Additional Mathematics
A-Level Pure Mathematics : Year 1

Let *a* be for algebra

T O P I C S in A L G E B R A
Quadratic Equations & Inequalities

Lesson 1

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics In Algebra

1.1 Revision of GCSE Algebra

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 60

Question 1
IGCSE Examination Question from November 2008, Paper 3H, Q2 (Edexcel)

(a) Factorise $7p - 21$

[1 mark]

(b) Solve $4(x + 5) = 12$
You must show sufficient working

[2 marks]

Question 2
IGCSE Examination Question from June 2011, Paper 3H, Q5 (Edexcel)

Show that $\frac{5}{6} - \frac{3}{4} = \frac{1}{12}$

[2 marks]

Question 3

IGCSE Examination Question from November 2008, Paper 3H, Q8 (Edexcel)

(a) Simplify

(i) $p^5 \times p$

(ii) $\dfrac{q^5}{q^3}$

[2 marks]

(b) Expand and simplify

$$3(4x - 1) - 4(2x - 3)$$

[2 marks]

(c) Expand and simplify

$$(y + 3)(y + 5)$$

[2 marks]

Question 4

*IGCSE Examination Question from May 2012, Paper 3H, Q3 (E*dexcel)

(a) Write as a single power of 2,

$$2^3 \times 2^6$$

[1 mark]

(b) Write as a single power of 3,

$$\frac{3^9}{3^4}$$

[1 mark]

(c)

$$\frac{5^n}{5^4 \times 5^6} = 5^3$$

Find the value of *n*.

[2 marks]

Question 5

IGCSE Examination Question from November 2008, Paper 3H, Q14 (Edexcel)

(a) Factorise completely

$$9ab - 12b^2$$

[2 marks]

(b) Simplify

$$\left(2ab^2\right)^3$$

[2 marks]

Question 6

IGCSE Examination Question from May 2012, Paper 3H, Q17 (Edexcel)

(a) Simplify

$$(3a^2 b)^4$$

[2 marks]

(b) Simplify

$$(9c^8)^{\frac{1}{2}}$$

[2 marks]

Question 7

(a) Factorise $x^2 - 8x + 12$

[2 marks]

(b) Factorise $x^2 - 81$

[1 mark]

Question 8

Simplify the following algebraic expressions by first factorising the quadratics:

$$\frac{x^2 + 3x - 88}{x^2 - 2x - 48}$$

[2 marks]

Question 9
Simplify fully

$$\frac{x^2 + 5x}{x^2 - 25}$$

[3 marks]

Question 10
Express as a single fraction
(i)

$$\frac{2(2x + 7)}{5} + \frac{5(4x + 1)}{3}$$

[3 marks]

(ii)

$$\frac{2(5x + 4)}{3} - \frac{3(3x - 2)}{4}$$

[3 marks]

Question 11

Simplify the following expression;

$$\frac{7}{(x+4)} + \frac{5}{(x+3)}$$

[3 marks]

Question 12

Beginning "LHS = " show that;

$$\frac{3}{(x+6)} + \frac{5}{(x-10)} = \frac{8x}{(x+6)(x-10)}$$

[3 marks]

Question 13

(a) Simplify

$$\frac{x^2}{x^2 - 5x}$$

[2 marks]

(b) Simplify

$$\frac{6}{2x - 9} - \frac{2}{2x + 3}$$

[4 marks]

Question 14

Solve

$$\frac{x - 3}{2} + \frac{x - 5}{3} = 6$$

[4 marks]

Question 15
Solve the equation;

$$\frac{x}{2} = \frac{2(x - 2)}{7}$$

[3 marks]

Question 16
Find the two solutions to the equation;

$$\frac{x}{x + 3} = \frac{10}{x - 3}$$

[4 marks]

1.2 Answers (Revision of GCSE Algebra)

Answer 1

(a) $7(p - 3)$

(b) -2

[1, 2 marks]

Answer 2

$$\begin{aligned} \text{LHS} &= \frac{5}{6} - \frac{3}{4} \qquad\qquad lcm\{6,\ 4\} = 12 \\ &= \frac{2}{2} \times \frac{5}{6} - \frac{3}{4} \times \frac{3}{3} \\ &= \frac{10}{12} - \frac{9}{12} \\ &= \frac{1}{12} \\ &= \text{RHS} \qquad\qquad \square \end{aligned}$$

[2 marks]

Answer 3

(a) **(i)** p^6

(ii) q^2

[1, 1 marks]

(b) $4x + 9$

[2 marks]

(c) $y^2 + 8y + 15$

[2 marks]

Answer 4

(a) 2^9

(b) 3^5

(c) $n = 13$

[1, 1, 2 marks]

Answer 5

(a) $3b\ (3a - 4b)$

(b) $8\,a^3\,b^6$

[2, 2 marks]

Answer 6

(a) $81\,a^8\,b^4$

(b) $3\,c^4$

[2, 2 marks]

Answer 7

(a) $(x-2)(x-6)$

(b) $(x+9)(x-9)$

[2, 1 marks]

Answer 8

$$\frac{x+11}{x+6}$$

[2 marks]

Answer 9

$$\frac{x}{x-5}$$

[3 marks]

Answer 10

(i) $\dfrac{112x+67}{15}$ **(ii)** $\dfrac{13x+50}{12}$

[3, 3 marks]

Answer 11

$$\frac{12x+41}{(x+4)(x+3)}$$

[3 marks]

Answer 12

$$\begin{aligned}
\text{LHS} &= \frac{3}{(x+6)} + \frac{5}{(x-10)} \\
&= \frac{(x-10)}{(x-10)} \times \frac{3}{(x+6)} + \frac{5}{(x-10)} \times \frac{(x+6)}{(x+6)} \\
&= \frac{3x-30+5x+30}{(x+6)(x-10)} \\
&= \frac{8x}{(x+6)(x-10)} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 13

(a) $\dfrac{x}{x - 5}$ **(b)** $\dfrac{8x + 36}{(2x + 3)(2x - 9)}$

[2, 4 marks]

Answer 14

$x = 11$

[4 marks]

Answer 15

$x = -\dfrac{8}{3}$ (-2.67)

[3 marks]

Answer 16

$x = -2, \; 15$

[4 marks]

Lesson 2

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics In Algebra

2.1 More Revision of GCSE Algebra (Homework)

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 60

Question 1

IGCSE Examination Question from November 2007, Paper 4H, Q2 (Edexcel)

(a) Factorise $5x - 20$

[1 mark]

(b) Factorise $y^2 + 6y$

[2 marks]

Question 2

IGCSE Examination Question from May 2007, Paper 3H, Q9 (Edexcel)

(a) Solve $5x - 4 = 2x + 7$

[2 marks]

(b) Solve $\dfrac{7 - 2y}{4} = 2y + 3$

[4 marks]

Question 3

IGCSE Examination Question from January 2013, Paper 4H, Q8 (Edexcel)

(a) Factorise $n^2 + 8n$

[2 marks]

(b) Expand and simplify

$$3(2x - 5) - 4(x + 3)$$

[2 marks]

(c) Expand and simplify

$$(y + 7)(y + 2)$$

[2 marks]

Question 4

IGCSE Examination Question from May 2008, Paper 4H, Q6 (Edexcel)

Show that $\dfrac{2}{3} + \dfrac{1}{4} = \dfrac{11}{12}$

[2 marks]

Question 5

IGCSE Examination Question from May 2007, Paper 3H, Q5 (Edexcel)

(a) Simplify, leaving your answers in index form

(i) $7^5 \times 7^3$ **(ii)** $5^9 \div 5^3$

[2 marks]

(b) Solve $\dfrac{2^9 \times 2^4}{2^n} = 2^8$

[2 marks]

Question 6

IGCSE Examination Question from May 2008, Paper 3H, Q14 (Edexcel)

(a) Factorise $10y - 15$

[1 mark]

(b) Factorise $9p^2q + 12pq^2$

[2 marks]

(c) **(i)** Factorise $x^2 + 6x - 16$

(ii) Solve $x^2 + 6x - 16 = 0$

[3 marks]

Question 7

IGCSE Examination Question from January 2013, Paper 4H, Q15 (Edexcel)

(a) Simplify $\dfrac{5x^5y^6}{x^2y^4}$

[2 marks]

(b) Simplify $\left(2n^4\right)^3$

[2 marks]

Question 8

IGCSE Examination Question from May 2006, Paper 3H, Q13 (a) (c) (Edexcel)

(a) Expand and simplify $(3x - 5)(4x + 7)$

[2 marks]

(b) Simplify $\left(64y^6\right)^{\frac{2}{3}}$

[2 marks]

Question 9

Simplify the following algebraic expressions by first factorising the quadratics:

$$\frac{x^2 + 2x - 24}{x^2 - 3x - 54}$$

[2 marks]

Question 10
IGCSE Examination Question from May 2004, Paper 3H, Q16 (Edexcel)
Express this algebraic fraction as simply as possible

$$\frac{2x^2 - 3x - 20}{x^2 - 16}$$

[3 marks]

Question 11
Express as a single fraction
(i)

$$\frac{(3x + 7)}{5} + \frac{(7x - 4)}{3}$$

[3 marks]

(ii)

$$\frac{5(4x + 1)}{2} - \frac{3(7x - 2)}{5}$$

[3 marks]

Question 12

Simplify the following expression;

$$\frac{1}{(x+4)} + \frac{4}{(x+5)}$$

[3 marks]

Question 13

Beginning "LHS = " show that;

$$\frac{8}{(x-4)} + \frac{2}{(x+6)} = \frac{10(x+4)}{(x-4)(x+6)}$$

[3 marks]

Question 14

IGCSE Examination Question from May 2006, Paper 4H, Q4 (Edexcel)

Arul had x sweets.

Nikos had four times as many sweets as Arul.

(a) Write down an expression, in terms of x, for the number of sweets Nikos had.

[1 mark]

Nikos gave 6 of his sweets to Arul.

Now they both have the same number of sweets.

(b) Use this information to form an equation in x.

[2 marks]

(c) Solve your equation to find the number of sweets that Arul had at the start.

[2 marks]

Question 15

(a) Simplify

$$\frac{x^2 - 7x}{x^2}$$

[2 marks]

(b) Simplify

$$\frac{2}{x + 1} - \frac{4}{2x + 3}$$

[4 marks]

Question 16
Solve

$$\frac{x - 3}{2} + \frac{x - 2}{5} = 10$$

[4 marks]

Question 17
IGCSE Examination Question from January 2013, 4H, Q15
Solve

$$\frac{2}{5x - 2} = \frac{3}{6x + 1}$$

[4 marks]

Question 18

Find the two solutions to the equation;

$$\frac{x}{x+5} = \frac{2}{x-7}$$

[4 marks]

2.2 Answers (More Revision of GCSE Algebra - Homework)

Answer 1

(a) $5(x - 4)$

(b) $y(y + 6)$

[1, 2 marks]

Answer 2

(a) $\frac{11}{3}$

(b) -0.5

[2, 4 marks]

Answer 3

(a) $n(n + 8)$

(b) $2x - 27$

(c) $y^2 + 9y + 14$

[2, 2, 2 marks]

Answer 4

$$\begin{aligned} \text{LHS} &= \frac{2}{3} + \frac{1}{4} \qquad LCM\{3,\ 4\} = 12 \\ &= \frac{4}{4} \times \frac{2}{3} + \frac{1}{4} \times \frac{3}{3} \\ &= \frac{8}{12} + \frac{3}{12} \\ &= \frac{11}{12} \\ &= \text{RHS} \qquad \square \end{aligned}$$

[2 marks]

Answer 5

(a) **(i)** 7^8

(ii) 5^6

(b) $n = 5$

[2, 2 marks]

Answer 6

(a) $5(2y - 3)$

(b) $3pq(3p + 4q)$

(c) **(i)** $(x + 8)(x - 2)$

(ii) $x = -8$ or $x = 2$

[1, 2, 3 marks]

Answer 7

(a) $5x^3y^2$ **(b)** $8n^{12}$

[2, 2 marks]

Answer 8

(a) $12x^2 + x - 35$ **(b)** $16y^4$

[2, 2 marks]

Answer 9

$$\frac{x - 4}{x - 9}$$

Answer 10

$$\frac{2x + 5}{x + 4}$$

[2, 3 marks]

Answer 11

(i) $\dfrac{44x + 1}{15}$ **(ii)** $\dfrac{58x + 37}{10}$

[3, 3 marks]

Answer 12

$$\frac{5x + 21}{(x + 5)(x + 4)}$$

[3 marks]

Answer 13

$$\begin{aligned}
\text{LHS} &= \frac{8}{(x - 6)} + \frac{2}{(x + 6)} \\
&= \frac{(x + 6)}{(x + 6)} \times \frac{8}{(x - 4)} + \frac{2}{(x + 6)} \times \frac{(x - 4)}{(x - 4)} \\
&= \frac{8x + 48 + 2x - 8}{(x + 6)(x - 4)} \\
&= \frac{10x + 40}{(x - 4)(x + 6)} \\
&= \frac{10\,(x + 4)}{(x - 4)(x + 6)} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 14

(a) $4x$ **(b)** $x + 6 = 4x - 6$ **(c)** $x = 4$

[1, 2, 2 marks]

Answer 15

(a) $\dfrac{x - 7}{x}$ **(b)** $\dfrac{2}{(2x + 3)\,(x + 1)}$

[2, 4 marks]

Answer 16

$x = 17$

Answer 17

$x = 2\frac{2}{3}$

Answer 18

$x = -1,\ 10$

[4, 4, 4 marks]

Lesson 3

3.1 Completing the square

Example

Solve this equation by completing the square, giving an exact answer.

$$x^2 - 6x = 2$$

[2 marks]

3.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 70

Question 1

Solve these equations by completing the square, giving exact answers.

(i) $x^2 - 8x = 1$

(ii) $x^2 + 2x = 5$

(iii) $x^2 - 12x = 5$

(iv) $x^2 + 14x + 30 = 0$

[8 marks]

Question 2

By completing the square on $x^2 + 6x = 11$ show that the equation has solutions of the form $a \pm b\sqrt{c}$ where a, b and c are integers and c is square free.

[3 marks]

Question 3

Solve these equations by completing the square, giving exact answers.

(i) $(x - 5)(x + 3) = 1$

(ii) $x + \dfrac{1}{x} = 4$

(iii) $x(x + 4) = 7$

(iv) $\dfrac{1}{(x + 1)} + x = 3$

[12 marks]

Question 4

The following equation is to be solved by the method of completing the square;

$$\frac{2}{x} + \frac{17}{x^2} = 1$$

Show that the exact solutions are of the form $x = a + b\sqrt{c}$

where a, b and c are integers and c is square free.

[4 marks]

Question 5

The following equation is to be solved by the method of completing the square;

$$\frac{5}{2x} = \frac{x - 20}{4}$$

Show that the exact solutions are of the form $x = a + b\sqrt{c}$

where a, b and c are integers and c is square free.

[4 marks]

Question 6

Examination question from May 2005, Q3

$$x^2 - 8x - 29 \equiv (x + a)^2 + b$$

where a and b are constants.

(a) Find the value of a and the value of b.

[2 marks]

(b) Hence, or otherwise, show that the roots of

$$x^2 - 8x - 29 = 0$$

are $c \pm d\sqrt{5}$ where c and d are integers to be found.

[3 marks]

Question 7

$$x^2 - 3x + 1 \equiv (x + a)^2 + b$$

where a and b are rational constants.

(a) Find the value of a and the value of b.

[2 marks]

(b) Hence, or otherwise, show that the roots of

$$x^2 - 3x + 1 = 0$$

are $c \pm d\sqrt{5}$ where c and d are rational constants to be found.

[3 marks]

Question 8

Find, as surds, the roots of the equation:

$$(x - 2)^2 = 2(x + 1)(x - 4)$$

[4 marks]

Question 9

The following equation is to be solved by the method of completing the square;

$$x^2 - 3x - 1 = 0$$

Show that the exact solutions are of the form;

$$x = \frac{a \pm \sqrt{b}}{2}$$

clearly stating the values of the integers, *a* and *b*.

[3 marks]

Question 10
Use the method of completing the square to solve the equation;

$$x - 2x^{\frac{1}{2}} - 1 = 0$$

Give your answers in the exact form;

$$x = a + b\sqrt{c}$$

where a, b and c are integers and c is square free.

[4 marks]

Question 11

Peter is investigating quadratic equations of the form;

$$x^2 + 2px + p = 0$$

where p is a constant and x is a variable.

