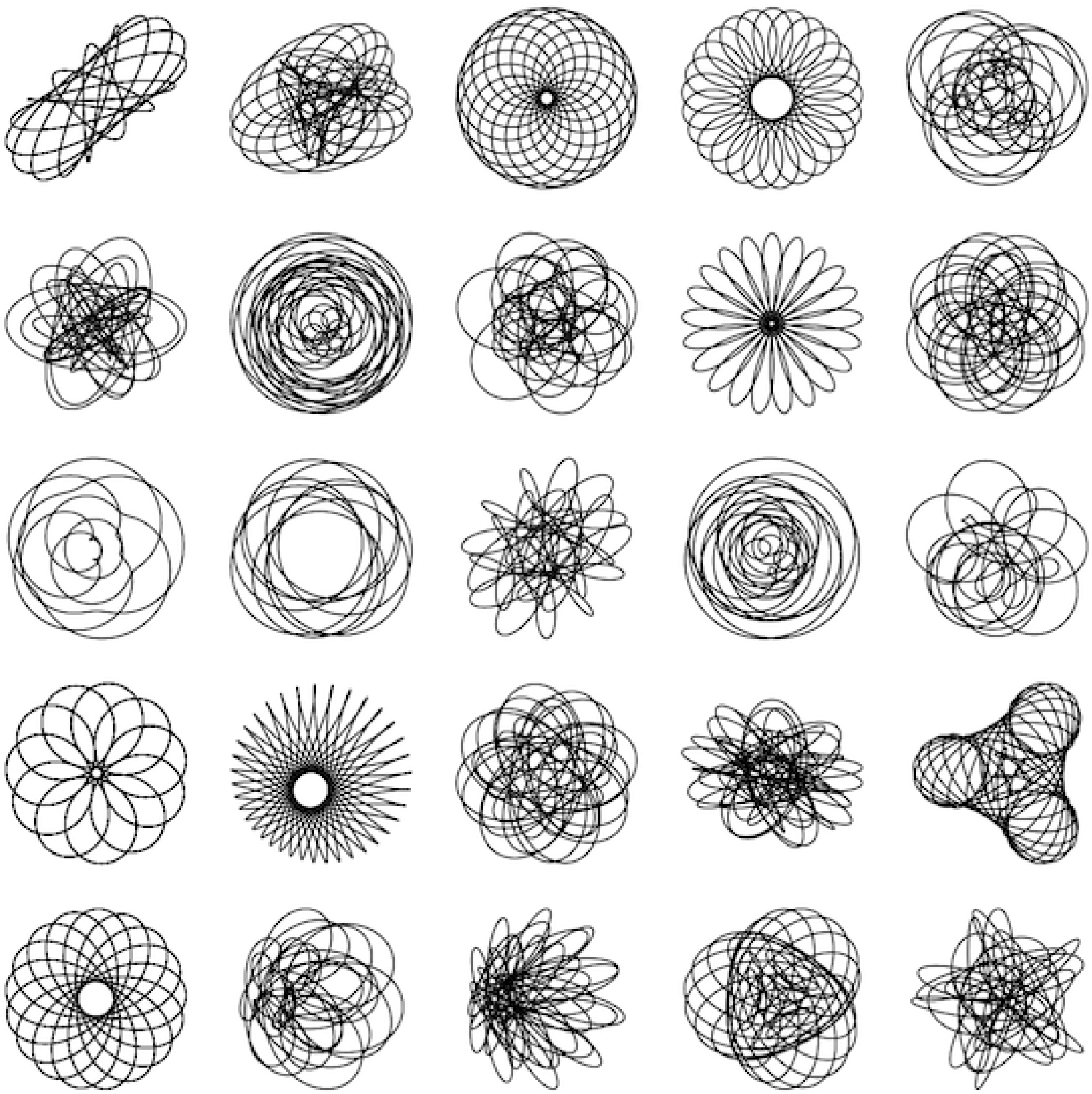
A-Level Pure Mathematics
Year 2

DIFFERENTIATION III



Chain Rule • Product Rule • Quotient Rule

1.1 The Chain Rule

The Chain Rule is a rule about differentiation that has similarities with the familiar Differentiation Power Rule for differentiating x raised to the power n.

The Differentiation Power Rule

If $y = x^n$ then $\dfrac{dy}{dx} = n\,x^{n-1}$ where x is a variable and n is a constant.

1.2 Example

Find the derivative of $y = (4x + 5)^2$ by,

(i) first expanding the brackets, then using "The Differentiation of x^n Rule"

(ii) using the "The Chain Rule"

Answer (i)

$$y = (4x + 5)^2$$

$$y = (4x + 5)(4x + 5)$$

$$y = 16x^2 + 40x + 25$$

$$\frac{dy}{dx} = 32x + 40$$

$$\frac{dy}{dx} = 8(4x + 5)$$

Answer (ii)

Teaching Video : http://www.NumberWonder.co.uk/v9028/1.mp4

After watching the video write out the chain rule method of solution here

☞

[2 marks]

1.3 Three For You To Do

Here is the formal statement of The Chain Rule that was given in the video,

---

The Chain Rule for $\boldsymbol{y = [\, f(x) \,]^n}$

$$\text{If } y = [\, f(x) \,]^n \text{ then } \frac{dy}{dx} = n\,[\, f(x) \,]^{n-1} f'(x)$$

Now try to apply The Chain Rule to these three further examples. The answers are on the following page so you can immediately check to see if you are correct.

Try 1 $\quad y = (4x + 5)^{20}$

[2 marks]

Try 2 $\quad y = (8x + 1)^5$

[2 marks]

Try 3 $\quad y = \left(4x^3 + 8\right)^4$

[2 marks]

Answer to Try 1

$$y = (4x + 5)^{20}$$
$$\frac{dy}{dx} = 20(4x + 5)^{19} \times 4$$
$$\frac{dy}{dx} = 80(4x + 5)^{19}$$

Answer to Try 2

$$y = (8x + 1)^{5}$$
$$\frac{dy}{dx} = 5(8x + 1)^{4} \times 8$$
$$\frac{dy}{dx} = 40(8x + 1)^{4}$$

Answer to Try 3

$$y = \left(4x^3 + 8\right)^4$$
$$\frac{dy}{dx} = 4\left(4x^3 + 8\right)^3 \times \left(12x^2\right)$$
$$\frac{dy}{dx} = 48x^2\left(4x^3 + 8\right)^3$$

1.4 Exercise

Marks Available : 35

Question 1

Differentiate $y = \left(5x^2 + 7\right)^3$

[2 marks]

Question 2

Differentiate $y = 5\left(11 - 6x^2\right)^5$

[2 marks]

Question 3

Differentiate $y = \sqrt{9 - 5x}$

[2 marks]

Question 4

Differentiate the following function,

$$f(x) = 4(9 + 14x)^{\frac{3}{2}}$$

[2 marks]

Question 5

Differentiate the following function,

$$f(x) = \frac{5}{(7x + 8)^3}$$

[3 marks]

Question 6

Find $\dfrac{dy}{dx}$ when $y = \dfrac{5}{3\,(7 - 2x)^5}$

[3 marks]

Question 7

Differentiate the following function;

$$y = 6 + \frac{1}{(3x^2 + 2)}$$

[3 marks]

Question 8

Consider the curve,

$$y = (2x - 3)^4$$

Find the equation of the tangent to the curve at the point (2, 1)

Give your answer in the form $y = mx + c$

[3 marks]

Question 9

(a) A curve has the following equation,

$$y = (x + 3)^3 - 4(x + 3)$$

Find the coordinates of points on the curve with gradient 8.

[3 marks]

(b) Repeat part (a) for the curve with the following equation,

$$y = x^3 - 4x$$

[3 marks]

(c) How are your part (a) and part (b) answers related ?

[1 mark]

Question 10

(a) Find the equation of the tangent to the curve,

$$y = \frac{2}{x^2 - 3}$$

at the point $\left(3, \ \frac{1}{3}\right)$

Give your answer in the form $ax + by + c = 0$

[4 marks]

(b) Find the normal to the curve at the same point.

Again, give your answer in the form $ax + by + c = 0$.

[4 marks]

1.5 Answers

1.5.1 Solution to 1.2 Example (Teaching Video)

$$y = (4x + 5)^2$$

$$\frac{dy}{dx} = 2(4x + 5)^1 \times 4$$

$$\frac{dy}{dx} = 8(4x + 5)$$

1.5.2 Solutions to 1.4 Exercise)

Answer 1

$$\frac{dy}{dx} = 30x\left(5x^2 + 7\right)^2$$

[2 marks]

Answer 2

$$\frac{dy}{dx} = -300x\left(11 - 6x^2\right)^4$$

[2 marks]

Answer 3

$$\frac{dy}{dx} = -\frac{5}{2}(9 - 5x)^{-\frac{1}{2}}$$

or

$$\frac{dy}{dx} = -\frac{5}{2\sqrt{9 - 5x}}$$

[2 marks]

Answer 4

$$f'(x) = 84(9 + 14x)^{\frac{1}{2}}$$

or

$$f'(x) = 84\sqrt{9 + 14x}$$

[2 marks]

Answer 5

$$f'(x) = -105(7x + 8)^{-4}$$

or

$$f'(x) = -\frac{105}{(7x + 8)^4}$$

[3 marks]

Answer 6

$$\frac{dy}{dx} = \frac{50}{3}(7 - 2x)^{-6}$$

or

$$\frac{dy}{dx} = \frac{50}{3(7 - 2x)^6}$$

[3 marks]

Answer 7

$$\frac{dy}{dx} = -\frac{6x}{(3x^2 + 2)^2}$$

[3 marks]

Answer 8

$$\frac{dy}{dx} = 8(2x - 3)^3$$

When $x = 2$, this gives,

$$\frac{dy}{dx} = 8$$

Thus,

$$y = 8x - 15$$

[3 marks]

Answer 9

(a) $$\frac{dy}{dx} = 3(x + 3)^2 - 4$$

So, need to solve, $\frac{dy}{dx} = 8$

Which gives the two solutions, $(-5, 0)$, $(-1, 0)$

[3 marks]

(b) $(-2, 0)$, $(2, 0)$

[3 marks]

(c) The first pair of answers is the second pair translated by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

[1 mark]

Answer 10

(a) $x + 3y - 4 = 0$

[4 marks]

(b) $9x - 3y - 26 = 0$

[4 marks]

2.1 The Product Rule

Given two functions, $u(x)$ and $v(x)$ that are multiplying each other, The Product Rule gives a method of obtaining the derivative of their product. It states that,

$$\big(\, u(x)\; v(x)\,\big)' \;=\; u(x)\; v'(x) \;+\; u'(x)\; v(x)$$

All of the x in brackets are considered to be unnecessary clutter and so the rule is more usually written in the following succinct and elegant form,

---

The Product Rule

If $f = uv$ then $f' = u\,v' + u'\,v$

2.2 Example

Differentiate the product of x^5 with x^3 by immediately applying The Product Rule.

Teaching Video : http://www.NumberWonder.co.uk/v9028/2.mp4

Watch the video and then write out the solution here

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[2 marks]

2.3 Exercise

Marks Available : 34

Question 1

(i) Show how to use The Product Rule to differentiate,

$$y = x^9 \times x^{11}$$

[2 marks]

(ii) Use algebra to simplify first, then differentiate,

$$y = x^9 \times x^{11}$$

[2 marks]

Question 2

(i) Show how to use The Product Rule to differentiate,

$$y = 3x^5 \times 4x^7$$

[2 marks]

(ii) Use algebra to simplify first, then differentiate,

$$y = 3x^5 \times 4x^7$$

[2 marks]

Question 3

(i) Show how to use The Product Rule to differentiate,

$$y = x \times x$$

[2 marks]

(ii) Use algebra to simplify first, then differentiate,

$$y = x \times x$$

[2 marks]

Question 4

(i) Show how to use The Product Rule to differentiate,

$$y = 8x^{\frac{3}{2}} \times 6x^{\frac{5}{2}}$$

[3 marks]

(ii) Use algebra to simplify first, then differentiate,

$$y = 8x^{\frac{3}{2}} \times 6x^{\frac{5}{2}}$$

[3 marks]

Question 5

(i) Show how to use The Product Rule to differentiate,

$$y = x^{-3} \times x^{8}$$

[2 marks]

(ii) Use algebra to simplify first, then differentiate,

$$y = x^{-3} \times x^{8}$$

[2 marks]

Question 6

(i) Show how to use The Product Rule to differentiate,

$$y = \left(x^{2} - 1 \right)\left(x^{2} + 1 \right)$$

[3 marks]

(ii) Use algebra to simplify first, then differentiate,

$$y = \left(x^{2} - 1 \right)\left(x^{2} + 1 \right)$$

[3 marks]

Question 7

Use the product rule to show that;

$$y = x^4 \left(3x^2 + 1 \right)$$

has a first derivative given by,

$$\frac{dy}{dx} = 2x^3 \left(9x^2 + 2 \right)$$

and a second derivative given by,

$$\frac{d^2y}{dx^2} = 6x^2 \left(15x^2 + 2 \right)$$

before determining the third derivative.

[6 marks]

2.4 Answers

2.4.1 Solution to 2.2 Example (Teaching Video)

$$y = x^5 \, x^3$$

$$\frac{dy}{dx} = x^5 \, 3x^2 + 5x^4 \, x^3$$

$$\frac{dy}{dx} = 3x^7 + 5x^7$$

$$\frac{dy}{dx} = 8x^7$$

[2 marks]

The sensible way to do this problem is to simplify first then differentiate. This allows the answer via The Chain Rule to be checked.

$$y = x^5 x^3$$

$$y = x^8$$

$$\frac{dy}{dx} = 8x^7$$

The first method allows practice of the product rule before having to use it on questions that can't be done using the second method.

2.4.2 Solutions to 2.3 Exercise

Answer 1

(i) Using the Product Rule;

$$y = x^9 \times x^{11}$$

$$\frac{dy}{dx} = x^9 \times 11x^{10} + 9x^8 \times x^{11}$$

$$\frac{dy}{dx} = 11x^{19} + 9x^{19}$$

$$\frac{dy}{dx} = 20x^{19}$$

[2 marks]

(ii) Using algebra to simplify first;

$$y = x^9 \times x^{11}$$

$$y = x^{20}$$

$$\frac{dy}{dx} = 20x^{19}$$

[2 marks]

Answer 2

(i) Using the product rule;

$$y = 3x^5 \times 4x^7$$
$$\frac{dy}{dx} = 3x^5 \times 28x^6 + 15x^4 \times 4x^7$$
$$\frac{dy}{dx} = 84x^{11} + 60x^{11}$$
$$\frac{dy}{dx} = 144x^{11}$$

[2 marks]

(ii) Using algebra to simplify first,

$$y = 3x^5 \times 4x^7$$
$$y = 12x^{12}$$
$$\frac{dy}{dx} = 144x^{11}$$

[2 marks]

Answer 3

(i) Using the product rule;

$$y = x \times x$$
$$\frac{dy}{dx} = x \times 1 + 1 \times x$$
$$\frac{dy}{dx} = 2x$$

[2 marks]

(ii) Using algebra to simplify first;

$$y = x \times x$$
$$y = x^2$$
$$\frac{dy}{dx} = 2x$$

[2 marks]

Answer 4

(i) Using the product rule;

$$y = 8x^{\frac{3}{2}} \times 6x^{\frac{5}{2}}$$
$$\frac{dy}{dx} = 8x^{\frac{3}{2}} \times 15x^{\frac{3}{2}} + 12x^{\frac{1}{2}} \times 6x^{\frac{5}{2}}$$
$$\frac{dy}{dx} = 120x^3 + 72x^3$$
$$\frac{dy}{dx} = 192x^3$$

[3 marks]

(ii) Using algebra to simplify first;

$$y = 8x^{\frac{3}{2}} \times 6x^{\frac{5}{2}}$$

$$y = 48x^4$$

$$\frac{dy}{dx} = 192x^3$$

[3 marks]

Answer 5

(i) Using the product rule,

$$y = x^{-3} \times x^8$$

$$\frac{dy}{dx} = x^{-3} \times 8x^7 - 3x^{-4} \times x^8$$

$$\frac{dy}{dx} = 8x^4 - 3x^4$$

$$\frac{dy}{dx} = 5x^4$$

[2 marks]

(ii) Using algebra to simplify first,

$$y = x^{-3}\ x^8$$

$$y = x^5$$

$$\frac{dy}{dx} = 5x^4$$

[2 marks]

Answer 6

(i) Using the product rule,

$$y = \left(x^2 - 1\right)\left(x^2 + 1\right)$$

$$\frac{dy}{dx} = \left(x^2 - 1\right) 2x + 2x\left(x^2 + 1\right)$$

$$\frac{dy}{dx} = 2x^3 - 2x + 2x^3 + 2x$$

$$\frac{dy}{dx} = 4x^3$$

[3 marks]

(ii) Using algebra to simplify first,

$$y = \left(x^2 - 1\right)\left(x^2 + 1\right)$$

$$y = x^4 - 1$$ Difference of two squares

$$\frac{dy}{dx} = 4x^3$$

[3 marks]

Answer 7

Using the product rule to show that;

$$y = x^4 (3x^2 + 1)$$

has a derivative given by;

$$\frac{dy}{dx} = 2x^3 (9x^2 + 2)$$

and a second derivative given by;

$$\frac{d^2y}{dx^2} = 6x^2 (15x^2 + 2)$$

before determining the third derivative.

$$\frac{d^3y}{dx^3} = 24x (15x^2 + 1) \qquad \text{or} \qquad \frac{d^3y}{dx^3} = 360x^3 + 24x$$

[6 marks]

3.1 Product In, Product Out (PIPO)

When using The Product Rule, the object before differentiation is a product. It's considered elegant to have an object after the differentiation that's also a product. Furthermore, in tackling optimisation problems in which local minima and maxima are sought, (which correspond to where the derivative, the gradient, is zero) having a product equalling zero (rather than a sum) is a desirable situation.
In short, initially applying The Product Rule is often only half of a question; manipulating the algebra to derive an answer in the form of a product is the other.

3.2 Example

Show that the derivative of $y = x^3(2x + 5)^3$ can be expressed as,

$$\frac{dy}{dx} = 3x^2(2x + 5)^2(4x + 5)$$

Teaching Video : http://www.NumberWonder.co.uk/v9028/3.mp4

Watch the video and
then write out the
solution here

☞

[5 marks]

3.3 One For You To Do

Often, in the middle of using The Product Rule, The Chain Rule is required.

The Chain Rule for $\boldsymbol{y = [\,f(x)\,]^n}$

$$\text{If } y = [\,f(x)\,]^n \text{ then } \frac{dy}{dx} = n\,[\,f(x)\,]^{n-1}\,f'(x)$$

The Product Rule

$$\text{If } f = uv \quad \text{then} \quad f' = u\,v' + u'\,v$$

Try this problem, then check your solution with mine on the following page.

Try 1 Use the product rule to show that the derivative of,

$$y = (\,2x + 3\,)^2\,(\,5x - 1\,)^3$$

is,

$$\frac{dy}{dx} = (\,2x + 3\,)(\,5x - 1\,)^2\,(\,50x + 41\,)$$

[5 marks]

Answer to Try 1

$$y = (2x+3)^2 (5x-1)^3$$

$$\frac{dy}{dx} = (2x+3)^2 \times 3(5x-1)^2 \times 5 + 2(2x+3)^1 \times 2 \times (5x-1)^3$$

$$\frac{dy}{dx} = (2x+3)(5x-1)^2 \{ 15(2x+3) + 4(5x-1) \}$$

$$\frac{dy}{dx} = (2x+3)(5x-1)^2 \{ 30x + 45 + 20x - 4 \}$$

$$\frac{dy}{dx} = (2x+3)(5x-1)^2 \{ 50x + 41 \}$$

3.3 Exercise

Marks Available : 35

Question 1

Use the product rule to show that the derivative of,

$$y = x^5 (x-1)^2$$

is,

$$\frac{dy}{dx} = x^4 (x-1)(7x-5)$$

[5 marks]

Question 2

Use the product rule to show that the derivative of,

$$y = x^7 (6x + 5)^3$$

is,

$$\frac{dy}{dx} = 5x^6 (6x + 5)^2 (12x + 7)$$

[5 marks]

Question 3

Use the product rule to show that the derivative of,

$$y = \left(x^2 - 3 \right)\left(x + 1 \right)^2$$

is,

$$\frac{dy}{dx} = 2\left(x^2 - 1 \right)\left(2x + 3 \right)$$

[5 marks]

Question 4

Use the product rule to show that the derivative of,

$$y = (4x + 1)^{\frac{3}{2}}(x^2 + 5)$$

is,

$$\frac{dy}{dx} = 2\sqrt{4x + 1}\,(7x^2 + x + 15)$$

[5 marks]

Question 5

Use the product rule to show that the derivative of,

$$y = (2x - 3)^3(x^2 + 1)^2$$

is,

$$\frac{dy}{dx} = 2(2x - 3)^2(x^2 + 1)(7x^2 - 6x + 3)$$

[5 marks]

Question 6

Use the product rule to find the derivative of,

$$y = x^3 (5x + 1)^2$$

Write your answer as a product.

