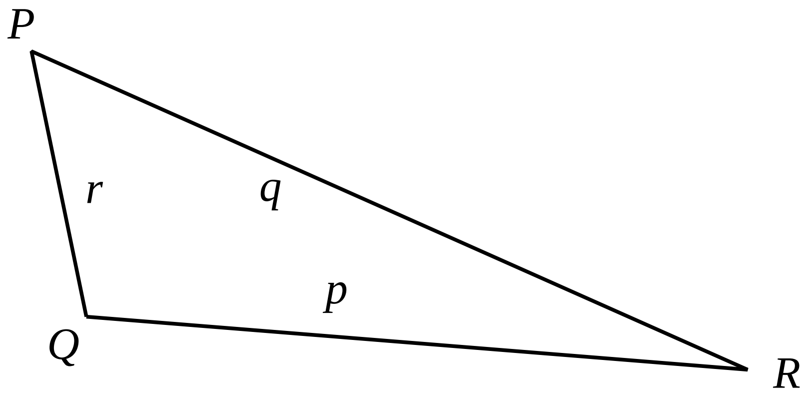
Find the length of side *ST* correct to three significant figures.

7.5 Answers

7.5.1 Solutions to 7.3 Exercise

Answer 1

(i)

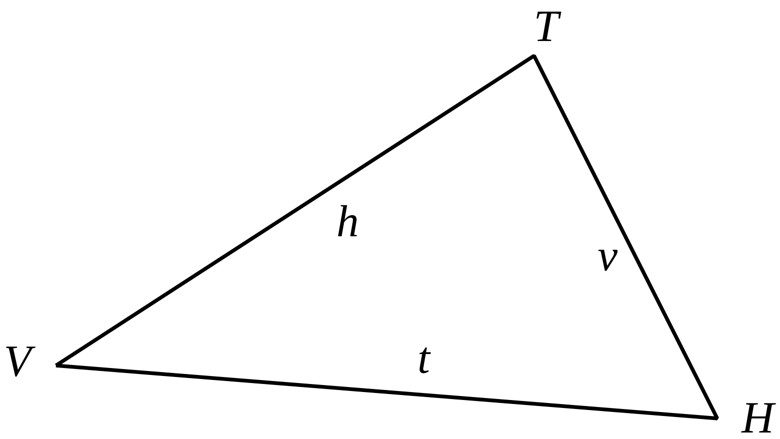


(ii)

$$\frac{p}{\sin P} = \frac{q}{\sin Q} = \frac{r}{\sin R}$$

Answer 2

(i)



(ii)

$$\frac{h}{\sin H} = \frac{t}{\sin T} = \frac{v}{\sin V}$$

Answer 3

(i)	173	**(ii)**	13.6
(iii)	1.17	**(iv)**	3890

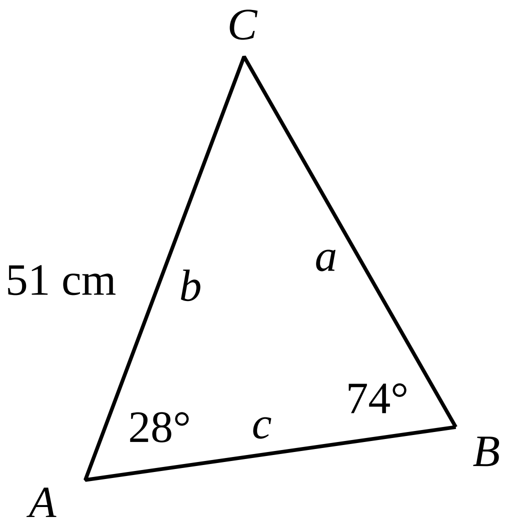
Answer 4

(i)	5.71	**(ii)**	222
(iii)	2350	**(iv)**	54.6

Answer 5

(i) $\angle C = 78°$

(ii)



(iii)

$$\frac{a}{\sin A} = \boxed{\frac{b}{\sin B}} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 28} = \frac{51}{\sin 74}$$

$$a = \frac{51 \times \sin 28}{\sin 74}$$

$= 24.9$ cm (3 significant figures asked for)

(iv)

$$Area\ \triangle = \frac{1}{2} ab \sin C \qquad (C \text{ is the included angle})$$

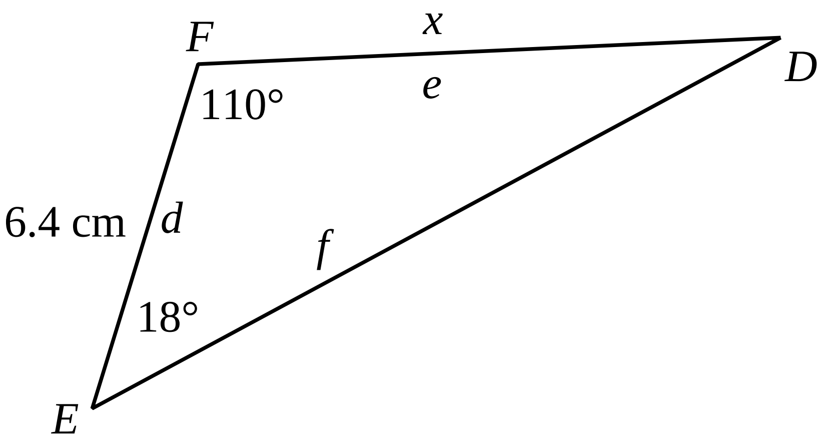
$$= \frac{1}{2} \times 24.9 \times 51 \times \sin 78$$

$= 621$ cm^2

Answer 6

(i) $\angle D = 52°$

(ii)



(iii)

$$\boxed{\frac{d}{\sin D}} = \frac{e}{\sin E} = \frac{f}{\sin F}$$

$$\frac{e}{\sin 18} = \frac{6.4}{\sin 52}$$

$$e = \frac{6.4 \times \sin 18}{\sin 52}$$

$$= 2.51 \text{ cm} \quad \text{(3 significant figures asked for)}$$

(iv)

$$Area\ \Delta = \frac{1}{2} de \sin F \qquad \text{(F is the included angle)}$$

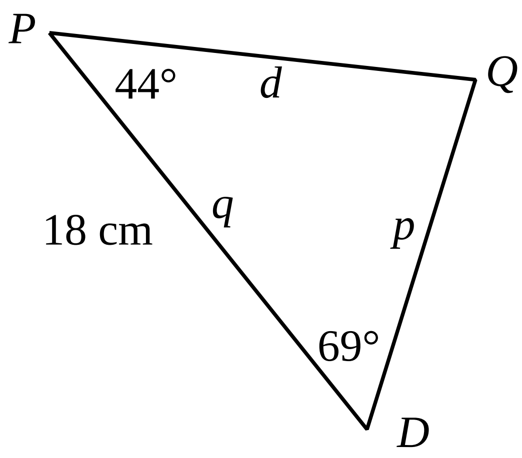
$$= \frac{1}{2} \times 6.4 \times 2.51 \times \sin 110$$

$$= 7.55 \text{ cm}^2$$

Answer 7

(i) $\angle Q = 67°$

(ii)



(iii)

$$\frac{p}{\sin P} = \frac{d}{\sin D} = \boxed{\frac{q}{\sin Q}}$$

$$\frac{p}{\sin 44} = \frac{18}{\sin 67}$$

$$p = \frac{18 \times \sin 44}{\sin 67}$$

$$= 13.6 \text{ cm} \quad \text{(3 significant figures asked for)}$$

(iv)

$$\boxed{\frac{p}{\sin P}} = \frac{d}{\sin D} = \boxed{\frac{q}{\sin Q}}$$

$$\frac{d}{\sin 69} = \frac{18}{\sin 67}$$

$$d = \frac{18 \times \sin 69}{\sin 67}$$

$$= 18.3 \text{ cm} \quad \text{(3 significant figures asked for)}$$

(v)

$$Area\ \Delta = \frac{1}{2} pd \sin Q \qquad \text{(alternatives can be used)}$$

$$= \frac{1}{2} \times 13.6 \times 18.3 \times \sin 67$$

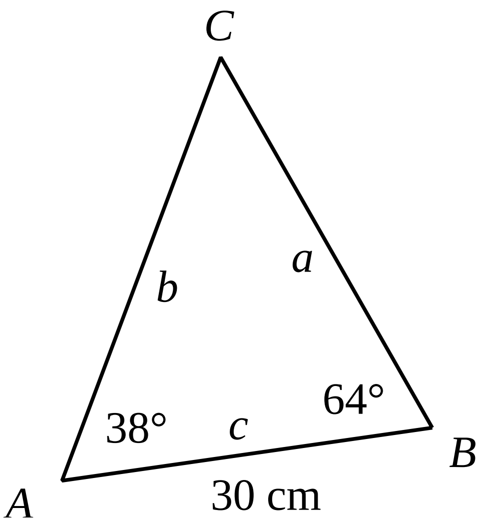
$$= 114.5 \text{ cm}^2$$

7.5.2 Solutions to 7.4 Homework

Answer 1

(i) $\angle C = 78°$

(ii)