(i) By completing the square, find the algebraic solution to all equations of this type.

[3 marks]

(ii) Use your part (i) answer to write down the equation that corresponds to $p = 10$, and its solution.

[1 mark]

(iii) Use your part (i) answer to write down the equation that corresponds to $p = 12$, and its solution.

[1 mark]

(iv) Use your part (i) answer to write down the equation that corresponds to $p = 0.5$

Explain why this equation has no solutions.

[2 marks]

(v) What relationship must hold for one of Peter's equations to have solutions ?

[2 marks]

Question 12

Peter's friend, Sophia, is investigating quadratic equations of the form;

$$x^2 + 2px + q = 0$$

where p and q are constants and x is a variable.

(i) By completing the square, find the algebraic solution to all equations of this type.

[3 marks]

(ii) Use your part (i) answer to write down the equation that corresponds to $p = 3$, $q = 5$, and its solution.

[1 mark]

(iii) Use your part (i) answer to write down the equation that corresponds to $p = -4$, $q = 11$, and its solution.

[1 mark]

(iv) Use your part **(i)** answer to write down the equation that corresponds to $p = 3$, $q = 10$
Explain why this equation has no solutions.

[2 marks]

(v) What relationship must hold for one of Sophia's equations to have solutions ?

[2 marks]

3.3 Answers

3.3.1 Solutions (3.1 Introductory Example)

$$x^2 - 6x = 2$$
$$(x - 3)^2 - 9 = 2$$
$$(x - 3)^2 = 11$$
$$x - 3 = \pm\sqrt{11}$$
$$x = 3 \pm \sqrt{11}$$

[2 marks]

3.3.2 Solutions (3.2 Exercise)

Answer 1

(i) $4 \pm \sqrt{17}$ **(ii)** $-1 \pm \sqrt{6}$

(iii) $6 \pm \sqrt{41}$ **(iv)** $-7 \pm \sqrt{19}$

[8 marks]

Answer 2

$-3 \pm 2\sqrt{5}$

[3 marks]

Answer 3

(i) $1 \pm \sqrt{17}$ **(ii)** $2 \pm \sqrt{3}$

(iii) $-2 \pm \sqrt{11}$ **(iv)** $1 \pm \sqrt{3}$

[12 marks]

Answer 4

$1 \pm 3\sqrt{2}$

Answer 5

$10 \pm \sqrt{110}$

[4, 4 marks]

Answer 6

(a) $a = -4, \quad b = -45$

(b) $x = 4 \pm 3\sqrt{5}, \quad c = 4, \quad d = 3$

[2, 3 marks]

Answer 7

(a) $a = -\dfrac{3}{2}, \quad b = -\dfrac{5}{4}$

(b) $x = \dfrac{3}{2} \pm \dfrac{\sqrt{5}}{2} \qquad c = \dfrac{3}{2} \qquad d = \dfrac{1}{2}$

[2, 3 marks]

Answer 8

$$x = 1 \pm \sqrt{13}$$

Answer 9

$$x = \frac{3 \pm \sqrt{13}}{2} \qquad a = 3, \; b = 13$$

[4, 3 marks]

Answer 10

$$x = 3 \pm 2\sqrt{2}$$

[4 marks]

Answer 11

(i)

$$\begin{aligned} \left[x^2 + 2px \right] + p &= 0 \\ \left[(x + p)^2 - p^2 \right] + p &= 0 \\ (x + p)^2 &= p^2 - p \\ x + p &= \pm\sqrt{p^2 - p} \\ x &= -p \pm \sqrt{p^2 - p} \end{aligned}$$

[3 marks]

(ii) With $p = 10$

$$\begin{aligned} x^2 + 20x + 10 &= 0 \\ x &= -10 \pm \sqrt{90} \\ &= -10 \pm 3\sqrt{10} \end{aligned}$$

[1 mark]

(iii) With $p = 12$

$$\begin{aligned} x^2 + 24x + 12 &= 0 \\ x &= -12 \pm \sqrt{132} \\ &= -12 \pm 2\sqrt{33} \end{aligned}$$

[1 mark]

(iv) With $p = 0.5$

$$\begin{aligned} x^2 + x + 0.5 &= 0 \\ x &= -0.5 \pm \sqrt{0.25 - 0.5} \\ &= -0.5 \pm \sqrt{-0.25} \end{aligned}$$

which has no solutions due to the negative square root

[2 marks]

(v)

$$p^2 \geqslant p \qquad \therefore \; p \leqslant -1 \;\; or \;\; p \geqslant 1$$

[2 marks]

Answer 12

(i)

$$\left[x^2 + 2px \right] + q = 0$$

$$\left[(x + p)^2 - p^2 \right] + q = 0$$

$$(x + p)^2 = p^2 - q$$

$$x + p = \pm\sqrt{p^2 - q}$$

$$x = -p \pm \sqrt{p^2 - q}$$

[3 marks]

(ii)

$$x^2 + 6x + 5 = 0$$

$$x = -3 \pm \sqrt{4}$$

$$= -1 \text{ or } -5$$

[1 mark]

(iii)

$$x^2 - 8x + 11 = 0$$

$$x = 4 \pm \sqrt{5}$$

[1 mark]

(iv)

$$x^2 + 6x + 10 = 0$$

$$x = -3 \pm \sqrt{9 - 10}$$

$$= -3 \pm \sqrt{-1}$$

which has no solutions due to the negative square root

[2 marks]

(v)

$$p^2 \geqslant q$$

[2 marks]

4.1 Completing The Square : Tougher Problems

4.1.1 Example

By completing the square, solve the equation :

$$2x^2 + 3x - 3 = 0$$

Give your answer in the form $a \pm b\sqrt{33}$ for some constants a and b.

4.1.2 Model Solution :

Pull out the coefficient of x^2 from the terms in x^2 and x :

$$2\left[x^2 + \frac{3}{2}x\right] - 3 = 0$$

Complete the square within the square brackets :

$$2\left[\left(x + \frac{3}{4}\right)^2 - \frac{9}{16}\right] - 3 = 0$$

Expand the square brackets :

$$2\left(x + \frac{3}{4}\right)^2 - \frac{9}{8} - 3 = 0$$

Tidy up :

$$2\left(x + \frac{3}{4}\right)^2 - \frac{9}{8} - \frac{24}{8} = 0$$

$$2\left(x + \frac{3}{4}\right)^2 - \frac{33}{8} = 0$$

$$\left(x + \frac{3}{4}\right)^2 = \frac{33}{16}$$

Square root both sides :

$$x + \frac{3}{4} = \pm\sqrt{\frac{33}{16}}$$

$$x = -\frac{3}{4} \pm \frac{\sqrt{33}}{4}$$

[4 marks]

4.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 36

Question 1

Without using a calculator, use the method of completing the square to solve these equations;

(i) $2x^2 + 7x + 1 = 0$

[4 marks]

(ii) $5x^2 + 7x + 1 = 0$

[4 marks]

(iii) $3x^2 + 5x + 1 = 0$

[4 marks]

(iv) $10x^2 + 3x - 2 = 0$

[4 marks]

Question 2

Without using a calculator, by completing the square, solve the equation;

$5x^2 + 4x - 2 = 0$

[4 marks]

Question 3

Without using a calculator, by completing the square, show that the solutions to the equation; $2x^2 - 12x + 17 = 0$ are;

$$x = 3 \pm \frac{\sqrt{2}}{2}$$

[4 marks]

Question 4

Without using a calculator, by completing the square, show that the solutions to the equation; $2x^2 + 2x - 1 = 0$ are;

$$x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

[4 marks]

Question 5

By completing the square, solve the equation; $9x^2 + 6x - 17 = 0$

[4 marks]

Question 6

By completing the square, solve the equation; $9x^2 - 6x - 26 = 0$

[4 marks]

Appendix : Completing the Square

How to solve quadratics by Martin Hansen

(Published in M500, The Open University Mathematics Magazine)

To solve the quadratic equation:

$$ax^2 + bx + c = 0$$

Multiply through by $4a$:

$$\left[\, 4a^2x^2 + 4abx \,\right] + 4ac = 0$$

Complete the square within the square brackets:

$$\left[\, (2ax + b)^2 - b^2 \,\right] + 4ac = 0$$

$$(2ax + b)^2 = b^2 - 4ac$$

Square-root both sides:

$$2ax + b = \pm\sqrt{b^2 - 4ac}$$

And the well-known formula is there for the taking:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

4.3 Answers (4.2 Exercise)

Answer 1

(i)

$$2x^2 + 7x + 1 = 0$$

$$2\left[x^2 + \frac{7}{2}x \right] + 1 = 0$$

$$2\left[\left(x + \frac{7}{4} \right)^2 - \frac{49}{16} \right] + 1 = 0$$

$$2\left(x + \frac{7}{4} \right)^2 - \frac{49}{8} + \frac{8}{8} = 0$$

$$2\left(x + \frac{7}{4} \right)^2 - \frac{41}{8} = 0$$

$$2\left(x + \frac{7}{4} \right)^2 = \frac{41}{8}$$

$$\left(x + \frac{7}{4} \right)^2 = \frac{41}{16}$$

$$x + \frac{7}{4} = \pm\sqrt{\frac{41}{16}} \Rightarrow x = -\frac{7}{4} \pm \frac{\sqrt{41}}{4}$$

[4 marks]

(ii)

$$5x^2 + 7x + 1 = 0$$

$$5\left[x^2 + \frac{7}{5}x \right] + 1 = 0$$

$$5\left[\left(x + \frac{7}{10} \right)^2 - \frac{49}{100} \right] + 1 = 0$$

$$5\left(x + \frac{7}{10} \right)^2 - \frac{49}{20} + \frac{20}{20} = 0$$

$$5\left(x + \frac{7}{10} \right)^2 - \frac{29}{20} = 0$$

$$5\left(x + \frac{7}{10} \right)^2 = \frac{29}{20}$$

$$\left(x + \frac{7}{10} \right)^2 = \frac{29}{100}$$

$$x + \frac{7}{10} = \pm\sqrt{\frac{29}{100}} \Rightarrow x = -\frac{7}{10} \pm \frac{\sqrt{29}}{10}$$

[4 marks]

(iii)

$$3x^2 + 5x + 1 = 0$$

$$3\left[x^2 + \frac{5}{3}x\right] + 1 = 0$$

$$3\left[\left(x + \frac{5}{6}\right)^2 - \frac{25}{36}\right] + 1 = 0$$

$$3\left(x + \frac{5}{6}\right)^2 - \frac{25}{12} + \frac{12}{12} = 0$$

$$3\left(x + \frac{5}{6}\right)^2 - \frac{13}{12} = 0$$

$$3\left(x + \frac{5}{6}\right)^2 = \frac{13}{12}$$

$$\left(x + \frac{5}{6}\right)^2 = \frac{13}{36}$$

$$x + \frac{5}{6} = \pm\sqrt{\frac{13}{36}}$$

$$x = -\frac{5}{6} \pm \frac{\sqrt{13}}{6}$$

[4 marks]

(iv)

$$10x^2 + 3x - 2 = 0$$

$$10\left[x^2 + \frac{3}{10}x\right] - 2 = 0$$

$$10\left[\left(x + \frac{3}{20}\right)^2 - \frac{9}{400}\right] - 2 = 0$$

$$10\left(x + \frac{3}{20}\right)^2 - \frac{9}{40} - \frac{80}{40} = 0$$

$$10\left(x + \frac{3}{20}\right)^2 - \frac{89}{40} = 0$$

$$10\left(x + \frac{3}{20}\right)^2 = \frac{89}{40}$$

$$\left(x + \frac{3}{20}\right)^2 = \frac{89}{400}$$

$$x + \frac{3}{20} = \pm\sqrt{\frac{89}{400}}$$

$$x = -\frac{3}{20} \pm \frac{\sqrt{89}}{20}$$

[4 marks]

Answer 2

$$5x^2 + 4x - 2 = 0$$

$$5\left[x^2 + \frac{4}{5}x\right] - 2 = 0$$

$$5\left[\left(x + \frac{2}{5}\right)^2 - \frac{4}{25}\right] - 2 = 0$$

$$5\left(x + \frac{2}{5}\right)^2 - \frac{4}{5} - \frac{10}{5} = 0$$

$$5\left(x + \frac{2}{5}\right)^2 - \frac{14}{5} = 0$$

$$5\left(x + \frac{2}{5}\right)^2 = \frac{14}{5}$$

$$\left(x + \frac{2}{5}\right)^2 = \frac{14}{25}$$

$$x + \frac{2}{5} = \pm\sqrt{\frac{14}{25}}$$

$$x = -\frac{2}{5} \pm \frac{\sqrt{14}}{5}$$

[4 marks]

Answer 3

$$2x^2 - 12x + 17 = 0$$

$$2\left[x^2 - 6x\right] + 17 = 0$$

$$2\left[(x - 3)^2 - 9\right] + 17 = 0$$

$$2(x - 3)^2 - 18 + 17 = 0$$

$$2(x - 3)^2 - 1 = 0$$

$$2(x - 3)^2 = 1$$

$$(x - 3)^2 = \frac{1}{2}$$

$$x - 3 = \pm\sqrt{\frac{1}{2}}$$

$$x = 3 \pm \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$x = 3 \pm \frac{\sqrt{2}}{2}$$

[4 marks]

Answer 4

$$2x^2 + 2x - 1 = 0$$

$$2\left[x^2 + x\right] - 1 = 0$$

$$2\left[\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}\right] - 1 = 0$$

$$2\left(x + \frac{1}{2}\right)^2 - \frac{1}{2} - \frac{2}{2} = 0$$

$$2\left(x + \frac{1}{2}\right)^2 - \frac{3}{2} = 0$$

$$2\left(x + \frac{1}{2}\right)^2 = \frac{3}{2}$$

$$\left(x + \frac{1}{2}\right)^2 = \frac{3}{4}$$

$$x + \frac{1}{2} = \pm\sqrt{\frac{3}{4}}$$

$$x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

[4 marks]

Answer 5

$$9x^2 + 6x - 17 = 0$$

$$9\left[x^2 + \frac{2}{3}x\right] - 17 = 0$$

$$9\left[\left(x + \frac{1}{3}\right)^2 - \frac{1}{9}\right] - 17 = 0$$

$$9\left(x + \frac{1}{3}\right)^2 - 1 - 17 = 0$$

$$9\left(x + \frac{1}{3}\right)^2 - 18 = 0$$

$$9\left(x + \frac{1}{3}\right)^2 = 18$$

$$\left(x + \frac{1}{3}\right)^2 = 2$$

$$x + \frac{1}{3} = \pm\sqrt{2}$$

$$x = -\frac{1}{3} \pm \sqrt{2}$$

[4 marks]

Answer 6

$$9x^2 - 6x - 26 = 0$$

$$9\left[x^2 - \frac{2}{3}x \right] - 26 = 0$$

$$9\left[\left(x - \frac{1}{3} \right)^2 - \frac{1}{9} \right] - 26 = 0$$

$$9\left(x - \frac{1}{3} \right)^2 - 1 - 26 = 0$$

$$9\left(x - \frac{1}{3} \right)^2 - 27 = 0$$

$$9\left(x - \frac{1}{3} \right)^2 = 27$$

$$\left(x - \frac{1}{3} \right)^2 = 3$$

$$x - \frac{1}{3} = \pm\sqrt{3}$$

$$x = \frac{1}{3} \pm \sqrt{3}$$

[4 marks]

5.1 Quadratic Equation Roots

A quadratic equation is of the form; $y = ax^2 + bx + c$

In this equation x and y are variables, whereas a, b and c are constants.
The graph of such an equation is a ***quadratic curve***, also called a ***parabola***.
It's a useful shape, used in, for example, car headlights, electric heaters, radio telescopes and solar furnesses.
Often the mathematical interested is where (if anywhere) a given quadratic curve crosses the x-axis. It does this when it has zero height; that is, when y is zero. When we talk about ***solving a quadratic equation*** we mean finding the values of x (often called the *roots*) for which $y = 0$.