[5 marks]

Question 7

Find the x component of the coordinates of the stationary points on the curve

$$y = \left(x^2 - 1 \right) \sqrt{1 + x}$$

[5 marks]

3.4 Answers

3.4.1 Solution to 3.2 Example (Teaching Video)

$$y = x^3(2x+5)^3$$

$$\frac{dy}{dx} = x^3 \times 3(2x+5)^2 \times 2 + 3x^2(2x+5)^3$$

$$\frac{dy}{dx} = 6x^3(2x+5)^2 + 3x^2(2x+5)^3$$

$$\frac{dy}{dx} = 3x^2(2x+5)^2\ \{2x + (2x+5)\}$$

$$\frac{dy}{dx} = 3x^2(2x+5)^2(4x+5)$$

[5 marks]

There is a clever way of checking this question via another method which can fill five minutes at the end of the lesson. At the start it's an unhelpful distraction! The idea is to convert the given expression into a form to which the chain rule can be applied.

$$y = x^3(2x+5)^3$$

$$\frac{dy}{dx} = \{x(2x+5)\}^3$$

$$\frac{dy}{dx} = \left\{2x^2 + 5x\right\}^3$$

$$\frac{dy}{dx} = 3\left\{2x^2 + 5x\right\}^2\left(4x+5\right)$$

$$\frac{dy}{dx} = \{x(2x+5)\}^2(4x+5)$$

$$\frac{dy}{dx} = 3x^2(2x+5)^2(4x+5)$$

3.4.2 Solutions to 3.3 Exercise

Answer 1

$$
\begin{aligned}
y &= x^5 (x - 1)^2 \\
\frac{dy}{dx} &= x^5 \times 2(x - 1)^1 \times 1 + 5x^4 (x - 1)^2 \\
&= 2x^5 (x - 1) + 5x^4 (x - 1)^2 \\
&= x^4 (x - 1)\{ 2x + 5(x - 1) \} \\
&= x^4 (x - 1)\{ 2x + 5x - 5 \} \\
&= x^4 (x - 1)(7x - 5) \qquad \square
\end{aligned}
$$

[5 marks]

Answer 2

$$
\begin{aligned}
y &= x^7 (6x + 5)^3 \\
\frac{dy}{dx} &= x^7 \times 3(6x + 5)^2 \times 6 + 7x^6 (6x + 5)^3 \\
&= 18x^7 (6x + 5)^2 + 7x^6 (6x + 5)^3 \\
&= x^6 (6x + 5)^2 \{ 18x + 7(6x + 5) \} \\
&= x^6 (6x + 5)^2 \{ 18x + 42x + 35 \} \\
&= x^6 (6x + 5)^2 \{ 60x + 35 \} \\
&= 5x^6 (6x + 5)^2 (12x + 7) \qquad \square
\end{aligned}
$$

[5 marks]

Answer 3

$$
\begin{aligned}
y &= \left(x^2 - 3\right)\left(x + 1\right)^2 \\
\frac{dy}{dx} &= \left(x^2 - 3\right) \times 2\left(x + 1\right)^1 \times 1 + 2x\left(x + 1\right)^2 \\
&= 2\left(x^2 - 3\right)\left(x + 1\right)^1 + 2x\left(x + 1\right)^2 \\
&= 2\left(x + 1\right)\left\{x^2 - 3 + x(x + 1)\right\} \\
&= 2\left(x + 1\right)\left\{x^2 - 3 + x^2 + x\right\} \\
&= 2\left(x + 1\right)\left\{2x^2 + x - 3\right\} \\
&= 2(x + 1)\{ (2x + 3)(x - 1) \} \\
&= 2\left(x^2 - 1\right)\left(2x + 3\right) \qquad \square
\end{aligned}
$$

[5 marks]

Answer 4

$$y = (4x+1)^{\frac{3}{2}}(x^2+5)$$

$$\frac{dy}{dx} = (4x+1)^{\frac{3}{2}} \times 2x + \frac{3}{2}(4x+1)^{\frac{1}{2}} \times 4 \times (x^2+5)$$

$$= 2x(4x+1)^{\frac{3}{2}} + 6(4x+1)^{\frac{1}{2}}(x^2+5)$$

$$= 2(4x+1)^{\frac{1}{2}}\{x(4x+1) + 3(x^2+5)\}$$

$$= 2(4x+1)^{\frac{1}{2}}\{4x^2 + x + 3x^2 + 15\}$$

$$= 2\sqrt{4x+1}\,\{7x^2 + x + 15\} \qquad \square$$

[5 marks]

Answer 5

$$y = (2x-3)^3(x^2+1)^2$$

$$\frac{dy}{dx} = (2x-3)^3 \times 2(x^2+1)^1 \times 2x + 3(2x-3)^2 \times 2 \times (x^2+1)^2$$

$$= 4x(2x-3)^3(x^2+1) + 6(2x-3)^2(x^2+1)^2$$

$$= 2(2x-3)^2(x^2+1)\{2x(2x-3) + 3(x^2+1)\}$$

$$= 2(2x-3)^2(x^2+1)\{4x^2 - 6x + 3x^2 + 3\}$$

$$= 2(2x-3)^2(x^2+1)(7x^2 - 6x + 3) \qquad \square$$

[5 marks]

Answer 6

$$y = x^3(5x+1)^2$$

$$\frac{dy}{dx} = x^3 \times 2(5x+1)^1 \times 5 + 3x^2(5x+1)^2$$

$$= 10x^3(5x+1) + 3x^2(5x+1)^2$$

$$= x^2(5x+1)\{10x + 3(5x+1)\}$$

$$= x^2(5x+1)\{10x + 15x + 3\}$$

$$= x^2(5x+1)(25x+3)$$

[5 marks]

Answer 7

$$y = \left(x^2 - 1\right)\sqrt{1 + x}$$

$$y = \left(x^2 - 1\right)\left(x + 1\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \left(x^2 - 1\right) \times \frac{1}{2}\left(x + 1\right)^{-\frac{1}{2}} \times 1 + 2x\left(x + 1\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}(x - 1)(x + 1) \times \frac{1}{\sqrt{1 + x}} + 2x\sqrt{1 + x}$$

$$= \frac{1}{2}(x - 1)\sqrt{1 + x} + 2x\sqrt{1 + x}$$

$$= \frac{1}{2}\sqrt{1 + x}\,\{(x - 1) + 4x\}$$

$$= \frac{1}{2}\sqrt{1 + x}\,(5x - 1)$$

For a stationary point,

$$\frac{dy}{dx} = 0$$

$$\frac{1}{2}\sqrt{1 + x}\,(5x - 1) = 0$$

$$\text{Either } \sqrt{1 + x} = 0 \text{ or } 5x - 1 = 0$$

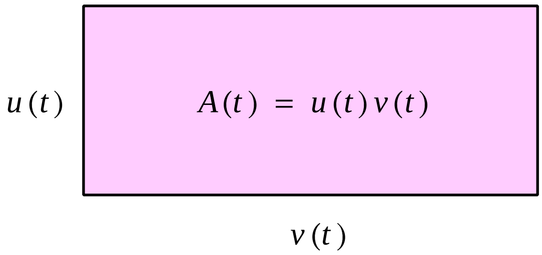
$$x = -1 \text{ or } x = \frac{1}{5}$$

[5 marks]

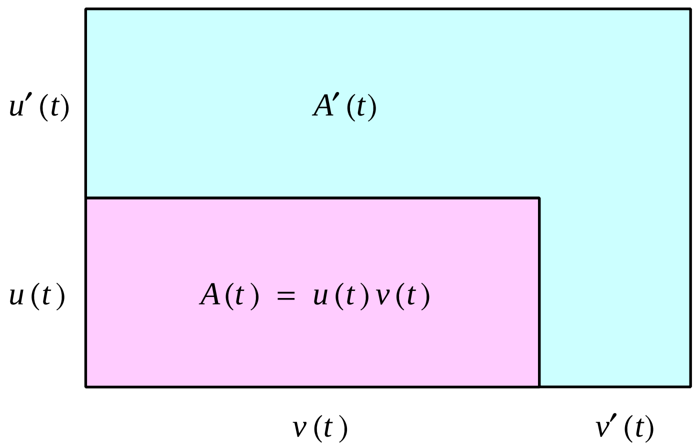
4.1 An Intuitive Proof for The Product Rule

At this stage of the A-Level course, the mathematics that is needed to rigorously prove The Product Rule has yet to be developed. However, why The Product Rule is of the form it is can be intuitively visualised.

The starting point is to imagine that a rectangle at some time, t, has a height given by a function $u(t)$ and a width given by a function $v(t)$.
The area of the rectangle, $A(t)$, is then given by $A(t) = u(t)\,v(t)$.



Some time later (for example 1 second), the size of the rectangle will have changed. Derivatives can be thought of as "rates of change". Let the rate of change in the height be given by $u'(t)$ and the rate of change in the width be $v'(t)$.

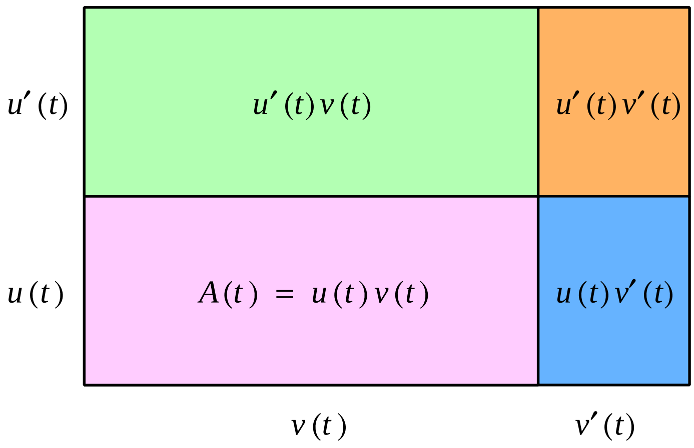


The key question is what is the rate of change of the area of the rectangle, $A'(t)$?

In the diagram above, this is the region shaded blue. It can be divided up into three rectangles one with area $u(t)\,v'(t)$, one with area $u'(t)\,v'(t)$ and one with area $u'(t)\,v(t)$.

In other words,

$$A'(t) = u(t)\,v'(t) + u'(t)\,v'(t) + u'(t)\,v(t)$$



The final step is to let the time interval tend towards zero; to get the “instantaneous rate of change”.

The interesting observation is that the contribution of the orange rectangle, representing $u'(t)\ v'(t)$ is becoming increasingly insignificant in comparison to the contributions from the blue rectangle, representing $u(t)\ v'(t)$, and the green rectangle, representing $u'(t)\ v(t)$.

Thus $\quad A'(t) = u(t)\ v'(t) + u'(t)\ v'(t) + u'(t)\ v(t)$

becomes $\quad A'(t) = u(t)\ v'(t) + u'(t)\ v(t)$

which is The Product Rule $\square$

4.2 Exercise

Marks Available : 40

Question 1

Given that, $f(x) = \left(2x^4 - 3x + 6\right)^5$ use The Chain Rule to show $f'(1) = 5^6$

[3 marks]

Question 2

Given that, $g(x) = \sqrt{5x^2 - 4}$ use The Chain Rule to show $g'(2) = 2.5$

[3 marks]

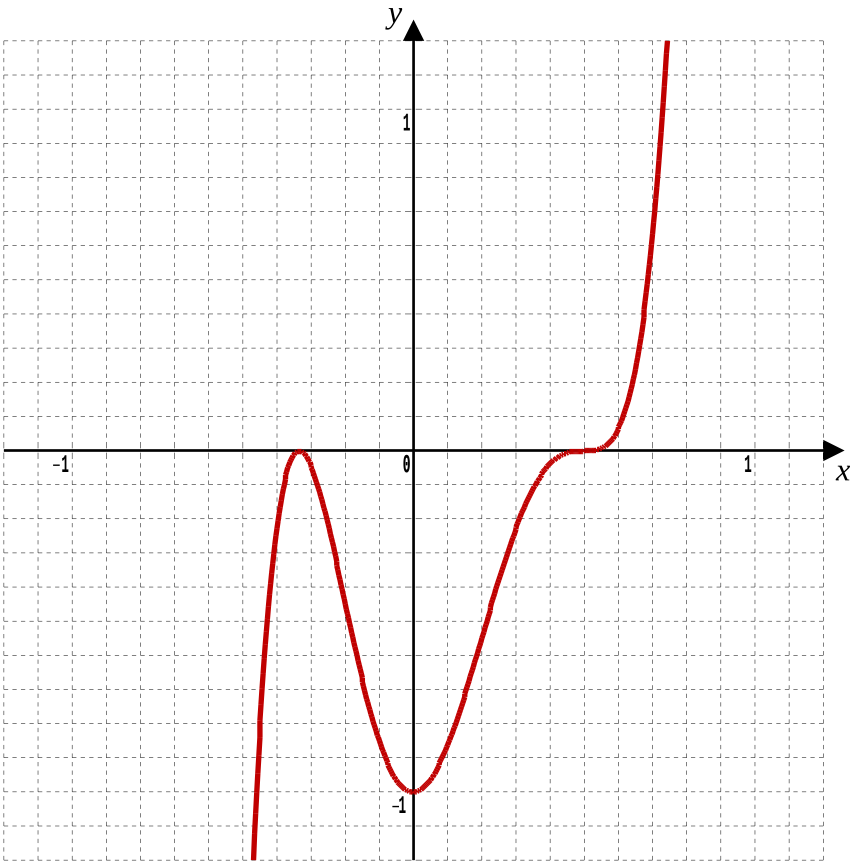
Question 3

A curve C has equation $y = (5 - 2x)^3$

Find the tangent to the curve at the point P with x-coordinate 2

[4 marks]

Question 4

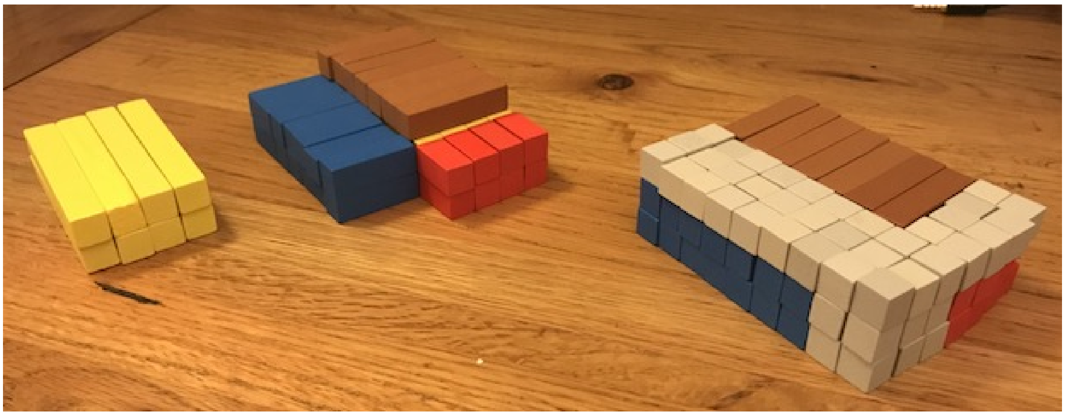


Use The Product Rule to find the stationary points of the curve with equation,

$$y = (2x - 1)^3 (3x + 1)^2$$

[6 marks]

Question 5



The yellow cuboid to the left of the photograph measures $7 \times 2 \times 4$. It's extended to measure $(7 + 2) \times (2 + 1) \times (4 + 3)$ as shown to the right of the photograph.

(i) Calculate the increase in the volume of the cuboid.

[2 marks]

(ii) Calculate the percentage of the extra volume that is given by the red, brown and blue parts.

[4 marks]

(iii) If the original $7 \times 2 \times 4$ cuboid had been extended to measure $(7 + 0.2) \times (2 + 0.1) \times (4 + 0.3)$ what percentage of the extra volume would be given by the new red, brown and blue parts.

[5 marks]

(iv) Explain the significance of the answers to parts (ii) and (iii) in relation to the circumstances in which the calculation of the extra volume can be approximated by just the red, brown and blue cuboids, without a significant loss of answer accuracy.

[2 marks]

(v) A function $V(x)$ is the product of three other functions,

$$V(x) = u(x)\ v(x)\ w(x)$$

Keeping in mind the earlier parts of this question, and the proof of The Product Rule at the start of this lesson, make an inspired guess as to what the formula for the derivative of $V(x)$ is likely to be.

[2 marks]

Question 6

Show that if $f(x) = x^2\sqrt{3x - 1}$ then $f'(x) = \dfrac{x(15x - 4)}{2\sqrt{3x - 1}}$

[5 marks]

Question 7

The curve C has equation $y = \dfrac{1}{(1 - 2x)^2}$ where $x \neq \dfrac{1}{2}$

The point A on C has x-coordinate $\dfrac{1}{4}$

Find an equation of the tangent to C at A in the form $y = mx + c$

[4 marks]

4.3 Answers

Answer 1

$$f(x) = \left(2x^4 - 3x + 6\right)^5$$
$$f'(x) = 5\left(2x^4 - 3x + 6\right)^4 \left(8x^3 - 3\right)$$
$$f'(1) = 5\,(2 - 3 + 6)^4\,(8 - 3)$$
$$= 5 \times 5^4 \times 5$$
$$= 5^6 \qquad \square$$

[3 marks]

Answer 2

$$g(x) = \sqrt{5x^2 - 4}$$
$$= \left(5x^2 - 4\right)^{\frac{1}{2}}$$
$$g'(x) = \frac{1}{2}\left(5x^2 - 4\right)^{-\frac{1}{2}} \times 10x$$
$$= \frac{5x}{\sqrt{5x^2 - 4}}$$
$$g'(2) = \frac{5 \times 2}{\sqrt{5 \times 2^2 - 4}}$$
$$= \frac{10}{\sqrt{16}}$$
$$= 2.5 \qquad \square$$

[3 marks]

Answer 3

$$y = (5 - 2x)^3$$
$$\frac{dy}{dx} = 3(5 - 2x)^2 \times (-2)$$
$$= -6\,(5 - 2x)^2$$
$$\text{When } x = 2, \quad \frac{dy}{dx} = -6 \text{ and } y = 1$$
$$y = mx + c$$
$$1 = -6 \times 2 + c$$
$$c = 13$$
$$\therefore\ y = -6x + 13$$

[4 marks]

Answer 4

$$y = (2x - 1)^3 (3x + 1)^2$$

$$\frac{dy}{dx} = (2x - 1)^3 \times 2(3x + 1)^1 \times 3 + 3(2x - 1)^2 \times 2 \times (3x + 1)^2$$

$$= 6(2x - 1)^3 (3x + 1) + 6(2x - 1)^2 (3x + 1)^2$$

$$= 6(2x - 1)^2 (3x + 1) \{2x - 1 + 3x + 1\}$$

$$= 6(2x - 1)^2 (3x + 1) (5x)$$

$$= 30x(2x - 1)^2 (3x + 1)$$

Stationary points occur when the gradient is zero

$$30x (2x - 1)^2 (3x + 1) = 0$$

$$\text{Either } x = 0, \quad \text{or } x = \frac{1}{2} \text{ or } x = -\frac{1}{3}$$

$$\text{Points are thus, } (0, -1), \left(\frac{1}{2}, 0\right), \left(-\frac{1}{3}, 0\right)$$

[6 marks]

Answer 5

(i) $\Delta V = (7 + 2) \times (2 + 1) \times (4 + 3) - 7 \times 2 \times 4$

$= 9 \times 3 \times 7 - 7 \times 2 \times 4$

$= 189 - 56$

$= 133$

[2 marks]

(ii) $V_{RED} = 4 \times 2 \times 2 = 16$

$V_{BROWN} = 7 \times 4 \times 1 = 28$

$V_{BLUE} = 7 \times 2 \times 3 = 42$ Total : 86

$$\text{Percentage} = \frac{86}{133} \times 100 = 64.7\%$$

[4 marks]

(iii) $\Delta V = (7 + 0.2) \times (2 + 0.1) \times (4 + 0.3) - 7 \times 2 \times 4$

$= 7.2 \times 2.1 \times 4.3 - 7 \times 2 \times 4$

$= 65.016 - 56$

$= 9.016$

$V_{RED} = 4 \times 2 \times 0.2 = 1.6$

$V_{BROWN} = 7 \times 4 \times 0.1 = 2.8$

$V_{BLUE} = 7 \times 2 \times 0.3 = 4.2$ Total : 8.6

$$\text{Percentage} = \frac{8.6}{9.016} \times 100 = 95.4\%$$

[5 marks]

(iv) It can be argued that the significant part of the extra volume for small extensions is given by the sum of the volumes of the red, brown and blue cuboids shown in the centre of the photograph, and that the white 'filler' parts can be ignored. This argument is strengthened by the calculation part of the question. Firstly, in part (ii) the volume from just the red plus brown plus blue volume answer is inaccurate. This is because the extensions are relatively large. But then, in part (iii), where the extensions are much smaller, a far more accurate answer is obtained.