(iii)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \boxed{\frac{c}{\sin C}}$$

$$\frac{a}{\sin 38} = \frac{30}{\sin 78}$$

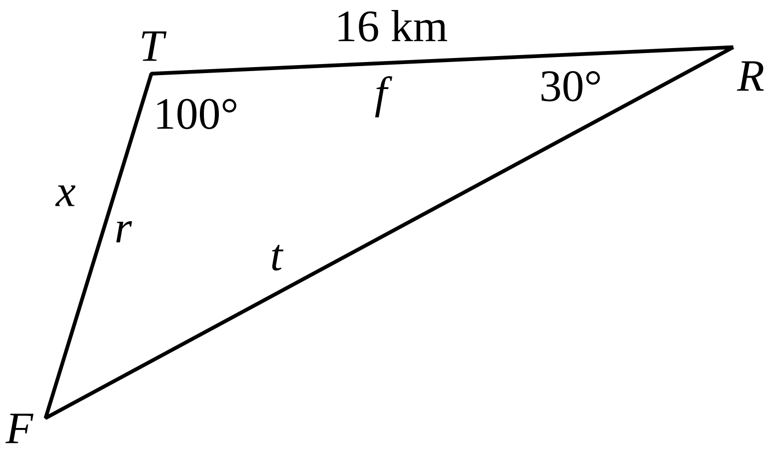
$$a = \frac{30 \times \sin 38}{\sin 78}$$

$$= 18.9 \text{ cm} \quad \text{(3 significant figures asked for)}$$

Answer 2

(i) $\angle F = 50°$

(ii)



(iii)

$$\boxed{\frac{f}{\sin F}} = \frac{r}{\sin R} = \frac{t}{\sin T}$$

$$\frac{r}{\sin 30} = \frac{16}{\sin 50}$$

$$r = \frac{16 \times \sin 30}{\sin 50}$$

$$= 10.4 \text{ km (3 significant figures asked for)}$$

Answer 3

$$Area\ \triangle = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} \times 12 \times 7 \times \sin 125$$

$$= 34.4 \text{ cm}^2$$

Answer 4

$$\frac{r}{\sin R} = \frac{s}{\sin S} = \boxed{\frac{t}{\sin T}}$$

$$r = \frac{12 \sin 34}{\sin 76}$$

$$= 6.92 \text{ cm}$$

8.1 The Upside Down Sine Rule

Previously, The Sine Rule was used to find the length of a triangle's side given

- another length

and
- two (and ∴ three) angles.

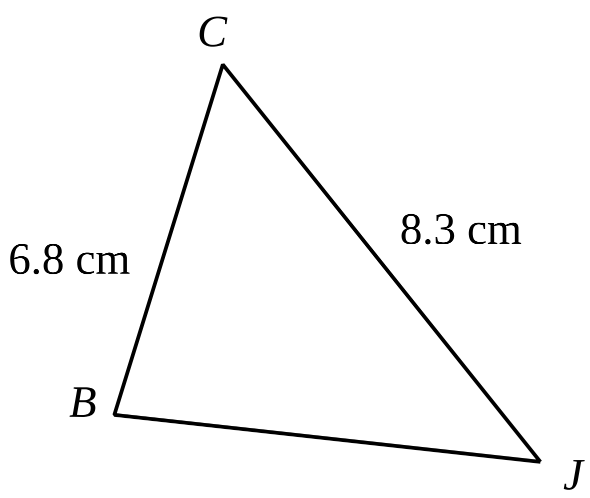
If we know
- the length of two of a triangle's sides

and
- an excluded angle

then The Sine Rule can be used 'upside down' to find the other excluded angle.

Example 1

(i) On the following triangle mark the two excluded angles, each with a *.

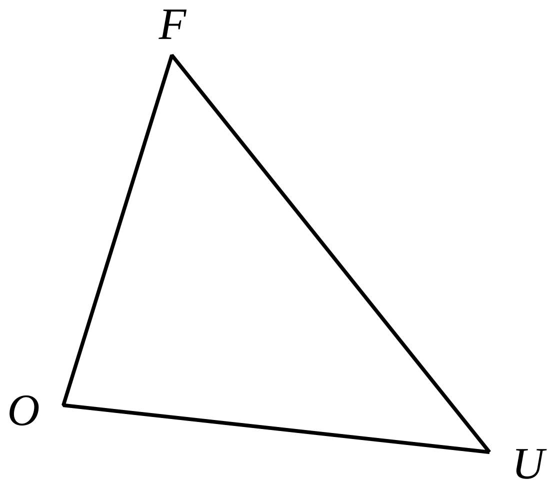


(ii) On the diagram place the letters *j*, *c* and *b*.

(iii) Write down The Sine Rule in terms of *J*, *j*, *C*, *c*, *B* and *b*.

(iv) Write down the corresponding 'upside down' version of The Sine Rule.

Example 2
A triangle, ΔUFO, is shown below.



(i) On the triangle place the letters *u*, *f* and *o*.

(ii) Onto the triangle add the facts that

- $\angle U = 53°$
- $UF = 8.4$ cm
- $FO = 7.6$ cm
- $\angle O = x$, the angle to be found.

(iii) On the triangle mark the two excluded angles, each with a *.

(iv) Write down the 'upside down' version of The Sine Rule for the triangle.

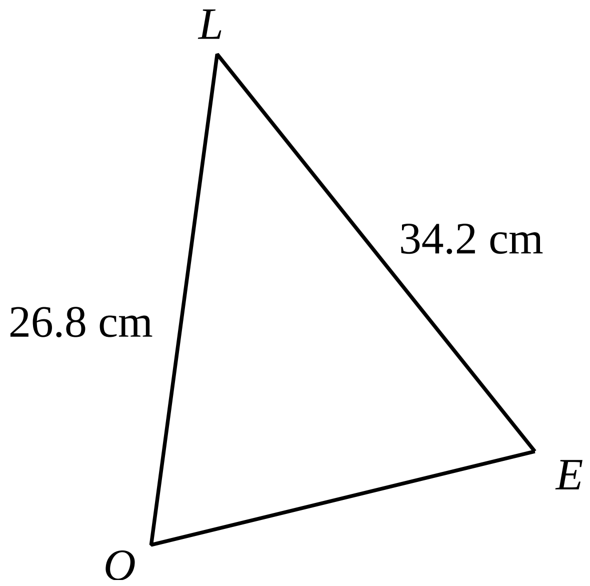
(v) Find angle *x*, in degrees and accurate to 3 significant figures.

HINT ☐ Circle which to find
☐ Box known pair

8.2 Exercise

Question 1

(i) On the following triangle mark the two excluded angles, each with a *.



(ii) On the diagram place the letters *e*, *l* and *o*.

(iii) Write down The Sine Rule in terms of *E, e, L, l, O* and *o*.

(iv) Write down the corresponding 'upside down' version of the sine rule.

Question 2

Which is the correct ending for the given sentence ?

I'd use The Sine Rule (upside down version) to find an excluded angle if I knew....

(a) ☐ The lengths of all three sides of a triangle, and no angles.

(b) ☐ The length of two sides and the included angle.

(c) ☐ The length of two sides and one excluded angle.

(d) ☐ All three angles, but not the length of any side.

Question 3

In each case, calculate the value of *x* in degrees, correct to three significant figures;

(i) $\sin x = 0.334$

(ii) $\sin x = \dfrac{67}{80}$

(iii)

$$\sin x = \frac{41.6 \times 0.5643}{34.2}$$

(iv)

$$\sin x = \frac{15.8 \times \sin 42°}{21.5}$$

(v)

$$\sin x = \frac{7.1 \times \sin 62°}{11.8}$$

(vi)

$$\sin x = \frac{28.6 \times \sin 82°}{45}$$

Question 4

Solve to find the values of *x* in degrees, correct to three significant figures.

(i)

$$\frac{\sin x}{5} = \frac{9}{62}$$

(ii)

$$\frac{\sin x}{55} = \frac{\sin 24°}{73}$$

(iii)

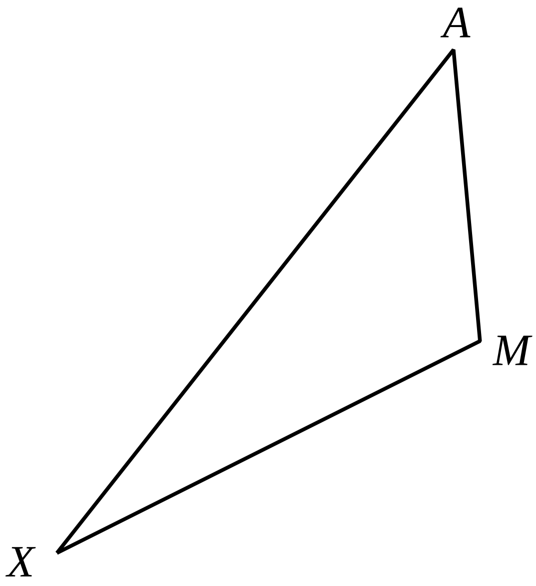
$$\frac{\sin x}{16} = \frac{\sin 35°}{23}$$

(iv)

$$\frac{\sin x}{5.4} = \frac{\sin 60}{4.9}$$

Question 5

A triangle, $\triangle MAX$, is shown below.