Quadratic equations, when written in the form, $ax^2 + bx + c = 0$ can be solved by

(i) Factorisation into two pairs of brackets,
(ii) Completing the square.
(iii) Using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

5.2 Example

Find the roots of the equation $x^2 + 14x + 40 = 0$ in each of the following ways;

(i) By factorisation into two pairs of brackets,

$$x^2 + 14x + 40 = 0$$

[2 marks]

(ii) By completing the square,

$$x^2 + 14x + 40 = 0$$

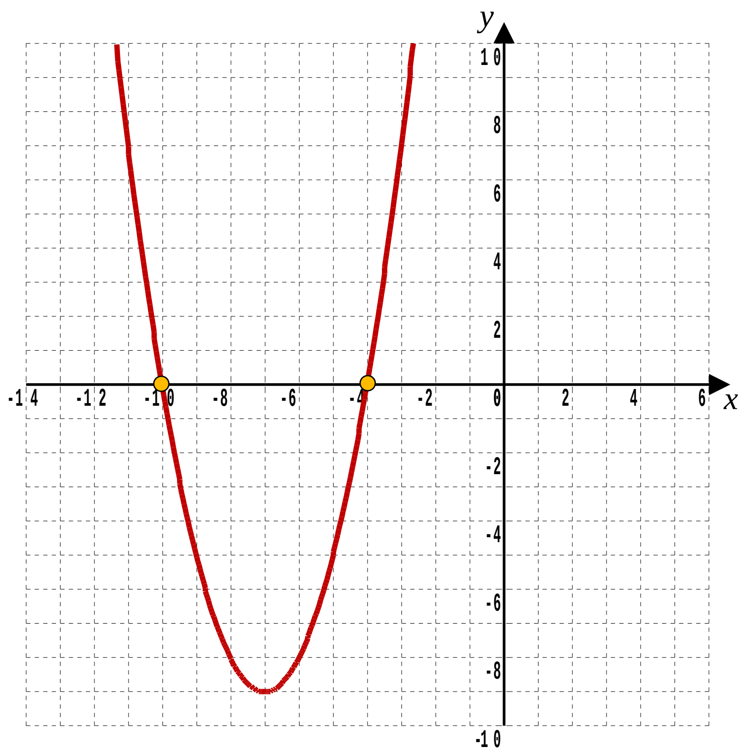
[2 marks]

(iii) Using the formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x^2 + 14x + 40 = 0$$

[2 marks]

The graph of $y = x^2 + 14x + 40$ is plotted below;



The roots $x = -10$, $x = -4$ correspond to where the curve cuts the x-axis.

5.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

By first factorising the quadratic, find the solutions to the following equations,

(i) $9x^2 - 16x + 7 = 0$

[3 marks]

(ii) $14x^2 + 11x + 2 = 0$

[3 marks]

Question 2

Use the method of completing the square to find the exact roots of the equation,

$$y = x^2 - 6x + 1$$

(The roots are the values of x that make $y = 0$)

[3 marks]

Question 3

Use the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the exact roots of the equation,

$$y = 7x^2 - 3x - 2$$

[3 marks]

Question 4

Simon has decided to use the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve the equation $x^2 + x + 1 = 0$ and find where the graph of $y = x^2 + x + 1$ crosses the *x*-axis.
Simon reports that the formula "goes wrong". Explain what he means by this and also what you can thus deduce about where the graph crosses the *x*-axis.

[4 marks]

Question 5

In general, the part of the formula for solving quadratic equations that sits under the square root symbol is called the discriminant, D.

Thus, $D = b^2 - 4ac$

For a particular quadratic equation, D can have a positive value, or a negative value or it can be equal to zero.

If D has a positive value, that is, $D > 0$, then

- the quadratic equation has two real roots.

If D has a value of exactly zero, that is $D = 0$, then

- the quadratic equation has one real root.
  (Annoyingly and confusingly this situation is often referred to as being of one repeated real root or two identical real roots)

If D has a negative value, that is, $D < 0$, then

- the quadratic equation has zero real roots.

For each of the following quadratic equations, calculate the value of the discriminant, D, and hence state the number of distinct real roots that equation has.

DO NOT SOLVE THE EQUATIONS !

(i) $x^2 + 4x - 5 = 0$

(ii) $x^2 + 4x + 5 = 0$

(iii) $2x^2 + x - 5 = 0$

(iv) $x^2 + 4x + 4 = 0$

(v) $3x^2 + 4x + 2 = 0$

(vi) $4x^2 - 25 = 0$

[6 marks]

Question 6

A-Level Examination Question from June 2009, C1, Q6 (Edexcel)

The equation $x^2 + 3px + p = 0$, where p is a non-zero constant, has equal roots.

Find the value of p.

[4 marks]

Question 7

A-Level Examination Question from January 2018, C12, Q4 (a)

The equation,

$$(p - 2)x^2 + 8x + (p + 4) = 0, \text{ where } p \text{ is a constant}$$

has no real roots.

Show that p satisfies $p^2 + 2p - 24 > 0$

[3 marks]

Question 8

A-Level Examination Question from January 2016, C12, Q13(a) (Edexcel)

The equation $k(3x^2 + 8x + 9) = 2 - 6x$, where k is a real constant, has no real roots. Show that k satisfies the inequality

$$11k^2 - 30k - 9 > 0$$

[4 marks]

Question 9

A-Level Examination Question from May 2016, C1, Q8 (Edexcel)

The straight line with equation $y = 3x - 7$ does not cross or touch the curve with equation $y = 2px^2 - 6px + 4p$, where p is a constant.

Show that, $4p^2 - 20p + 9 < 0$

[4 marks]

Question 10

The circle, C, with centre $(3, -5)$ and radius 5 has equation,

$$x^2 + y^2 - 6x + 10y + 9 = 0$$

The line with equation $y = kx$, where k is a positive constant, cuts C at a single point.

Find the exact value of k.

[6 marks]

5.4 Answers

5.4.1 Solutions to 5.2 Example

(i)

$$x^2 + 14x + 40 = 0$$

$$(x + 4)(x + 10) = 0$$

Either $x + 4 = 0$ in which case $x = -4$

or $x + 10 = 0$ in which case $x = -10$

[2 marks]

(ii)

$$x^2 + 14x + 40 = 0$$

$$\left[x^2 + 14x \right] + 40 = 0$$

$$\left[(x + 7)^2 - 49 \right] + 40 = 0$$

$$(x + 7)^2 - 9 = 0$$

$$(x + 7)^2 = 9$$

$$x + 7 = \pm 3$$

$$x = -7 \pm 3$$

That is, $x = -10, -4$

[2 marks]

(iii) $x^2 + 14x + 40 = 0$

$$x = \frac{-14 \pm \sqrt{14^2 - 4 \times 1 \times 40}}{2}$$

$$= \frac{-14 \pm \sqrt{196 - 160}}{2}$$

$$= \frac{-14 \pm \sqrt{36}}{2}$$

$$= \frac{-14 \pm 6}{2}$$

$$= -10, -4$$

[2 marks]

Notice:
On the graph observe the x-axis crossing points are at $(-10, 0)$ or $(-4, 0)$
Also, from the rewrite of the curve's equation

$$y = x^2 + 14x + 40$$

as

$$y = (x + 7)^2 - 9$$

it can be noticed that the minimum value is at $(-7, -9)$

5.4.2 Solutions to 5.3 Exercise

Answer 1

(i)

$$9x^2 - 16x + 7 = 0$$

$$(9x - 7)(x - 1) = 0$$

Either $9x - 7 = 0$ in which case $x = \dfrac{7}{9}$

or $x - 1 = 0$ in which case $x = 1$

[3 marks]

(ii)

$$14x^2 + 11x + 2 = 0$$

$$(7x + 2)(2x + 1) = 0$$

Either $7x + 2 = 0$ in which case $x = -\dfrac{2}{7}$

or $2x + 1 = 0$ in which case $x = -\dfrac{1}{2}$

[3 marks]

Answer 2

$$x^2 - 6x + 1 = 0$$

$$\left[x^2 - 6x\right] + 1 = 0$$

$$\left[(x - 3)^2 - 9\right] + 1 = 0$$

$$(x - 3)^2 - 8 = 0$$

$$(x - 3)^2 = 8$$

$$x - 3 = \pm 2\sqrt{2}$$

$$x = 3 \pm 2\sqrt{2}$$

[3 marks]

Answer 3

The roots occur when, $7x^2 - 3x - 2 = 0$

$$x = \frac{3 \pm \sqrt{3^2 - 4 \times 7 \times (-2)}}{2 \times 7}$$

$$= \frac{3 \pm \sqrt{65}}{14}$$

[3 marks]

Answer 4
Simon has found that the formula yields an expression containing the square root of a negative number which he calls "going wrong".

That is $\quad x = \dfrac{-1 \pm \sqrt{-3}}{2}$

This means there are zero real solutions, or roots, to this equation.
The number of roots corresponds to the number of *x*-axis crossing points should the graph of the given quadratic equation be plotted.

So the maths is telling Simon that the graph of $y = x^2 + x + 1$ does not cross the *x*-axis.

[4 marks]

Answer 5

(i)	$D = 36$ which is +ve	$\therefore$ 2 solutions
(ii)	$D = -4$ which is –ve	$\therefore$ 0 solutions
(iii)	$D = 41$ which is +ve	$\therefore$ 2 solutions
(iv)	$D = 0$	$\therefore$ 1 solutions
(v)	$D = -8$ which is –ve	$\therefore$ 0 solutions
(vi)	$D = 400$ which is +ve	$\therefore$ 2 solutions

[6 marks]

Answer 6
For equal roots the discriminant must be zero.

$$D = 0$$

$$(3p)^2 - 4 \times 1 \times p = 0$$

$$9p^2 - 4p = 0$$

As told *p* is not zero, divide by *p*

$$9p - 4 = 0$$

$$p = \frac{4}{9}$$

[4 marks]

Answer 7

$$(p - 2)x^2 + 8x + (p + 4) = 0$$

For no real roots the discriminant of this equation must be less than zero;

$$8^2 - 4 \times (p - 2) \times (p + 4) < 0$$

$$64 - 4\left(p^2 + 2p - 8\right) < 0$$

$$64 - 4p^2 - 8p + 32 < 0$$

Multiply through by – 1 and reverse the inequality

$$4p^2 + 8p - 96 > 0$$

Divide through by 4

$$p^2 + 2p - 24 > 0 \quad \square$$

[3 marks]

Answer 8

$$k\left(3x^2 + 8x + 9\right) = 2 - 6x$$

$$3kx^2 + (8k + 6)x + (9k - 2) = 0$$

For no real roots the discriminant of this equation must be less than zero;

$$(8k + 6)^2 - 4 \times 3k \times (9k - 2) < 0$$

$$64k^2 + 96k + 36 - 108k^2 + 24k < 0$$

$$-44k^2 + 120k + 36 < 0$$

Multiply through by – 1 and reverse the inequality

$$44k^2 - 120k - 36 > 0$$

Divide through by 4

$$11k^2 - 30k - 9 > 0 \quad \square$$

[4 marks]

Answer 9

The line and the curve would have intersected when,

$$2px^2 - 6px + 4p = 3x - 7$$

$$2px^2 - (6p + 3)x + (4p + 7) = 0$$

For the line and the curve not to intersect, this must have no solutions.
In other words, its discriminant must be less than zero.

$$D < 0$$

$$(-1)^2(6p + 3)^2 - 4 \times 2p \times (4p + 7) < 0$$

$$36p^2 + 36p + 9 - 32p^2 - 56p < 0$$

$$4p^2 - 20p + 9 < 0 \quad \square$$

[4 marks]

Answer 10

$$x^2 + y^2 - 6x + 10y + 9 = 0$$

$$x^2 + (kx)^2 - 6x + 10(kx) + 9 = 0$$

$$\left(1 + k^2\right)x^2 + (10k - 6)x + 9 = 0$$

To have a single point of intersection there must be only one solution to this.

$\therefore$ The discriminant must be equal to zero.

$$(10k - 6)^2 - 4 \times \left(1 + k^2\right) \times 9 = 0$$

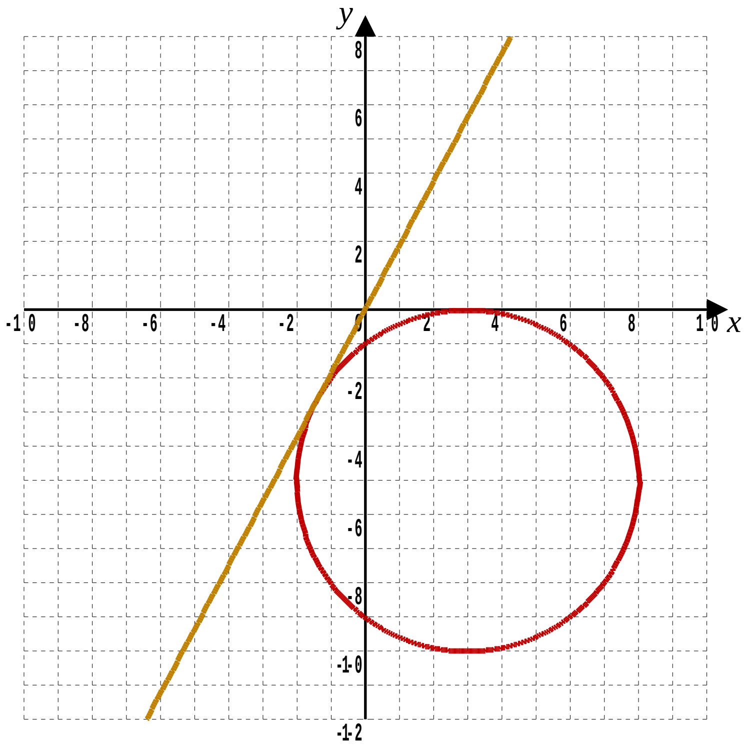
$$100k^2 - 120k + 36 - 36 - 36k^2 = 0$$

$$64k^2 - 120k = 0$$

As $k \neq 0$, divide through by k

$$64k - 120 = 0$$

$$\therefore\ k = \frac{15}{8}$$



The graph shows the circle plotted in red and the line $y = \dfrac{15}{8}x$ plotted in gold.

They intersect at a single point

6.1 The Discriminant

In lesson 5 it was noted that, when solving quadratic equations, the number of roots (solutions) is not always two.
Here is a recap of the theory giving rise to this observation;

The starting point, from GCSE, is to recall that a generalised quadratic;

$$a x^2 + bx + c = 0$$

has solutions given by;

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The piece of formula under the square root sign determines the number of roots.
This crucial piece of formula is termed the discriminant;

$$D = b^2 - 4ac$$

Here are three examples to illustrate the dramatic effect the discriminant has on the number of roots:

$x^2 + x + 1 = 0$	$x^2 + 2x + 1 = 0$	$x^2 + x - 2 = 0$
$a = 1$	$a = 1$	$a = 1$
$b = 1$	$b = 2$	$b = 1$
$c = 1$	$c = 1$	$c = -2$
$x = \dfrac{-1 \pm \sqrt{-3}}{2}$	$x = \dfrac{-2 \pm \sqrt{0}}{2}$	$x = \dfrac{-1 \pm \sqrt{9}}{2}$
$x = \{ \}$	$x = \{-1\}$	$x = \{-2,\ 1\}$
No Roots	One Root	Two Roots
$D < 0$	$D = 0$	$D > 0$
D is Negative	D is Zero	D is Positive

Notes

- Sometimes *One Root* is referred to as *a repeated root* or *repeated roots.*
- Geometrically the roots are where the graph of the quadratic crosses the x-axis.
- The theory is often used the other way around. For example, if told that an equation has a repeated root then it can immediately be deduced that,

$$D = 0$$

That is, $\quad b^2 - 4ac = 0$

6.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 46

Question 1

For each of the following equations,

(a) Calculate the value of the discriminant,

(b) State if the equation has 0, 1 or 2 roots.

DO NOT SOLVE THE EQUATIONS !

(i) $x^2 + 4x - 3 = 0$

(ii) $x^2 - 2x + 8 = 0$

(iii) $x^2 + 6x + 9 = 0$

(iv) $(x + 4)(x + 3) + 1 = 0$

(v) $\dfrac{(x - 2)}{(x + 1)} = x + 3$

(vi) $9x^2 + 30x + 25 = 0$

[6 marks]

Question 2

The following equation has no real roots

$$x^2 - 3x - k = 0$$

where k is some fixed real constant.

Show that k must be less than -2.25

[3 marks]

Question 3

The following equation has two distinct real roots;

$$x^2 + kx + 4 = 0$$

where k is some fixed real constant.

(i) Show that $k^2 > 16$

[2 marks]

(ii) Which of the following equations will have two distinct real roots ?