[2 marks]

(v)

The Triple Product Rule

If $V = uvw$ then $V' = u\,v\,w' + u\,v'\,w + u'\,v\,w$

[2 marks]

Answer 6

$$
\begin{aligned}
f(x) &= x^2\sqrt{3x-1} \\
&= x^2(3x-1)^{\frac{1}{2}} \\
f'(x) &= x^2 \times \frac{1}{2}(3x-1)^{-\frac{1}{2}} \times 3 + 2x(3x-1)^{\frac{1}{2}} \\
&= \frac{3x^2}{2\sqrt{3x-1}} + 2x\sqrt{3x-1} \times \frac{2\sqrt{3x-1}}{2\sqrt{3x-1}} \\
&= \frac{3x^2}{2\sqrt{3x-1}} + \frac{4x(3x-1)}{2\sqrt{3x-1}} \\
&= \frac{3x^2 + 12x^2 - 4x}{2\sqrt{3x-1}} \\
&= \frac{15x^2 - 4x}{2\sqrt{3x-1}} \\
&= \frac{x(15x-4)}{2\sqrt{3x-1}} \qquad \square
\end{aligned}
$$

[5 marks]

Answer 7

$$
\begin{aligned}
y &= \frac{1}{(1-2x)^2} \\
&= (1-2x)^{-2} \\
\frac{dy}{dx} &= -2(1-2x)^{-3} \times (-2) \\
&= \frac{4}{(1-2x)^3}
\end{aligned}
$$

When $x = \frac{1}{4}$, $y = 4$, $\frac{dy}{dx} = 32$

$$
\begin{aligned}
y &= mx + c \\
4 &= 32 \times \frac{1}{4} + c \\
4 &= 8 + c \\
c &= -4
\end{aligned}
$$

$\therefore$ Tangent is $y = 32x - 4$

[4 marks]

Lesson 5

5.1 The Quotient Rule

Given two functions, $u(x)$ and $v(x)$, the first divided by the second, The Quotient Rule gives a method of obtaining the derivative of the division. It states that,

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{v(x)\ u'(x) - v'(x)\ u(x)}{(v(x))^2}$$

All of the x in brackets are considered to be unnecessary clutter and so the rule is more usually written in the following succinct and elegant form,

---

The Quotient Rule

$$\text{If } f = \frac{u}{v} \quad \text{then} \quad f' = \frac{v\,u' - v'\,u}{v^2}$$

5.2 Example

Differentiate $y = \dfrac{x+4}{x+5}$ by immediately applying The Quotient Rule

Teaching Video : http://www.NumberWonder.co.uk/v9028/5.mp4

Watch the video and then write out the solution here

☞

[3 marks]

5.3 Exercise

Marks Available : 40

Question 1

Given that $y = \dfrac{4x}{x + 3}$

use The Quotient Rule to show that the derivative is given by $\dfrac{dy}{dx} = \dfrac{12}{(x + 3)^2}$

[3 marks]

Question 2

Given that $y = \dfrac{x^2}{(x + 5)}$ use The Quotient Rule to show that

the derivative is given by $\dfrac{dy}{dx} = \dfrac{x(x + 10)}{(x + 5)^2}$

[3 marks]

Question 3

Given that $y = \dfrac{5x - 2}{3x + 1}$ use The Quotient Rule to show that

the derivative is given by $\dfrac{dy}{dx} = \dfrac{11}{(3x + 1)^2}$

[3 marks]

Question 4

Given that $y = \dfrac{x^2 + 1}{x^2 + 4}$ use The Quotient Rule to show that

the derivative is given by $\dfrac{dy}{dx} = \dfrac{6x}{(x^2 + 4)^2}$

[3 marks]

Question 5

Given that $y = \dfrac{x^5}{(2x + 1)^3}$ use The Quotient Rule to show that

the derivative is given by $\dfrac{dy}{dx} = \dfrac{x^4(4x + 5)}{(2x + 1)^4}$

[4 marks]

Question 6

Given that $y = \dfrac{x^7}{(3x + 2)^5}$ use The Quotient Rule to show that

the derivative is given by $\dfrac{dy}{dx} = \dfrac{2x^6(3x + 7)}{(3x + 2)^6}$

[5 marks]

Question 7

Given that $y = \dfrac{2(x+3)^3}{\sqrt{x}}$ use The Quotient Rule to

show that $\dfrac{dy}{dx} = \dfrac{(x+3)^2(5x-3)}{x^{\frac{3}{2}}}$

[4 marks]

Question 8

Given that $y = x^2\sqrt{x+5}$

use The Product Rule to show that $\dfrac{dy}{dx} = \dfrac{5x(x+4)}{2\sqrt{x+5}}$

[5 marks]

Question 9

$$f(x) = \frac{2x}{x + 5} + \frac{6x}{x^2 + 7x + 10} \qquad x > 0$$

(a) Show that

$$f(x) = \frac{2x}{x + 2}$$

(b) Hence find $f'(3)$

[5 marks]

Question 10

Given that the function $f(x) = \dfrac{x}{x^2 + 2}$ is increasing on the interval $[-k, \ k]$ find the largest possible value of k.

[5 marks]

5.4 Answers

5.4.1 Solution to 5.2 Example (Teaching Video)

$$y = \frac{x+4}{x+5}$$

$$\frac{dy}{dx} = \frac{(x+5)\times 1 - 1\times(x+4)}{(x+5)^2}$$

$$= \frac{x+5-x-4}{(x+5)^2}$$

$$= \frac{1}{(x+5)^2}$$

[3 marks]

There is a clever way of checking this question via an algebraic method.

$$y = \frac{x+4}{x+5}$$

$$= \frac{x+4+1-1}{x+5}$$

$$= \frac{x+5}{x+5} - \frac{1}{x+5}$$

$$= 1 - (x+5)^{-1}$$

$$\frac{dy}{dx} = -(-1)(x+5)^{-2}\times 1$$

$$= (x+5)^{-2}$$

$$= \frac{1}{(x+5)^2}$$

5.4.2 Solutions to 5.3 Exercise

Answer 1

$$y = \frac{4x}{x+3}$$

$$\frac{dy}{dx} = \frac{(x+3) \times 4 - 1 \times 4x}{(x+3)^2}$$

$$= \frac{4x + 12 - 4x}{(x+3)^2}$$

$$= \frac{12}{(x+3)^2} \qquad \square$$

[3 marks]

Answer 2

$$y = \frac{x^2}{(x+5)}$$

$$\frac{dy}{dx} = \frac{(x+5) \times 2x - 1 \times x^2}{(x+5)^2}$$

$$= \frac{2x^2 + 10x - x^2}{(x+5)^2}$$

$$= \frac{x^2 + 10x}{(x+5)^2}$$

$$= \frac{x(x+10)}{(x+5)^2} \qquad \square$$

[3 marks]

Answer 3

$$y = \frac{5x - 2}{3x + 1}$$

$$\frac{dy}{dx} = \frac{(3x+1) \times 5 - 3 \times (5x - 2)}{(3x+1)^2}$$

$$= \frac{15x + 5 - 15x + 6}{(3x+1)^2}$$

$$= \frac{11}{(3x+1)^2} \qquad \square$$

[3 marks]

Answer 4

$$y = \frac{x^2 + 1}{x^2 + 4}$$

$$\frac{dy}{dx} = \frac{\left(x^2 + 4\right) \times 2x - 2x \times \left(x^2 + 1\right)}{(x^2 + 4)^2}$$

$$= \frac{2x^3 + 8x - 2x^3 - 2x}{(x^2 + 4)^2}$$

$$= \frac{6x}{(x^2 + 4)^2} \qquad \square$$

[3 marks]

Answer 5

$$y = \frac{x^5}{(2x + 1)^3}$$

$$\frac{dy}{dx} = \frac{(2x + 1)^3 \times 5x^4 - 3(2x + 1)^2 \times 2 \times x^5}{(2x + 1)^6}$$

$$= \frac{5x^4(2x + 1)^3 - 6x^5(2x + 1)^2}{(2x + 1)^6}$$

$$= \frac{x^4(2x + 1)^2 \{5(2x + 1) - 6x\}}{(2x + 1)^6}$$

$$= \frac{x^4\{10x + 5 - 6x\}}{(2x + 1)^4}$$

$$= \frac{x^4(4x + 5)}{(2x + 1)^4} \qquad \square$$

[4 marks]

Answer 6

$$y = \frac{x^7}{(3x+2)^5}$$

$$\frac{dy}{dx} = \frac{(3x+2)^5 \times 7x^6 - 5(3x+2)^4 \times 3 \times x^7}{(3x+2)^{10}}$$

$$= \frac{7x^6(3x+2)^5 - 15x^7(3x+2)^4}{(3x+2)^{10}}$$

$$= \frac{x^6(3x+2)^4\ \{7(3x+2) - 15x\}}{(3x+2)^{10}}$$

$$= \frac{x^6\{21x + 14 - 15x\}}{(3x+2)^6}$$

$$= \frac{2x^6(3x+7)}{(3x+2)^6} \qquad \square$$

[5 marks]

Answer 7

$$y = \frac{2(x+3)^3}{\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{2}} \times 6(x+3)^2 \times 1 - \frac{1}{2}x^{-\frac{1}{2}} \times 2(x+3)^3}{x} \times \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

$$= \frac{6x(x+3)^2 - (x+3)^3}{x^{\frac{3}{2}}}$$

$$= \frac{(x+3)^2\ \{6x - (x+3)\}}{x^{\frac{3}{2}}}$$

$$= \frac{(x+3)^2(5x-3)}{x^{\frac{3}{2}}} \qquad \square$$

[4 marks]

Answer 8

$$y = x^2\sqrt{x+5} = x^2(x+5)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = x^2 \times \frac{1}{2}(x+5)^{-\frac{1}{2}} + 2x(x+5)^{\frac{1}{2}}$$

$$= \frac{x^2}{2\sqrt{x+5}} + 2x\sqrt{x+5} \times \frac{2\sqrt{x+5}}{2\sqrt{x+5}}$$

$$= \frac{x^2 + 4x(x+5)}{2\sqrt{x+5}}$$

$$= \frac{x^2 + 4x^2 + 20x}{2\sqrt{x+5}}$$

$$= \frac{5x(x+4)}{2\sqrt{x+5}} \qquad \square$$

[5 marks]

Answer 9

(a)

$$f(x) = \frac{2x}{x+5} + \frac{6x}{x^2+7x+10}$$

$$= \frac{2x}{x+5} + \frac{6x}{(x+5)(x+2)}$$

$$= \frac{2x}{x+5} \times \frac{(x+2)}{(x+2)} + \frac{6x}{(x+5)(x+2)}$$

$$= \frac{2x^2 + 4x + 6x}{(x+5)(x+2)}$$

$$= \frac{2x(x+5)}{(x+5)(x+2)}$$

$$= \frac{2x}{x+2} \qquad \square$$

(b)

$$f'(x) = \frac{(x+2) \times 2 - 1 \times 2x}{(x+2)^2}$$

$$= \frac{2x + 4 - 2x}{(x+2)^2}$$

$$= \frac{4}{(x+2)^2}$$

$$f'(3) = \frac{4}{(3+2)^2}$$

$$= \frac{4}{25}$$

[5 marks]

Answer 10

$$f(x) = \frac{x}{x^2 + 2}$$

$$f'(x) = \frac{(x^2 + 2) \times 1 - 2x \times x}{(x^2 + 2)^2}$$

$$= \frac{x^2 + 2 - 2x^2}{(x^2 + 2)^2}$$

$$= \frac{2 - x^2}{(x^2 + 2)^2}$$

We require $\quad f'(x) \geqslant 0$

$$\frac{2 - x^2}{(x^2 + 2)^2} \geqslant 0$$

$$2 - x^2 \geqslant 0$$

$$-\sqrt{2} \leqslant x \leqslant \sqrt{2}$$

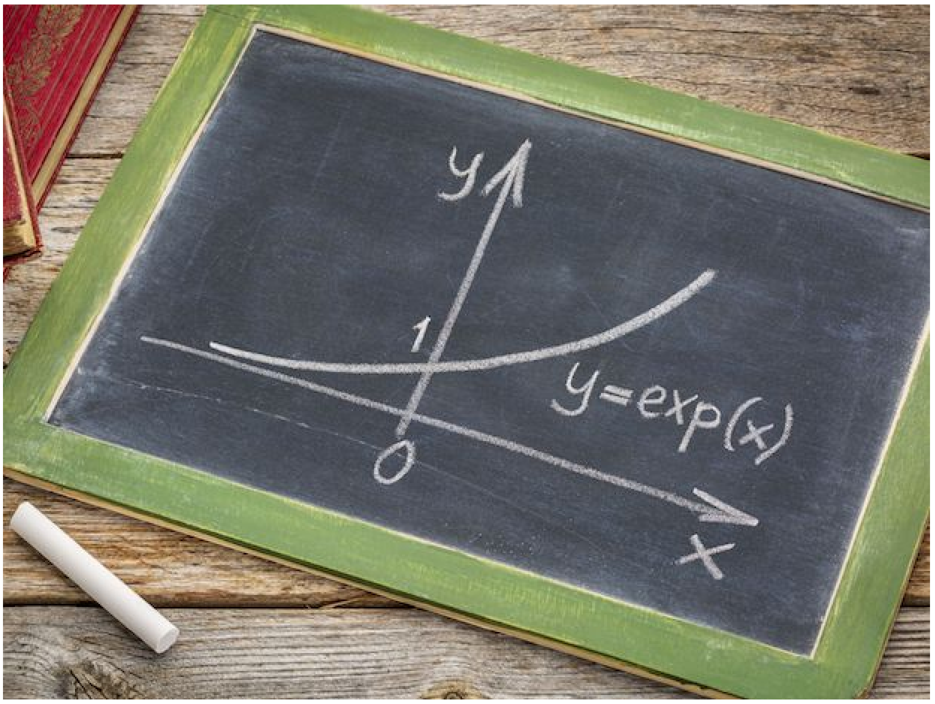
That is, $k = \sqrt{2}$

[5 marks]

Lesson 6

A-Level Pure Mathematics : Year 2
Differentiation III

6.1 The Exponential Function



The function $y = e^x$ has the remarkable property of being its own derivative.

The Derivative of $y = e^x$

$$\text{If } y = e^x \text{ then } \frac{dy}{dx} = e^x$$

6.2 The Value of e

e is the number 2.71828 18284 59045 23536 02875 ...
Like π this is an irrational number and like π it crops up in many surprising places throughout mathematics.

6.3 Differentiating Exponentiated Functions

The Product Rule and The Quotient Rule can be applied to situations where the exponential function is involved. So too can The Chain Rule, as follows,

---

The Chain Rule for $y = e^{f(x)}$

$$\text{If } y = e^{f(x)} \text{ then } \frac{dy}{dx} = e^{f(x)} f'(x)$$

6.4 Examples

Differentiate each of the following,

(i) $y = e^{8x^3 + 5x^2}$ (Chain Rule Example)

(ii) $y = x\,e^{3x}$ (Product Rule Example)

(iii) $y = \dfrac{x}{e^x}$ (Quotient Rule Example)

Teaching Video : http://www.NumberWonder.co.uk/v9028/6.mp4

Watch the video and
then write out the
solutions here

☞

[3, 3, 3 marks]

6.5 Exercise

Marks Available : 50

Question 1

Differentiate each of the following with respect to *x*;

(i) $y = e^{5x}$ **(ii)** $y = e^{3x^5}$

[2, 2 marks]

(iii) $y = 8e^{6x} + 6e^{5x}$ **(iv)** $y = 14e^{x}$

[2, 2 marks]

(v) $y = e^{-7x}$ **(vi)** $y = \dfrac{4}{e^{5x}}$

[2, 2 marks]

Question 2

Remembering that when dividing same base indices subtract, differentiate the following with respect to *x*,

$$y = \frac{e^{7x}}{e^{3x}}$$

[2 marks]

Question 3

By first expanding the brackets, differentiate with respect to *x*,

$$y = e^{3x}\left(e^{4x} + 7e^{-9x}\right)$$

[3 marks]

Question 4

Use The Product Rule to differentiate each of the following with respect to *x*.
Simplify to obtain an elegant answers.

(i) $y = x^3 e^x$ **(ii)** $y = x e^{-x}$

[3, 3 marks]

Question 5

Use The Quotient Rule to differentiate each of the following with respect to *x*.
Simplify to obtain elegant answers.

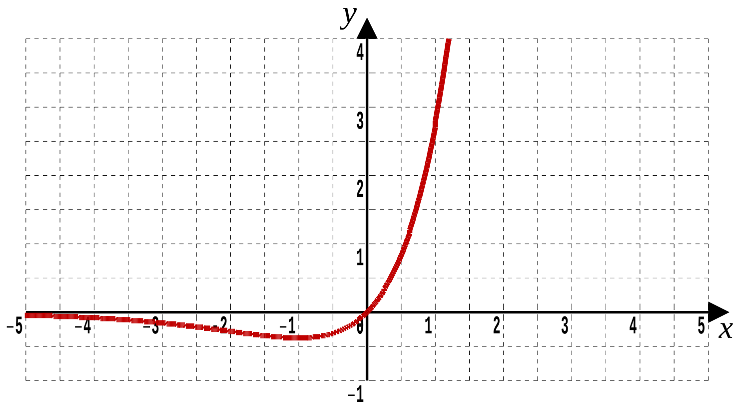
(i) $y = \dfrac{e^x}{x}$

[3 marks]

(ii) $y = \dfrac{2 - x^2}{e^{2x}}$

[3 marks]

Question 6



A curve has equation $y = x\,e^{x}$

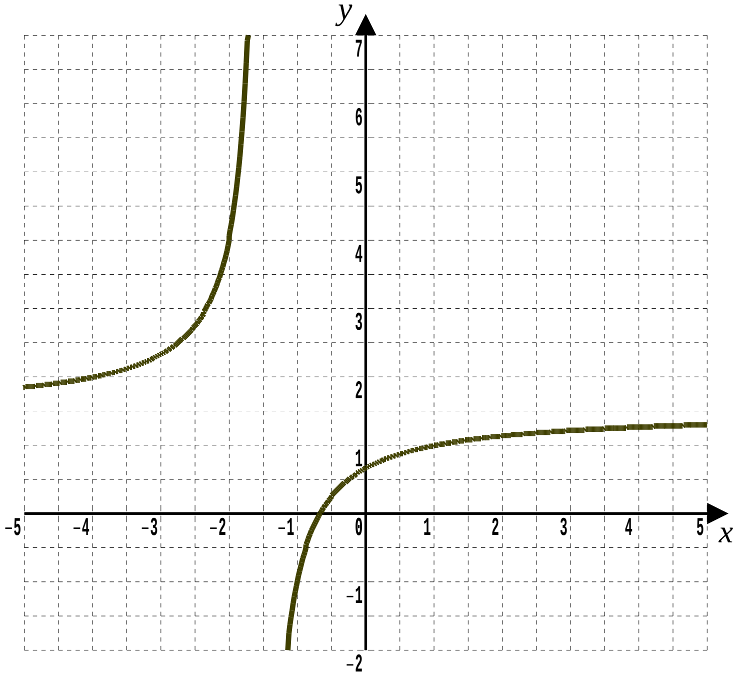
(i) Find $\dfrac{dy}{dx}$

[3 marks]

(ii) Find the equation of the tangent to the curve when $x = 0$
Draw this tangent onto the graph, above.

