14.2 Answers

Answer 1

(i) $y = e^x \quad \Rightarrow \quad \frac{dy}{dx} = e^x$

(ii) $y = x^e \quad \Rightarrow \quad \frac{dy}{dx} = e\,x^{e-1}$

(iii) $y = e^e \quad \Rightarrow \quad \frac{dy}{dx} = 0$

[6 marks]

Answer 2

(i) $y = \ln x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$

(ii) $y = \ln x^2$

$= 2\ln x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2}{x}$

(iii) $y = \ln\left(\frac{1}{x}\right)$

$= \ln x^{-1}$

$= -\ln x \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x}$

[6 marks]

Answer 3

$$y = e^{\frac{1}{2}x}$$

$$\frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x}$$

The curve intercepts the y-axis when $x = 0$

so $(0,\ 1)$ is on the tangent which has a gradient of $\frac{1}{2}$

$$\therefore y = \frac{1}{2}x + 1$$

$$\Leftrightarrow 2y - x - 2 = 0$$

[6 marks]

Answer 4

$$y = 5\sqrt{x} - \frac{1}{2}\ln x$$

$$= 5x^{\frac{1}{2}} - \frac{1}{2}\ln x$$

$$\frac{dy}{dx} = \frac{5}{2}x^{-\frac{1}{2}} - \frac{1}{2} \times \frac{1}{x}$$

$$= \frac{5}{2\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}} - \frac{1}{2x}$$

$$= \frac{5\sqrt{x} - 1}{2x}$$

To find where the gradient is 3

$$\frac{5\sqrt{x} - 1}{2x} = 3$$

$$5\sqrt{x} - 1 = 6x$$

$$6x - 5\sqrt{x} + 1 = 0$$

Let $z = \sqrt{x} \Rightarrow z^2 = x$ and the equation becomes,

$$6z^2 - 5z + 1 = 0$$

$$(3z - 1)(2z - 1) = 0$$

$$\therefore \text{ Either } \sqrt{x} = \frac{1}{3} \text{ or } \sqrt{x} = \frac{1}{2}$$

$$\text{Either } x = \frac{1}{9} \text{ or } x = \frac{1}{4}$$

$$\text{For } x = \frac{1}{9},\ \ y = 5\sqrt{\frac{1}{9}} - \frac{1}{2}\ln\left(\frac{1}{9}\right)$$

$$= \frac{5}{3} - \frac{1}{2}\ln 3^{-2}$$

$$= \frac{5}{3} + \ln 3$$

$$\text{For } x = \frac{1}{4},\ \ y = 5\sqrt{\frac{1}{4}} - \frac{1}{2}\ln\left(\frac{1}{4}\right)$$

$$= \frac{5}{2} - \frac{1}{2}\ln 2^{-2}$$

$$= \frac{5}{2} + \ln 2$$

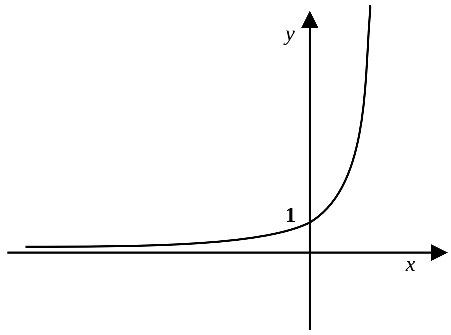
The points, correct to three significant figures, are $(0.111,\ 2.77)$, $(0.250,\ 3.19)$

[10 marks]

Answer 5

(i) $f(x) = e^{2x} - e^{x} + 1 \Rightarrow f'(x) = 2e^{2x} - e^{x}$

(ii) The graph of $y = e^{x}$ has the x-axis as an asymptote so the curve never has an intersection with the x-axis. $\therefore\ e^{x} = 0$ has no solution. $\square$



(iii)

$$2e^{2x} - e^{x} = 0$$

$$e^{x}(2e^{x} - 1) = 0$$

Either $e^{x} = 0$ which has no solution

Or $2e^{x} - 1 = 0$

$$e^{x} = \frac{1}{2}$$

$$x = ln\left(\frac{1}{2}\right)$$

(iv)

$$y = e^{2x} - e^{x} + 1$$

When $x = ln\left(\frac{1}{2}\right)$, $y = e^{2\,ln\left(\frac{1}{2}\right)} - e^{ln\left(\frac{1}{2}\right)} + 1$

$$= e^{ln\left(\frac{1}{2}\right)^{2}} - e^{ln\left(\frac{1}{2}\right)} + 1$$

$$= \frac{1}{4} - \frac{1}{2} + 1$$

$= \frac{3}{4}$ That is, $a = \frac{1}{2}$ and $b = \frac{3}{4}$

(v) $f''(x) = 4e^{2x} - e^{x} \Rightarrow f''\left(ln\left(\frac{1}{2}\right)\right) = 4e^{2ln\left(\frac{1}{2}\right)} - e^{ln\left(\frac{1}{2}\right)} = 4 \times \frac{1}{4} - \frac{1}{2}$

so $f''(x) = \frac{1}{2}$ at the turning point

(vi) As $f''(x)$ is positive the turning point is a minimum

[12 marks]