(a) $x^2 + 5x + 4 = 0$

(b) $x^2 + 3x + 4 = 0$

(c) $x^2 - x + 4 = 0$

(d) $x^2 - 7x + 4 = 0$

[2 marks]

(iii) Solve by completing the square;

$$x^2 + 6x + 4 = 0$$

[2 marks]

Question 4
The following equation has a repeated root;

$$x^2 + (2k + 10)x + \left(k^2 + 5\right) = 0$$

where k is some fixed constant.
Determine the value of k.

[3 marks]

Question 5
A-Level Examination Question from January 2005, C1, Q3 (Edexcel)
Given that the equation

$$kx^2 + 12x + k = 0$$

where k is a positive constant, has equal roots, find the value of k.

[4 marks]

Question 6

A-Level Examination Question from May 2006, C1, Q8 (Edexcel)

The equation

$$x^2 + 2px + (3p + 4) = 0$$

where p is a positive constant, has equal roots.

(a) Find the value of p.

[4 marks]

(b) For this value of p, solve the equation

$$x^2 + 2px + (3p + 4) = 0$$

[2 marks]

Question 7

Explain why

$$(x + 4)^2 + 1$$

is always positive, no matter what the value of x.

[2 marks]

Question 8

By completing the square, show that the expression

$$x^2 + 2x + 5$$

is positive for all real values of x.

[4 marks]

Question 9

$$f(x) = x^2 + (k + 5)x + 4k$$

where k is a real constant.

(i) Show that the discriminant can be expressed in the form

$$(k - a)^2 + b$$

where a and b are positive integers.

[3 marks]

(ii) Explain how your part **(i)** answer reveals that $f(x) = 0$ will always have two real roots.

[2 marks]

Question 10
Without solving the equation, how many roots does the following equation have ?

$$6x^2 + 7x - 5 = 0$$

Justify your answer.

[3 marks]

Question 11
Consider the equation

$$(k + 1)x^2 + kx + k + 1 = 0$$

where k is a constant.

This equation has a repeated root.

Determine the possible values of k.

[4 marks]

6.3 Answers

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics In Algebra

6.3.1 Solutions to 6.2 Exercise

Answer 1

(i)	$D = 28$ which is + ve	$\therefore$ 2 distinct real roots
(ii)	$D = -28$ which is – ve	$\therefore$ 0 real roots
(iii)	$D = 0$	$\therefore$ 1 real root
(iv)	$D = -3$ which is – ve	$\therefore$ 0 real roots
(v)	$D = -11$ which is – ve	$\therefore$ 0 real roots
(vi)	$D = 0$	$\therefore$ 1 real root

[6 marks]

Answer 2

$$\begin{aligned}
\text{No real roots implies that } D &< 0 \\
(-3)^2 - 4 \times 1 \times (-k) &< 0 \\
9 + 4k &< 0 \\
4k &< -9 \\
k &< -\frac{9}{4} \\
k &< -2.25
\end{aligned}$$

[3 marks]

Answer 3

(i)

$$\begin{aligned}
\text{Two distinct real roots implies that } D &> 0 \\
k^2 - 4 \times 1 \times 4 &> 0 \\
k^2 - 16 &> 0 \\
k^2 &> 16
\end{aligned}$$

[2 marks]

(ii)

$k = 5 \Rightarrow k^2 = 25$ for $x^2 + 5x + 4 = 0$ so, *y*es, this has 2 real roots

$k = 3 \Rightarrow k^2 = 9$ for $x^2 + 3x + 4 = 0$ so, no, this does not has 2 real roots

$k = -1 \Rightarrow k^2 = 1$ for $x^2 - x + 4 = 0$ so, no, this does not have 2 real roots

$k = -7 \Rightarrow k^2 = 49$ for $x^2 - 7x + 4 = 0$ so, yes, this has 2 roots

[2 marks]

(iii) $x = -3 \pm \sqrt{5}$

(Must be obtained by completing the square)

[2 marks]

Answer 4

A repeated root implies that, $D = 0$

$$(2k + 10)^2 - 4 \times 1 \times \left(k^2 + 5\right) = 0$$

$$4k^2 + 40k + 100 - 4k^2 - 20 = 0$$

$$40k + 80 = 0$$

$$k = -2$$

[3 marks]

Answer 5

A repeated root implies that, $D = 0$

$$144 - 4k^2 = 0$$

$$k^2 = 36$$

$$k = \pm 6$$

$k = 6$ as the question states that k is positive

[4 marks]

Answer 6

(a) $$x^2 + 2px + (3p + 4) = 0$$

Equal roots implies that, $D = 0$

$$(2p)^2 - 4 \times 1 \times (3p + 4) = 0$$

$$4p^2 - 12p - 16 = 0$$

Divide through by 4

$$p^2 - 3p - 4 = 0$$

$$(p - 4)(p - 1) = 0$$

Either $p = 4$ or $p = -1$ (Rejected as p is a positive constant)

$$\therefore \ p = 4$$

[4 marks]

(b) The equation becomes, $x^2 + 8x + 16 = 0$

$$(x + 4)^2 = 0$$

$$x = -4$$

[2 marks]

Answer 7

Obviously, $(x + 4)^2 \geqslant 0$ as any real number when squared is non-negative.

Even when $(x + 4)^2$ has its least value of zero, 1 is added on.

$\therefore \ (x + 4)^2 + 1 > 0$ That is, is always positive. ☐

[2 marks]

Answer 8

$$x^2 + 2x + 5 = \left[x^2 + 2x \right] + 5$$
$$= \left[(x + 1)^2 - 1 \right] + 5$$
$$= (x + 1)^2 + 4$$

As $(x + 1)^2$ is always greater than or equal to zero

it follows that $(x + 1)^2 + 4$ is always positive

$\therefore\ x^2 + 2x + 5$ is always positive ☐

[4 marks]

Answer 9

(i)

$$D = (k + 5)^2 - 4 \times 1 \times 4k$$
$$= k^2 + 10k + 25 - 16k$$
$$= k^2 - 6k + 25$$
$$= (k - 3)^2 - 9 + 25$$
$$= (k - 3)^2 + 16$$

That is $a = 3$ and $b = 16$

[3 marks]

(ii) Obviously, $(k - 3)^2 \geqslant 0$ as any real number squared is non-negative.

Even when $(k - 3)^2$ has its least value of zero, 16 is added on.

$\therefore\ (x - 3)^2 + 16 > 0$ That is, is always positive.

$\therefore\ f(x) = 0$ will always have two distinct real roots. ☐

[2 marks]

Answer 10

The discriminant, $D = 169$.
As this is positive, the equation will have 2 distinct real roots.

[3 marks]

Answer 11

Having two repeated roots implies that, $D = 0$

$$k^2 - 4 \times (k + 1)(k + 1) = 0$$

$$k^2 - 4\left(k^2 + 2k + 1\right) = 0$$

$$k^2 - 4k^2 - 8k - 4 = 0$$

$$-3k^2 - 8k - 4 = 0$$

$$3k^2 + 8k + 4 = 0$$

$$(3k + 2)(k + 2) = 0$$

$$\therefore \; k = -\frac{2}{3} \quad \text{or} \quad k = -2$$

[4 marks]

7.1 Sketching Quadratic Curves

Given a quadratic curve, our various algebraic techniques are useful in quickly getting an approximate idea of what its graph looks like, without going to the time and effort of plotting an accurate graph.

7.2 Example

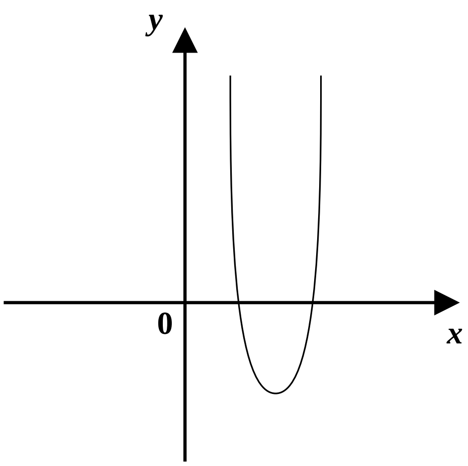
Consider the curve:

$$y = x^2 - 14x + 46$$

(i) Rewrite the equation of the curve in completed square form.

[1 mark]

(ii) Hence add the coordinates of the minimum point to the sketch graph.



[1 mark]

(iii) What is the minimum value of the function $f(x) = x^2 - 14x + 46$?

[1 mark]

(iv) For what value of x does the function $f(x) = x^2 - 14x + 46$ have minimum value ?

[1 mark]

7.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 45

Question 1

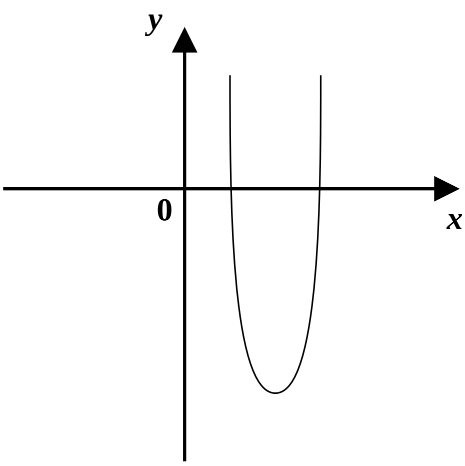
Consider the curve:

$$y = x^2 - 8x + 5$$

(i) Rewrite the equation of the curve in completed square form.

[1 mark]

(ii) Hence add the coordinates of the minimum point to the sketch graph.



[1 mark]

(iii) What is the minimum value of the function $f(x) = x^2 - 8x + 5$?

[1 mark]

Question 2

Given that $f(x) = 3x^2 + 12x + 19$ find the value of the discriminant and explain what it tells you about the graph of $y = f(x)$

[2 marks]

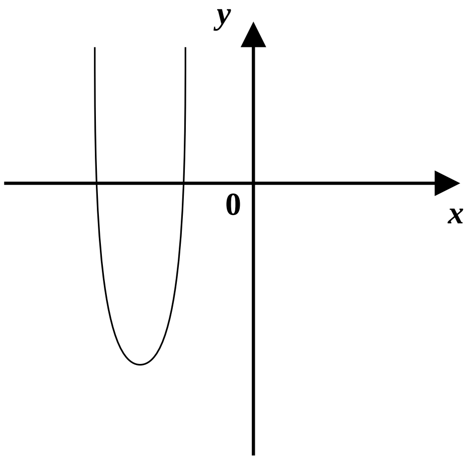
Question 3
Consider the curve:

$$y = x^2 + 10x + 18$$

(i) Rewrite the equation of the curve in completed square form.

[1 mark]

(ii) Hence add the coordinates of the minimum point to the sketch graph.



[1 mark]

(iii) For what value of x does the function $f(x) = x^2 + 10x + 18$ have its minimum value ?

[1 mark]

Question 4
Given that $f(x) = 3x^2 + 12x + 19$ write $f(x)$ in the form $p(x + q)^2 + r$, where p, q and r are integers to be found.

[3 marks]

Question 5
Without sketching a graph, for each function state its minimum value.

(i) $f(x) = (x - 3)^2 + 11$

(ii) $f(z) = \left(z + \frac{1}{4}\right)^2 + \frac{3}{8}$

(iii) $g(x) = 8(x + 7)^2 - 1$

(iv) $f(w) = 4\left(w - \frac{5}{2}\right)^2 - \frac{3}{4}$

[4 marks]

Question 6
Without sketching a graph, for each function state the value of *x* for which the function has minimum value.

(i) $f(x) = (x - 8)^2 + 13$

(ii) $f(x) = \left(x - \frac{2}{5}\right)^2 + \frac{4}{7}$

(iii) $g(x) = 7(x + 2.4)^2 - 3.6$

(iv) $f(x) = \frac{3}{10}\pi\,(x - 8)^2 - 32$

[4 marks]

Question 7

Given that $k > 0$, by considering $\left(\sqrt{k} - 1\right)^2$, or otherwise, use algebra to show that $\dfrac{k + 1}{\sqrt{k}}$ has a least value of 2.

[4 marks]

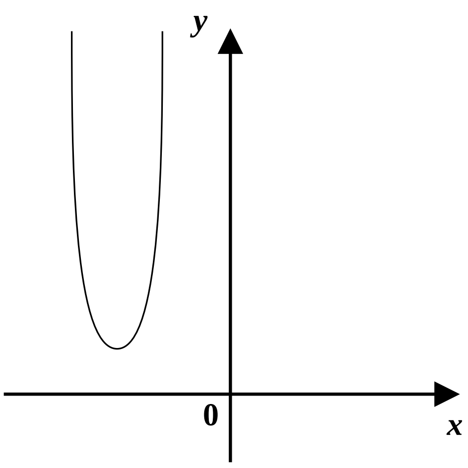
Question 8

Consider the curve, $y = 3x^2 + 30x + 76$

(i) Rewrite the equation of the curve in completed square form.

[3 marks]

(ii) Hence add the coordinates of the minimum point to the sketch graph.



[1 mark]

(iii) What is the minimum value of the function $f(x) = 3x^2 + 30x + 76$?

[1 mark]

Question 9

Show that the quadratic equation $x^2 + (2k + 3)x + k^2 + 3k + 1 = 0$ has two distinct real roots in x, for all values of the constant k.

[4 marks]

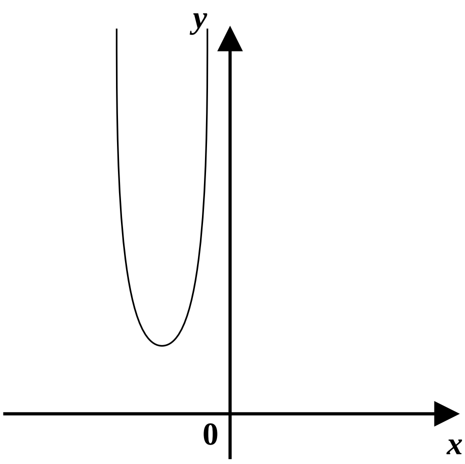
Question 10

Consider the curve, $y = x^2 + x + 1$

(i) Rewrite the equation of the curve in completed square form.
Your answer will have some fractions in it.

[1 mark]

(ii) Hence add the coordinates of the minimum point to the sketch graph.



[1 mark]

(iii) For what value of x does the function $f(x) = x^2 + x + 1$ have its minimum value ?

[1 mark]

Question 11

The cubic curve $y = x^3 - 6x^2 + 12x + B$ can be written in the form

$y = (x - A)^3 - 4$, where A and B are non-zero constants.

Find the value of A and the value of B.

[5 marks]

Question 12
Consider the curve;

$$y = x^2 - 5x - 24$$

In this question we are NOT interested in finding the minimum point.
Instead we wish to know where the graph of this curve crosses the *x*-axis.

(i) Write the equation of the curve in factorised form.

[1 mark]

(ii) Hence state the coordinates of the two points where the graph of the curve crosses the *x*-axis.

[2 marks]

(iii) Without further calculation, sketch the curve.

[2 marks]

7.4 Homework

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics in Algebra

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks available : 50

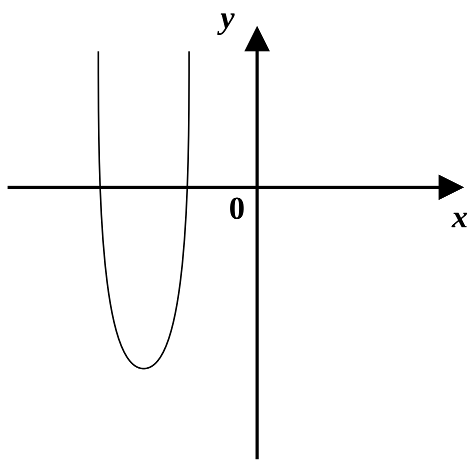
Question 1

Consider the curve, $y = x^2 + 10x + 13$

(i) Rewrite the equation of the curve in completed square form.

[1 mark]

(ii) Hence add the coordinates of the minimum point to the sketch graph.



[1 mark]

(iii) What is the minimum value of the function $f(x) = x^2 + 10x + 13$?

[1 mark]

Question 2

Factorise completely;

(i) $x^2 + 11x + 28$

(ii) $x^2 + x - 30$

(iii) $3x^2 - 14x + 16$

(iv) $2x^2 + x - 36$

[1, 1, 2, 2 marks]

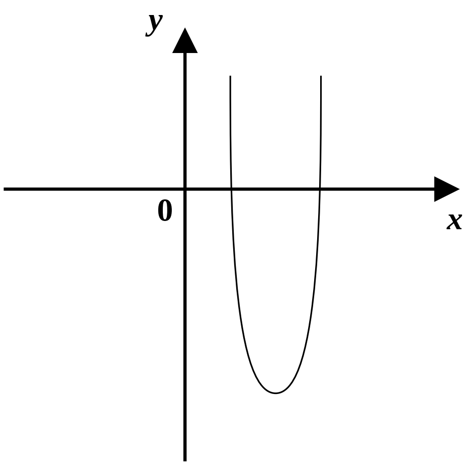
Question 3

Consider the curve, $y = x^2 - 12x - 21$

(i) Rewrite the equation of the curve in completed square form.