[4 marks]

Question 7



A curve has equation, $y = \dfrac{3x + 2}{2x + 3}$

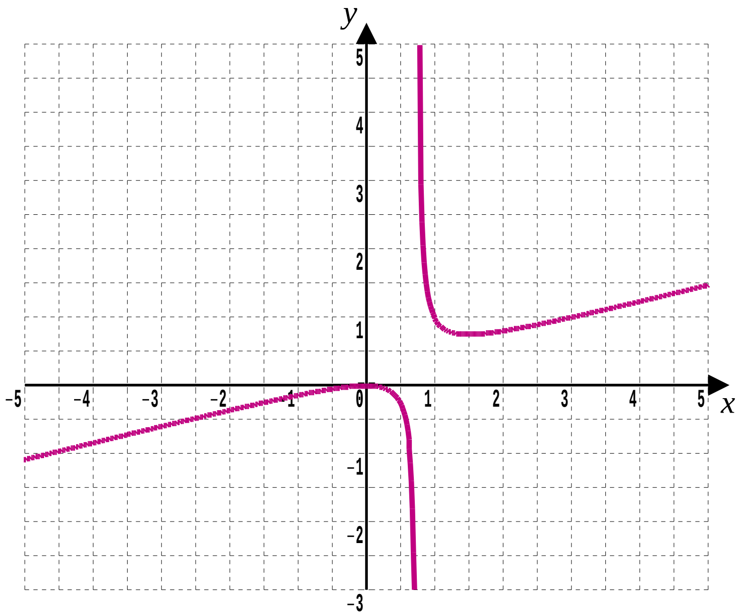
(i) Find $\dfrac{dy}{dx}$

[3 marks]

(ii) Find the equation of the normal to the curve when $x = 1$
Draw this normal onto the graph, above.

[4 marks]

Question 8



A curve has equation $y = \dfrac{x^2}{(4x - 3)}$

(i) Find an expression for $\dfrac{dy}{dx}$

[3 marks]

(ii) Find the equation of the tangent to the curve when $x = 1$
Draw this tangent onto the graph, above.

[4 marks]

6.6 Answers

6.6.1 Solutions to 6.4 Examples (Teaching Video)

(i) $y = e^{8x^3 + 5x^2}$ (Chain Rule Example)

$$\frac{dy}{dx} = e^{8x^3 + 5x^2}\left(24x^2 + 10x \right)$$

[3 marks]

(ii) $y = x\,e^{3x}$ (Product Rule Example)

$$\begin{aligned} \frac{dy}{dx} &= x \times e^{3x} \times 3 + 1 \times e^{3x} \\ &= 3x\,e^{3x} + e^{3x} \\ &= e^{3x}(\,3x + 1\,) \end{aligned}$$

[3 marks]

(iii) $y = \dfrac{x}{e^x}$ (Quotient Rule Example)

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^x \times 1 - e^x \times x}{e^x \times e^x} \\ &= \frac{e^x(\,1 - e^x\,)}{e^x \times e^x} \\ &= \frac{1 - e^x}{e^x} \end{aligned}$$

[3 marks]

6.6.2 Solutions to 6.5 Exercise

Answer 1

(i) $y = e^{5x} \quad \Rightarrow \quad \dfrac{dy}{dx} = 5\,e^{5x}$

(ii) $y = e^{3x^5} \quad \Rightarrow \quad \dfrac{dy}{dx} = 15\,x^4\,e^{3x^5}$

(iii) $y = 8\,e^{6x} + 6\,e^{5x} \quad \Rightarrow \quad \dfrac{dy}{dx} = 48\,e^{6x} + 30\,e^{5x}$

(iv) $y = 14\,e^{x} \quad \Rightarrow \quad \dfrac{dy}{dx} = 14\,e^{x}$

(v) $y = e^{-7x} \quad \Rightarrow \quad \dfrac{dy}{dx} = -7\,e^{-7x}$

(vi) $y = 4\,e^{-5x} \quad \Rightarrow \quad \dfrac{dy}{dx} = -20\,e^{-5x}$

[12 marks]

Answer 2

$$y = \frac{e^{7x}}{e^{3x}}$$

$$= e^{4x}$$

$$\frac{dy}{dx} = 4\,e^{4x}$$

[2 marks]

Answer 3

$$y = e^{3x}\left(e^{4x} + 7\,e^{-9x}\right)$$

$$= e^{7x} + 7\,e^{-6x}$$

$$\frac{dy}{dx} = 7\,e^{7x} - 42\,e^{-6x}$$

[3 marks]

Answer 4

(i)

$$y = x^3\,e^x$$

$$\frac{dy}{dx} = x^3\,e^x + 3\,x^2\,e^x$$

$$= x^2\,e^x\,(\,x + 3\,)$$

[3 marks]

(ii)

$$y = x\,e^{-x}$$

$$\frac{dy}{dx} = x\left(-\,e^{-x}\right) + 1 \times e^{-x}$$

$$= e^{-x}\,(-x + 1\,)$$

$$= \frac{1 - x}{e^x}$$

[3 marks]

Answer 5

(i)

$$y = \frac{e^x}{x}$$

$$\frac{dy}{dx} = \frac{x\,e^x - 1 \times e^x}{x^2}$$

$$= \frac{e^x(x - 1)}{x^2}$$

[3 marks]

(ii)

$$y = \frac{2 - x^2}{e^{2x}}$$

$$\frac{dy}{dx} = \frac{e^{2x} \times (-2x) - e^{2x} \times 2 \times (2 - x^2)}{e^{2x} \times e^{2x}}$$

$$= \frac{e^{2x}(-2x - 4 + 2x^2)}{e^{2x} \times e^{2x}}$$

$$= \frac{2(x^2 - x - 2)}{e^{2x}}$$

$$= \frac{2\,(x - 2)(x + 1)}{e^{2x}}$$

[3 marks]

Answer 6

(i)

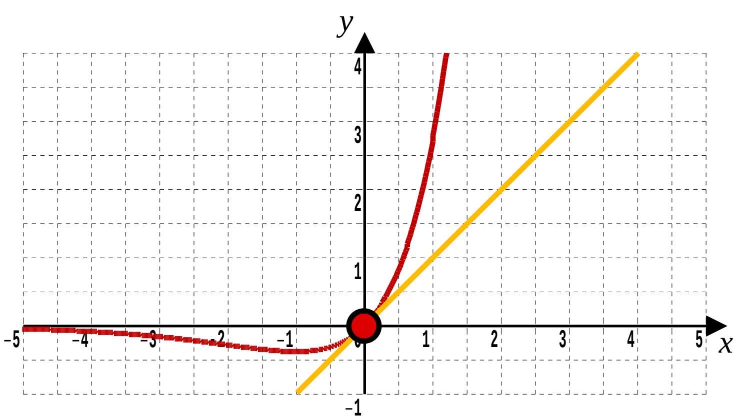
$$y = x\,e^x$$

$$\frac{dy}{dx} = x\,e^x + 1 \times e^x$$

$$= e^x(x + 1)$$

[3 marks]

(ii) When $x = 0$, $y = 0$ and $\frac{dy}{dx} = 1 \Rightarrow y = x$ is the tangent



[4 marks]

Answer 7

(i)

$$y = \frac{3x + 2}{2x + 3}$$

$$\frac{dy}{dx} = \frac{(2x + 3) \times 3 - 2 \times (3x + 2)}{(2x + 3)^2}$$

$$= \frac{6x + 9 - 6x - 4}{(2x + 3)^2}$$

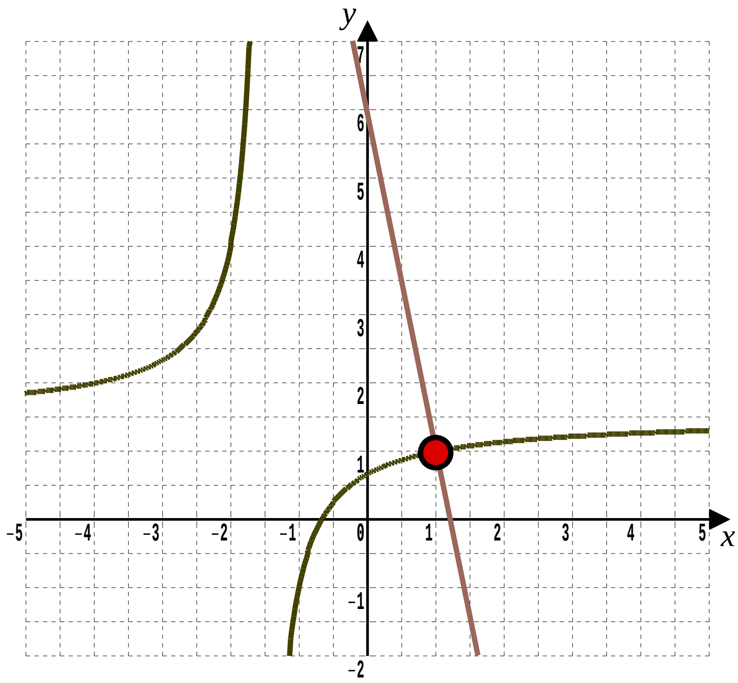
$$= \frac{5}{(2x + 3)^2}$$

[3 marks]

(ii) When $x = 1$, $y = 1$ and $\frac{dy}{dx} = \frac{1}{5}$

Thus normal's gradient is $-\ 5$

$\therefore\ y = -5x + 6$ is the normal's equation.



[4 marks]

Answer 8

(i)

$$y = \frac{x^2}{(4x - 3)}$$

$$\frac{dy}{dx} = \frac{(4x - 3) \times 2x - 4x^2}{(4x - 3)^2}$$

$$= \frac{8x^2 - 6x - 4x^2}{(4x - 3)^2}$$

$$= \frac{4x^2 - 6x}{(4x - 3)^2}$$

$$= \frac{2x(2x - 3)}{(4x - 3)^2}$$

[3 marks]

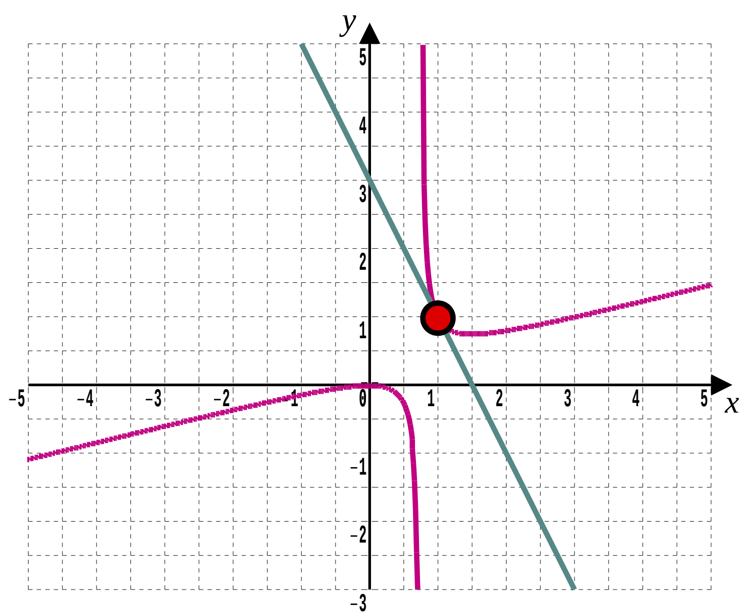
(ii) When $x = 1$, $y = 1$ and $\dfrac{dy}{dx} = -2$

$$y = mx + c$$

$$1 = -2 \times 1 + c$$

$$c = 3$$

$\therefore\ y = -2x + 3$ is the tangent



[4 marks]

Lesson 7

A-Level Pure Mathematics : Year 2
Differentiation III

7.1 The Logarithm Function

The inverse of the exponential function is the natural logarithm function.
It has a surprising derivative.

The Derivative of $y = \ln x$

$$\text{If } y = \ln x \text{ then } \frac{dy}{dx} = \frac{1}{x}$$

Proof :

$$y = \ln x$$

exponentiate both sides

$$e^y = e^{\ln x}$$

$$e^y = x$$

$$x = e^y$$

$$\frac{dx}{dy} = e^y$$

$$\frac{dx}{dy} = x$$

$$\frac{dy}{dx} = \frac{1}{x} \qquad \square$$

Notes on the Proof

The proof relied upon knowing that the exponential function is its own derivative.
Proving that from first principles is beyond A-Level as it needs the result that,

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln(a)$$

"Easy" proofs that the derivative of e^x is e^x need the result that the derivative of $\ln x$ is $\frac{1}{x}$ but proving that from first principles is not any easier.

7.2 Differentiating Logarithm Functions

The Product Rule and The Quotient Rule can be applied to situations where the natural logarithm function is involved. So too can The Chain Rule, as follows,

---

The Chain Rule for $y = \ln(f(x))$

$$\text{If } y = \ln(f(x)) \text{ then } \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

7.3 Examples

Differentiate each of the following,

(i) $y = 5\ln\left(x^2 + e^x\right)$ (Chain Rule Example)

(ii) $y = x^3 \ln x^2$ (Product Rule Example)

(iii) $y = \dfrac{\ln x}{2x}$ (Quotient Rule Example)

Teaching Video : http://www.NumberWonder.co.uk/v9028/7.mp4

Watch the video and
then write out the
solutions here

☞

[3, 3, 3 marks]

7.4 Exercise

Marks Available : 50

Question 1

Differentiate each of the following with respect to x;

(i) $y = 3\,ln\,x$

(ii) $y = 4\,ln\,x^3$

[2, 2 marks]

(iii) $y = ln(5x)$

(iv) $y = 6\,ln(7x)$

[2, 2 marks]

(v) $y = ln\left(4x^3 + 3\right)$

(vi) $y = (ln\,x)^3$

[2, 2 marks]

(vii) $y = (ln\,x)^{-4}$

(viii) $y = \dfrac{4}{ln\,x}$

[2, 2 marks]

Question 2
Use The Product Rule to differentiate the following with respect to *x*,

$$y = x \ln x^7$$

[3 marks]

Question 3
Use The Quotient Rule to differentiate the following with respect to *x*;

$$y = \frac{x}{\ln x}$$

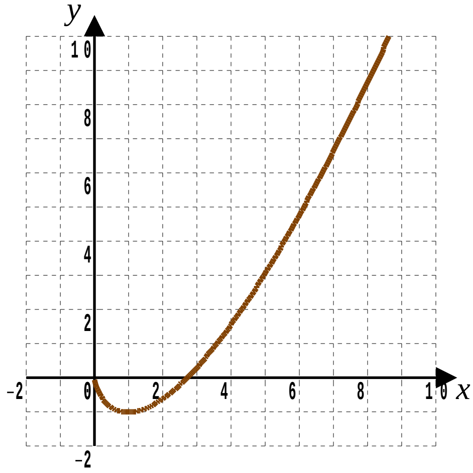
[3 marks]

Question 4
Use The Quotient Rule to differentiate the following with respect to *x*,

$$y = \frac{\ln x^3}{x^3}$$

[3 marks]

Question 5



The graph is of the function

$$f(x) = x(\ln(x) - 1) \qquad x > 0$$

(i) Find the exact coordinates of the point, *A*, where the gradient is 1

[3 marks]

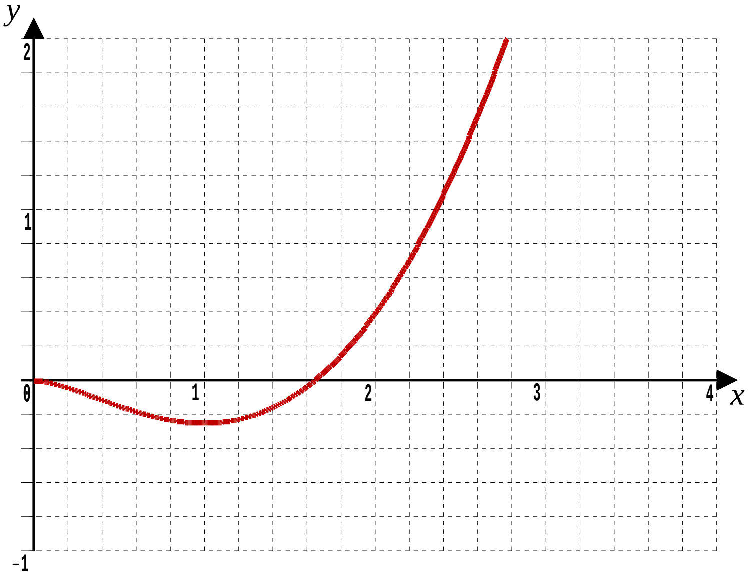
(ii) Find the exact coordinates of the turning point, *B*, and prove that it is a minimum.

[3 marks]

(iii) Find the exact coordinates of where the tangent to the curve at point *A*, meets the tangent to the curve at point *B*

[3 marks]

Question 6



The graph is of the function

$$g(x) = \frac{1}{4}x^2(2\ln(x) - 1) \qquad x > 0$$

(i) Find the exact value of $g'(e)$

[4 marks]

(ii) Find the exact coordinates of the turning point and prove that it is a minimum.