(i) On the triangle place the letters m, a and x.

(ii) Onto the triangle add the facts that

- $\angle M = 104°$
- $MX = 7.8$ cm
- $XA = 12.7$ cm
- $\angle A = x$, the angle to be found.

(iii) On the triangle mark the two excluded angles, each with a *.

(iv) Write down the 'upside down' version of The Sine Rule for the triangle.

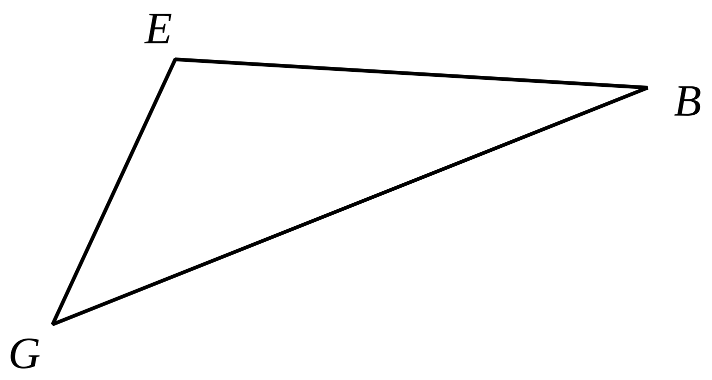
(v) Find angle x, in degrees and accurate to 3 significant figures.

HINT ☐ Circle which to find

☐ Box known pair

Question 6

A triangle, $\triangle BEG$, is shown below.



(i) On $\triangle BEG$ place the letters b, e and g.

(ii) On $\triangle BEG$ add the facts that

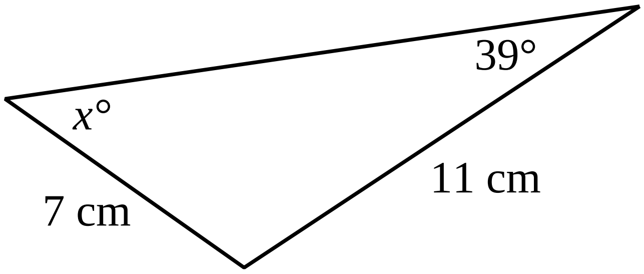
- $\angle B = x°$, the angle to be found.
- $BG = 17.8$ cm
- $GE = 14.9$ cm
- $\angle E = 130°$.

(iii) On $\triangle BEG$ mark the two excluded angles, each with a *.

(iv) Write down the 'upside down' version of the sine rule for $\triangle BEG$.

(v) Find angle x, in degrees and accurate to 3 significant figures.

Question 7



Find the size of the angle marked x, in degrees and accurate to 3 significant figures.

Question 8

This is a tough question.

$\triangle XYZ$ has XY of length 13.2 cm and length XZ of length 27.5 cm.

The angle at vertex Z is 24°.

Find the included angle at vertex X.

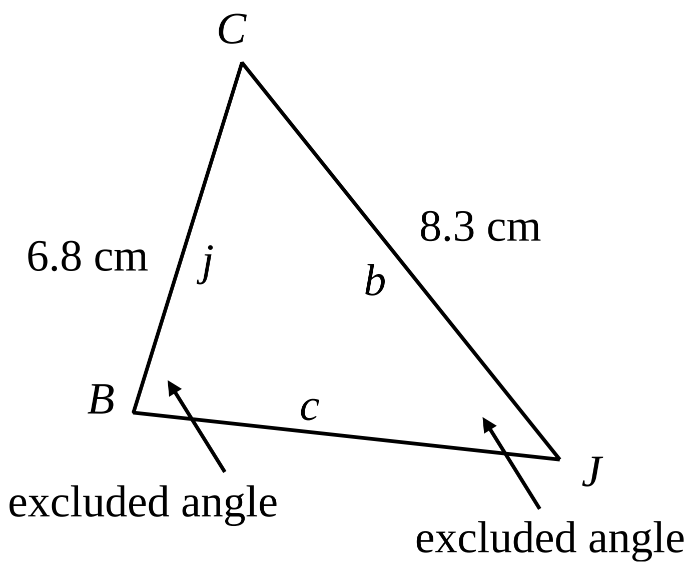
HINT : Find the excluded angle at vertex Y first, as this can be obtained using the 'upside down' version of The Sine Rule.

8.3 Answers

8.3.1 Solutions to 8.1 Examples

Answer 1

(i) and **(ii)**



(iii)

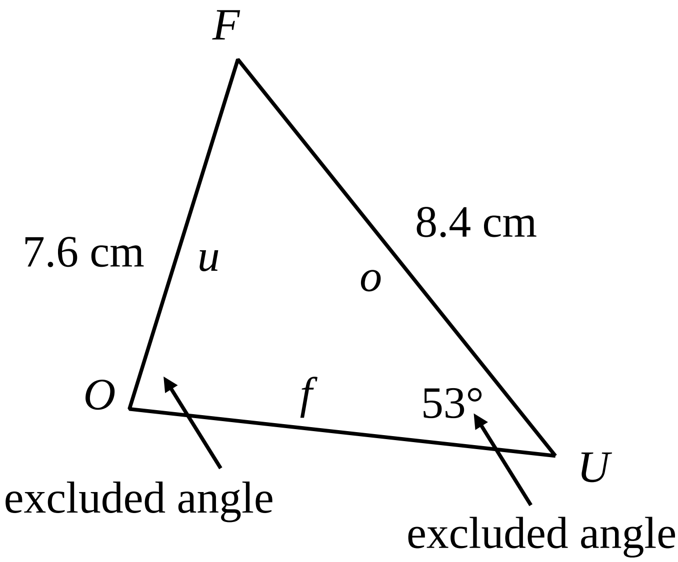
$$\frac{j}{\sin J} = \frac{c}{\sin C} = \frac{b}{\sin B}$$

(iv)

$$\frac{\sin J}{j} = \frac{\sin C}{c} = \frac{\sin B}{b}$$

Answer 2

(i) and **(ii)**



(iii)

$$\frac{\sin U}{u} = \frac{\sin F}{f} = \frac{\sin O}{o}$$

(iv)

$$\boxed{\frac{\sin U}{u}} = \frac{\sin F}{f} = \frac{\sin O}{o}$$

(v)

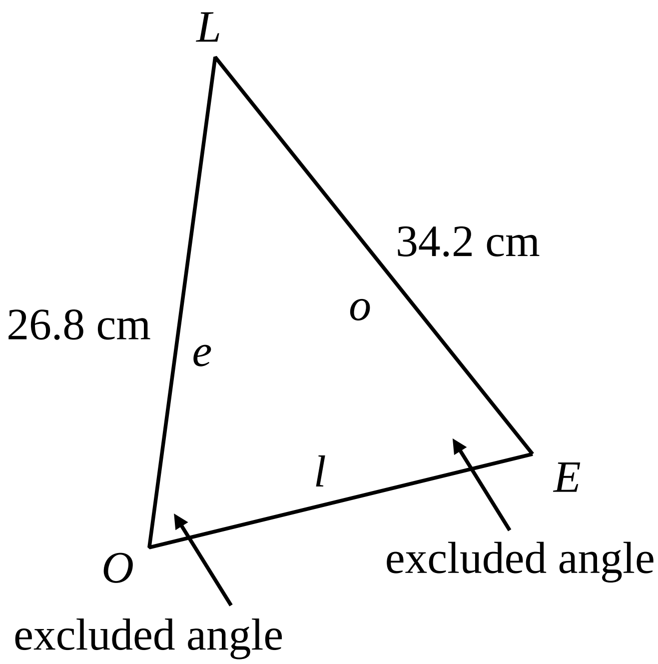
$$\frac{\sin O}{8.4} = \frac{\sin 53}{7.6}$$

$$\begin{aligned} \sin O &= \frac{8.4 \times \sin 53}{7.6} \\ &= 0.883 \\ O &= \arcsin 0.883 \\ &= 62.0° \end{aligned}$$

8.3.2 Solutions to 8.2 Exercise

Answer 1

(i) and **(ii)**



(iii)

$$\frac{e}{\sin E} = \frac{l}{\sin L} = \frac{o}{\sin O}$$

(iv)

$$\frac{\sin E}{e} = \frac{\sin L}{l} = \frac{\sin O}{o}$$

Answer 2

I'd use The Sine Rule (upside down version) to find an excluded angle if I knew....