[1 mark]

(ii) Hence add the coordinates of the minimum point to the sketch graph.



[1 mark]

(iii) For what value of x does the function $f(x) = x^2 - 12x - 21$ have its minimum value ?

[1 mark]

Question 4

Solve the following equations;

(i) $x^2 + 7x - 8 = 0$ **(ii)** $x^2 - 2x - 35 = 0$

(iii) $3x^2 + 17x + 10 = 0$ **(iv)** $5x^2 + 4x - 33 = 0$

[2, 2, 3, 3 marks]

Question 5

Consider the curve, $y = 4x^2 + 24x + 11$

(i) Rewrite the equation of the curve in completed square form.

[4 marks]

(ii) Hence sketch the graph of the curve with the coordinates of its minimum point clearly marked

[3 marks]

Question 6

Find the value of,

(i) $25^{\frac{3}{2}}$ **(ii)** $27^{\frac{2}{3}}$ **(iii)** $\left(\frac{3}{2}\right)^{-4}$

[1, 1, 1 marks]

Question 7

Write each of the following in the form $a\sqrt{b}$ where a and b are integers and b is also square free;

(i) $\sqrt{45}$ **(ii)** $\frac{6}{\sqrt{2}}$ **(iii)** $\frac{\sqrt{108}}{\sqrt{6}}$

[1, 1, 1 marks]

Question 8

Showing your method, rationalise the denominator of, $\dfrac{5 + \sqrt{3}}{3 - \sqrt{3}}$ and present your final answer in the form $a + b\sqrt{3}$ where a and b are constants to be found

[5 marks]

Question 9

(i) Find the values of a, b and c for which

$$5x^2 - 40x + 81 \equiv a(x + b)^2 + c$$

[4 marks]

(ii) Hence state the minimum value of the function,

$$f(x) = 5x^2 - 40x + 81$$

[1 mark]

Question 10

(i) Find the values of *a*, *b* and *c* for which

$$4x^2 - 12x + 25 \equiv a(x + b)^2 + c$$

[4 marks]

(ii) Hence state the minimum value of the function,

$$f(x) = 4x^2 - 12x + 25$$

[1 mark]

7.5 Answers

7.5.1 Solution to 7.2 Example (Introductory Example)

(i) $y = (x - 7)^2 - 3$

(ii) $(7, -3)$ **(iii)** -3 **(iv)** 7

[4 marks]

7.5.2 Solutions to 7.3 Exercise

Answer 1

(i) $y = (x - 4)^2 - 11$ **(ii)** $(4, -11)$ **(iii)** -11

[3 marks]

Answer 2

$D = 12^2 - 4 \times 3 \times 19 = -84$

As this is negative the graph does not cross the x-axis.

[2 marks]

Answer 3

(i) $y = (x + 5)^2 - 7$ **(ii)** $(-5, -7)$ **(iii)** -5

[3 marks]

Answer 4

$$\begin{aligned} 3x^2 + 12x + 19 &= 3\left[x^2 + 4x \right] + 19 \\ &= 3\left[(x + 2)^2 - 4 \right] + 19 \\ &= 3(x + 2)^2 - 12 + 19 \\ &= 3(x + 2)^2 + 7 \end{aligned}$$

$$\therefore\ p = 3,\ q = 2 \text{ and } r = 7$$

[3 marks]

Answer 5

(i) 11 **(ii)** $\frac{3}{8}$ **(iii)** -1 **(iv)** $-\frac{3}{4}$

[4 marks]

Answer 6

(i) 8 **(ii)** $\frac{2}{5}$ **(iii)** -2.4 **(iv)** 8

[4 marks]

Answer 7

$$\left(\sqrt{k} - 1\right)^2 \geqslant 0$$
$$k - 2\sqrt{k} + 1 \geqslant 0$$
$$k + 1 \geqslant 2\sqrt{k}$$
$$\frac{k+1}{\sqrt{k}} \geqslant 2 \qquad \square$$

[4 marks]

Answer 8

(i) $y = 3(x+5)^2 + 1$ (ii) $(-5,\ 1)$ (iii) 1

[3, 1, 1 marks]

Answer 9

$$x^2 + (2k+3)x + k^2 + 3k + 1 = 0$$
$$D = (2k+3)^2 - 4 \times 1 \times \left(k^2 + 3k + 1\right)$$
$$= 4k^2 + 12k + 9 - 4k^2 - 12k - 4 > 0$$
$$= 5$$

As the discriminant is positive the quadratic equation
has two distinct real roots $\square$

[4 marks]

Answer 10

(i) $y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ **(ii)** $\left(-\frac{1}{2}, \frac{3}{4}\right)$ (iii) $-\frac{1}{2}$

[3 marks]

Answer 11

Knowing the binomial theorem is helpful although the question can be done without.
The coefficient matching technique will be new to some students.

$$(x - A)^3 - 4 = x^3 - 6x^2 + 12x + B$$
$$x^3 - 3x^2A + 3xA^2 - A^3 - 4 = x^3 - 6x^2 + 12x + B$$
$$\therefore\ 3A = 6 \Rightarrow A = 2$$
$$-A^3 - 4 = B \Rightarrow B = -12$$

[5 marks]

Answer 12

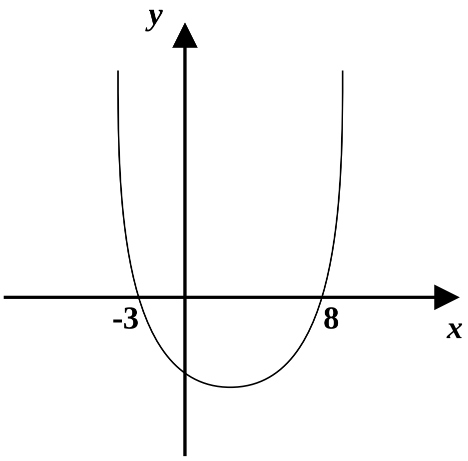
(i) $y = (x + 3)(x - 8)$

[1 mark]

(ii) $(-3,\ 0),\ (8,\ 0)$

[2 marks]

(iii)



[2 marks]

7.5.3 Solutions to 7.4 Homework

Answer 1

(i) $y = (x + 5)^2 - 12$ **(ii)** $(-5, -12)$ **(iii)** -12

[3 marks]

Answer 2

(i) $x^2 + 11x + 28$

$= (x + 7)(x + 4)$

(ii) $x^2 + x - 30$

$= (x + 6)(x - 5)$

(iii) $3x^2 - 14x + 16$

$= (3x - 8)(x - 2)$

(iv) $2x^2 + x - 36$

$= (2x + 9)(x - 4)$

[1, 1, 2, 2 marks]

Answer 3

(i) $y = (x - 6)^2 - 57$ **(ii)** $(6, -57)$ **(iii)** 6

[3 marks]

Answer 4

(i) $x^2 + 7x - 8 = 0$

$(x + 8)(x - 1) = 0$

$x = -8$ or $x = 1$

(ii) $x^2 - 2x - 35 = 0$

$(x + 5)(x - 7) = 0$

$x = -5$ or $x = 7$

(iii) $3x^2 + 17x + 10 = 0$

$(x + 5)(3x + 2) = 0$

$x = -5$ or $x = -\dfrac{2}{3}$

(iv) $5x^2 + 4x - 33 = 0$

$(x + 3)(5x - 11) = 0$

$x = -3$ or $x = \dfrac{11}{5}$

[2, 2, 3, 3 marks]

Answer 5

(i) $y = 4(x + 3)^2 - 25$

[4 marks]

(ii) Parabola with minimum at $(-3, -25)$

[3 marks]

Answer 6

(i) $25^{\frac{3}{2}}$

$= 125$

(ii) $27^{\frac{2}{3}}$

$= 9$

(iii) $\left(\dfrac{3}{2}\right)^{-4}$

$= \dfrac{16}{81}$

[1, 1, 1 marks]

Answer 7

(i) $\sqrt{45}$

$= 3\sqrt{5}$

(ii) $\dfrac{6}{\sqrt{2}}$

$= 3\sqrt{2}$

(iii) $\dfrac{\sqrt{108}}{\sqrt{6}}$

$= 3\sqrt{2}$

[1, 1, 1 marks]

Answer 8

$$\frac{5+\sqrt{3}}{3-\sqrt{3}} = \frac{5+\sqrt{3}}{3-\sqrt{3}} \times \frac{3+\sqrt{3}}{3+\sqrt{3}}$$
$$= \frac{15+8\sqrt{3}+3}{9-3}$$
$$= \frac{18+8\sqrt{3}}{6}$$
$$= \frac{9+4\sqrt{3}}{3}$$
$$= 3+\frac{4}{3}\sqrt{3}$$

[5 marks]

Answer 9

(i) $5x^2 - 40x + 81 \equiv a(x+b)^2 + c$

$$5x^2 - 40x + 81 = 5\left[x^2 - 8x\right] + 81$$
$$= 5\left[(x-4)^2 - 16\right] + 81$$
$$= 5(x-4)^2 - 80 + 81$$
$$= 5(x-4)^2 + 1$$
$$\therefore\ a = 5,\ b = -4 \text{ and } c = 1$$

[4 marks]

(ii) The minimum value will be 1

[1 mark]

Answer 10

(i) $$4x^2 - 12x + 25 \equiv a(x + b)^2 + c$$

$$\begin{aligned} 4x^2 - 12x + 25 &= 4\left[x^2 - 3x\right] + 25 \\ &= 4\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right] + 25 \\ &= 4\left(x - \frac{3}{2}\right)^2 - 9 + 25 \\ &= 4\left(x - \frac{3}{2}\right)^2 + 16 \end{aligned}$$

$$\therefore\ a = 4,\ b = -\frac{3}{2} \text{ and } c = 16$$

[4 marks]

(ii) The minimum value will be 16

[1 mark]

Lesson 8

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics in Algebra

8.1 Inequalities

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 45

"Together" Examples

Solve these inequalities:

(i) $5x + 2 \leqslant 9$ **(ii)** $7 - 3x \geqslant 13$

[1, 2 marks]

(iii) $-2 < 4x + 3 \leqslant 9$ **(iv)** $\dfrac{4}{x} < \dfrac{5}{9}$

[2, 2 marks]

This next question is where the focus falls in the A-Level course.
It's crucial to draw a graph in order to understand properly what is going on.

(v) $x^2 - 2x - 15 < 0$

[3 marks]

8.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 45

Question 1

Solve these inequalities;

(i) $7x - 3 > 25$ **(ii)** $13 - 4x < 11$

[1, 2 marks]

(iii) $3x - 5 > 19 - 5x$ **(iv)** $3(2x + 3) - 4(x - 2) < 11$

[2, 2 marks]

(v) $17 + 7(x - 6) > 45$ **(vi)** $13 < 5x + 3 < 21$

[2, 2 marks]

(vii) $-3 < 5 - 4x \leqslant 25$ **(viii)** $\frac{10}{x} < \frac{5}{12}$

[2, 2 marks]

Question 2

Solve these quadratic inequalities by;

- ☐ First factorising the quadratic into two pairs of brackets.
- ☐ Clearly stating the 'critical values' of the quadratic.
- ☐ Sketching the quadratic.
- ☐ Presenting your final solution with inequality signs clearly displayed.

(i) $x^2 + 9x + 20 \leqslant 0$

(ii) $x^2 + 3x - 18 < 0$

[4, 4 marks]

(iii) $x^2 + 13x + 12 > 0$

(iv) $x^2 - 11x + 24 \geqslant 0$

[4, 4 marks]

Question 3

The discriminant of the quadratic equation $a x^2 + bx + c = 0$ is $D = b^2 - 4ac$

Each of the following equations is to have two distinct real roots. $\therefore \ D > 0$

By considering this inequality, find the range of k for each of the following,

(i) $x^2 + 6x + k = 0$ **(ii)** $k x^2 + 4x + 2 = 0$

[4, 4 marks]

(iii) $x^2 - 2x + k = 0$ **(iv)** $k x^2 - 6x + 3 = 0$

[4, 4 marks]

Question 4

Consider the following equation;

$$kx^2 - 2x + k = 0$$

This equation has two distinct real roots.

(i) By considering the discriminant show that this implies,

$$(k - 1)(k + 1) < 0$$

[4 marks]

(ii) Hence state the range of values of *k*.

[4 marks]

Question 5

(i) By completing the square, write the algebraic expression

$$2x^2 - 28x + 87$$

in the form

$$a(x + b)^2 + c$$

where a, b and c are numbers to be found.

[3 marks]

(ii) Hence show that the equation

$$2x^2 - 28x + 87 = 0$$

has solutions of the form

$$x = c \pm d\sqrt{22}$$

where c and d are numbers to be found.

[3 marks]

8.3.1 Solutions to 8.1 "Together" Examples

(i) $5x + 2 \leqslant 9$

$5x \leqslant 7$

$x \leqslant \dfrac{7}{5}$

(ii) $7 - 3x \geqslant 13$

$-3x \geqslant 6$

$x \leqslant -2$

[1, 2 marks]

(iii) $-2 < 4x + 3 \leqslant 9$

$-5 < 4x \leqslant 6$

$-\dfrac{5}{4} < x \leqslant \dfrac{6}{4}$

(iv) $\dfrac{4}{x} < \dfrac{5}{9}$

$\dfrac{x}{4} > \dfrac{9}{5}$

$x > \dfrac{36}{5}$

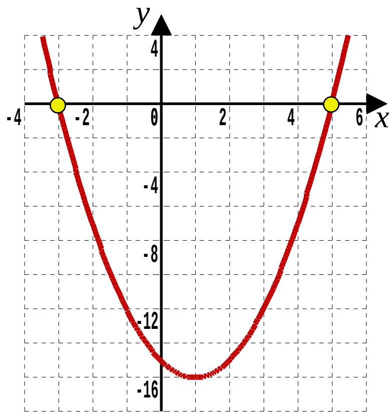
[2, 2 marks]

(v) $x^2 - 2x - 15 < 0$

Finding the critical values...

$(x - 5)(x + 3) = 0$

$x = -3$ or $x = 5$



From the graph, the solution to the inequality $x^2 - 2x - 15 < 0$ is $-3 < x < 5$

EXTRA :

Had the inequality been $x^2 - 2x - 15 > 0$ the solutions, from the same graph, would have been $x < -3$ or $x > 5$

[3 marks]

8.3.2 Solutions to 8.2 Exercise

Answer 1

(i)

$$\begin{aligned} 7x - 3 &> 25 \\ 7x &> 28 \\ x &> 4 \end{aligned}$$

(ii)

$$\begin{aligned} 13 - 4x &< 11 \\ -4x &< -2 \\ x &> 0.5 \end{aligned}$$

[1, 2 marks]

(iii)

$$\begin{aligned} 3x - 5 &> 19 - 5x \\ 8x &> 24 \\ x &> 3 \end{aligned}$$

(iv)

$$\begin{aligned} 3(2x + 3) - 4(x - 2) &< 11 \\ 6x + 9 - 4x + 8 &< 11 \\ 2x &< -6 \\ x &< -3 \end{aligned}$$

[2, 2 marks]

(v)

$$\begin{aligned} 17 + 7(x - 6) &> 45 \\ 17 + 7x - 42 &> 45 \\ 7x &> 70 \\ x &> 10 \end{aligned}$$

(vi)

$$\begin{aligned} 13 < 5x + 3 &< 21 \\ 10 < 5x &< 18 \\ 2 < x &< \frac{18}{5} \end{aligned}$$

[2, 2 marks]

(vii)

$$\begin{aligned} -3 < 5 - 4x &\leqslant 25 \\ -8 < -4x &\leqslant 20 \\ -2 < -x &\leqslant 5 \\ 2 > x &\geqslant -5 \\ -5 \leqslant x &< 2 \end{aligned}$$

(viii)

$$\begin{aligned} \frac{10}{x} &< \frac{5}{12} \\ \frac{x}{10} &> \frac{12}{5} \\ x &> 24 \end{aligned}$$

[2, 2 marks]

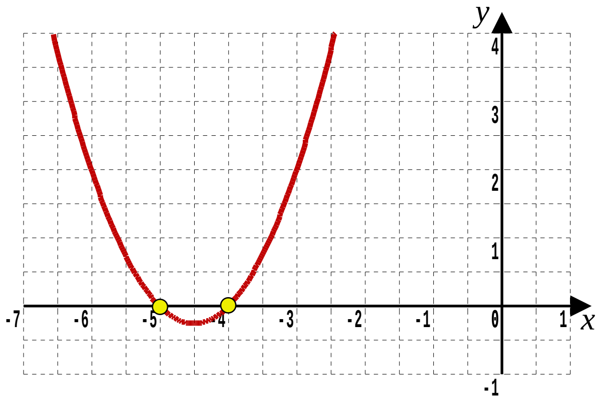
Answer 2

(i) $x^2 + 9x + 20 \leqslant 0$

Finding the critical values...

