7.5 Answers

7.5.1 Solutions to 7.3 Examples (Teaching Video)

(i) $y = 5\,ln\left(x^2 + e^x\right)$ (Chain Rule Example)

$$\frac{dy}{dx} = 5 \times \frac{2x + e^x}{x^2 + e^x}$$

[3 marks]

(ii) $y = x^3\,ln\,x^2$ (Product Rule Example)

$$= 2\,x^3\,ln\,x$$

$$\frac{dy}{dx} = 2\left(x^3 \times \frac{1}{x} + 3\,x^2\,ln\,x\right)$$

$$= 2\left(x^2 + 3\,x^2\,ln\,x\right)$$

$$= 2\,x^2\,(\,1 + 3\,ln\,x\,)$$

[3 marks]

(iii) $y = \frac{ln\,x}{2x}$ (Quotient Rule Example)

$$= \frac{1}{2}\left(\frac{ln\,x}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{x \times \frac{1}{x} - 1 \times ln\,x}{x^2}$$

$$= \frac{1}{2} \times \frac{1 - ln\,x}{x^2}$$

$$= \frac{1 - ln\,x}{2\,x^2}$$

[3 marks]

7.5.2 Solutions to 7.4 Exercise

Answer 1

(i) $y = 3\, ln\, x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{3}{x}$

(ii) $y = 4\, ln\, x^3$

$= 12\, ln\, x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{12}{x}$

(iii) $y = ln(5x)$

$= ln\, 5 + ln\, x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{x}$

(iv) $y = 6\, ln(7x)$

$= 6\, ln\, 7 + 6\, ln\, x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{6}{x}$

(v) $y = ln\left(4x^3 + 3\right) \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{12\, x^2}{4x^3 + 3}$

(vi) $y = (ln\, x)^3 \quad \Rightarrow \quad \dfrac{dy}{dx} = 3(ln\, x)^2 \times \dfrac{1}{x}$

$= \dfrac{3\,(ln\, x)^2}{x}$

(vii) $y = (ln\, x)^{-4} \quad \Rightarrow \quad \dfrac{dy}{dx} = -4\,(ln\, x)^{-5} \times \dfrac{1}{x}$

$= -\dfrac{4}{x\,(ln\, x)^5}$

(viii) $y = \dfrac{4}{ln\, x}$

$= 4\,(ln\, x)^{-1} \quad \Rightarrow \quad \dfrac{dy}{dx} = 4 \times (-1) \times (ln\, x)^{-2} \times \dfrac{1}{x}$

$= -\dfrac{4}{x\,(ln\, x)^2}$

[16 marks]

Answer 2

$$y = x\, ln\, x^7$$
$$= 7\, x\, ln\, x$$
$$\frac{dy}{dx} = 7\left(x \times \frac{1}{x} + 1 \times ln\, x \right)$$
$$= 7\,(1 + ln\, x)$$

[3 marks]

Answer 3

$$y = \frac{x}{ln\, x}$$
$$\frac{dy}{dx} = \frac{ln\, x \times 1 - \frac{1}{x} \times x}{(ln\, x)^2}$$
$$= \frac{ln\, x - 1}{(ln\, x)^2}$$

[3 marks]

Answer 4

$$y = \frac{ln\, x^3}{x^3}$$
$$= 3 \times \frac{ln\, x}{x^3}$$
$$\frac{dy}{dx} = 3\left(\frac{x^3 \times \frac{1}{x} - 3x^2\, ln\, x}{x^6} \right)$$
$$= \frac{3\, x^2}{x^6}\left(1 - 3\, ln\, x \right)$$
$$= \frac{3\,(1 - 3\, ln\, x)}{x^4}$$

[3 marks]

Answer 5

(i)

$$f(x) = x(\ln(x) - 1)$$

$$f'(x) = x \times \frac{1}{x} + 1 \times (\ln(x) - 1)$$

$$= 1 + \ln(x) - 1$$

$$= \ln x$$

Asked to find where $f'(x) = 1$

$$\ln x = 1$$

$$x = e^1$$

$$= e \qquad \text{(About 2.72)}$$

When $x = e,\ y = e(\ln e - 1)$

$$= 0$$

The gradient is 1 at the point $A(e,\ 0)$ where the graph crosses the x-axis.

[3 marks]

(ii) For a turning point,

$$f'(x) = 0$$

$$\ln x = 0$$

$$x = e^0$$

$$= 1$$

When $x = 1,\ y = 1 \times (\ln(1) - 1)$

$$= -1$$

$\therefore$ Turning Point $B(1, -1)$

$$f'(x) = \ln x \Rightarrow f''(x) = \frac{1}{x} \quad \therefore f''(1) = 1$$

As $f''(1)$ is positive the turning point is a minimum.

[3 marks]

(iii) The tangent from the turning point will have equation $y = -1$

For the tangent from A, $y = mx + c$

$$0 = 1 \times e + c \quad \therefore c = -e$$

$$y = x - e$$

Intersection of tangents is where $x - e = -1$

$$x = e - 1$$

$\therefore$ Point of intersection is $(e - 1, -1)$

[3 marks]

Answer 6

(i)

$$g(x) = \frac{1}{4}x^2\left(2\,ln(x) - 1\right)$$

$$g'(x) = \frac{1}{4}\left(x^2\left(\frac{2}{x}\right) + 2x\left(2ln(x) - 1\right)\right)$$

$$= \frac{1}{4}\left(2x + 2x\left(2\,ln(x) - 1\right)\right)$$

$$= \frac{2x}{4}\left(1 + 2\,ln(x) - 1\right)$$

$$= \frac{x}{2}\left(2\,ln(x)\right)$$

$$= x\,ln\,x$$

$$\therefore g'(e) = e\,ln\,e$$

$$= e$$

[4 marks]

(ii) For a turning point,

$$f'(x) = 0$$

$$x\,ln\,x = 0$$

Either $x = 0$ (Rejected as $x > 0$)

or $ln\,x = 0$

$$x = e^0$$

$$= 1$$

$$g(1) = \frac{1}{4} \times 1^2 \times \left(2\,ln\,1 - 1\right)$$

$$= -\frac{1}{4}$$

$\therefore$ Turning Point $\left(1, -\frac{1}{4}\right)$

$$f'(x) = x\,ln\,x \Rightarrow f''(x) = x \times \frac{1}{x} + 1 \times ln\,x$$

$$\therefore f''(1) = 1$$

As $f''(1)$ is positive the turning point is a minimum.

[3 marks]

Answer 7

$$y = e^x \ln x$$

$$\frac{dy}{dx} = e^x \times \frac{1}{x} + e^x \ln x$$

$$= e^x \left(\frac{1}{x} + \ln x \right)$$

When $x = 1$, $y = 0$ and $\frac{dy}{dx} = e$

Tangent : $y = mx + c$

$$0 = e \times 1 + c \Rightarrow c = -e$$

The tangent has the equation, $y = e(x - 1)$

[3 marks]

Answer 8

(i)

$$y = \frac{\ln x}{x^2}$$

$$\frac{dy}{dx} = \frac{x^2 \times \frac{1}{x} - 2x \ln x}{x^4}$$

$$= \frac{x - 2x \ln x}{x^4}$$

$$= \frac{x(1 - 2\ln x)}{x^4}$$

$$= \frac{1 - 2\ln x}{x^3}$$

[3 marks]

(ii) When $x = 1$, $y = 0$, $\frac{dy}{dx} = 1$

$$y = mx + c$$

$$0 = 1 + c \Rightarrow c = -1$$

Equation of the tangent is $y = x - 1$

[3 marks]