[3 marks]

Question 7

Find the exact equation of the tangent to the following curve where $x = 1$

$$y = e^x \ln x$$

[3 marks]

Question 8

A curve has equation $y = \dfrac{\ln x}{x^2}$

(i) Find an expression for $\dfrac{dy}{dx}$

[3 marks]

(ii) Find the equation of the tangent to the curve when $x = 1$

[3 marks]

7.5 Answers

7.5.1 Solutions to 7.3 Examples (Teaching Video)

(i) $y = 5\,ln\left(x^2 + e^x\right)$ (Chain Rule Example)

$$\frac{dy}{dx} = 5 \times \frac{2x + e^x}{x^2 + e^x}$$

[3 marks]

(ii) $y = x^3\,ln\,x^2$ (Product Rule Example)

$$= 2\,x^3\,ln\,x$$

$$\frac{dy}{dx} = 2\left(x^3 \times \frac{1}{x} + 3\,x^2\,ln\,x\right)$$

$$= 2\left(x^2 + 3\,x^2\,ln\,x\right)$$

$$= 2\,x^2\,(\,1 + 3\,ln\,x\,)$$

[3 marks]

(iii) $y = \dfrac{ln\,x}{2x}$ (Quotient Rule Example)

$$= \frac{1}{2}\left(\frac{ln\,x}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{x \times \frac{1}{x} - 1 \times ln\,x}{x^2}$$

$$= \frac{1}{2} \times \frac{1 - ln\,x}{x^2}$$

$$= \frac{1 - ln\,x}{2\,x^2}$$

[3 marks]

7.5.2 Solutions to 7.4 Exercise

Answer 1

(i) $y = 3\,ln\,x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{3}{x}$

(ii) $y = 4\,ln\,x^3$

$= 12\,ln\,x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{12}{x}$

(iii) $y = ln\,(5x)$

$= ln\,5 + ln\,x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{x}$

(iv) $y = 6\,ln\,(7x)$

$= 6\,ln\,7 + 6\,ln\,x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{6}{x}$

(v) $y = ln\left(4x^3 + 3\right) \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{12\,x^2}{4\,x^3 + 3}$

(vi) $y = (\,ln\,x\,)^3 \quad \Rightarrow \quad \dfrac{dy}{dx} = 3\,(\,ln\,x\,)^2 \times \dfrac{1}{x}$

$= \dfrac{3\,(\,ln\,x\,)^2}{x}$

(vii) $y = (\,ln\,x\,)^{-4} \quad \Rightarrow \quad \dfrac{dy}{dx} = -4\,(\,ln\,x\,)^{-5} \times \dfrac{1}{x}$

$= -\dfrac{4}{x\,(\,ln\,x\,)^5}$

(viii) $y = \dfrac{4}{ln\,x}$

$= 4\,(\,ln\,x\,)^{-1} \quad \Rightarrow \quad \dfrac{dy}{dx} = 4 \times (-1) \times (\,ln\,x\,)^{-2} \times \dfrac{1}{x}$

$= -\dfrac{4}{x\,(\,ln\,x\,)^2}$

[16 marks]

Answer 2

$$y = x \ln x^7$$
$$= 7x \ln x$$
$$\frac{dy}{dx} = 7\left(x \times \frac{1}{x} + 1 \times \ln x \right)$$
$$= 7(1 + \ln x)$$

[3 marks]

Answer 3

$$y = \frac{x}{\ln x}$$
$$\frac{dy}{dx} = \frac{\ln x \times 1 - \frac{1}{x} \times x}{(\ln x)^2}$$
$$= \frac{\ln x - 1}{(\ln x)^2}$$

[3 marks]

Answer 4

$$y = \frac{\ln x^3}{x^3}$$
$$= 3 \times \frac{\ln x}{x^3}$$
$$\frac{dy}{dx} = 3\left(\frac{x^3 \times \frac{1}{x} - 3x^2 \ln x}{x^6} \right)$$
$$= \frac{3x^2}{x^6}(1 - 3\ln x)$$
$$= \frac{3(1 - 3\ln x)}{x^4}$$

[3 marks]

Answer 5

(i)

$$f(x) = x(\ ln(x) - 1\ )$$

$$f'(x) = x \times \frac{1}{x} + 1 \times (\ ln(x) - 1\ )$$

$$= 1 + ln(x) - 1$$

$$= ln\, x$$

Asked to find where $f'(x) = 1$

$$ln\, x = 1$$

$$x = e^1$$

$$= e \qquad \text{(About 2.72)}$$

When $x = e,\ y = e\ (\ ln\, e - 1\ )$

$$= 0$$

The gradient is 1 at the point $A(\ e,\ 0\ )$ where the graph crosses the x-axis.

[3 marks]

(ii) For a turning point,

$$f'(x) = 0$$

$$ln\, x = 0$$

$$x = e^0$$

$$= 1$$

When $x = 1,\ y = 1 \times (\ ln(1) - 1\ )$

$$= -1$$

$\therefore$ Turning Point $B(\ 1, -1\ )$

$$f'(x) = ln\, x \Rightarrow f''(x) = \frac{1}{x} \quad \therefore f''(1) = 1$$

As $f''(1)$ is positive the turning point is a minimum.

[3 marks]

(iii) The tangent from the turning point will have equation $y = -1$

For the tangent from A, $y = mx + c$

$$0 = 1 \times e + c \quad \therefore c = -e$$

$$y = x - e$$

Intersection of tangents is where $x - e = -1$

$$x = e - 1$$

$\therefore$ Point of intersection is $(e - 1, -1\ )$

[3 marks]

Answer 6

(i)

$$g(x) = \frac{1}{4}x^2\left(2\,ln(x) - 1\right)$$

$$g'(x) = \frac{1}{4}\left(x^2\left(\frac{2}{x}\right) + 2x\left(2ln(x) - 1\right)\right)$$

$$= \frac{1}{4}\left(2x + 2x\left(2\,ln(x) - 1\right)\right)$$

$$= \frac{2x}{4}\left(1 + 2\,ln(x) - 1\right)$$

$$= \frac{x}{2}\left(2\,ln(x)\right)$$

$$= x\,ln\,x$$

$$\therefore g'(e) = e\,ln\,e$$

$$= e$$

[4 marks]

(ii) For a turning point,

$$f'(x) = 0$$

$$x\,ln\,x = 0$$

Either $x = 0$ (Rejected as $x > 0$)

or $ln\,x = 0$

$$x = e^0$$

$$= 1$$

$$g(1) = \frac{1}{4} \times 1^2 \times \left(2\,ln\,1 - 1\right)$$

$$= -\frac{1}{4}$$

$\therefore$ Turning Point $\left(1, -\frac{1}{4}\right)$

$$f'(x) = x\,ln\,x \Rightarrow f''(x) = x \times \frac{1}{x} + 1 \times ln\,x$$

$$\therefore f''(1) = 1$$

As $f''(1)$ is positive the turning point is a minimum.

[3 marks]

Answer 7

$$y = e^x \ln x$$

$$\frac{dy}{dx} = e^x \times \frac{1}{x} + e^x \ln x$$

$$= e^x \left(\frac{1}{x} + \ln x \right)$$

When $x = 1$, $y = 0$ and $\frac{dy}{dx} = e$

Tangent : $y = mx + c$

$$0 = e \times 1 + c \Rightarrow c = -e$$

The tangent has the equation, $y = e(x - 1)$

[3 marks]

Answer 8

(i)

$$y = \frac{\ln x}{x^2}$$

$$\frac{dy}{dx} = \frac{x^2 \times \frac{1}{x} - 2x \ln x}{x^4}$$

$$= \frac{x - 2x \ln x}{x^4}$$

$$= \frac{x(1 - 2 \ln x)}{x^4}$$

$$= \frac{1 - 2 \ln x}{x^3}$$

[3 marks]

(ii) When $x = 1$, $y = 0$, $\frac{dy}{dx} = 1$

$$y = mx + c$$

$$0 = 1 + c \Rightarrow c = -1$$

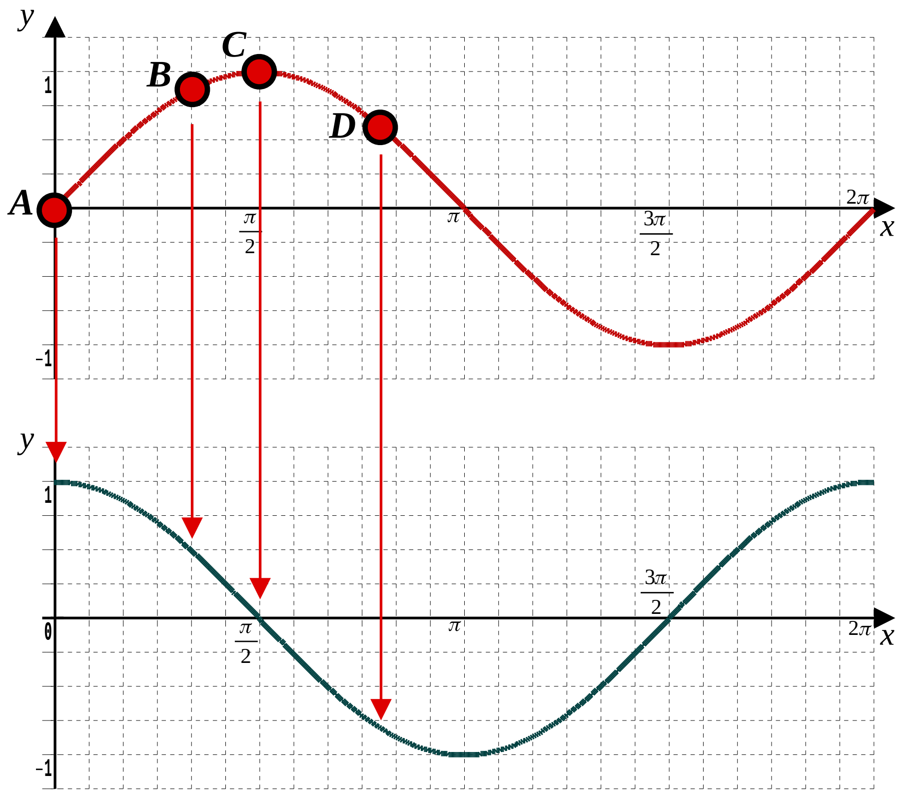
Equation of the tangent is $y = x - 1$

[3 marks]

Lesson 8

8.1 The Sine Function

The upper graph shows the curve $y = \sin x$ with x measured in radians.
Of interest is the gradient of this graph.



Consider what the gradient of the upper graph is doing at the four points labelled.
A : The curve at the origin seems to be sloping like the line $y = x$, gradient 1.
B : The gradient between A and B is positive and reducing in magnitude.
C : At this maximum turning point the gradient has fallen to zero.
D : The gradient between C and D is negative and increasing in magnitude.

Continuing in this manner, and plotting the gradients as a separate graph, the lower graph is obtained. It looks very much like the graph of cosine !

This argument, although not a proof, is an intuitive reasoning of the important result that, provided radians are used,

---

Derivative of Sine

If $y = \sin x$ then $\dfrac{dy}{dx} = \cos x$ where x is in radians

---

8.2 Differentiating Sine Functions

The Product Rule and The Quotient Rule can be applied to situations where the sine function is involved. So too can The Chain Rule, as follows,

---

The Chain Rule for $\boldsymbol{y = sin(f(x))}$

$$\text{If } y = sin(f(x)) \text{ then } \frac{dy}{dx} = cos(f(x)) \times f'(x)$$

8.3 Examples

Differentiate each of the following,

(i) $y = sin^4 3x$ (Chain Rule Example)

(ii) $y = x\,sin\left(x^2\right)$ (Product Rule Example)

(iii) $y = \dfrac{sin\,(2x)}{e^{2x}}$ (Quotient Rule Example)

Teaching Video : http://www.NumberWonder.co.uk/v9028/8.mp4

Watch the video and then write out the solutions here

☞

[3, 3, 3 marks]

8.4 Exercise

Marks Available : 40

Question 1

Differentiate each of the following with respect to *x*,

(i) $y = 7\,sin\,(5x)$

(ii) $y = 6\,sin\left(3\,x^3\right)$

[2, 2 marks]

(iii) $y = 2\,sin^3 x$

(iv) $y = \sqrt{sin\,x}$

[2, 2 marks]

(v) $y = 11\,e^{sin\,x}$

(vi) $y = 3\,e^{2\,sin(5x)}$

[2, 2 marks]

(vii) $y = ln\,(sin\,x)$

(viii) $y = sin\,(ln\,x)$

[2, 2 marks]

Question 2

By writing $y = csc\, x$ as $y = (\sin x)^{-1}$ and using The Chain Rule, show that,

---

Derivative of *csc x*

If $y = csc\, x$ then $\dfrac{dy}{dx} = -\,csc\, x \cot x$ where x is in radians

---

[4 marks]

Question 3

$$f(x) = x \sin x$$

(i) Remembering to use radians, find the exact value of $f\left(\dfrac{\pi}{6}\right)$

[2 marks]

(ii) Use The Product Rule to differentiate $f(x)$

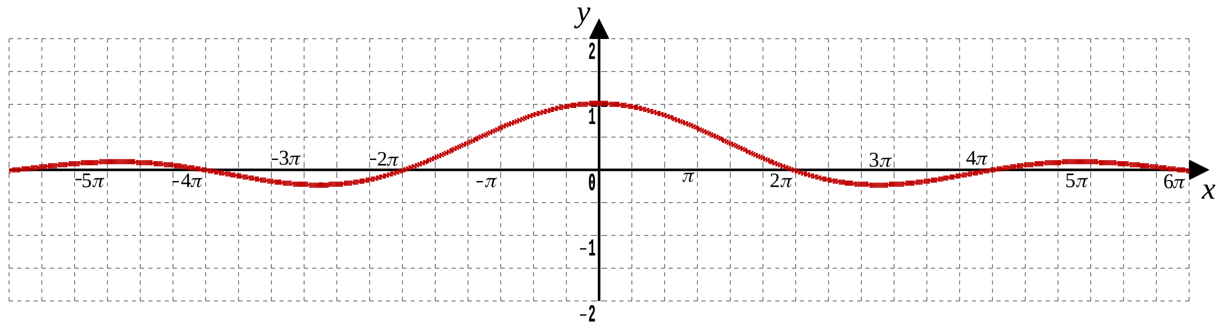
[2 marks]

(iii) Show that $f'\left(\dfrac{\pi}{6}\right) = \dfrac{6 + \sqrt{3}\,\pi}{12}$

[2 marks]

Question 4

The graph is of the important function, $f(x) = \dfrac{\sin x}{x}$



(i) The graph suggests a key result about the value of $\lim_{x \to 0} \dfrac{\sin x}{x}$

Explain where one should look on the graph for the result and what it is.

[2 marks]

(ii) Use The Quotient Rule to differentiate $f(x)$

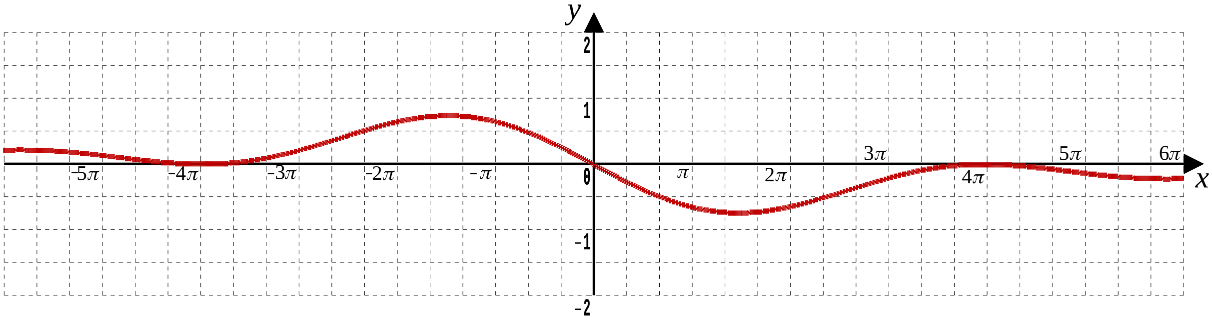
[2 marks]

(iii) Determine the exact value of $f'(\pi)$

[2 marks]

Question 5

The graph is of another important function, $f(x) = \dfrac{\cos x - 1}{x}$



Use the graph to deduce a key result for $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x}$

Explain where you were looking on the graph to deduce your result.

[2 marks]

Question 6

Prove from first principles that the derivative of *sin x* is *cos x*

Make use of the limit studied in Question 4 and the limit studied in Question 5 along with the following video from "The Math Sorcerer"

Teaching Video : http://www.NumberWonder.co.uk/v9028/8b.mp4

[6 marks]

8.5 Answers

8.5.1 Solutions to 8.3 Examples (Teaching Video)

(i) $y = \sin^4 3x$ (Chain Rule Example)

$$= (\sin 3x)^4$$

$$\frac{dy}{dx} = 4(\sin 3x)^3 \times (\cos 3x) \times 3$$

$$= 12 \sin^3 3x \cos 3x$$

[3 marks]

(ii) $y = x \sin\left(x^2\right)$ (Product Rule Example)

$$\frac{dy}{dx} = x \cos\left(x^2\right) \times 2x + 1 \times \sin\left(x^2\right)$$

$$= 2x^2 \cos\left(x^2\right) + \sin\left(x^2\right)$$

[3 marks]

(iii) $y = \dfrac{\sin(2x)}{e^{2x}}$ (Quotient Rule Example)

$$\frac{dy}{dx} = \frac{e^{2x} \times \cos(2x) \times 2 - 2e^{2x} \times \sin(2x)}{e^{2x} \times e^{2x}}$$

$$= \frac{2e^{2x}(\cos(2x) - \sin(2x))}{e^{2x} \times e^{2x}}$$

$$= \frac{2(\cos(2x) - \sin(2x))}{e^{2x}}$$

[3 marks]

8.5.2 Solutions to 8.4 Exercise

Answer 1

(i) $y = 7\,sin\ (5x) \quad \Rightarrow \quad \dfrac{dy}{dx} = 35\,cos\ (5x)$

(ii) $y = 6\,sin\left(3x^3\right) \quad \Rightarrow \quad \dfrac{dy}{dx} = 54\,x^2\,cos\left(3x^3\right)$

(iii) $y = 2\,sin^3 x$

$$= 2\,(\,sin\,x\,)^3 \quad \Rightarrow \quad \frac{dy}{dx} = 6\ (\,sin\,x\,)^2\ cos\,x$$

$$= 6\,sin^2 x\,cos\,x$$

(iv) $y = \sqrt{sin\,x}$

$$= (\,sin\,x\,)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\,(\,sin\,x\,)^{-\frac{1}{2}}\ cos\,x$$

$$= \frac{cos\,x}{2\sqrt{sin\,x}}$$

(v) $y = 11\,e^{sin\,x} \quad \Rightarrow \quad \dfrac{dy}{dx} = 11\,e^{sin\,x}\,cos\,x$

(vi) $y = 3\,e^{2sin\,(5x)} \quad \Rightarrow \quad \dfrac{dy}{dx} = 3\,e^{2\,sin\,(5x)} \times 2\,cos\ (5x) \times 5$

$$= 30\,e^{2\,sin\,(5x)}\,cos\ (5x)$$

(vii) $y = ln\,(sin\,x) \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{sin\,x} \times cos\,x$

$$= cot\,x$$

(viii) $y = sin\,(ln\,x\,) \quad \Rightarrow \quad \dfrac{dy}{dx} = cos\,(\,ln\,x\,) \times \dfrac{1}{x}$

$$= \frac{cos\,(\,ln\,x\,)}{x}$$

[16 marks]

Answer 2

$$y = csc\,x$$

$$= \frac{1}{sin\,x}$$

$$= (\,sin\,x\,)^{-1}$$

$$\frac{dy}{dx} = (-1)\ (\,sin\,x\,)^{-2}\ cos\,x$$

$$= -\frac{1}{sin\,x} \times \frac{cos\,x}{sin\,x}$$

$$= -\,csc\,x\,cot\,x \qquad \square$$

[4 marks]

Answer 3

(i)

$$f(x) = x \sin x$$

$$f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} \times \sin\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{6} \times \frac{1}{2}$$

$$= \frac{\pi}{12}$$

[2 marks]

(ii)

$$f'(x) = x \cos x + \sin x$$

[2 marks]

(iii)

$$f'\left(\frac{\pi}{6}\right) = \frac{\pi}{6} \cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{6} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{6}{6}$$

$$= \frac{6 + \sqrt{3}\,\pi}{12} \qquad \square$$

[2 marks]

Answer 4

(i) Look at the graph as it approaches the *y*-axis from both the left and the right. In both cases it's smoothly heading towards the number 1 on the *y*-axis.

[2 marks]

(ii)

$$f(x) = \frac{\sin x}{x}$$

$$f'(x) = \frac{x \cos x - \sin x}{x^2}$$

[2 marks]

(iii)

$$f'(\pi) = \frac{\pi \cos \pi - \sin \pi}{\pi^2}$$

$$= \frac{\pi \times (-1) - 0}{\pi^2}$$

$$= \frac{-\pi}{\pi^2}$$

$$= -\frac{1}{\pi}$$

[2 marks]

Answer 5
Look at the graph as it approaches the *y*-axis from both the left and the right.
In both cases it's smoothly heading towards the number 0 on the *y*-axis.

