10.3 Answers

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics in Algebra

Answer 1

(a)

$$4x - 5 > 15 - x$$
$$5x > 20$$
$$x > 4$$

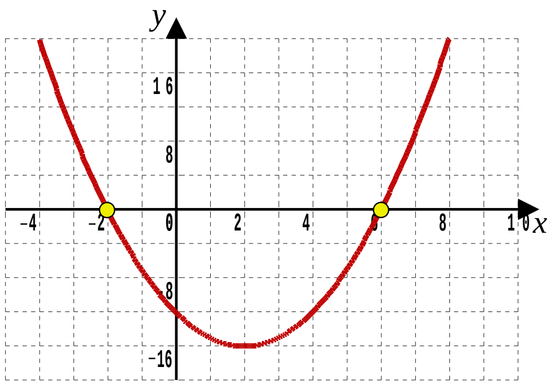
[2 marks]

(b)

$$x\,(x - 4) > 12$$
$$x^2 - 4x - 12 > 0$$

Finding the critical values...

$$(x + 2)(x - 6) = 0$$
$$x = -2 \text{ or } x = 6$$



From the graph, the solution to the inequality $x\,(x - 4) > 12$

is $x < -2$ or $x > 6$

[4 marks]

Answer 2

$$x^3 - 9x = x\left(x^2 - 3^2\right)$$
$$= x(x - 3)(x + 3)$$

[3 marks]

Answer 3

(a)

$$4x - 3 > 7 - x$$
$$5x > 10$$
$$x > 2$$

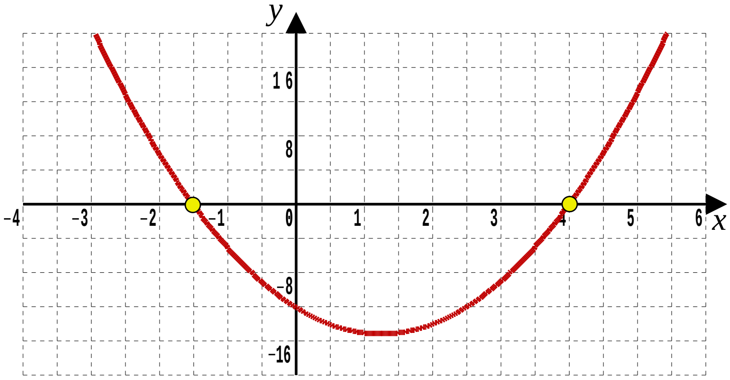
[2 marks]

(b)

$$2x^2 - 5x - 12 < 0$$

Finding the critical values...

$$(2x + 3)(x - 4) = 0$$
$$x = -\frac{3}{2} \text{ or } x = 4$$



From the graph, the solution to the inequality $2x^2 - 5x - 12 < 0$ is $-\frac{3}{2} < x < 4$

[4 marks]

(c) $2 < x < 4$

[1 mark]

Answer 4

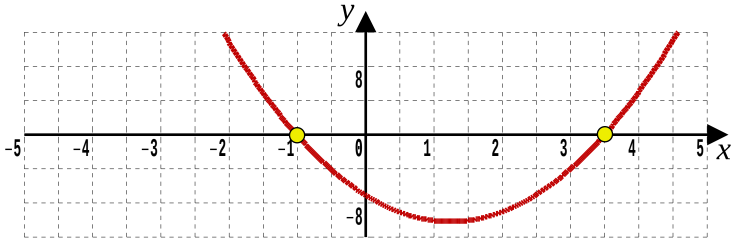
(a)

$$3(x - 2) < 8 - 2x$$
$$3x - 6 < 8 - 2x$$
$$5x < 14$$
$$x < \frac{14}{5}$$

[2 marks]

(b) Finding the critical values...

$$(2x - 7)(1 + x) = 0$$
$$x = -1 \text{ or } x = \frac{7}{2}$$



From the graph, the solution to the inequality $(2x - 7)(1 + x) < 0$ is $-1 < x < \frac{7}{2}$

[4 marks]

(c) $-1 < x < \frac{14}{5}$

[1 mark]

Answer 5

(a)

$$x^2 + 6x + 11 = \left[x^2 + 6x \right] + 11$$
$$= \left[(x + 3)^2 - 9 \right] + 11$$
$$= (x + 3)^2 + 2$$

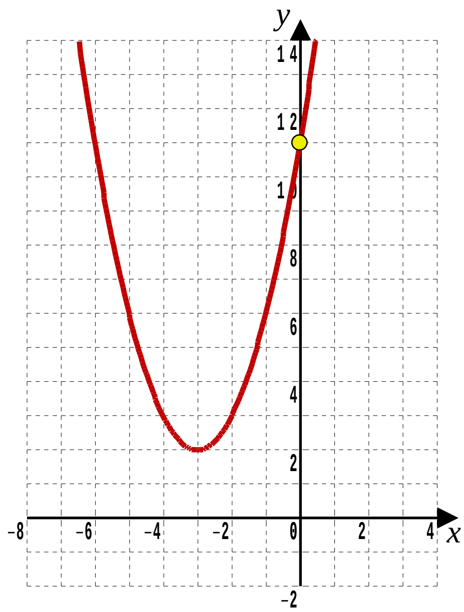
That is, $p = 3$ and $q = 2$

[2 marks]

(b) From part (a) the parabola has a minimum turning point at $(-3,\ 2)$

Anticipating part (c), notice that the discriminant is negative, confirming there are not any x-axis crossing points.