(c) ☐ The length of two sides and one excluded angle.

Answer 3

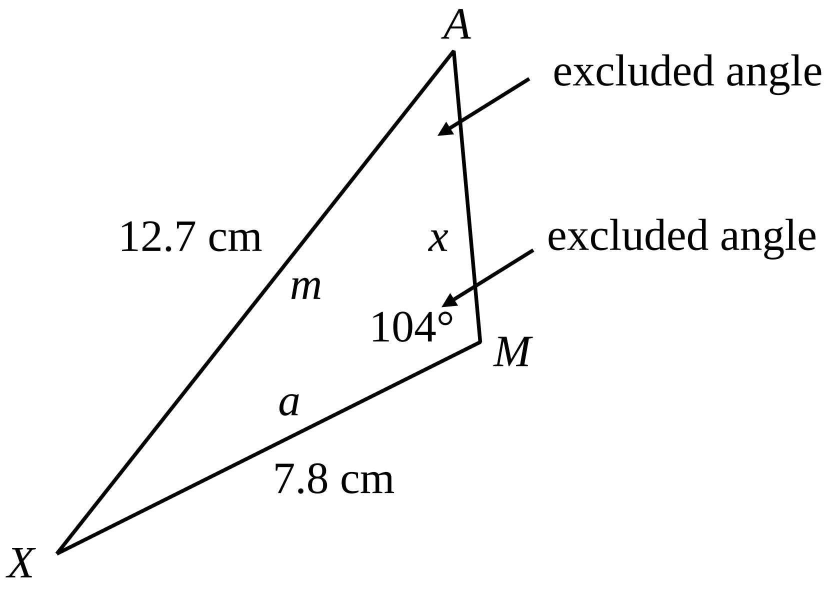
(i)	19.5°	**(ii)**	56.9°
(iii)	43.3°	**(iv)**	29.5°
(v)	32.1°	**(vi)**	39.0°

Answer 4

(i)	46.5°	**(ii)**	17.8°
(iii)	23.5°	**(iv)**	72.6°

Answer 5

(i) (ii) and **(iii)**



(iv)

$$\frac{\sin M}{m} = \frac{\sin A}{a} = \frac{\sin X}{x}$$

(v)

$$\boxed{\frac{\sin M}{m}} = \frac{\sin A}{a} = \frac{\sin X}{x}$$

$$\frac{\sin A}{7.8} = \frac{\sin 104}{12.7}$$

$$\sin A = \frac{7.8 \times \sin 104}{12.7}$$

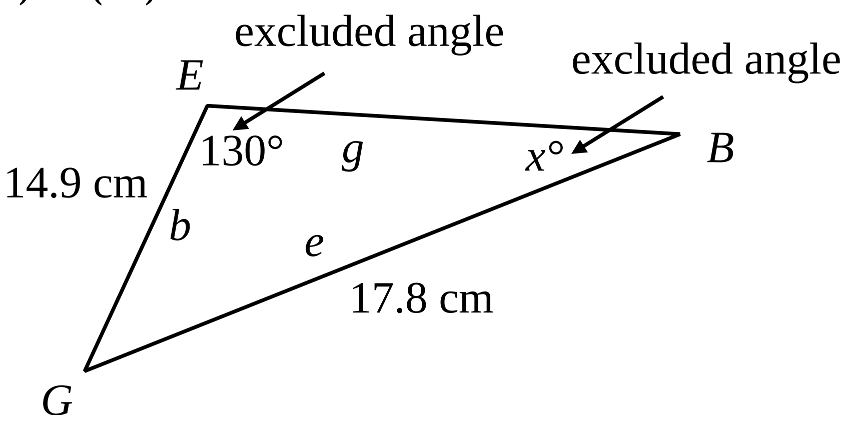
$$= 0.596$$

$$A = \arcsin 0.596$$

$$= 36.6°$$

Answer 6

(i) (ii) and **(iii)**



(iv)

$$\frac{\sin B}{b} = \frac{\sin E}{e} = \frac{\sin G}{g}$$

(v)

$$\frac{\sin B}{b} = \boxed{\frac{\sin E}{e}} = \frac{\sin G}{g}$$

$$\frac{\sin B}{14.9} = \frac{\sin 130}{17.8}$$

$$\sin B = \frac{14.9 \times \sin 130}{17.8}$$

$$= 0.641$$

$$A = \arcsin 0.641$$

$$= 39.9°$$

Answer 8

$x = 81.5°$

Answer 9

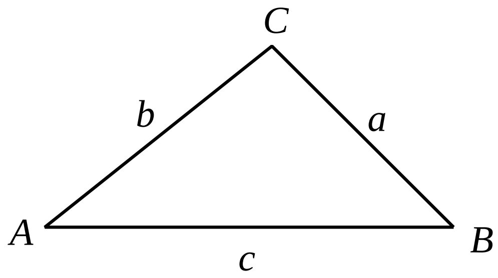
$$Y = \arcsin\left(\frac{27.5 \times \sin 24}{13.2}\right)$$

$$= 57.9°$$

$$\therefore X = 98.1°$$

9.1 Summary

Trigonometry
Non-Right Angled Triangles



The Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Known : One side length & two (and $\therefore$ three) angles
Seeking : Any side length

The Upside Down Sine Rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Known : Two side lengths & an excluded angle
Seeking : The other excluded angle

The Cosine Rule

$$a^2 = b^2 + c^2 - 2\,b\,c\cos A$$

Known : Two side lengths & the included angle
Seeking : The third side length

The Reversed Cosine Rule

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c}$$

Known : Three side lengths
Seeking : Any angle

Useful Area of a Triangle Formula

$$Area\ \Delta = \frac{1}{2}\,a\,b\sin C$$

9.2 Exercise

Question 1

In $\triangle ABC$, two of the angles are, $A = 53°$, and $B = 85°$.

Opposite the angle, A, is a side of length $a = 16$ cm.

(i) What is the size of angle C, the third angle ?

(ii) Sketch the triangle, not to scale, and mark on all known lengths and angles.

(iii) Find the length of each missing side, stating which is b and which is c.

Question 2

In $\triangle ABC$, two of the lengths are, $a = 4$ cm, $b = 12$ cm.

The angle between these two known lengths is $C = 47°$.

(i) Sketch the triangle, not to scale, and mark on all known lengths and angles.

(ii) Find the length of the third side, c.

(iii) Find the area of the triangle.

Question 3

A triangle has sides of length 18 cm, 12 cm and 10 cm.

(i) Find the size of the angle opposite the side of length 18 cm.

(ii) Find the area of the triangle.

Question 4

In $\triangle LMN$, the lengths l and n are 4.5 cm and 2.7cm respectively.

Angle M is obtuse, 112°.

(i) Sketch the triangle, not to scale, and mark on all known lengths and angles.

(ii) Find the length of m, the side opposite angle M.

Question 5

In $\triangle XYZ$, find the length of sides x and y if angles X and Y are $134°$ and $27°$ respectively, and the side, z, is 14 cm.

Question 6

In $\triangle FGH$, the sides are of lengths, $f = 3.6$ cm, $g = 4.2$ cm and $h = 5.1$ cm.
Find the size of the angle opposite side h.

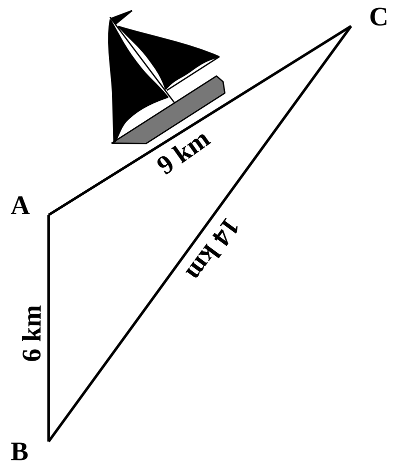
9.3 Homework

GCSE Mathematics

Non-Right Angled Trigonometry

Question 1

A yacht sails 6 km due south from *A* to *B*, changes course and sails 14 km to *C*.
Finally, it sails 9 km back to its starting position.
The journey is shown, in sketch form, below. (NOT to scale)



Calculate the bearing that the yacht sails on when moving from *B* to *C*.

Question 2

In $\triangle ABC$, side a is of length 7 cm and side b of length 15 cm.

$\angle C$ *is* ***obtuse*** and measures 147°.

Calculate the length of the longest side in $\triangle ABC$.

Question 3

A ship sails due south from A for a distance, c, to point B.

It then sails for 5.6 km on bearing 098° until it is at point C which is south-east of A.

(i) Sketch the triangle, not to scale, and mark on all known lengths and angles.