[2 marks]

Answer 6
By definition,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore (\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h}$$

$$= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

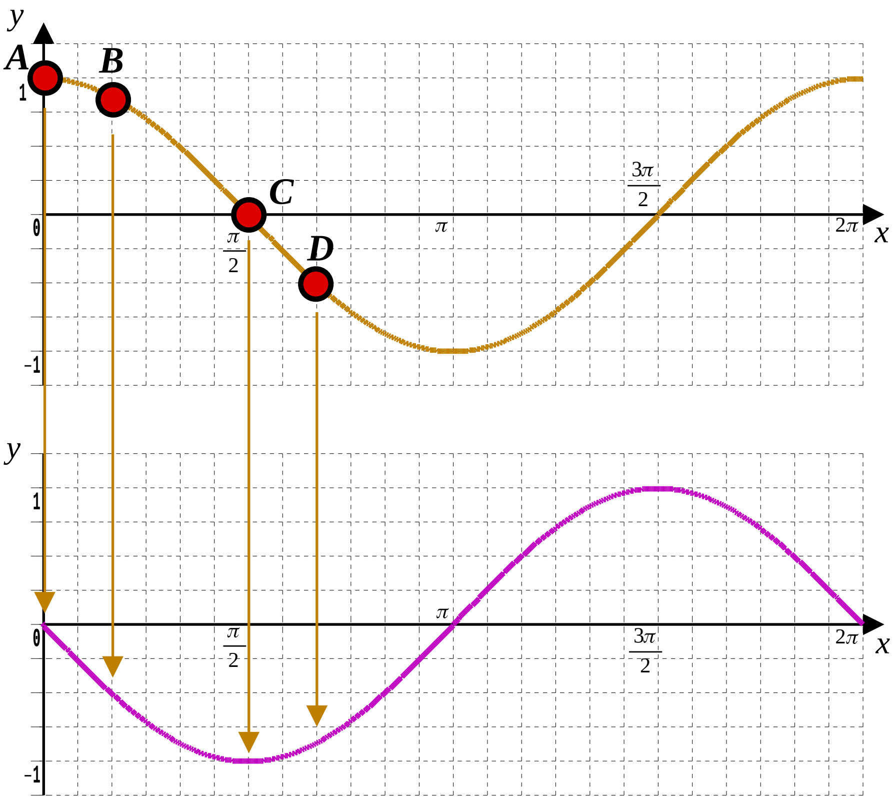
$$= \sin x \times 0 + \cos x \times 1$$

$$= \cos x \qquad \square$$

[6 marks]

9.1 The Cosine Function

The upper graph shows the curve $y = \cos x$ with x measured in radians.
Of interest is the gradient of this graph.



Consider what the gradient of the upper graph is doing at the four points labelled.
A : At this maximum turning point the gradient is zero.
B : The gradient between A and B is negative and increasing in magnitude.
C : The curve seems to be sloping like the line $y = -x$, gradient -1
D : The gradient between C and D is negative and decreasing in magnitude.

Continuing in this manner, and plotting the gradients as a separate graph, the lower graph is obtained. It looks very much like the graph of $-\sin x$

This argument, although not a proof, is an intuitive reasoning of the important result that, provided radians are used,

---

Derivative of Cosine

If $y = \cos x$ then $\dfrac{dy}{dx} = -\sin x$ where x is in radians

---

9.2 Differentiating Cosine Functions

The Product Rule and The Quotient Rule can be applied to situations where the cosine function is involved. So too can The Chain Rule, as follows,

---

The Chain Rule for $\boldsymbol{y = cos\,(f\,(x))}$

$$\text{If } y = cos\,(\,f\,(x)\,) \text{ then } \frac{dy}{dx} = -\,sin\,(\,f\,(x)\,) \times f'(x)$$

9.3 Example

Prove that a consequence of the derivative of $sin\,x$ being $cos\,x$ is that the derivative of $cos\,x$ is $(-\,sin\,x)$

Teaching Video : http://www.NumberWonder.co.uk/v9028/9.mp4

Watch the video and then write out the proof here

☞

[6 marks]

9.4 Exercise

Marks Available : 40

Question 1

Differentiate each of the following with respect to *x*,

(i) $y = cos\left(3x^5 + 2e^{3x} \right)$ **(ii)** $y = 5x^7 + cos^4(3x)$

[2, 2 marks]

(iii) $y = cos\left(\frac{5}{x} + ln\, x \right), \quad x > 0$ **(iv)** $y = sin\left(1 - cos\, x \right)$

[2, 2 marks]

Question 2

By writing $y = sec\, x$ as $y = (cos\, x)^{-1}$ and using The Chain Rule, show that,

---

Derivative of *sec x*

If $y = sec\, x$ then $\frac{dy}{dx} = sec\, x\, tan\, x$ where *x* is in radians

---

[3 marks]

Question 3

By writing $y = tan\,x$ as $y = \dfrac{sin\,x}{cos\,x}$ and using The Quotient Rule, show that,

---

Derivative of *tan x*

If $y = tan\,x$ then $\dfrac{dy}{dx} = sec^2\,x$ where x is in radians

---

[3 marks]

Question 4

By writing $y = cot\,x$ as $y = \dfrac{cos\,x}{sin\,x}$ and using The Quotient Rule, show that,

---

Derivative of *cot x*

If $y = cot\,x$ then $\dfrac{dy}{dx} = -\,csc^2\,x$ where x is in radians

---

[3 marks]

Question 5

Table of Derivatives of Trigonometric Functions

$f(x)$	$f'(x)$	Told in Exam ?
$\sin x$	$\cos x$	No
$\cos x$	$-\sin x$	No
$\csc x$	$-\csc x \cot x$	Yes
$\sec x$	$\sec x \tan x$	Yes
$\tan x$	$\sec^2 x$	Yes
$\cot x$	$-\csc^2 x$	Yes

Use the above table of standard derivatives to differentiate each of the following with respect to x,

(i) $y = \tan(5x)$

(ii) $y = \sec\left(3x^4 + x^2\right)$

[2, 2 marks]

(iii) $y = \cot\left(5 - \ln x\right)$

(iv) $y = e^{\tan 4x}$

[2, 2 marks]

(v) $y = \ln(\sec x + \tan x)$

Simplify your answer as much as possible

[2 marks]

Question 6

From the small angle approximations it is known that,

- $\sin\theta \approx \theta$
- $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$
- $\tan\theta \approx \theta$

This allows a couple of useful deductions to be made, firstly that,

$$\underset{h\to 0}{limit}\ \frac{\sin h}{h} = \underset{h\to 0}{limit}\ \frac{h}{h} = 1$$

and secondly, that,

$$\underset{h\to 0}{limit}\ \frac{\cos h - 1}{h} = \underset{h\to 0}{limit}\ \frac{1 - \frac{1}{2}h^2 - 1}{h} = \underset{h\to 0}{limit}\left(-\frac{1}{2}h\right) \to 0$$

(i) Given that $f(x) = \cos x$, show from first principles that,

$$f'(x) = \underset{h\to 0}{limit}\left(\left(\frac{\cos h - 1}{h}\right)\cos x - \frac{\sin h}{h}\sin x\right)$$

[4 marks]

(ii) Hence prove that $f'(x) = -\sin x$

[2 marks]

Question 7

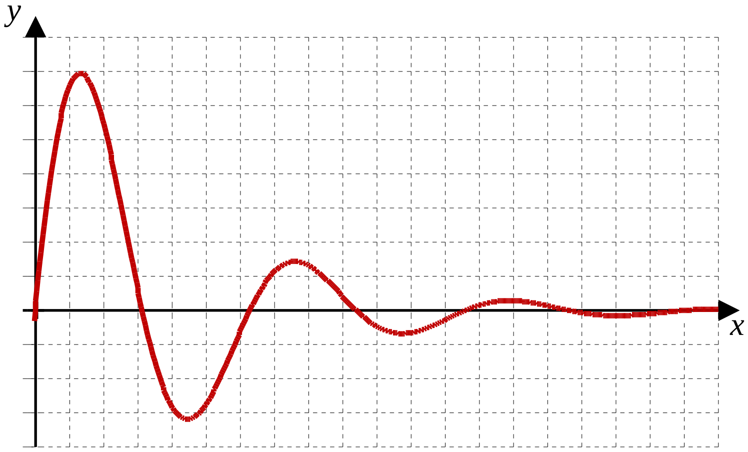
Given that $f(x) = csc\,x$ find the exact value of $f'\left(\frac{\pi}{6}\right)$

Simplify your answer.

[3 marks]

Question 8

A-Level Examination Question from June 2019, Paper 1, Q12(a)



$$f(x) = 10\,e^{-0.25x}\,sin\,x \qquad\qquad x \geqslant 0$$

Show that the x coordinates of the turning points of the curve with equation $y = f(x)$ satisfy the equation $tan\,x = 4$

[4 marks]

9.5 Answers

9.5.1 Solution to 9.3 Example (Teaching Video)

$$y = \cos x$$

$$y = (1 - \sin^2 x)^{\frac{1}{2}} \qquad \text{Using } \cos^2 x + \sin^2 x = 1$$

$$\frac{dy}{dx} = \frac{1}{2}(1 - \sin^2 x)^{-\frac{1}{2}}(-2\sin x)(\cos x)$$

$$\frac{dy}{dx} = -\frac{\sin x \cos x}{\sqrt{1 - \sin^2 x}}$$

$$\frac{dy}{dx} = -\frac{\sin x \cos x}{\cos x}$$

$$\frac{dy}{dx} = -\sin x \qquad \square$$

[6 marks]

9.5.2 Solutions to 9.4 Exercise

Answer 1

(i) $y = \cos(3x^5 + 2e^{3x})$

$$\Rightarrow \frac{dy}{dx} = -(15x^4 + 6e^{3x})\sin(3x^5 + 2e^{3x})$$

(ii) $y = 5x^7 + \cos^4(3x)$

$$= 5x^7 + (\cos(3x))^4$$

$$\Rightarrow \frac{dy}{dx} = 35x^6 + 4(\cos(3x))^3(-\sin(3x))\,3$$

$$= 35x^6 - 12\cos^3(3x)\sin(3x)$$

(iii) $y = \cos\left(\frac{5}{x} + \ln x\right)$

$$\Rightarrow \frac{dy}{dx} = -\sin\left(\frac{5}{x} + \ln x\right)\left(-5x^{-2} + \frac{1}{x}\right)$$

$$= -\sin\left(\frac{5}{x} + \ln x\right)\left(-\frac{5}{x^2} + \frac{1}{x}\right)$$

(iv) $y = \sin(1 - \cos x)$

$$\Rightarrow \frac{dy}{dx} = \cos(1 - \cos x)(\sin x)$$

[2, 2, 2, 2 marks]

Answer 2

$$\begin{aligned} y &= sec\,x \\ &= \frac{1}{cos\,x} \\ &= (\,cos\,x\,)^{-1} \\ \frac{dy}{dx} &= (-1)\;(\,cos\,x\,)^{-2}(-\,sin\,x\,) \\ &= \frac{1}{cos\,x} \times \frac{sin\,x}{cos\,x} \\ &= sec\,x\,tan\,x \qquad \square \end{aligned}$$

[3 marks]

Answer 3

$$\begin{aligned} y &= tan\,x \\ &= \frac{sin\,x}{cos\,x} \\ \frac{dy}{dx} &= \frac{cos\,x\,cos\,x\; - \;(-\,sin\,x\,)\;sin\,x}{cos^2\,x} \\ &= \frac{cos^2\,x \;+\; sin^2\,x}{cos^2\;x} \\ &= \frac{1}{cos^2\,x} \\ &= sec^2\,x \qquad \square \end{aligned}$$

[3 marks]

Answer 4

$$\begin{aligned} y &= cot\,x \\ &= \frac{cos\,x}{sin\,x} \\ \frac{dy}{dx} &= \frac{sin\,x\;(-\,sin\,x\,)\; - \;cos\,x\,cos\,x}{sin^2\,x} \\ &= \frac{(-\,1\,)\left(\,sin^2\,x \;+\; cos^2\,x\,\right)}{sin^2\,x} \\ &= -\,\frac{1}{sin^2\,x} \\ &= -\,csc^2\,x \qquad \square \end{aligned}$$

[3 marks]

Answer 5

(i) $y = tan\ (5x)$

$$\Rightarrow \frac{dy}{dx} = 5\, sec^2 (5x)$$

(ii) $y = sec\left(3x^4 + x^2 \right)$

$$\Rightarrow \frac{dy}{dx} = \left(12x^3 + 2x \right) sec\left(3x^4 + x^2 \right) tan\left(3x^4 + x^2 \right)$$

(iii) $y = cot\left(5 - ln\, x \right)$

$$\Rightarrow \frac{dy}{dx} = -csc^2 \left(5 - ln\, x \right) \times \left(-\frac{1}{x} \right)$$

$$= \frac{csc^2 \left(5 - ln\, x \right)}{x}$$

(iv) $y = e^{tan\, 4x}$

$$\Rightarrow \frac{dy}{dx} = e^{tan\, 4x}\, sec^2 (4x)\ 4$$

$$= 4\, sec^2 (4x)\ e^{tan\, 4x}$$

(v) $y = ln\left(sec\, x + tan\, x \right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{sec\, x + tan\, x} \times \left(sec\, x\, tan\, x + sec^2\, x \right)$$

$$= \frac{1}{sec\, x + tan\, x} \times sec\, x \left(tan\, x + sec\, x \right)$$

$$= sec\, x$$

[10 marks]

Answer 6

(i) Starting from the definition,

$$f'(x) = \underset{h \to 0}{limit} \frac{f(x + h) - f(x)}{h}$$

$$\therefore \left(sin\, x \right)' = \underset{h \to 0}{limit} \frac{cos(x + h) - cos(x)}{h}$$

$$= \underset{h \to 0}{limit} \frac{cos\, x\, cos\, h - sin\, x\, sin\, h - cos\, x}{h}$$

$$= \underset{h \to 0}{limit} \left(\frac{cos\, x\, cos\, h - cos\, x}{h} - \frac{sin\, x\, sin\, h}{h} \right)$$

$$= \underset{h \to 0}{limit} \left(\left(\frac{cos\, h - 1}{h} \right) cos\, x - \frac{sin\, h}{h}\, sin\, x \right)$$

[4 marks]

(ii)

$$= \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \cos x \times 0 - \sin x \times 1$$

$$= -\sin x \qquad \square$$

[2 marks]

Answer 7

$$f(x) = \csc x$$

$$f'(x) = -\csc x \cot x$$

$$f'\left(\frac{\pi}{6}\right) = -\frac{1}{\sin\left(\frac{\pi}{6}\right)} \times \frac{1}{\tan\left(\frac{\pi}{6}\right)}$$

$$= -2 \times \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= -2\sqrt{3}$$

[3 marks]

Answer 8

$$f(x) = 10\, e^{-0.25x} \sin x$$

$$f'(x) = 10\, e^{-0.25x} \cos x - 2.5\, e^{-0.25x} \sin x$$

For turning points, $f'(x) = 0$

$$\therefore\ 10\, e^{-0.25x} \cos x - 2.5\, e^{-0.25x} \sin x = 0$$

$$2.5\, e^{-0.25x} \left(4 \cos x - \sin x \right) = 0$$

Either $2.5\, e^{-0.25x} = 0$ which has no solutions,

or $\quad 4 \cos x - \sin x = 0$

$$\sin x = 4 \cos x$$

That is, $\quad \tan x = 4 \qquad \square$

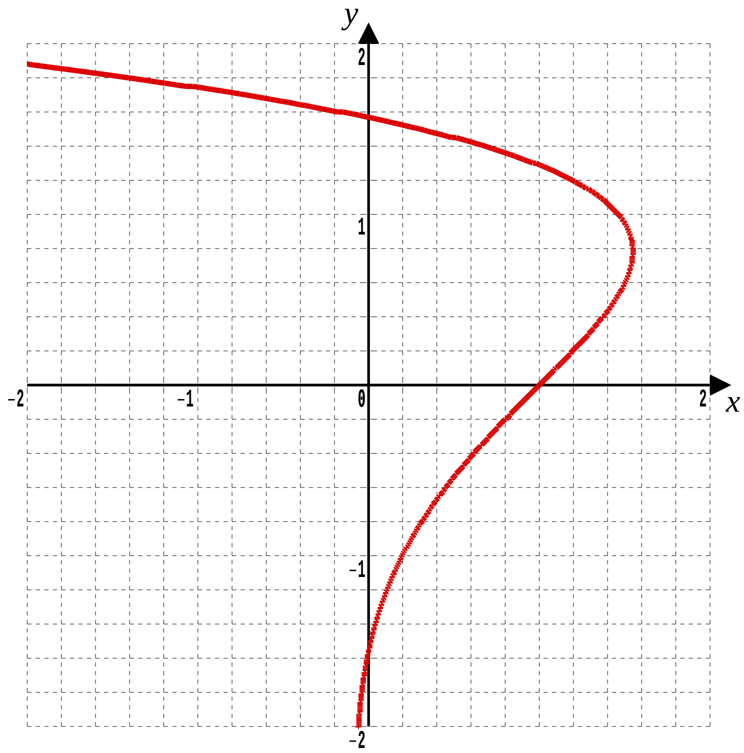
[4 marks]

Lesson 10

A-Level Pure Mathematics : Year 2
Differentiation III

10.1 Differentiating $x = f(y)$

The graph is of the curve with equation $x = e^{y} \cos y$ with y in radians.



Video : http://www.NumberWonder.co.uk/v9028/10a.mp4

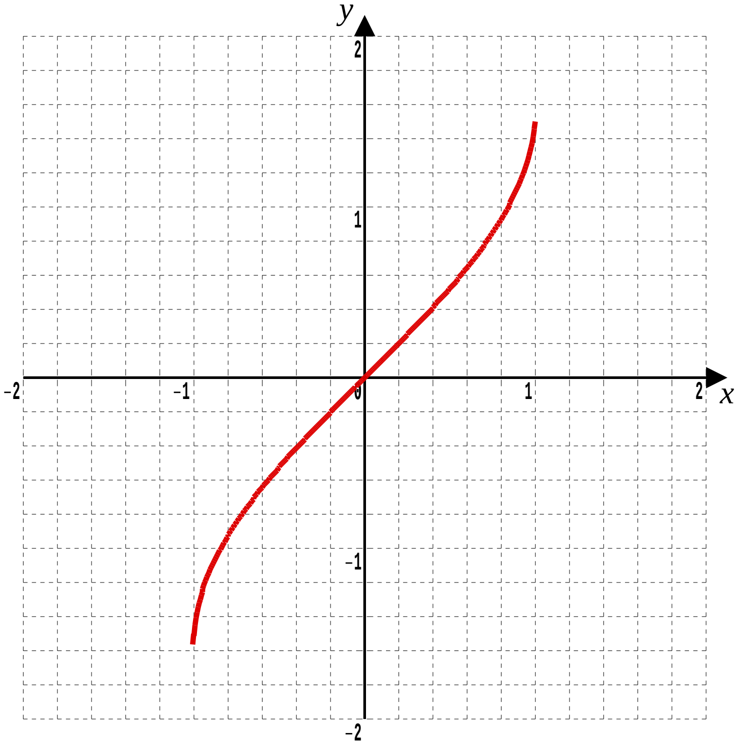
(i) Obtain an expression for $\dfrac{dy}{dx}$ in terms of y

(ii) What is the equation of the normal to the curve when $y = 0$?

(iii) Add your part (ii) normal onto the graph above

[2, 2, 2 marks]

10.2 Differentiating *arcsin x*



The graph of $y = arcsin\ x$

It is the inverse of a one-to-one piece of the *sin x* function

---

$$y = arcsin\, x \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Proof

Teaching Video : http://www.NumberWonder.co.uk/v9028/10b.mp4

Watch the video and then write out the proof here

☞

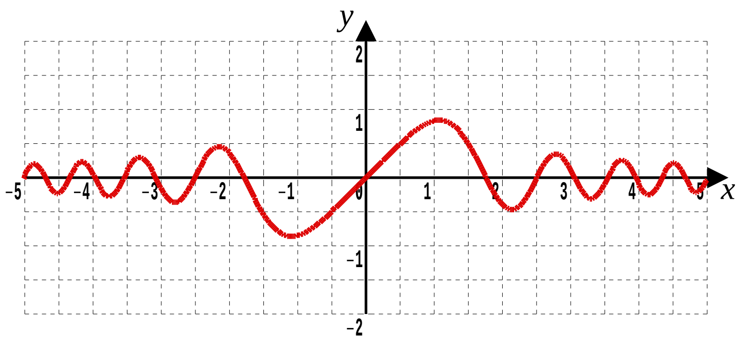
[6 marks]

10.3 Exercise

Marks Available : 40

Question 1

The graph is of the curve with equation $y = \dfrac{\sin(x^2)}{x}$ with x in radians.