$(x + 5)(x - 4) = 0$

$x = -5$ or $x = -4$



From the graph, the solution to the inequality $x^2 + 9x + 20 < 0$

is $-5 \leqslant x \leqslant -4$

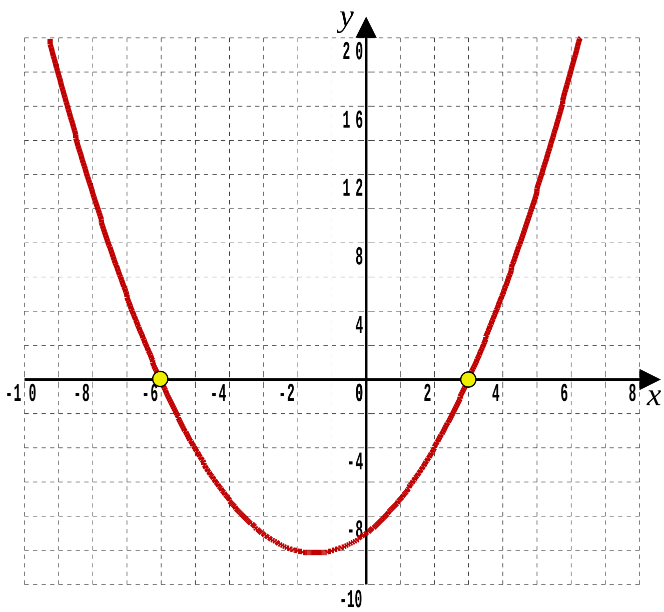
[3 marks]

(ii) $x^2 + 3x - 18 < 0$

Finding the critical values...

$(x + 6)(x - 3) = 0$

$x = -6$ or $x = 3$



From the graph, the solution to the inequality $x^2 + 3x - 18 < 0$

is $-6 < x < 3$

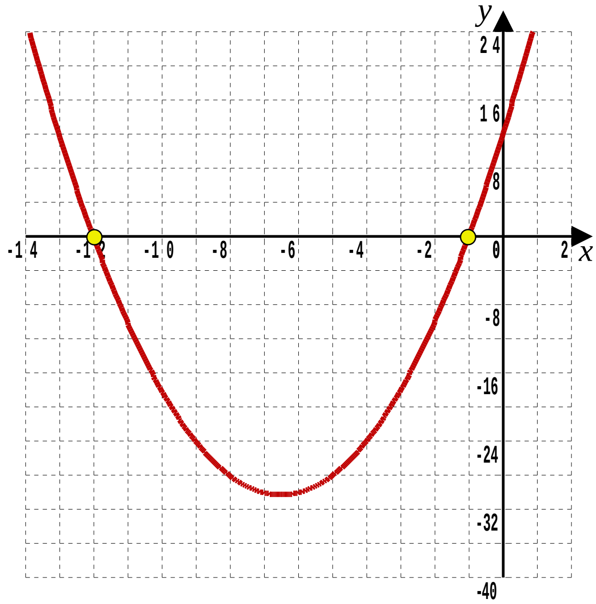
[3 marks]

(iii) $x^2 + 13x + 12 > 0$

Finding the critical values...

$(x + 12)(x + 1) = 0$

$x = -12$ or $x = -1$



From the graph, the solution to the inequality $x^2 + 13x + 12 > 0$

is $x < -12$ or $x > -1$

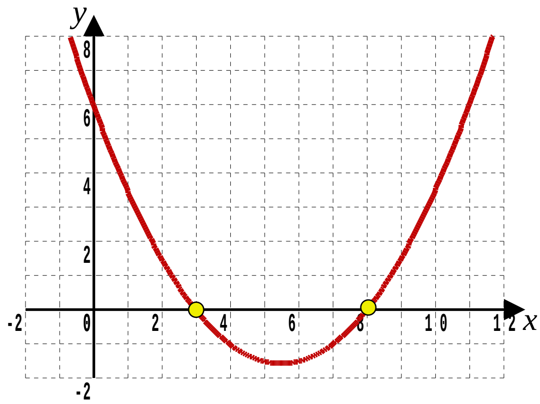
[3 marks]

(iv) $x^2 - 11x + 24 \geqslant 0$

Finding the critical values...

$(x - 3)(x - 8) = 0$

$x = 3$ or $x = 8$



From the graph, the solution to the inequality $x^2 - 11x + 24 \geqslant 0$

is $x \leqslant 3$ or $x \geqslant 8$

[3 marks]

Answer 3

(i) $x^2 + 6x + k = 0$

$$D > 0$$
$$36 - 4 \times 1 \times k > 0$$
$$-4k > -36$$
$$k < 9$$

(ii) $kx^2 + 4x + 2 = 0$

$$D > 0$$
$$16 - 4 \times k \times 2 > 0$$
$$-8k > -16$$
$$k < 2$$

[4, 4 marks]

(iii) $x^2 - 2x + k = 0$

$$D > 0$$
$$4 - 4 \times 1 \times k > 0$$
$$-4k > -4$$
$$k < 1$$

(iv) $kx^2 - 6x + 3 = 0$

$$D > 0$$
$$36 - 4 \times k \times 3 > 0$$
$$-12k > -36$$
$$k < 3$$

[4, 4 marks]

Answer 4

(i) $kx^2 - 2x + k = 0$

Two distinct real roots implies that $D > 0$

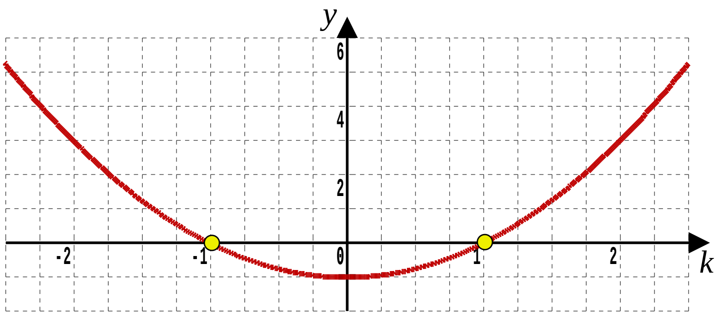
$$\therefore 4 - 4 \times k \times k > 0$$
$$4 - 4k^2 > 0$$
$$4k^2 - 4 < 0$$

divide through by 4

$$k^2 - 1 < 0$$
$$(k - 1)(k + 1) < 0 \qquad \square$$

[4 marks]

(ii) Let $y = (k - 1)(k + 1)$ in which case, for $y < 0$, the critical values are $k = -1$ or $k = 1$ and the graph is,



From the graph, the solution to the inequality $(k - 1)(k + 1) < 0$ is $-1 < k < 1$

[4 marks]

Answer 5

(i)

$$2x^2 - 28x + 87 = 2\left[x^2 - 14x \right] + 87$$
$$= 2\left[(x - 7)^2 - 49 \right] + 87$$
$$= 2(x - 7)^2 - 98 + 87$$
$$= 2(x - 7)^2 - 11$$
$$\therefore\ a = 2,\ b = -7 \text{ and } c = -11$$

[3 marks]

$$2x^2 - 28x + 87 = 0$$
$$\therefore\ 2(x - 7)^2 - 11 = 0 \quad \text{from part (i)}$$
$$2(x - 7)^2 = 11$$
$$(x - 7)^2 = \frac{11}{2}$$

square root both sides

$$x - 7 = \pm\frac{\sqrt{11}}{\sqrt{2}}$$
$$= 7 \pm \frac{\sqrt{11}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= 7 \pm \frac{1}{2}\sqrt{22}$$

That is, $c = 7$ and $d = \frac{1}{2}$

[3 marks]

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics in Algebra

9.1 Simultaneous Equations (One linear, one Quadratic)

Solving simultaneous equations where one equation is linear and the other quadratic often come up in examinations because they test several algebraic manipulation skills all in one question:

- Expanding brackets, FOIL
- Gathering together like terms
- Rearranging equations into the form $f(x) = 0$
- Factorising quadratics
- Solving quadratic equations

9.2 Example : The Question

Solve the simultaneous equations,

$$y = 2x - 3 \qquad \text{(line)}$$

$$x^2 + y^2 = 2 \qquad \text{(circle)}$$

9.3 Example : The Solution

We begin by using *the method of substitution.*

$$x^2 + y^2 = 2$$

$$x^2 + (2x - 3)^2 = 2$$

- Expanding brackets, FOIL

$$x^2 + (2x - 3)(2x - 3) = 2$$

$$x^2 + 4x^2 - 6x - 6x + 9 = 2$$

- Gathering together like terms

$$5x^2 - 12x + 9 = 2$$

- Rearranging equations into the form $f(x) = 0$

$$5x^2 - 12x + 7 = 0$$

- Factorising quadratics

$$(5x - 7)(x - 1) = 0$$

- Solving quadratic equations

$$\text{Either } 5x - 7 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = 1.4 \qquad\qquad x = 1$$

Geometrically, the answers are points where the line intersects the circle.
So complete the solution by using the equation of the line, $y = 2x - 3$, to work out the y values that correspond to x being 1.4, or 1

The points of intersection are $(1.4, -0.2)$ or $(1, -1)$

[5 marks]

9.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

Solve the simultaneous equations

$$y = x - 4$$
$$x^2 + y^2 = 58$$

[5 marks]

Question 2

GCSE Examination Question from June 2007, Paper 3H, Q19 (Edexcel)

Solve the simultaneous equations

$$y = 3x - 1$$
$$x^2 + y^2 = 5$$

[5 marks]

Question 3

A-Level Examination Question from May 2011, C1, Q4 (Edexcel)

Solve the simultaneous equations

$$x + y = 2$$
$$4y^2 - x^2 = 11$$

[5 marks]

Question 4

Solve the simultaneous equations

$$y = x - 7$$

$$x^2 + y^2 = 109$$

[5 marks]

Question 5

A-Level Examination Question from January 2010, C1, Q5 (Edexcel)

Solve the simultaneous equations,

$$y - 3x + 2 = 0$$
$$y^2 - x - 6x^2 = 0$$

[5 marks]

Question 6

Solve the simultaneous equations,

$$y = 2x - 3$$
$$x^2 + y^2 = 18$$

[5 marks]

9.5 Answers

Answer 1
Starting with the equations,

$$y = x - 4$$
$$x^2 + y^2 = 58$$

$$x^2 + (x - 4)^2 = 58$$
$$x^2 + x^2 - 8x + 16 = 58$$
$$2x^2 - 8x - 42 = 0$$

divide through by 2

$$x^2 - 4x - 21 = 0$$
$$(x + 3)(x - 7) = 0$$

Either $x = -3$ or $x = 7$

Using $y = x - 4$ then gives the solutions as,

$(-3, -7)$ or $(7, 3)$

[5 marks]

Answer 2
Starting with the equations,

$$y = 3x - 1$$
$$x^2 + y^2 = 5$$

$$x^2 + (3x - 1)^2 = 5$$
$$x^2 + 9x^2 - 6x + 1 = 5$$
$$10x^2 - 6x - 4 = 0$$

divide through by 2

$$5x^2 - 3x - 2 = 0$$
$$(5x + 2)(x - 1) = 0$$

Either $x = -\frac{2}{5}$ or $x = 1$

Using $y = 3x - 1$ then gives the solutions as,

$\left(-\frac{2}{5}, -\frac{11}{5}\right)$ or $(1, 2)$

[5 marks]

Answer 3

METHOD 1
Starting with the equations,

$$x + y = 2$$
$$4y^2 - x^2 = 11$$

$$4y^2 - \left[(2 - y)^2 \right] = 11$$
$$4y^2 - \left[4 - 4y + y^2 \right] = 11$$
$$4y^2 - 4 + 4y - y^2 = 11$$
$$3y^2 + 4y - 15 = 0$$
$$(3y - 5)(y + 3) = 0$$
$$\text{Either } y = \frac{5}{3} \text{ or } y = -3$$

Using $x = 2 - y$ then gives the solutions as,

$$\left(\frac{1}{3}, \frac{5}{3} \right) \text{ or } (5, -3)$$

[5 marks]

METHOD 2
Starting with the equations,

$$x + y = 2$$
$$4y^2 - x^2 = 11$$

$$4\left[(2 - x)^2 \right] - x^2 = 11$$
$$4\left[4 - 4x + x^2 \right] - x^2 = 11$$
$$16 - 16x + 4x^2 - x^2 = 11$$
$$3x^2 - 16x + 5 = 0$$
$$(3x - 1)(x - 5) = 0$$
$$\text{Either } x = \frac{1}{3} \text{ or } x = 5$$

Using $y = 2 - x$ then gives the solutions as,

$$\left(\frac{1}{3}, \frac{5}{3} \right) \text{ or } (5, -3)$$

[5 marks]

Answer 4

Starting with the equations,

$$y = x - 7$$
$$x^2 + y^2 = 109$$

$$x^2 + (x - 7)^2 = 109$$
$$x^2 + x^2 - 14x + 49 = 109$$
$$2x^2 - 14x - 60 = 0$$

divide through by 2

$$x^2 - 7x - 30 = 0$$
$$(x + 3)(x - 10) = 0$$

Either $x = -3$ or $x = 10$

Using $y = x - 7$ then gives the solutions as,

$(-3, -10)$ or $(10, 3)$

[5 marks]

Answer 5

Starting with the equations,

$$y - 3x + 2 = 0 \Leftrightarrow y = 3x - 2$$
$$y^2 - x - 6x^2 = 0$$

$$(3x - 2)^2 - x - 6x^2 = 0$$
$$9x^2 - 12x + 4 - x - 6x^2 = 0$$
$$3x^2 - 13x + 4 = 0$$
$$(3x - 1)(x - 4) = 0$$

Either $x = \dfrac{1}{3}$ or $x = 4$

Using $y = 3x - 2$ then gives the solutions as,

$\left(\dfrac{1}{3}, -1\right)$ or $(4, 10)$

[5 marks]

Answer 6

Starting with the equations,

$$y = 2x - 3$$

$$x^2 + y^2 = 18$$

$$x^2 + (2x - 3)^2 = 18$$

$$x^2 + 4x^2 - 12x + 9 = 18$$

$$5x^2 - 12x - 9 = 0$$

$$(5x + 3)(x - 3) = 0$$

$$\text{Either } x = -\frac{3}{5} \text{ or } x = 3$$

Using $y = 2x - 3$ then gives the solutions as,

$$\left(-\frac{3}{5}, -\frac{21}{5}\right) \text{ or } (3, 3)$$

[5 marks]

Lesson 10

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics in Algebra

10.1 Past Paper Work

Topics in Algebra has looked at some "standard, routine techniques".
The next exercise is constructed entirely from old examination questions and shows how the material may be formally tested.

10.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 74

Question 1

A-Level Examination Question from January 2012, C1, Q3 (Edexcel)

Find the set of values of x for which

(a) $4x - 5 > 15 - x$

[2 marks]

(b) $x (x - 4) > 12$

[4 marks]

Question 2

A-Level Examination Question from June 2008, C1, Q2 (Edexcel)

Factorise completely,

$$x^3 - 9x$$

[3 marks]

Question 3

A-Level Examination Question from June 2009, C1, Q4 (Edexcel)

Find the set of values of x for which

(a) $4x - 3 > 7 - x$

[2 marks]

(b) $2x^2 - 5x - 12 < 0$

[4 marks]

(c) ***both*** $4x - 3 > 7 - x$ ***and*** $2x^2 - 5x - 12 < 0$

[1 mark]

Question 4

A-Level Examination Question from May 2010, C1, Q3 (Edexcel)

Find the set of values of x for which

(a) $3(x - 2) < 8 - 2x$

[2 marks]

(b) $(2x - 7)(1 + x) < 0$

[3 marks]

(c) ***both*** $3(x - 2) < 8 - 2x$ ***and*** $(2x - 7)(1 + x) < 0$

[1 mark]

Question 5

A-Level Examination Question from May 2010, C1, Q4 (Edexcel)

(a) Show that $x^2 + 6x + 11$ can be written as,

$$(x + p)^2 + q$$

where p and q are integers to be found.