(ii) How far, as the crow flies, is the ship from A ?

Question 4

In $\triangle ABC$, a = 4.3 cm, b = 6.2 cm and c = 5.8 cm.

Find angle B.

Question 5
Captain Fisheye sails due south from *A* for 7.3 km looking for Fish Fingers.
He finds none, and so alters course to head on bearing 042° for 5.4 km.
Frozen-fingered he then decides to head back to *A*.
What is the total distance sailed in this fishless jaunt ?

Question 6

In $\triangle FGH$, $G = 13°$, $H = 32°$ and $f = 8.4$ cm.
Find the lengths of sides g and h.

9.4 Answers

9.4.1 Solutions to 9.2 Exercise

Answer 1

(i) $\angle C = 42°$

(ii) Sketch of triangle

(iii) $$b = \frac{16 \times 0.996}{0.799} = 19.94\,cm$$

$$c = \frac{16 \times 0.669}{0.799} = 13.40\,cm$$

Answer 2

(i) Sketch of triangle

(ii)

$$c^2 = 4^2 + 12^2 - 2 \times 4 \times 12 \times \cos 47$$
$$= 94.528$$
$$c = 0.72 \text{ cm}$$

(iii)

$$Area\,\Delta = \frac{1}{2} \times 4 \times 12 \times \sin 47$$
$$= 17.6 \text{ cm}^2$$

Answer 3

(i)

$$\cos A = \frac{12^2 + 10^2 - 18^2}{2 \times 12 \times 10}$$
$$= -0.333$$
$$A = 109.5°$$

(ii)

$$Area\,\Delta = \frac{1}{2} \times 12 \times 10 \times \sin 109.5$$
$$= 56.6 \text{ cm}^2$$

Answer 4

(i) Sketch of triangle

(ii)

$$m^2 = 4.5^2 + 2.7^2 - 2 \times 4.5 \times 2.7 \times \cos 112$$
$$= 36.6525$$
$$m = 6.05 \text{ cm}$$

Answer 5

$$x = \frac{14 \times \sin 134}{\sin 19} = 30.93\ cm$$

$$y = \frac{14 \times \sin 27}{\sin 19} = 19.52\ cm$$

Answer 6

$$\cos H = \frac{3.6^2 + 4.2^2 - 5.1^2}{2 \times 3.6 \times 4.2}$$

$$= 0.152$$

$$H = 81.3°$$

9.4.2 Solutions to 9.3 Homework

Answer 1

$$\cos B = \frac{14^2 + 6^2 - 9^2}{2 \times 14 \times 6}$$
$$= 0.899$$
$$B = 26°$$

Answer 2

$$c^2 = 7^2 + 15^2 - 2 \times 7 \times 15 \times \cos 147$$
$$= 450.19$$
$$c = 21.22 \text{ cm}$$

Answer 3

(i) Sketch of triangle

(ii)

$$b = \frac{5.6 \times \sin 98}{\sin 45} = 7.84 \text{ km}$$

Answer 4

$$\cos B = \frac{4.3^2 + 5.8^2 - 6.2^2}{2 \times 4.3 \times 5.8}$$
$$= 0.274$$
$$B = 74.1°$$

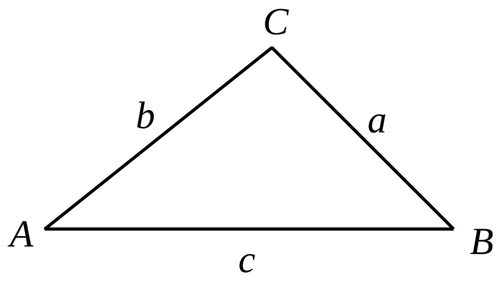
Answer 5

$$x^2 = 5.4^2 + 7.3^2 - 2 \times 5.4 \times 7.3 \times \cos 42°$$
$$= 23.87$$
$$x = 4.9 \text{ km}$$
$$\therefore \; Total\ Distance = 17.6 \text{ km}$$

Answer 6

$$g = \frac{8.4 \times \sin 13}{\sin 135} = 2.7 \text{ cm}$$
$$h = \frac{8.4 \times \sin 32}{\sin 135} = 6.3 \text{ cm}$$

Trigonometry
Non-Right Angled Triangles



The Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Known : One side length & two (and $\therefore$ three) angles
Seeking : Any side length

The Upside Down Sine Rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Known : Two side lengths & an excluded angle
Seeking : The other excluded angle

The Cosine Rule

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Known : Two side lengths & the included angle
Seeking : The third side length

The Reversed Cosine Rule

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c}$$

Known : Three side lengths
Seeking : Any angle

Useful Area of a Triangle Formula

$$Area\ \triangle = \frac{1}{2}\,a\,b \sin C$$

Lesson 10

GCSE Mathematics
Non-Right Angled Trigonometry

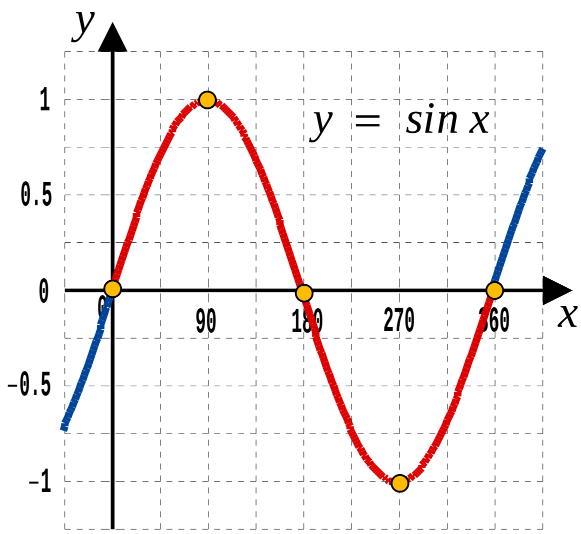
10.1 Revision for TEST

Question 1

(i) Use your calculator to find a solution to the equation;

$$y = \sin 50°$$

(ii) A section of the graph of $y = \sin x$ is shown below;



Show, by drawing on the graph, where your part **(i)** solution is.

(iii) Use your calculator to find a solution to the equation;

$$y = \sin 260°$$

(iv) Show, by drawing on the graph, where your part **(iii)** solution is.

[8 marks]

Question 2

Use your calculator to calculate in one go, the value of the following;

$$\frac{\sin 90° + \sin 45°}{\sin 90° - \sin 45°}$$

Give the answer correct to 4 decimal places.

[4 marks]

Question 3

(i) Solve the equation $A = arccos\,(-0.627)$
Give your answer to 1 decimal place.

(ii) If $cos\,B = 0.751$ what is B ?
Give your answer in degrees, to 1 decimal place.

(iii) Calculate the value of x correct to 3 significant figures;

$$x = \frac{5.8 \times sin\,76^\circ}{sin\,13^\circ}$$

(iv) Solve the following equation.
Give your answer correct to 3 significant figures.

$$\frac{x}{sin\,28^\circ} = \frac{723}{sin\,51^\circ}$$

[8 marks]

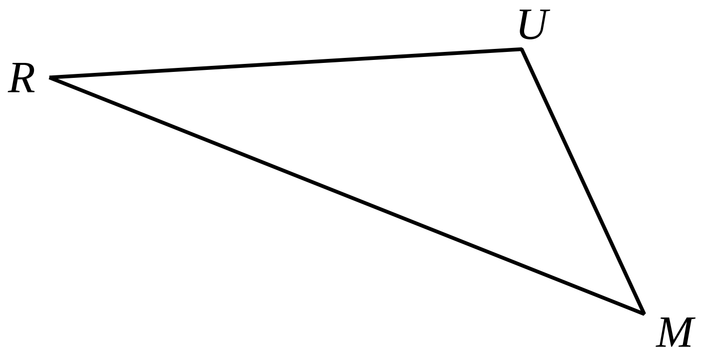
Question 4

Give a rough sketch of the graph of $y = tan\,x$ in the space below.
Your sketch should be over the interval $0^\circ \leqslant x \leqslant 360^\circ$

[4 marks]

Question 5

A triangle, $\triangle RUM$, is shown below.



(i) On $\triangle RUM$ place the letters r, u and m.

[1 mark]

(ii) On $\triangle RUM$ add the facts that

- $\angle R = x°$, the angle to be found.
- RM = 31.2 cm
- MU = 29.4 cm
- $\angle U = 115°$.

[1 mark]

(iii) On $\triangle RUM$ mark the two excluded angles, each with a *.

[1 mark]

(iv) Write down the 'upside down' version of the sine rule for $\triangle RUM$.