The only axis crossing point is $(0,\ 11)$



[2 marks]

(c)

$$D = b^2 - 4ac$$
$$= 6^2 - 4 \times 1 \times 11$$
$$= -8$$

[2 marks]

Answer 6

$$2qx^2 + qx - 1 = 0$$

(a) For this to have no real roots, the discriminant must be negative.

$$D < 0$$

$$q^2 - 4 \times 2q \times (-1) < 0$$

$$q^2 + 8q < 0$$

[2 marks]

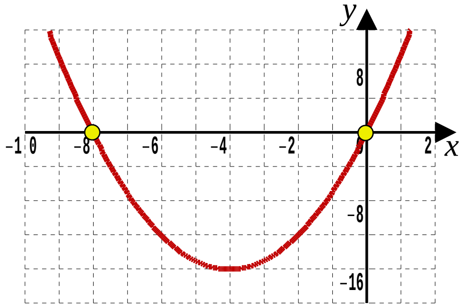
(b) Finding the critical values...

$$q^2 + 8q = 0$$

$$q(q + 8) = 0$$

$$\text{Either } q = 0 \text{ or } q = -8$$

Let $y = q^2 + 8q$.



From the graph, the solution to the inequality $q^2 + 8q < 0$

is $-8 < q < 0$

[3 marks]

Answer 7

$$f(x) = x^2 + (k + 3)x + k$$

(a) $D = (k + 3)^2 - 4 \times 1 \times k$

$= k^2 + 6k + 9 - 4k$

$= k^2 + 2k + 9$

[2 marks]

(b) $D = (k + 1)^2 + 8 \qquad \therefore\ a = 1$ and $b = 8$

[2 marks]

(c) Obviously $(k + 1)^2 \geqslant 0$ as any real number when squared is non-negative.

Even when $(k + 1)^2$ has its least value of zero, 8 is added on.

$\therefore\ (k + 1)^2 + 8 > 0$ That is, the discriminant is always positive.

Thus $f(x)$ always has two distinct real roots $\square$

[2 marks]

Answer 8

$$kx^2 + 4x + (5 - k) = 0$$

(a) To have 2 different real solutions the discriminant must be positive.

$$D > 0$$

$$4^2 - 4k(5 - k) > 0$$

$$16 - 20k + 4k^2 > 0$$

$$4k^2 - 20k + 16 > 0$$

divide through by 4

$$k^2 - 5k + 4 > 0 \qquad \square$$

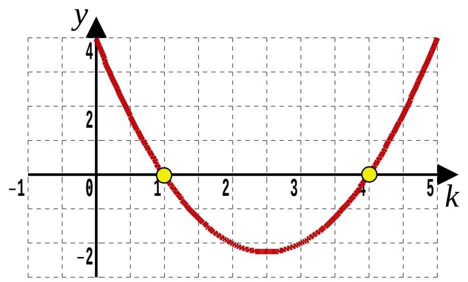
[3 marks]

(b) Finding the critical values...

$$(k - 1)(k - 4) = 0$$

Either $k = 1$ or $k = 4$

Let $y = k^2 - 5k + 4$



From the graph, the solution to the inequality $k^2 - 5k + 4 > 0$

is $k < 1$ or $k > 4$

[4 marks]

Answer 9

Starting with the equations,

$$y = x - 2$$

$$y^2 + x^2 = 10$$

$$(x - 2)^2 + x^2 = 10$$

$$x^2 - 4x + 4 + x^2 = 10$$

$$2x^2 - 4x - 6 = 0$$

divide through by 2

$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

Either $x = -1$ or $x = 3$

Using $y = x - 2$ then gives the solutions as,

$(-1, -3)$ or $(3, 1)$

[7 marks]

Answer 10

$$x^2 + kx + 8 = k$$

$$x^2 + kx + (8 - k) = 0$$

(a) To have no real solutions the discriminant must be negative.

$$D < 0$$

$$k^2 - 4 \times 1 \times (8 - k) < 0$$

$$k^2 + 4k - 32 < 0 \qquad \square$$

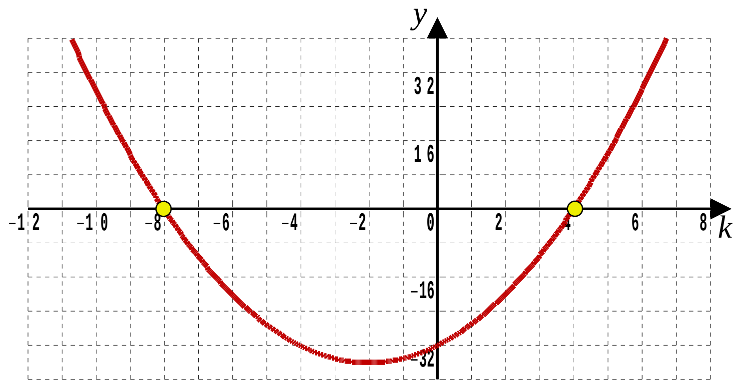
[3 marks]