[2 marks]

(b) Sketch the curve with equation $y = x^2 + 6x + 11$ showing clearly any intersections with the coordinate axes.

[2 marks]

(c) Find the value of the discriminant of $x^2 + 6x + 11$

[2 marks]

Question 6

A-Level Examination Question from June 2008, C1, Q8 (Edexcel)

Given that $2q\,x^2 + qx - 1 = 0$, where q is a constant, has no real roots,

(a) Show that $q^2 + 8q < 0$

[2 marks]

(b) Hence find the set of possible values of q

[3 marks]

Question 7

A-Level Examination question from May 2011, C1, Q7 (Edexcel)

$$f\,(x) = x^2 + (\,k + 3\,)\,x + k \qquad \text{where } k \text{ is a real constant}$$

(a) Find the discriminant of $f\,(x)$ in terms of k

[2 marks]

(b) Show that the discriminant of $f\,(x)$ can be expressed in the form $(\,k + a\,)^2 + b$ where a and b are integers to be found.

[2 marks]

(c) Show that, for all values of k, the equation $f\,(x) = 0$ has real roots.

[2 marks]

Question 8

A-Level Examination Question from January 2009, C1, Q7 (Edexcel)

The equation $kx^2 + 4x + (5 - k) = 0$, where k is a constant, has 2 different real solutions for x.

(a) Show that k satisfies, $k^2 - 5k + 4 > 0$

[3 marks]

(b) Hence find the set of possible values of k

[4 marks]

Question 9

A-Level Examination question from January 2007, C1, Q4 (Edexcel)

Solve the simultaneous equations

$$y = x - 2$$
$$y^2 + x^2 = 10$$

[7 marks]

Question 10

A-Level Examination Question from January 2008, C1, Q8 (Edexcel)

The equation $x^2 + kx + 8 = k$ has no real solutions for x

(a) Show that k satisfies $k^2 + 4k - 32 < 0$

[3 marks]

(b) Hence find the set of possible values of k.

[4 marks]

Question 11

A-Level Examination Question from January 2007, C1, Q5 (Edexcel)

The equation $2x^2 - 3x - (k + 1) = 0$, where k is a constant, has no real roots.

Find the set of possible values of k.

[4 marks]

Question 12

A-Level Examination Question from January 2010, C1, Q10 (Edexcel)

$$f(x) = x^2 + 4kx + (3 + 11k) \quad \text{where } k \text{ is a constant.}$$

(a) Express $f(x)$ in the form $(x + p)^2 + q$, where p and q are constants to be found in terms of k

[3 marks]

Given that the equation $f(x) = 0$ has no real roots,

(b) Find the set of possible values of k.

[4 marks]

Given that $k = 1$,

(c) Sketch the graph of $y = f(x)$, showing the coordinates of any point at which the graph crosses a coordinate axis.

[3 marks]

10.3 Answers

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics in Algebra

Answer 1

(a)

$$4x - 5 > 15 - x$$
$$5x > 20$$
$$x > 4$$

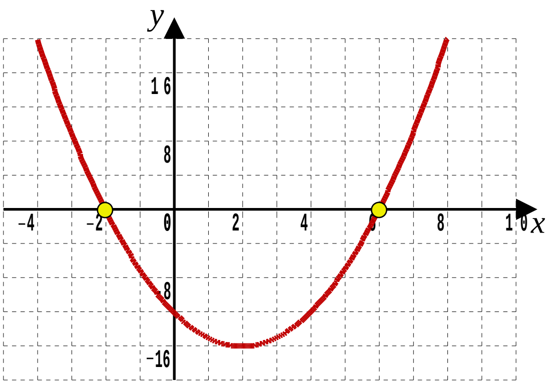
[2 marks]

(b)

$$x\ (x - 4) > 12$$
$$x^2 - 4x - 12 > 0$$

Finding the critical values...

$$(x + 2)(x - 6) = 0$$
$$x = -2 \text{ or } x = 6$$



From the graph, the solution to the inequality $x\ (x - 4) > 12$

is $x < -2$ or $x > 6$

[4 marks]

Answer 2

$$x^3 - 9x = x\left(x^2 - 3^2\right)$$
$$= x(x - 3)(x + 3)$$

[3 marks]

Answer 3

(a)

$$4x - 3 > 7 - x$$

$$5x > 10$$

$$x > 2$$

[2 marks]

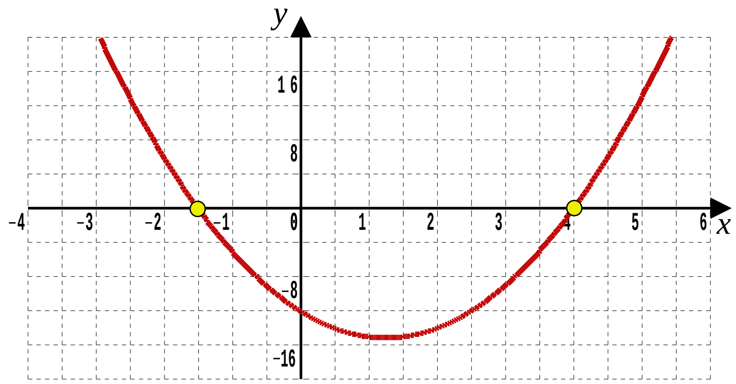
(b)

$$2x^2 - 5x - 12 < 0$$

Finding the critical values...

$$(2x + 3)(x - 4) = 0$$

$$x = -\frac{3}{2} \text{ or } x = 4$$



From the graph, the solution to the inequality $2x^2 - 5x - 12 < 0$ is $-\frac{3}{2} < x < 4$

[4 marks]

(c) $2 < x < 4$

[1 mark]

Answer 4

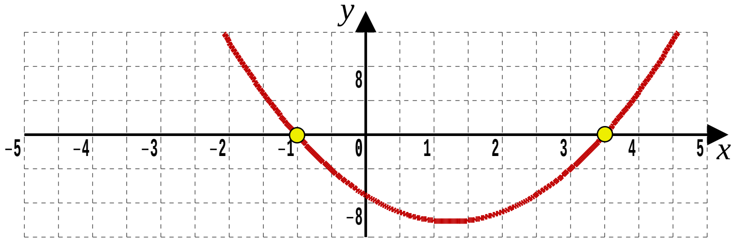
(a)

$$3(x-2) < 8-2x$$
$$3x-6 < 8-2x$$
$$5x < 14$$
$$x < \frac{14}{5}$$

[2 marks]

(b) Finding the critical values...

$$(2x-7)(1+x) = 0$$
$$x = -1 \text{ or } x = \frac{7}{2}$$



From the graph, the solution to the inequality $(2x-7)(1+x) < 0$ is $-1 < x < \frac{7}{2}$

[4 marks]

(c) $-1 < x < \frac{14}{5}$

[1 mark]

Answer 5

(a)

$$x^2 + 6x + 11 = \left[x^2 + 6x \right] + 11$$
$$= \left[(x + 3)^2 - 9 \right] + 11$$
$$= (x + 3)^2 + 2$$

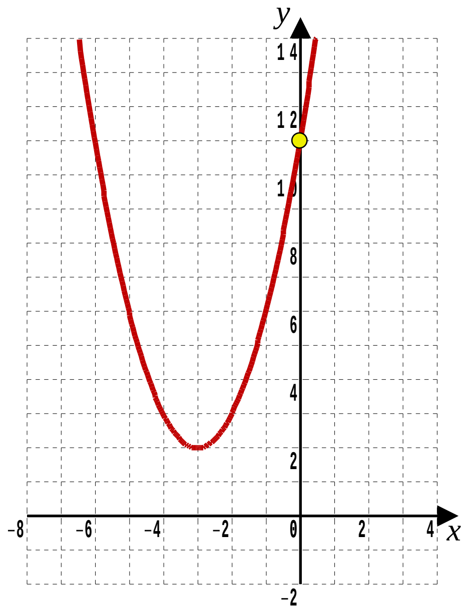
That is, $p = 3$ and $q = 2$

[2 marks]

(b) From part (a) the parabola has a minimum turning point at $(-3,\ 2)$

Anticipating part (c), notice that the discriminant is negative, confirming there are not any x-axis crossing points.

The only axis crossing point is $(0,\ 11)$



[2 marks]

(c)

$$D = b^2 - 4ac$$
$$= 6^2 - 4 \times 1 \times 11$$
$$= -8$$

[2 marks]

Answer 6

$$2qx^2 + qx - 1 = 0$$

(a) For this to have no real roots, the discriminant must be negative.

$$D < 0$$

$$q^2 - 4 \times 2q \times (-1) < 0$$

$$q^2 + 8q < 0$$

[2 marks]

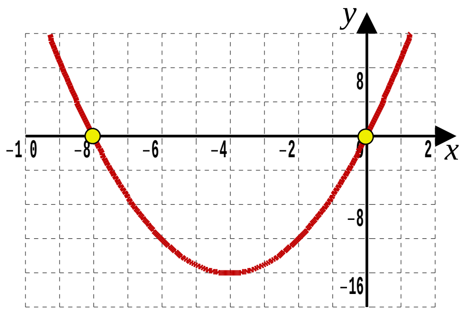
(b) Finding the critical values...

$$q^2 + 8q = 0$$

$$q(q + 8) = 0$$

$$\text{Either } q = 0 \text{ or } q = -8$$

Let $y = q^2 + 8q$.



From the graph, the solution to the inequality $q^2 + 8q < 0$

is $-8 < q < 0$

[3 marks]

Answer 7

$$f(x) = x^2 + (k + 3)x + k$$

(a) $D = (k + 3)^2 - 4 \times 1 \times k$

$= k^2 + 6k + 9 - 4k$

$= k^2 + 2k + 9$

[2 marks]

(b) $D = (k + 1)^2 + 8 \qquad \therefore \; a = 1$ and $b = 8$

[2 marks]

(c) Obviously $(k + 1)^2 \geqslant 0$ as any real number when squared is non-negative.

Even when $(k + 1)^2$ has its least value of zero, 8 is added on.

$\therefore \; (k + 1)^2 + 8 > 0$ That is, the discriminant is always positive.

Thus $f(x)$ always has two distinct real roots $\qquad \square$

[2 marks]

Answer 8

$$k x^2 + 4x + (5 - k) = 0$$

(a) To have 2 different real solutions the discriminant must be positive.

$$D > 0$$

$$4^2 - 4k(5 - k) > 0$$

$$16 - 20k + 4k^2 > 0$$

$$4k^2 - 20k + 16 > 0$$

divide through by 4

$$k^2 - 5k + 4 > 0 \qquad \square$$

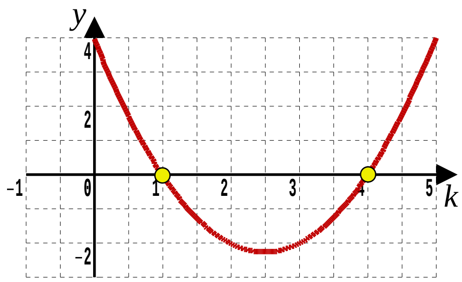
[3 marks]

(b) Finding the critical values...

$$(k - 1)(k - 4) = 0$$

Either $k = 1$ or $k = 4$

Let $y = k^2 - 5k + 4$



From the graph, the solution to the inequality $k^2 - 5k + 4 > 0$

is $k < 1$ or $k > 4$

[4 marks]

Answer 9

Starting with the equations,

$$y = x - 2$$

$$y^2 + x^2 = 10$$

$$(x - 2)^2 + x^2 = 10$$

$$x^2 - 4x + 4 + x^2 = 10$$

$$2x^2 - 4x - 6 = 0$$

divide through by 2

$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

Either $x = -1$ or $x = 3$

Using $y = x - 2$ then gives the solutions as,

$(-1, -3)$ or $(3, 1)$

[7 marks]

Answer 10

$$x^2 + kx + 8 = k$$

$$x^2 + kx + (8 - k) = 0$$

(a) To have no real solutions the discriminant must be negative.

$$D < 0$$

$$k^2 - 4 \times 1 \times (8 - k) < 0$$

$$k^2 + 4k - 32 < 0 \qquad \square$$

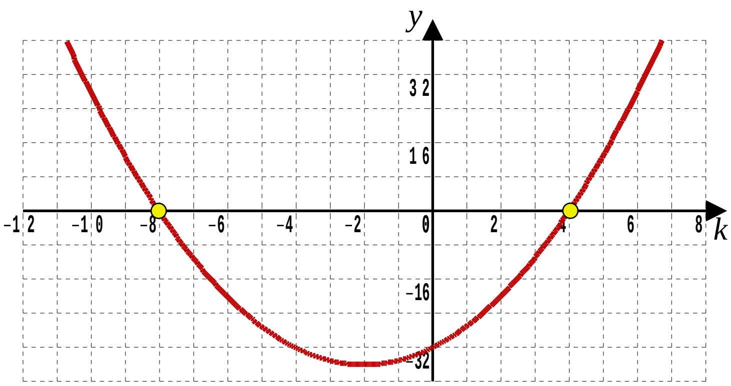
[3 marks]

(b) Finding the critical values...

$$(k + 8)(k - 4) = 0$$

$$\text{Either } k = -8 \text{ or } k = 4$$

Let $y = k^2 + 4k - 32$



From the graph, the solution to the inequality $k^2 + 4k - 32 < 0$

is $-8 < k < 4$

[4 marks]

Answer 11

$$2x^2 - 3x - (k + 1) = 0$$

To have no real roots the discriminant must be negative.

$$D < 0$$

$$(-3)^2 - 4 \times 2 \times [-(k + 1)] < 0$$

$$9 + 8k + 8 < 0$$

$$8k + 17 < 0$$

$$k < -\frac{17}{8}$$

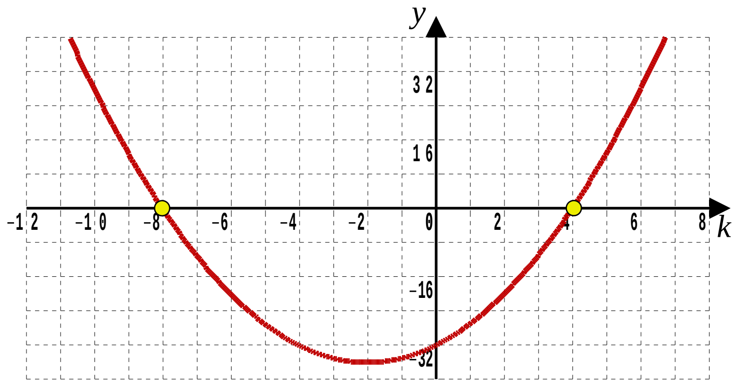
[3 marks]

(b) Finding the critical values...

$$(k + 8)(k - 4) = 0$$

$$\text{Either } k = -8 \text{ or } k = 4$$

Let $y = k^2 + 4k - 32$



From the graph, the solution to the inequality $k^2 + 4k - 32 < 0$

is $-8 < k < 4$

[4 marks]

Answer 12

(a)

$$x^2 + 4kx + (3 + 11k) = \left[x^2 + 4kx\right] + (3 + 11k)$$

$$= \left[(x + 2k)^2 - 4k^2\right] + (3 + 11k)$$

$$= (x + 2k)^2 + \left(3 + 11k - 4k^2\right)$$

$$\therefore\ p = 2k \text{ and } q = 3 + 11k - 4k^2$$

[3 marks]

(b) To have no real roots the discriminant must be negative.

$$D < 0$$

$$(4k)^2 - 4 \times 1 \times (3 + 11k) < 0$$

$$16k^2 - 44k - 12 < 0$$

divide through by 4

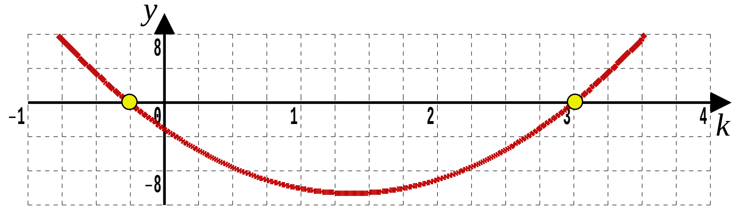
$$4k^2 - 11k - 3 < 0$$

Finding the critical values....

$$(4k + 1)(k - 3) = 0$$

$$\text{Either } k = -\frac{1}{4} \text{ or } k = 3$$

Let $y = 4k^2 - 11k - 3$



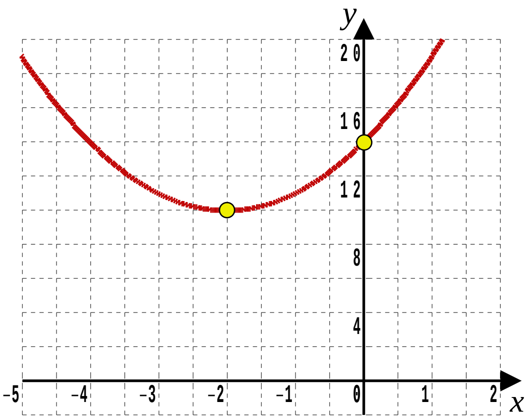
From the graph, the solution to the inequality $4k^2 - 11k - 3 < 0$

is $-\frac{1}{4} < k < 3$

[4 marks]

(c) With $k = 1$, $f(x) = x^2 + 4x + 14 = (x + 2)^2 + 10$

Now let $y = f(x)$



Notice the minimum at $(-2,\ 10)$ any the y-axis intercept at $(0,\ 14)$

[3 marks]

Lesson 11

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics in Algebra

11.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 60

Question 1

GCSE Examination Question from June 2021, Paper 1H, Q7 (Edexcel)

(a) Write down all possible values of y if

$$-4 \leqslant 2y < 6 \text{ and } y \text{ is an integer}$$

[2 marks]

(b) Solve the inequality, $7t - 3 \leqslant 2t + 31$

[2 marks]

Question 2

GCSE Examination Question from June 2021, Paper 1H, Q9 (Edexcel)

(a) Factorise fully, $25a^4c^7d + 45a^9c^3h$

[2 marks]

(b) Solve $(2x + 5)^2 = (2x + 3)(2x - 1)$

[3 marks]

Question 3

A-Level Examination question from May 2014, Paper C1, Q3 (Edexcel)

Find the set of values of x for which

(a) $3x - 7 > 3 - x$

[2 marks]

(b) $x^2 - 9x \leqslant 36$

[3 marks]

(c) both $3x - 7 > 3 - x$ and $x^2 - 9x \leqslant 36$

[1 mark]

Question 4

Use the method of completing the square to solve the equation $x^2 + 4 = 6x$

Give your solutions in surd form.