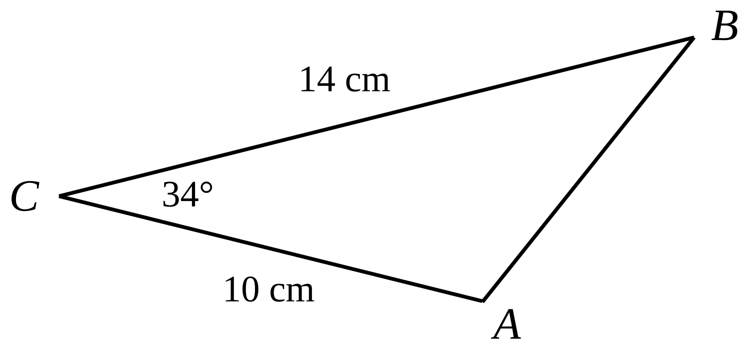
[1 mark]

(v) Find angle x, in degrees and accurate to 3 significant figures.

[5 marks]

Question 6

In $\triangle ABC$, shown below, $\angle C = 34°$ and lies between $CB = 14$ cm and $CA = 10$ cm.



(i) Use ***The Cosine Rule*** to calculate the length of side AB.

(ii) Use ***The Useful Area Of A Triangle Formula*** to calculate the triangle's area.

In both cases, clearly state the units of your answer.

[8 marks]

Question 7

A triangle has sides of length 12 cm, 15 cm and 18 cm.
Find the size of the angle opposite the side of length 15 cm.

[8 marks]

Question 8

In $\triangle ABC$, two of the angles are, $A = 48°$, and $C = 24°$.

Opposite the angle, B, is a side of length 17.3 cm.

(i) Use a well known fact about the sum of the angles in a triangle to determine the size of angle B.

(ii) Sketch the triangle, not to scale, and mark on all known lengths and angles.

(iii) Find the length of each missing side, stating which is BC and which is AB.

[10 marks]

Question 9

In $\triangle ELF$, $e = 8.6$ cm, $f = 18.4$ cm and $L = 72°$.

Find length l.

[8 marks]

Question 10

The area of any regular polygon is given by the formula;

$$A = \frac{n \ r^2 \sin\left(\frac{360}{n}\right)}{2}$$

where n is the number of sides,
and r is the 'radius' of the polygon, the length of a spoke from it's centre to a vertex.

(i) With r fixed at 2.5 cm, complete the following table

n	3	4	5	6	7	8	9	10	11	12
A										

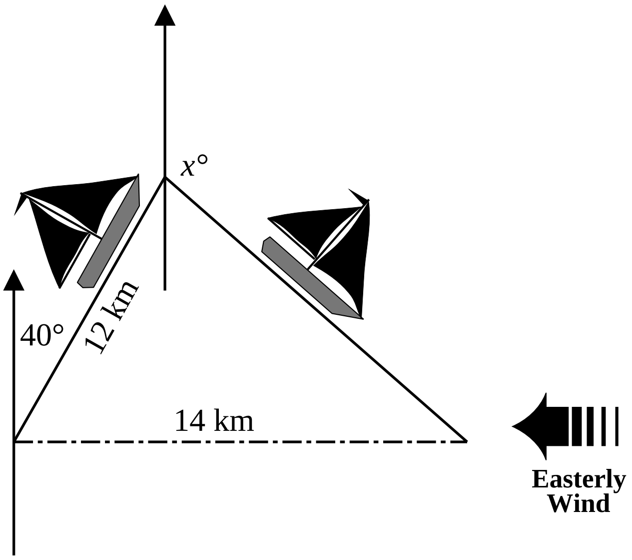
(ii) Plot a graph of your results with n in the x-axis and area A on the y-axis

(iii) What would you expect to happen to A as n continues to increase ?

[12 marks]

Question 11

In order to make progress into a due East headwind, a sailing yacht tacks 12 km on a bearing of 040° before changing tack and sailing a further distance as shown below.



It has at this point made 14 km of effective progress directly to the East.
Determine the bearing on which the yacht progressed , marked $x°$, when sailing on the second tack.

[10 marks]

Question 12

(i) For any triangle, $\triangle ABC$, explain why $\frac{1}{2}\ bc \sin A = \frac{1}{2}\ ab \sin C$

(ii) Hence prove that

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

(iii) $\triangle ABC$ has a = 17 cm, c = 15 cm and $\angle C$ = 54°.
Find the acute angle, $\angle A$.

(iv) Had you been told that $\angle A$ was obtuse, what would $\angle A$ have been ?

[11 marks]

10.2 Answers (Revision for TEST)

Answer 1

(i) 0.766

(ii) On graph : Vertical line from 50°, Horizontal line to 0.766

(iii) – 0.985

(iv) On graph : Vertical line from 260°, Horizontal line to – 0.985

Answer 2

5.8284 must be to 4 decimal places

Answer 3

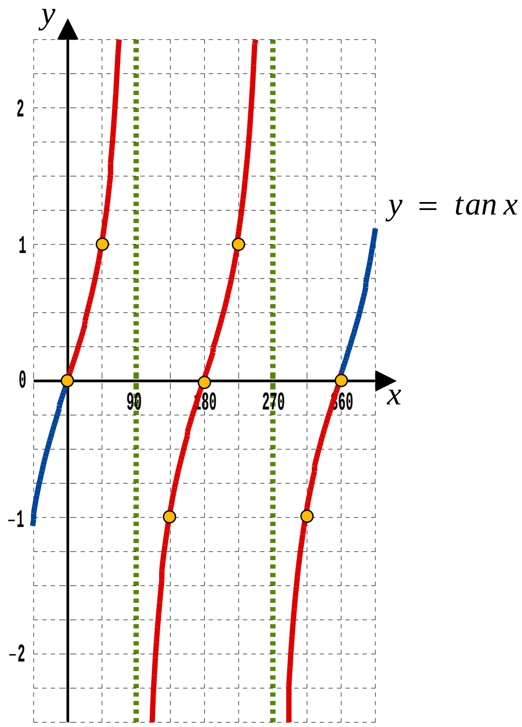
(i) 128.8° must be to 1 decimal place

(ii) 41.3° must be to 1 decimal place

(iii) 25.0° must be to 3 significant figures

(iv) 437 must be to 3 significant figures

Answer 4



Answer 5

(i) Check *r*, *u* and *m* placed correctly on diagram.

(ii) Check *x*, 31.2, 29.4 and 115° correctly placed on diagram.

(iii) Check star placed at *R* and *U*.

(iv) $\frac{\sin R}{r} = \frac{\sin U}{u} = \frac{\sin M}{m}$

(v) 58.7° must be to 3 significant figures

Answer 6

(i) 7.99 cm

(ii) 39.14 cm^2

Answer 7

55.7°

Answer 8

(i) 108°

(ii) Check sketch of triangle

(iii) $a = BC = 13.5$ cm

(iv) $c = AB = 7.40$ cm

Answer 9

17.7 cm

Answer 10

(i)

n	3	4	5	6	7	8	9	10	11	12
A	8.1	12.5	14.9	16.2	17.1	17.7	18.1	18.4	18.6	18.8

(ii) Graph's scales numbered and labelled.

(iii) $Area \rightarrow \pi r^2$ That is, 19.6, as the polygon becomes more like a circle

Answer 11

180 – 74.4 = 146° Bearing given to nearest degree

Answer 12

(i) Both are the area of the same triangle, ΔABC

(ii)

$$\frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A$$

$$ab \sin C = bc \sin A \qquad \text{multiply both sides by 2}$$

$$a \sin C = c \sin A \qquad \text{divide both sides by } b$$

$$\sin C = \frac{c \sin A}{a} \qquad \text{divide both sides by } a$$

$$\frac{\sin C}{c} = \frac{\sin A}{a} \qquad \text{divide both sides by } c$$

(iii) 66.5°

(iv) 113.5°

Lesson 11

GCSE Mathematics
Non-Right Angled Trigonometry

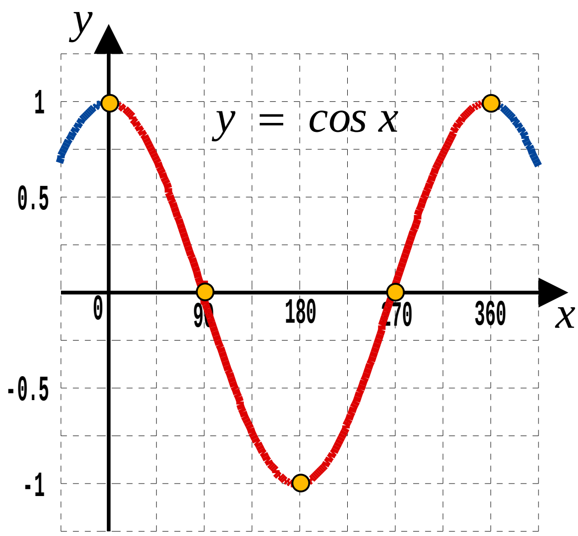
11.1 Test

Question 1

(i) Use your calculator to find a solution to the equation;

$$y = \cos 35°$$

(ii) A section of the graph of $y = \cos x$ is shown below;



Show, by drawing on the graph, where your part **(i)** solution is.