(i) Obtain an expression for $\dfrac{dy}{dx}$ in terms of x

[3 marks]

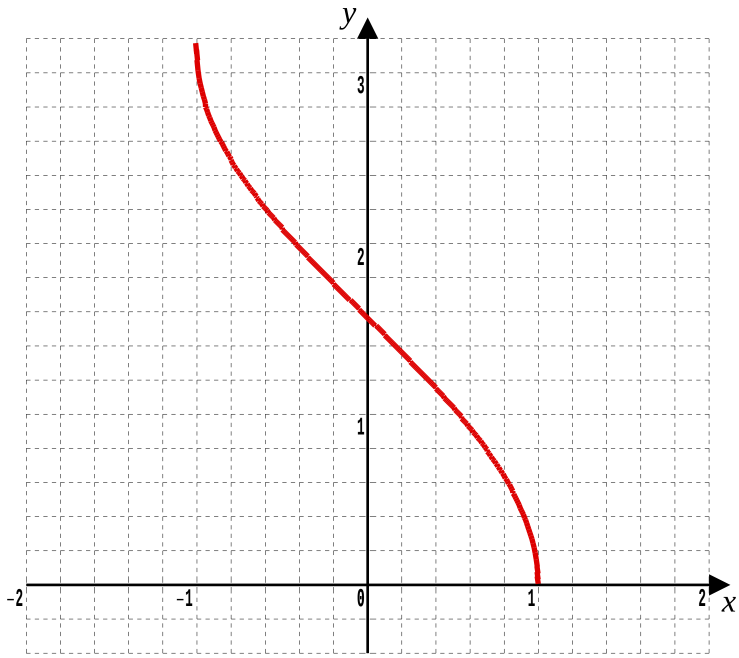
(ii) What is the ***exact*** equation of the tangent to the curve when $x = \sqrt{\pi}$

[5 marks]

(iii) Add your part (ii) tangent onto the graph above

[1 mark]

Question 2



The graph of $y = arccos\ x$

It is the inverse of a one-to-one piece of the *cos x* function

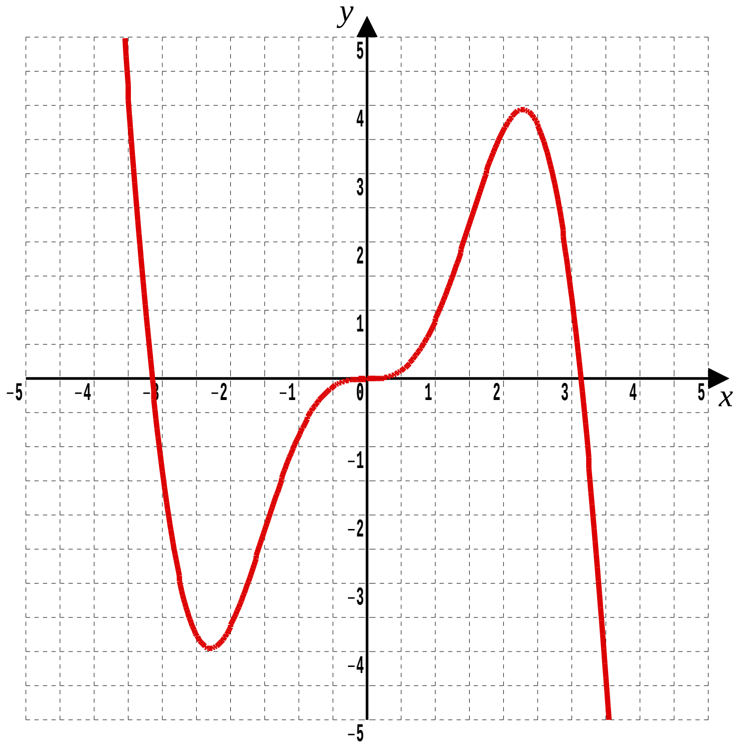
$$y = arccos\, x \Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

Assuming standard results for *sin x* and *cos x* prove the above result.

[7 marks]

Question 3

The graph is of a piece of the curve with equation $y = x^2 \sin x$



(i) Obtain an expression for $\dfrac{dy}{dx}$ in terms of x

[3 marks]

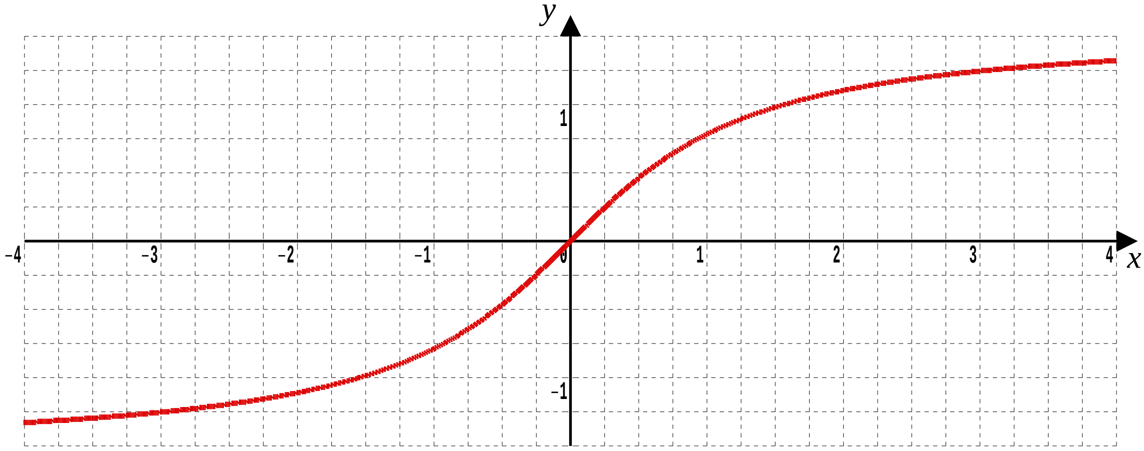
(ii) What is the ***exact*** equation of the normal to the curve when $x = \pi$

[6 marks]

(iii) Add your part (ii) normal onto the graph above

[1 mark]

Question 4



The graph of $y = arctan\ x$

It is the inverse of a one-to-one piece of the tan x function

$$y = arctan\ x \Rightarrow \frac{dy}{dx} = \frac{1}{1 + x^2}$$

(i) Show that if $y = tan\ x$ then $\frac{dy}{dx} = sec^2\ x$ by using the derivatives of $sin\ x$ and $cos\ x$ and the quotient rule.

[2 marks]

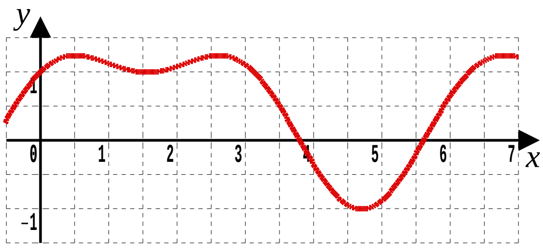
(ii) Hence prove that if $y = arctan\ x$ then $\frac{dy}{dx} = \frac{1}{1 + x^2}$

[6 marks]

Question 5

A curve has equation, $y = cos^2 x + sin\,x \qquad 0 < x < 2\pi$

Find the coordinates of its stationary points.



[6 marks]

10.4 Answers

10.4.1 Solutions to 10.1 Example (First Teaching Video)

(i)

$$x = e^{y}\cos y$$

$$\frac{dx}{dy} = e^{y}(-\sin y) + e^{y}\cos y \qquad \text{By The Product Rule}$$

$$\frac{dx}{dy} = e^{y}(\cos y - \sin y)$$

$$\frac{dy}{dx} = \frac{1}{e^{y}(\cos y - \sin y)}$$

(ii) When $y = 0$

$x = e^{0}\cos 0 = 1$ so $(1,\ 0)$ is a point on the normal

Also, when $y = 0$

$$\frac{dy}{dx} = \frac{1}{e^{0}(\cos 0 - \sin 0)} = 1$$

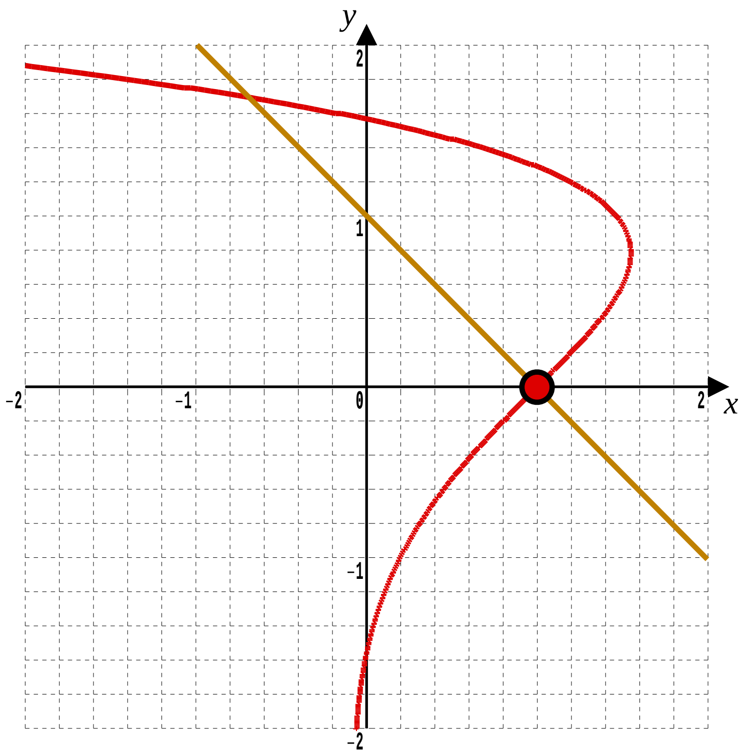
$\therefore$ Gradient of normal will be the "sign changed reciprocal" of (-1)

$$y = mx + c$$

$$0 = -1 \times 1 + c \qquad \Rightarrow c = 1$$

Normal has equation $y = -x + 1$ (That is, $y = 1 - x$)

(iii)



[2, 2, 2 marks]

10.4.2 Proof for 10.2 Differentiating *arcsin x*

Reciprocal Gradients Rule

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1 \qquad \Leftrightarrow \qquad \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

$$y = arcsin\, x$$

$$\therefore x = sin\, y$$

$$\frac{dx}{dy} = cos\, y$$

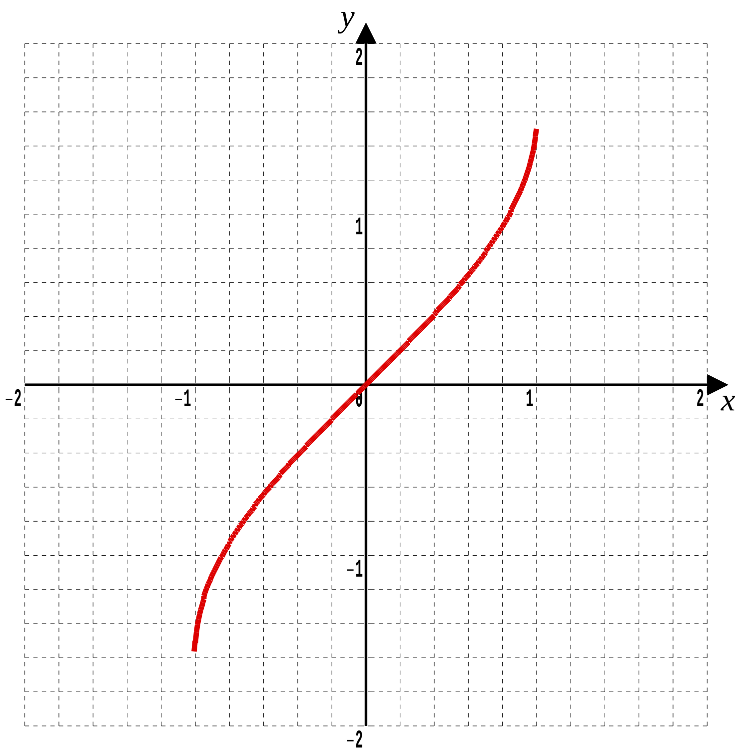
Now use the fact that $cos^2 y + sin^2 y = 1$ to get...

$$\frac{dx}{dy} = \pm\sqrt{1 - sin^2 y}$$

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1 - x^2}}$$

From the graph of $y = arcsin\, x$ we can see that the gradient is always positive

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \qquad \square$$



[6 marks]

10.4.3 Solutions to 10.3 Exercise

Answer 1

(i)

$$y = \frac{\sin(x^2)}{x}$$

$$\frac{dy}{dx} = \frac{x\cos(x^2)\,2x - \sin(x^2)}{x^2} \qquad \text{By The Quotient Rule}$$

$$\frac{dy}{dx} = \frac{2x^2\cos(x^2) - \sin(x^2)}{x^2}$$

[3 marks]

(ii) When $x = \sqrt{\pi}$

$$y = \frac{\sin(\pi)}{\sqrt{\pi}} = 0 \quad \text{so } \left(\sqrt{\pi},\ 0\right) \text{ is a point on the tangent}$$

Also, when $x = \sqrt{\pi}$

$$\frac{dy}{dx} = \frac{2\pi\cos(\pi) - \sin(\pi)}{\pi} = -2$$

$\therefore$ Gradient of tangent will be -2

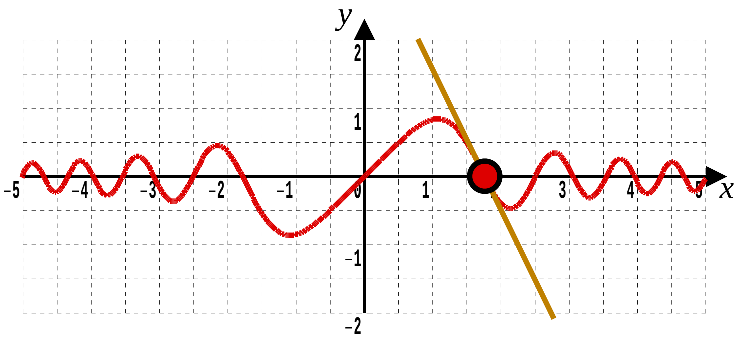
$$y = mx + c$$

$$0 = -2 \times \sqrt{\pi} + c \qquad \Rightarrow \quad c = 2\sqrt{\pi}$$

Tangent has equation $y = -2x + 2\sqrt{\pi}$

[5 marks]

(iii)



[1 mark]

Answer 2

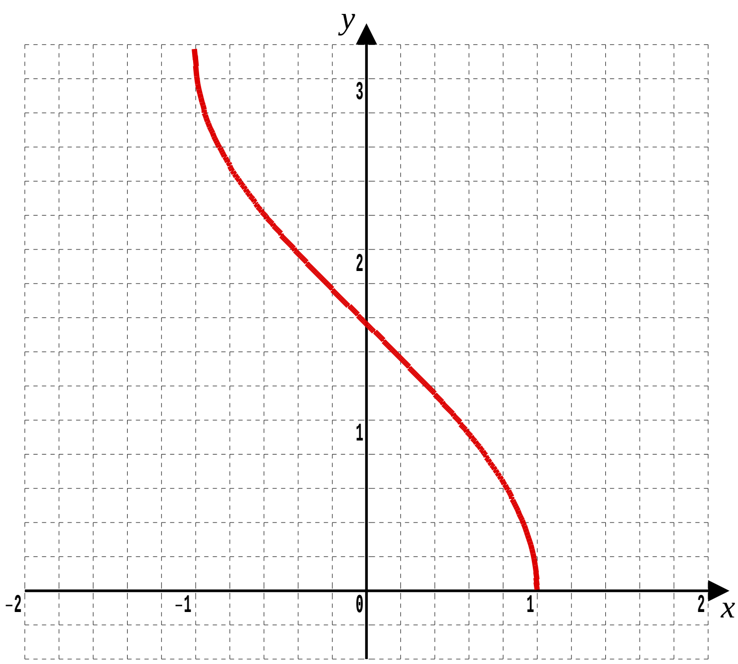
$$y = arccos\,x$$
$$\therefore x = cos\,y$$
$$\frac{dx}{dy} = -sin\,y$$

Now use the trigonometry identity $cos^2 y + sin^2 y = 1$ to get...

$$\frac{dx}{dy} = \pm\sqrt{1 - cos^2 y}$$
$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1 - x^2}}$$

From the graph of $y = arccos\,x$ it can be seen that the gradient is always negative

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}} \qquad \square$$



[7 marks]

Answer 3

(i)

$$y = x^2 \sin x$$

$$\frac{dy}{dx} = x^2 \cos x + 2x \sin x \qquad \text{by The Product Rule}$$

[3 marks]

(ii) When $x = \pi$

$$y = \pi^2 \sin \pi = 0 \quad \text{so} \ (\pi, \ 0) \ \text{is a point on the normal}$$

Also, when $x = \pi$

$$\frac{dy}{dx} = \pi^2 \cos \pi + 2\pi \sin \pi = -\pi^2$$

$\therefore$ Gradient of normal will be the "sign changed reciprocal", $\frac{1}{\pi^2}$

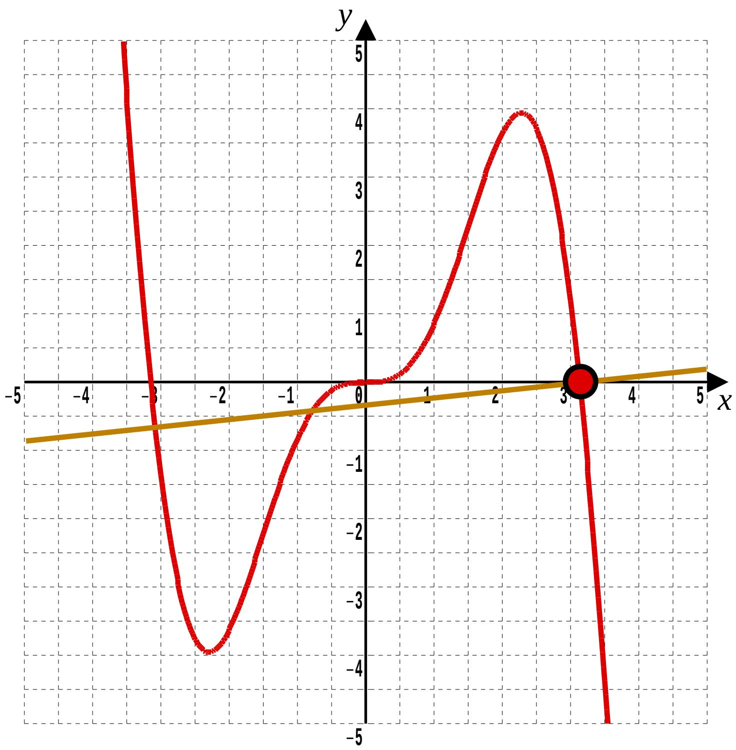
$$y = mx + c$$

$$0 = \frac{1}{\pi^2} \times \pi + c \qquad \Rightarrow c = -\frac{1}{\pi}$$

Normal has equation $y = \frac{x}{\pi^2} - \frac{1}{\pi}$

[6 marks]

(iii)



[1 mark]

Answer 4

(i)

$$y = \tan x$$

$$y = \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = \frac{\cos x \ \cos x \ - \ (-\sin x) \ \sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x \qquad \square$$

[2 marks]

(ii)

$$y = \arctan x$$

$$\therefore x = \tan y$$

$$\frac{dx}{dy} = \sec^2 y$$

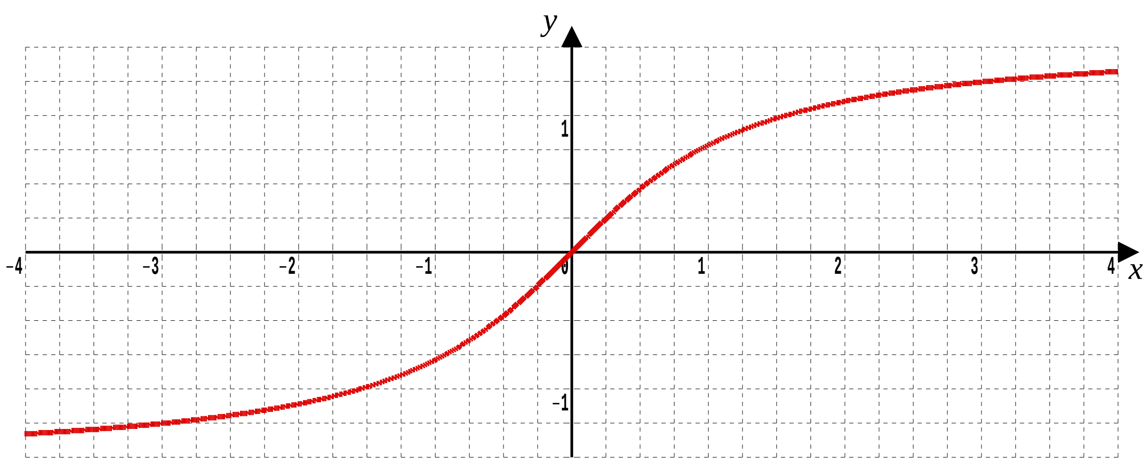
Now use the fact that $1 + \tan^2 y = \sec^2 y$ to get...

$$\frac{dx}{dy} = 1 + \tan^2 y$$

$$\frac{dx}{dy} = 1 + x^2$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2} \qquad \square$$

[6 marks]



Answer 5

$$y = cos^2 x + sin\,x$$

$$\frac{dy}{dx} = 2\,cos\,x\ (-\,sin\,x) + cos\,x$$

$$= cos\,x\ (\ 1 - 2\,sin\,x\ )$$

For stationary points...