(b) Finding the critical values...

$$(k + 8)(k - 4) = 0$$

$$\text{Either } k = -8 \text{ or } k = 4$$

Let $y = k^2 + 4k - 32$



From the graph, the solution to the inequality $k^2 + 4k - 32 < 0$

is $-8 < k < 4$

[4 marks]

Answer 11

$$2x^2 - 3x - (k + 1) = 0$$

To have no real roots the discriminant must be negative.

$$D < 0$$

$$(-3)^2 - 4 \times 2 \times [-(k + 1)] < 0$$

$$9 + 8k + 8 < 0$$

$$8k + 17 < 0$$

$$k < -\frac{17}{8}$$

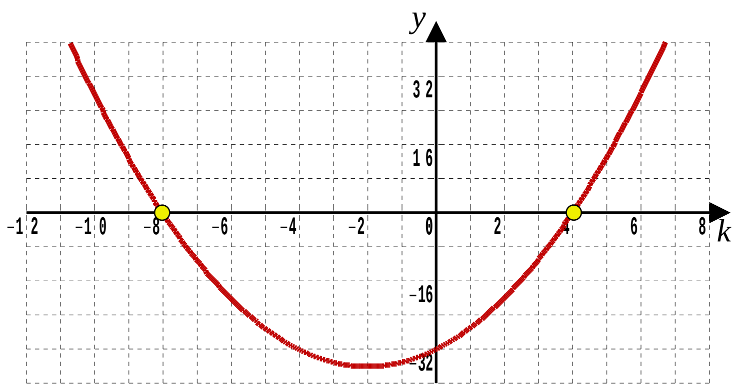
[3 marks]

(b) Finding the critical values...

$$(k + 8)(k - 4) = 0$$

$$\text{Either } k = -8 \text{ or } k = 4$$

Let $y = k^2 + 4k - 32$



From the graph, the solution to the inequality $k^2 + 4k - 32 < 0$

is $-8 < k < 4$

[4 marks]

Answer 12

(a) $x^2 + 4kx + (3 + 11k) = [x^2 + 4kx] + (3 + 11k)$

$$= [(x + 2k)^2 - 4k^2] + (3 + 11k)$$

$$= (x + 2k)^2 + (3 + 11k - 4k^2)$$

$$\therefore\ p = 2k \text{ and } q = 3 + 11k - 4k^2$$

[3 marks]

(b) To have no real roots the discriminant must be negative.

$$D < 0$$

$$(4k)^2 - 4 \times 1 \times (3 + 11k) < 0$$

$$16k^2 - 44k - 12 < 0$$

divide through by 4

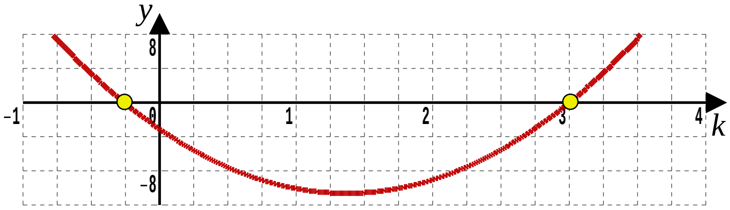
$$4k^2 - 11k - 3 < 0$$

Finding the critical values....

$$(4k + 1)(k - 3) = 0$$

$$\text{Either } k = -\frac{1}{4} \text{ or } k = 3$$

Let $y = 4k^2 - 11k - 3$



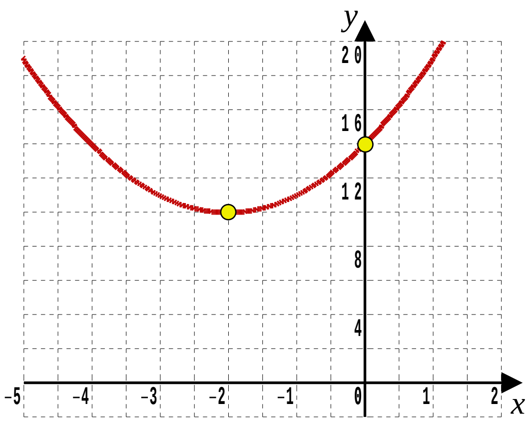
From the graph, the solution to the inequality $4k^2 - 11k - 3 < 0$

is $-\frac{1}{4} < k < 3$

[4 marks]

(c) With $k = 1$, $f(x) = x^2 + 4x + 14 = (x + 2)^2 + 10$

Now let $y = f(x)$



Notice the minimum at $(-2, \ 10)$ any the y-axis intercept at $(0, \ 14)$

[3 marks]