(iii) Use your calculator to find a solution to the equation;

$$y = \cos 240°$$

(iv) Show, by drawing on the graph, where your part **(iii)** solution is.

[8 marks]

Question 2

Use your calculator to calculate in one go, the value of the following;

$$\frac{\cos 30° - \cos 180°}{\cos 60° + \cos 45°}$$

Give the answer correct to 4 decimal places.

[4 marks]

Question 3

(i) Solve the equation $A = arccos\,(-0.769)$
Give your answer to 1 decimal place.

(ii) If $cos\,B = 0.238$ what is B ?
Give your answer in degrees, to 1 decimal place.

(iii) Calculate the value of x correct to 3 significant figures;

$$x = \frac{6.3 \times sin\,66°}{sin\,11°}$$

(iv) Solve the following equation.
Give your answer correct to 3 significant figures.

$$\frac{x}{sin\,25°} = \frac{432}{sin\,53°}$$

[8 marks]

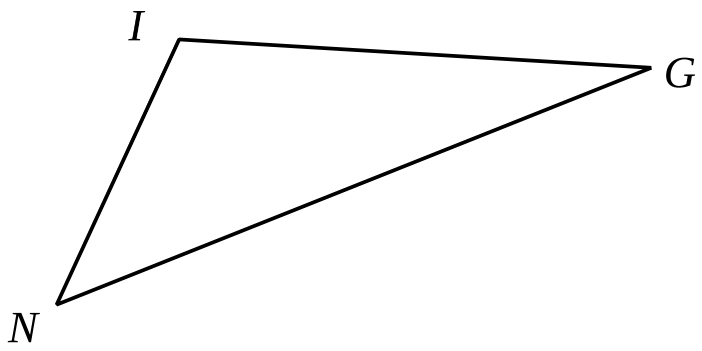
Question 4

Give a rough sketch of the graph of $y = tan\,x$ in the space below.
Your sketch should be over the interval $0° \leqslant x \leqslant 360°$

[4 marks]

Question 5

A triangle, $\triangle GIN$, is shown below.



(i) On $\triangle GIN$ place the letters g, i and n.

[1 mark]

(ii) On $\triangle GIN$ add the facts that

- $\angle G = x°$, the angle to be found.
- $GN = 43.4$ cm
- $NI = 37.8$ cm
- $\angle I = 104°$.

[1 mark]

(iii) On $\triangle GIN$ mark the two excluded angles, each with a *.

[1 mark]

(iv) Write down the 'upside down' version of the sine rule for $\triangle GIN$.

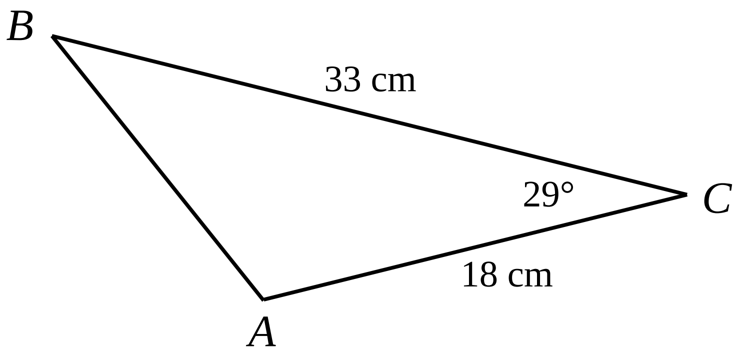
[1 mark]

(v) Find angle x, in degrees and accurate to 3 significant figures.

[5 marks]

Question 6

In $\triangle CAB$, shown below, $\angle C = 29°$ and lies between $CB = 33$ cm and $CA = 18$ cm.



(i) Use ***The Cosine Rule*** to calculate the length of side AB.

(ii) Use ***The Useful Area Of A Triangle Formula*** to calculate the triangle's area.

In both cases, clearly state the units of your answer.

[8 marks]

Question 7
A triangle has sides of length 214 m, 287 m and 316 m.
Find the size of the angle opposite the longest side.

[8 marks]

Question 8

In $\triangle ABC$, two of the angles are, $A = 68°$, and $C = 34°$.

Opposite the angle, B, is a side of length 9.4 cm.

(i) Use a well known fact about the sum of the angles in a triangle to determine the size of angle B.

(ii) Sketch the triangle, not to scale, and mark on all known lengths and angles.

(iii) Find the length of each missing side, stating which is BC and which is AB.

[10 marks]

Question 9
In $\triangle ORC$, o = 4.3 cm, c = 9.2 cm and R = 82°.
Find length r.

[8 marks]

Question 10

The perimeter of any regular polygon is given by the formula;

$$P = nr\sqrt{2\left(1 - cos\left(\frac{360}{n}\right)\right)}$$

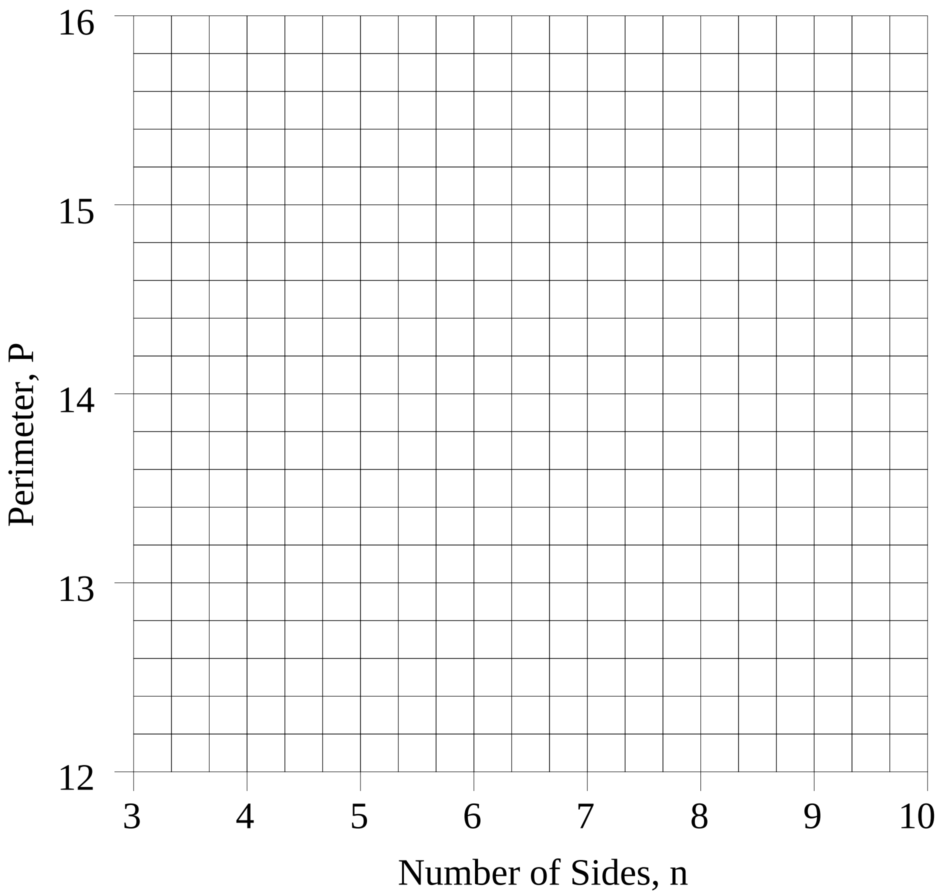
where n is the number of sides,

and r is the 'radius' of the polygon, the length of a spoke from its centre to a vertex.

(i) With r fixed at 2.5 cm, complete the following table, working to two decimal places,

n	3	4	5	6	7	8	9	10
P								

(ii) Plot a graph of your results with n in the x-axis and perimeter P on the y-axis

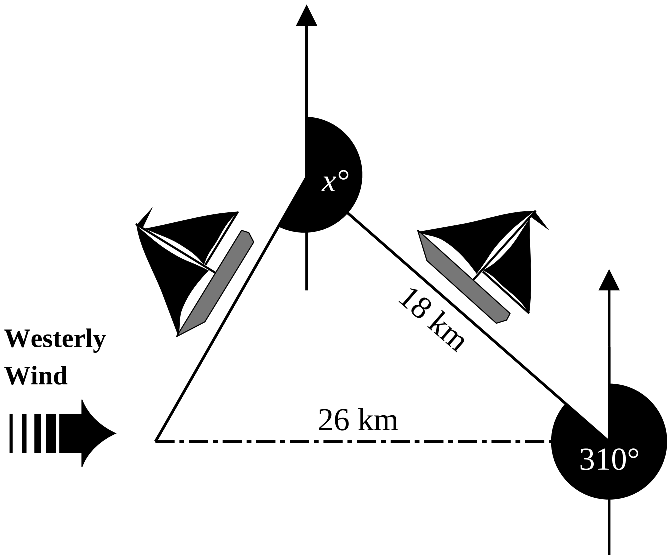


(iii) What would you expect to happen to P as n continues to increase ?