[2 marks]

Question 5

A-Level Examination Question from June 2019, Paper PM1, Q5(a)(b) (Edexcel)

$$f(x) = 2x^2 + 4x + 9$$

(a) Write $f(x)$ in the form $a(x + b)^2 + c$

where a, b and c are integers to be found.

[3 marks]

(b) Sketch the curve with equation $y = f(x)$

Show any points of intersection with the coordinate axes and the coordinates of any turning point.

[3 marks]

Question 6

GCSE Examination Question from January 2020, Paper 2HR, Q23, (Edexcel)

Express $7 - 12x - 2x^2$ in the form $a + b(x + c)^2$ where a, b and c are integers.

[3 marks]

Question 7

A-Level Examination Question from January 2011, Paper C1, Q8 (Edexcel)

The equation

$$x^2 + (k - 3)x + (3 - 2k) = 0$$

where k is a constant, has two distinct real roots.

(a) Show that k satisfies $k^2 + 2k - 3 > 0$

[3 marks]

(b) Find the set of possible values of k

[4 marks]

Question 8

A-Level Examination Question from May 2007, Paper C1, Q6 (Edexcel)

(a) By eliminating y from the equations

$$y = x - 4$$

$$2x^2 - xy = 8$$

show that

$$x^2 + 4x - 8 = 0$$

[2 marks]

(b) Hence, or otherwise, solve the simultaneous equations

$$y = x - 4$$

$$2x^2 - xy = 8$$

giving your answers in the form $a \pm b\sqrt{3}$, where a and b are integers.

[5 marks]

Question 9

Additional Mathematics Examination Question from June 2019, Paper 1, Q10 (OCR)

You are given that the line $y = 2x + k$ cuts the circle $x^2 + y^2 = 5$ in two points, *A* and *B*.

(a) Show that the *x* coordinates of *A* and *B* satisfy the equation

$$5x^2 + 4kx + \left(k^2 - 5\right) = 0$$

[2 marks]

(b) Hence find the values of *k* for which the line is a tangent to the circle.

[2 marks]

Question 10

A-Level Examination Question from May 2002, Paper C1, Q4 (Edexcel)

(a) By completing the square, find in terms of *k* the roots of the equation

$$x^2 + 2kx - 7 = 0$$

[4 marks]

(b) Prove that, for all real values of *k*, the roots of $x^2 + 2kx - 7 = 0$ are real and different.

[2 marks]

(c) Given that $k = \sqrt{2}$ find the exact roots of the equation.

[2 marks]

Question 11

The width of a rectangular sports pitch is x metres, $x > 0$.
The length of the pitch is 20 metres more than its width.
The perimeter of the pitch must be less than 300 metres.

(a) Form a linear inequality in x

[2 marks]

Given that the area of the pitch must be greater than 4800 m^2;

(b) Form a quadratic inequality in x

[2 marks]

(c) By solving your inequalities, find the set of possible values of x

[4 marks]

11.2 Answers

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics in Algebra

Answer 1

(a) $-4 \leqslant 2y < 6$

divide through by 2

$-2 \leqslant y < 3$

$\therefore y \in \{-2, -1, 0, 1, 2\}$

[2 marks]

(b) $7t - 3 \leqslant 2t + 31$

$5t \leqslant 34$

$t \leqslant \dfrac{34}{5}$

[2 marks]

Answer 2

(a) $25a^4c^7d + 45a^9c^3h = 5a^4c^3\left(5c^4d + 9a^5h\right)$

[2 marks]

(b)

$$\begin{aligned}(2x+5)^2 &= (2x+3)(2x-1)\\ 4x^2 + 20x + 25 &= 4x^2 + 4x - 3\\ 16x &= -28\\ x &= -\frac{28}{16}\\ &= -\frac{7}{4}\end{aligned}$$

[3 marks]

Answer 3

(a)

$$3x - 7 > 3 - x$$

$$4x > 10$$

$$x > \frac{5}{2}$$

[2 marks]

(b)

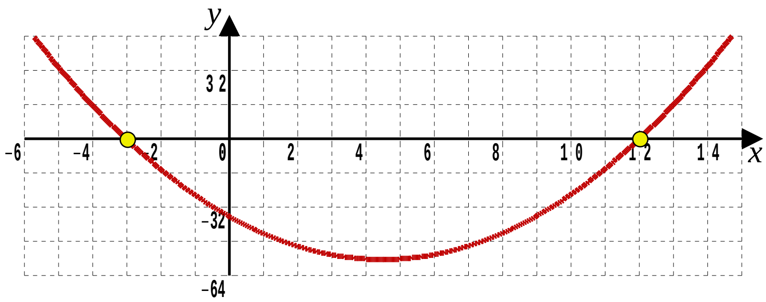
$$x^2 - 9x \leqslant 36$$

$$x^2 - 9x - 36 \leqslant 0$$

Finding the critical values...

$$(x + 3)(x - 12) = 0$$

$$x = -3 \text{ or } x = 12$$



From the graph, the solution to the inequality $x^2 - 9x \leqslant 36$ is $-3 \leqslant x \leqslant 12$

[3 marks]

(c) $\frac{5}{2} < x \leqslant 12$

[1 mark]

Answer 4

$$x^2 + 4 = 6x$$
$$\left[x^2 - 6x \right] + 4 = 0$$
$$\left[(x - 3)^2 - 9 \right] + 4 = 0$$
$$(x - 3)^2 = 5$$
$$x - 3 = \pm\sqrt{5}$$
$$x = 3 \pm \sqrt{5}$$

[2 marks]

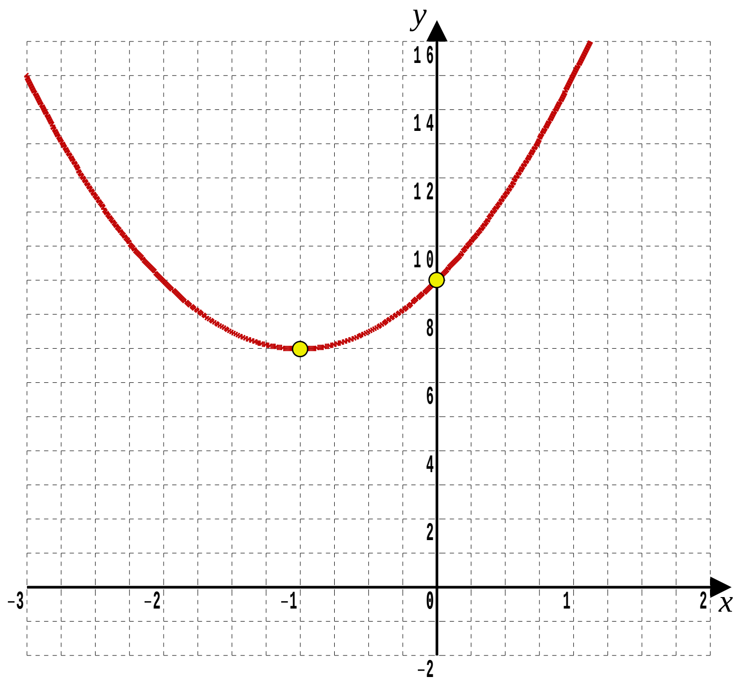
Answer 5

(a)

$$f(x) = 2x^2 + 4x + 9$$
$$= 2\left[x^2 + 2x \right] + 9$$
$$= 2\left[(x + 1)^2 - 1 \right] + 9$$
$$= 2(x + 1)^2 - 2 + 9$$
$$= 2(x + 1)^2 + 7$$
$$\therefore \quad a = 2, \; b = 1 \text{ and } c = 7$$

[3 marks]

(b)



The only point of coordinate axes intersection is at $(0, \; 9)$

There is a minimum turning point at $(-1, \; 7)$

[3 marks]

Answer 6

$$7 - 12x - 2x^2 = (-2)\left[x^2 + 6x\right] + 7$$
$$= (-2)\left[(x + 3)^2 - 9\right] + 7$$
$$= (-2)(x + 3)^2 + 18 + 7$$
$$= 25 + (-2)(x + 3)^2$$
$$\therefore\ a = 25,\ b = -2 \text{ and } c = 3$$

[3 marks]

Answer 7

$$x^2 + (k - 3)x + (3 - 2k) = 0$$

(a) To have 2 distinct real roots the discriminant must be positive.

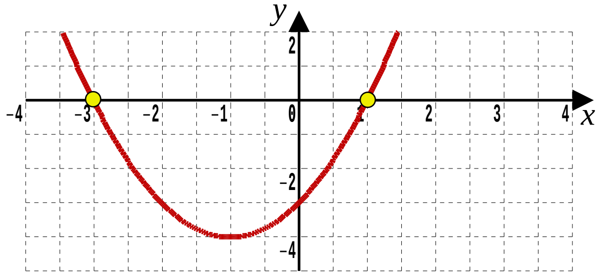
$$D > 0$$
$$(k - 3)^2 - 4 \times 1 \times (3 - 2k) > 0$$
$$k^2 - 6k + 9 - 12 + 8k > 0$$
$$k^2 + 2k - 3 > 0 \qquad \square$$

[3 marks]

(b) Finding the critical values...

$$(k + 3)(k - 1) = 0$$
$$\text{Either } k = -3 \text{ or } k = 1$$

Let $y = k^2 + 2k - 3$



From the graph, the solution to the inequality $k^2 + 2k - 3 > 0$

is $k < -3$ or $k > 1$

[4 marks]

Answer 8

(a) Starting with the equations,

$$y = x - 4$$
$$2x^2 - xy = 8$$

$$2x^2 - x(x - 4) = 8$$
$$2x^2 - x^2 + 4x - 8 = 0$$
$$x^2 + 4x - 8 = 0 \qquad \square$$

[2 marks]

(b)

$$\left[x^2 + 4x \right] - 8 = 0$$
$$\left[(x + 2)^2 - 4 \right] - 8 = 0$$
$$(x + 2)^2 = 12$$
$$x + 2 = \pm 2\sqrt{3}$$
$$x = -2 \pm 2\sqrt{3}$$
$$\therefore a = -2 \text{ and } b = 2$$

Using $y = x - 4$ then gives the solutions as,

$$\left(-2 - 2\sqrt{3}, -6 - 2\sqrt{3}\right) \text{ or } \left(-2 + 2\sqrt{3}, -6 + 2\sqrt{3}\right)$$

[5 marks]

Answer 9

(a) Starting with the equations,

$$y = 2x + k$$
$$x^2 + y^2 = 5$$

$$x^2 + (2x + k)^2 = 5$$
$$x^2 + 4x^2 + 4kx + k^2 = 5$$
$$5x^2 + 4kx + \left(k^2 - 5\right) = 0 \qquad \square$$

[2 marks]

(b) A tangent requires that there be one distinct root.
This in turn requires that the discriminant equal zero.

$$(4k)^2 - 4 \times 5 \times \left(k^2 - 5\right) = 0$$
$$16k^2 - 20k^2 + 100 = 0$$
$$4k^2 = 100$$
$$k = \pm 5$$

[2 marks]

Answer 10

(a)

$$x^2 + 2kx - 7 = 0$$

$$\left[x^2 + 2kx \right] - 7 = 0$$

$$\left[(x + k)^2 - k^2 \right] - 7 = 0$$

$$(x + k)^2 = 7 + k^2$$

$$x + k = \pm \sqrt{7 + k^2}$$

$$x = -k \pm \sqrt{7 + k^2}$$

[4 marks]

(b) The discriminant of $x^2 + 2kx - 7 = 0$ is,

$$(2k)^2 - 4 \times 1 \times (-7) = 4k^2 + 28$$

For all real values of k, $k^2 \geqslant 0$

Even when k^2 has its minimum value of 0, the discriminant, $4k^2 + 28$, is positive and so the roots of $x^2 + 2kx - 7$ are always real and different. □

[2 marks]

(c) From part (a) with $k = \sqrt{2}$, $x = -\sqrt{2} \pm \sqrt{7 + (\sqrt{2})^2}$

$\therefore$ The exact roots are $x = \pm 3 - \sqrt{2}$

[2 marks]

Answer 11

(a)

$$2x + 2(x + 20) < 300$$
$$4x + 40 < 300$$
$$4x < 260$$
$$x < 65$$

[2 marks]

(b)

$$x(x + 20) > 4800$$
$$x^2 + 20x - 4800 > 0$$

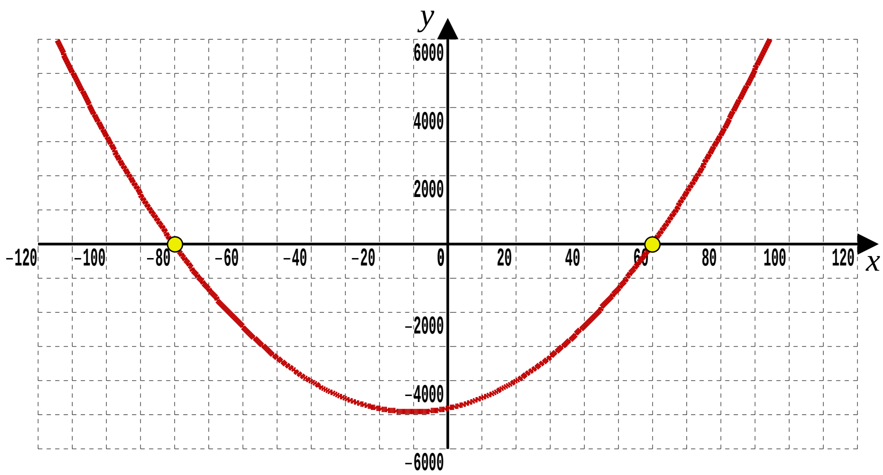
[2 marks]

(c) Finding the critical values for the inequality of part (b)...

$$(x + 80)(x - 60) = 0$$

Either $x = -80$ or $x = 60$

Let $y = x^2 + 20x - 4800$



From the graph, the solution to the inequality $x^2 + 20x - 4800 > 0$

is $x < -80$ or $x > 60$

In combination with part (a), and the fact that x has to be positive, the set of possible values for x is thus, $60 < x < 65$

[4 marks]