[12 marks]

Question 11

In order to make progress into a due West headwind, a sailing yacht tacks 18 km on a bearing of 310° before changing tack and sailing a further distance as shown below.



It has at this point made 26 km of effective progress directly to the West.
Determine the bearing on which the yacht progressed , marked x°, when sailing on the second tack.

[10 marks]

Question 12

(i) For any triangle, $\triangle ABC$, explain why $\frac{1}{2}\ ac \sin B = \frac{1}{2}\ bc \sin A$

(ii) Hence prove that

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

(iii) $\triangle ABC$ has a = 26 cm, b = 24 cm and $\angle B$ = 62°.

Find the acute angle, $\angle A$.

(iv) Had you been told that $\angle A$ was obtuse, what would $\angle A$ have been ?

[11 marks]

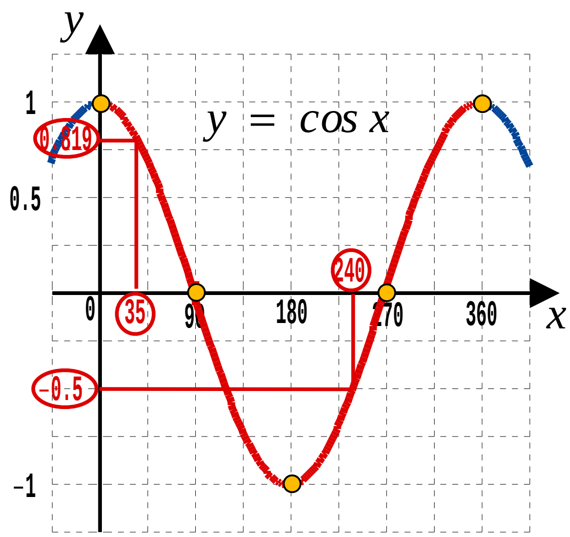
11.2 Answers

Answer 1

(i) $y = 0.819$

(iii) $y = -0.5$

(ii) and **(iv)**



Answer 2

1.5459

Answer 3

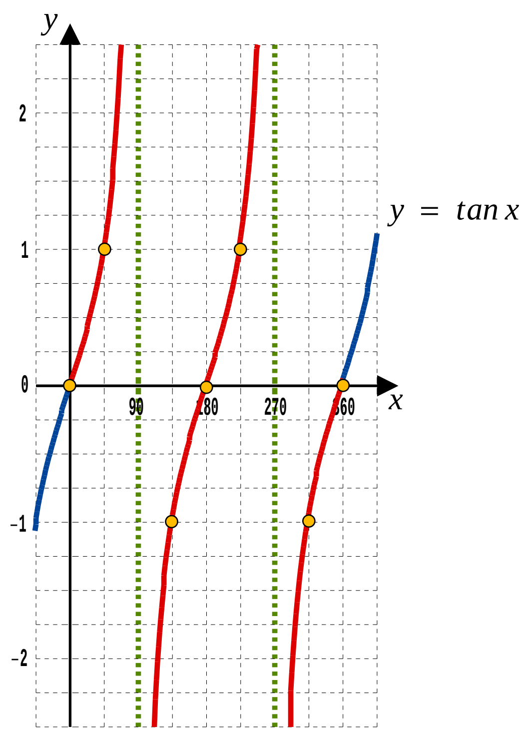
(i) $A = 140.3°$ (1 decimal place asked for)

(ii) $B = 76.2°$ (1 decimal place asked for)

(iii) $x = 30.2$ (3 significant figures asked for)

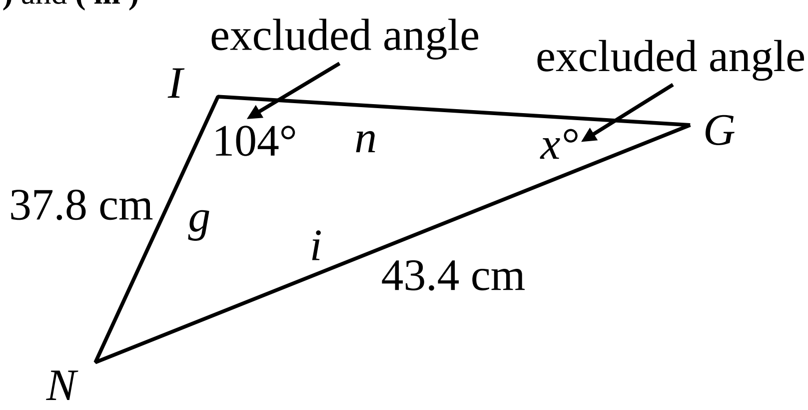
(iv) $x = 229$ (3 significant figures asked for)

Answer 4



Answer 5

(i) (ii) and **(iii)**



(iv)

$$\frac{\sin G}{g} = \boxed{\frac{\sin I}{i}} = \frac{\sin N}{n}$$

(v)

$$\begin{aligned} \sin G &= \frac{g \times \sin I}{i} \\ &= \frac{37.8 \times \sin 104}{43.4} \\ &= 0.845 \\ G &= \arcsin 0.845 \\ &= 57.7° \qquad \text{(3 significant answers asked for)} \end{aligned}$$

Answer 6

(i)

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ &= 33^2 + 18^2 - 2 \times 33 \times 18 \times \cos 29 \\ &= 373.95 \\ c &= 19.3 \text{ cm} \quad \text{(units asked for)} \end{aligned}$$

(ii)

$$\begin{aligned} Area\,\Delta &= \frac{1}{2} a\, b \sin C \\ &= \frac{1}{2} \times 33 \times 18 \times \sin 29 \\ &= 144 \text{ cm}^2 \qquad \text{(units asked for)} \end{aligned}$$

Answer 7

Let x be the angle opposite the longest side, in which case,

$$\begin{aligned} \cos x &= \frac{214^2 + 287^2 - 316^2}{2 \times 214 \times 287} \\ &= 0.230 \\ x &= \arccos 0.230 \\ &= 76.7° \end{aligned}$$

Answer 8

(i) $\angle B = 78°$

(ii) Sketch of triangle

(iii)

$$a = \frac{9.4 \times \sin 68}{\sin 78} = 8.91 \text{ cm} \quad \text{(This is } BC\text{)}$$

$$c = \frac{9.4 \times \sin 34}{\sin 78} = 5.37 \text{ cm} \quad \text{(This is } AB\text{)}$$

Answer 9

$$\begin{aligned} r^2 &= o^2 + c^2 - 2oc \cos R \\ &= 4.3^2 + 9.2^2 - 2 \times 4.3 \times 9.2 \times \cos 82 \\ &= 92.1 \\ r &= 9.6 \text{ cm} \end{aligned}$$

Answer 10

(i)

n	3	4	5	6	7	8	9	10
P	12.99	14.14	14.69	15.00	15.19	15.31	15.39	15.45

(ii)

(iii) As $n \to \infty$, the regular polygon becomes more like a circle.

$$C = 2\pi r$$

$$= 15.708\ldots \text{ cm}$$

That is, $P \to 15.708\ldots$ *as* $n \to \infty$

Answer 11

Strategy : the cosine rule avoids ambiguous case which occurs with the sine rule.

To get the length of the unknown side, u,

$$\begin{aligned} u^2 &= 18^2 + 26^2 - 2 \times 18 \times 26 \times \cos 40 \\ &= 283 \\ u &= 16.8 \text{ km} \end{aligned}$$

Now to get the angle inside the triangle, $W°$, near $x°$

$$\begin{aligned} \cos W &= \frac{16.8^2 + 18^2 - 26^2}{2 \times 16.8 \times 18} \\ &= -0.115 \\ W &= 96.6° \end{aligned}$$

So,

$$\begin{aligned} x &= 90 + 40 + 97 \\ &= 227° \end{aligned}$$

Answer 12

(i) Both $\frac{1}{2}\, ac \sin B$ and $\frac{1}{2}\, bc \sin A$ are the area of $\triangle ABC$

$\therefore$ They equal each other.

(ii)

$$\begin{aligned} \frac{1}{2}\, ac \sin B &= \frac{1}{2}\, bc \sin A \\ ac \sin B &= bc \sin A \\ a \sin B &= b \sin A \\ \frac{\sin A}{a} &= \frac{\sin B}{b} \end{aligned}$$

(iii)

$$\begin{aligned} \sin A &= \frac{26 \times \sin 62}{24} \\ &= 0.957 \\ A &= 73° \end{aligned}$$

(iv) 107° using symmetry of sine curve in mirror line at 90°