Either $cos\,x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ (maxima)

or $sin\,x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$ (minima)

[6 marks]

Lesson 11

A-Level Pure Mathematics : Year 2
Differentiation III

11.1 Revision

Marks Available : 40

Table of Standard Derivatives

$f(x)$	$f'(x)$	In Formula Book ?
x^n	$n x^{n-1}$	No
e^x	e^x	No
$\ln x$	$\frac{1}{x}$	No
$\sin x$	$\cos x$	No
$\cos x$	$-\sin x$	No
$\tan x$	$\sec^2 x$	Yes
$\csc x$	$-\csc x \cot x$	Yes
$\sec x$	$\sec x \tan x$	Yes
$\cot x$	$-\csc^2 x$	Yes
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	Yes
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	Yes
$\arctan x$	$\frac{1}{1+x^2}$	Yes

Question 1

Differentiate each of the following with respect to *x*,

(i) $y = 7x^4$ **(ii)** $y = 11\sqrt{x}$

[2 marks]

Question 2

By first expanding the brackets, differentiate each of the following with respect to *x*,

(i) $y = (x + 6)(2x - 5)$ **(ii)** $y = \sqrt{x}\left(\frac{1}{\sqrt{x}} + 3\sqrt{x}\right)$

[4 marks]

Question 3

Use The Chain Rule to differentiate each of the following with respect to *x*,

(i) $y = 4(x^3 + 3)^5$ **(ii)** $y = 4\,cos\,(2x)$

(iii) $y = sec^3 x$ **(iv)** $y = e^{sin\,x} + e^{cos\,x}$

[8 marks]

Question 4

(i) Use The Product Rule to find $\frac{dy}{dx}$ if $y = x\,ln\,x$

[2 marks]

(ii) Find also $\frac{d^2y}{dx^2}$

[1 mark]

Question 5
Consider the function

$$f(x) = tan(3x)$$

Determine the value of

$$f'\left(\frac{\pi}{18}\right)$$

[4 marks]

Question 6

Use The Quotient Rule to show that if $y = \dfrac{x^3 - 1}{x^3 + 1}$

$$\text{then } \frac{dy}{dx} = \frac{6x^2}{(x^3 + 1)^2}$$

[4 marks]

Question 7

Find the equation of the tangent to the curve $y = \dfrac{11}{x^2 - 3}$ when $x = 5$

Give the answer in the form $ax + by + c = 0$
where a, b and c are integers to be found.

[6 marks]

Question 8

The function $f\,(x)$ is given below

$$f\,(x) \;=\; \frac{x^2\, sin\,(2x)}{9\pi}$$

(i) Find $f\,'(x)$

(ii) Show that $f\,'\left(\dfrac{\pi}{4}\right) \;=\; \dfrac{1}{18}$

[5 marks]

Question 9

If

$$\frac{d}{dx}\left(ln\sqrt{ax + b} \right) = \frac{4}{ax + 1}$$

Find the values of *a* and *b*.

[4 marks]

11.2 Answers

Answer 1

(i) $y = 7x^4 \quad \Rightarrow \quad \frac{dy}{dx} = 28x^3$

(ii) $y = 11\sqrt{x}$

$= 11x^{\frac{1}{2}} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{11}{2}x^{-\frac{1}{2}}$

$= \frac{11}{2\sqrt{x}}$

[2 marks]

Answer 2

(i) $y = (x + 6)(2x - 5)$

$= 2x^2 + 7x - 30 \quad \Rightarrow \quad \frac{dy}{dx} = 4x + 7$

(ii) $y = \sqrt{x}\left(\frac{1}{\sqrt{x}} + 3\sqrt{x}\right)$

$= 1 + 3x \quad \Rightarrow \quad \frac{dy}{dx} = 3$

[4 marks]

Answer 3

(i) $y = 4(x^3 + 3)^5 \quad \Rightarrow \quad \frac{dy}{dx} = 20(x^3 + 3)^4(3x^2)$

$= 60x^2(x^3 + 3)^4$

(ii) $y = 4\cos(2x) \quad \Rightarrow \quad \frac{dy}{dx} = -8\sin(2x)$

(iii) $y = \sec^3 x$

$= (\sec x)^3 \quad \Rightarrow \quad \frac{dy}{dx} = 3(\sec x)^2 \sec x \tan x$

$= 3\sec^3 x \tan x$

(iv) $y = e^{\sin x} + e^{\cos x} \quad \Rightarrow \quad \frac{dy}{dx} = e^{\sin x}\cos x + e^{\cos x}(-\sin x)$

$= e^{\sin x}\cos x - e^{\cos x}\sin x$

[8 marks]

Answer 4

(i)

$$y = x \ln x$$

$$\frac{dy}{dx} = x \times \frac{1}{x} + 1 \times \ln x$$

$$= 1 + \ln x$$

[2 marks]

(ii)

$$\frac{d^2y}{dx^2} = \frac{1}{x}$$

[1 marks]

Answer 5

$$f(x) = \tan(3x)$$

$$f'(x) = 3\sec^2(3x)$$

$$f'\left(\frac{\pi}{18}\right) = 3\sec^2\left(\frac{\pi}{6}\right)$$

$$= 3 \times \frac{1}{\left(\cos\left(\frac{\pi}{6}\right)\right)^2}$$

$$= 3 \times \left(\frac{2}{\sqrt{3}}\right)^2$$

$$= 3 \times \frac{4}{3}$$

$$= 4$$

[4 marks]

Answer 6

$$y = \frac{x^3 - 1}{x^3 + 1}$$

$$\frac{dy}{dx} = \frac{\left(x^3 + 1\right)\left(3x^2\right) - \left(3x^2\right)\left(x^3 - 1\right)}{\left(x^3 + 1\right)^2}$$

$$= \frac{\left(3x^2\right)\left\{x^3 + 1 - x^3 + 1\right\}}{\left(x^3 + 1\right)^2}$$

$$= \frac{\left(3x^2\right)\{2\}}{\left(x^3 + 1\right)^2}$$

$$= \frac{6x^2}{\left(x^3 + 1\right)^2} \qquad \square$$

[4 marks]

Answer 7

$$y = \frac{11}{x^2 - 3}$$

$$= 11\left(x^2 - 3\right)^{-1}$$

$$\frac{dy}{dx} = -11\left(x^2 - 3\right)^{-2} \times 2x$$

$$= -\frac{22x}{(x^2 - 3)^2}$$

When $x = 5, \quad y = \frac{11}{5^2 - 3} = \frac{11}{22} = \frac{1}{2}$

$\therefore$ Point on curve where we want the tangent is $(5,\ 0.5)$

When $x = 5, \quad \frac{dy}{dx} = -\frac{22 \times 5}{(5^2 - 3)^2} = -\frac{22 \times 5}{22 \times 22} = -\frac{5}{22}$

For the tangent, $y = mx + c$

$$0.5 = -\frac{5}{22} \times 5 + c$$

$$c = \frac{1}{2} \times \frac{11}{11} + \frac{25}{22}$$

$$= \frac{36}{22}$$

Equation of the tangent, $y = -\frac{5}{22}x + \frac{36}{22}$

$$22y = -5x + 36$$

$$5x + 22y - 36 = 0$$

[6 marks]

Answer 8

(i)

$$f(x) = \frac{1}{9\pi}\left(x^2 \sin(2x)\right)$$

$$f'(x) = \frac{1}{9\pi}\left(2x^2 \cos(2x) + 2x \sin(2x)\right)$$

$$= \frac{2x}{9\pi}\left(x \cos(2x) + \sin(2x)\right)$$

(ii)

$$f'\left(\frac{\pi}{4}\right) = \frac{2\pi}{9\pi \times 4}\left(\frac{\pi}{4} \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{18}\left(\frac{\pi}{4} \times 0 + 1\right)$$

$$= \frac{1}{18} \qquad \square$$

[5 marks]

Answer 9

$$y = ln(ax + b)^{\frac{1}{2}}$$

$$= \frac{1}{2} ln\left(ax + b \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{ax + b} \times a$$

$$\therefore \frac{a}{2(ax + b)} = \frac{4}{ax + 1}$$

$$\therefore \frac{a}{2} = 4, \qquad b = 1$$

$$\text{Answer : } a = 8, \; b = 1$$

[4 marks]

Lesson 12

A-Level Pure Mathematics : Year 2
Differentiation III

12.1 Test

Marks Available : 40

Table of Standard Derivatives

$f(x)$	$f'(x)$	In Formula Book ?
x^n	$n x^{n-1}$	No
e^x	e^x	No
$\ln x$	$\frac{1}{x}$	No
$\sin x$	$\cos x$	No
$\cos x$	$-\sin x$	No
$\tan x$	$\sec^2 x$	Yes
$\csc x$	$-\csc x \cot x$	Yes
$\sec x$	$\sec x \tan x$	Yes
$\cot x$	$-\csc^2 x$	Yes
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	Yes
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	Yes
$\arctan x$	$\frac{1}{1+x^2}$	Yes

Question 1

Differentiate each of the following with respect to x,

(i) $y = 13x^5$ **(ii)** $y = \frac{24}{x}$

[2 marks]

Question 2

By first expanding the brackets, differentiate the following with respect to x,

(i) $y = (x + 7)(3x - 2)$ **(ii)** $y = \frac{1}{\sqrt{x}}\left(3\sqrt{x} + x\right)$

[4 marks]

Question 3

Use The Chain Rule to find the derivative of each of the following with respect to x,

(i) $y = (5x^3 + 1)^4$

(ii) $y = tan(5x)$

(iii) $y = csc^4 x$

(iv) $y = ln(2x) + e^{-3x}$

[8 marks]

Question 4

(i) Use The Product Rule to find $\dfrac{dy}{dx}$ if $y = x e^x$

[2 marks]

(ii) Find also $\dfrac{d^2y}{dx^2}$

[1 marks]

Question 5

Consider the function

$$f(x) = cot(6x)$$

Determine the value of

$$f'\left(\frac{\pi}{18}\right)$$

[4 marks]

Question 6

Use The Quotient Rule to show that if $y = \dfrac{x^5 - 2}{x^5 + 2}$

then $\dfrac{dy}{dx} = \dfrac{20x^4}{(x^5 + 2)^2}$

[4 marks]

Question 7

Find the equation of the tangent to the curve $y = \dfrac{28}{x^2 - 2}$ when $x = 3$

Give the answer in the form $ax + by + c = 0$
where a, b and c are integers to be found.

[6 marks]

Question 8

The function $f(x)$ is given below

$$f(x) = \frac{x^2 \, sin\,(3x)}{9\pi}$$

(i) Find $f'(x)$

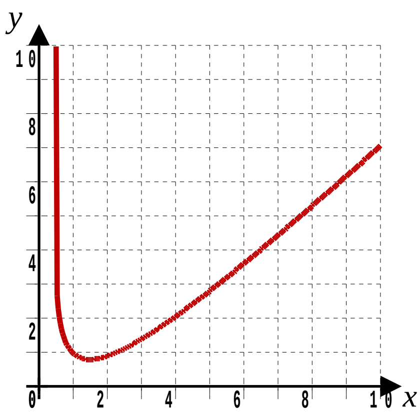
(ii) Show that

$$f'\left(\frac{\pi}{3}\right) = -\frac{\pi}{27}$$

[5 marks]

Question 9

$$f(x) = x - \ln(2x - 1)$$



Show that the exact coordinates of the turning point are $\left(\frac{3}{2}, \ \frac{3}{2} - \ln 2\right)$

[4 marks]

12.2 Answers

Answer 1

(i) $y = 13x^5 \quad \Rightarrow \quad \frac{dy}{dx} = 65x^4$

(ii) $y = \frac{24}{x}$

$= 24x^{-1} \quad \Rightarrow \quad \frac{dy}{dx} = -24x^{-2}$

$= -\frac{24}{x^2}$

[2 marks]

Answer 2

(i) $y = (x + 7)(3x - 2)$

$= 3x^2 + 19x - 14 \quad \Rightarrow \quad \frac{dy}{dx} = 6x + 19$

(ii) $y = \frac{1}{\sqrt{x}}\left(3\sqrt{x} + x\right)$

$= 3 + x^{\frac{1}{2}} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

[4 marks]

Answer 3

(i) $y = \left(5x^3 + 1\right)^4 \quad \Rightarrow \quad \frac{dy}{dx} = 4\left(5x^3 + 1\right)^3\left(15x^2\right)$

$= 60x^2\left(5x^3 + 1\right)^3$

(ii) $y = \tan(5x) \quad \Rightarrow \quad \frac{dy}{dx} = 5\sec^2(5x)$

(iii) $y = \csc^4 x \quad \Rightarrow \quad \frac{dy}{dx} = 4\csc^3 x(-\csc x \cot x)$

$= -4\csc^4 x \cot x$

$= -\frac{4\cos x}{\sin^5 x}$

(iv) $y = \ln(2x) + e^{-3x}$

$= \ln 2 + \ln x + e^{-3x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} - 3e^{-3x}$

[8 marks]

Answer 4

(i) $$y = x e^x \Rightarrow \frac{dy}{dx} = x e^x + e^x$$

[2 marks]

(ii) $$\frac{d^2y}{dx^2} = x e^x + e^x + e^x$$

$$= x e^x + 2 e^x$$

[1 marks]

Out of interest : $\dfrac{d^ny}{dx^n} = e^x (x + n)$

(Proof by induction) $\dfrac{d^{n+1}y}{dx^{n+1}} = e^x + e^x (x + n) = e^x (x + n + 1)$

Answer 5

$$f(x) = cot(6x)$$

$$f'(x) = -6\, csc^2(6x)$$

$$f'\left(\frac{\pi}{18}\right) = -6\, csc^2\left(\frac{\pi}{3}\right)$$

$$= -6 \times \frac{1}{\left(sin\left(\frac{\pi}{3}\right)\right)^2}$$

$$= -6 \times \left(\frac{2}{\sqrt{3}}\right)^2$$

$$= -6 \times \frac{4}{3}$$

$$= -8$$

[4 marks]

Answer 6

$$y = \frac{x^5 - 2}{x^5 + 2}$$

$$\frac{dy}{dx} = \frac{\left(x^5 + 2 \right)\left(5x^4 \right) - \left(5x^4 \right)\left(x^5 - 2 \right)}{(x^5 + 2)^2}$$

$$= \frac{\left(5x^4 \right)\left\{ x^5 + 2 - x^5 + 2 \right\}}{(x^5 + 2)^2}$$

$$= \frac{(5x^4)\{ 4 \}}{(x^5 + 2)^2}$$

$$= \frac{20x^4}{(x^5 + 2)^2} \qquad \square$$

[4 marks]

Answer 7

$$y = \frac{28}{x^2 - 2}$$

$$= 28\left(x^2 - 2\right)^{-1}$$

$$\frac{dy}{dx} = -28\left(x^2 - 2\right)^{-2} \times 2x$$

$$= -\frac{56x}{(x^2 - 2)^2}$$

When $x = 3, \quad y = \frac{28}{3^2 - 2} = \frac{28}{7} = 4$

$\therefore$ Point on curve where we want the tangent is $(3,\ 4)$

When $x = 3, \quad \frac{dy}{dx} = -\frac{56 \times 3}{(3^2 - 2)^2} = -\frac{56 \times 3}{49} = -\frac{24}{7}$

For the tangent, $y = mx + c$

$$4 = -\frac{24}{7} \times 3 + c$$

$$c = 4 \times \frac{7}{7} + \frac{72}{7}$$

$$= \frac{100}{7}$$

Equation of the tangent, $y = -\frac{24}{7}x + \frac{100}{7}$

$$7y = -24x + 100$$

$$24x + 7y - 100 = 0$$

[6 marks]

Answer 8

(i)

$$f(x) = \frac{1}{9\pi}\left(x^2 \sin(3x)\right)$$

$$f'(x) = \frac{1}{9\pi}\left(3x^2 \cos(3x) + 2x \sin(3x)\right)$$

$$= \frac{x}{9\pi}\left(3x \cos(3x) + 2 \sin(3x)\right)$$

(ii)

$$f'\left(\frac{\pi}{3}\right) = \frac{\pi}{9\pi \times 3}\left(\frac{3\pi}{3}\cos(\pi) + 2\sin(\pi)\right)$$

$$= \frac{1}{27}\left(\pi \times (-1) + 2 \times 0\right)$$

$$= -\frac{\pi}{27} \qquad \square$$

[5 marks]

Answer 9

$$
\begin{aligned}
f(x) &= x - ln(2x - 1) \\
f'(x) &= 1 - \frac{1}{2x - 1} \times 2 \\
&= \frac{(2x - 1) - 2}{2x - 1} \\
&= \frac{2x - 3}{2x - 1}
\end{aligned}
$$

For a turning point, $f'(x) = 0$

$$
\begin{aligned}
\frac{2x - 3}{2x - 1} &= 0 \\
2x - 3 &= 0 \\
2x &= 3 \\
x &= \frac{3}{2}
\end{aligned}
$$

Thus, $f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right) - ln\left(2 \times \frac{3}{2} - 1\right)$

$= 1.5 - ln\,2$

$\therefore$ The turning point is at $\left(\frac{3}{2}, \frac{3}{2} - ln\,2\right)$ ☐

[4 marks]

Lesson 13

A-Level Pure Mathematics : Year 2

Differentiation III

13.1 Later Date Revision

Marks Available : 40

Table of Standard Derivatives

$f(x)$	$f'(x)$	In Formula Book ?
x^n	$n x^{n-1}$	No
e^x	e^x	No
$ln\ x$	$\frac{1}{x}$	No
$sin\ x$	$cos\ x$	No
$cos\ x$	$-\ sin\ x$	No
$tan\ x$	$sec^2\ x$	Yes
$csc\ x$	$-\ csc\ x\ cot\ x$	Yes
$sec\ x$	$sec\ x\ \ tan\ x$	Yes
$cot\ x$	$-\ csc^2\ x$	Yes
$arcsin\ x$	$\frac{1}{\sqrt{1-x^2}}$	Yes
$arccos\ x$	$-\frac{1}{\sqrt{1-x^2}}$	Yes
$arctan\ x$	$\frac{1}{1+x^2}$	Yes

Question 1

Show that the derivative with respect to x of

$$y = sec\ x\ tan\ x$$

is

$$\frac{dy}{dx} = sec\ x\ (2\ sec^2\ x - 1)$$

[4 marks]

Question 2

Show that the derivative with respect to *x* of;

$$y = csc\,x\,cot\,x$$

is

$$\frac{dy}{dx} = csc\,x\left(1 - 2\,csc^2 x\right)$$

[4 marks]

Question 3

Consider the function;

$$f(x) = \frac{8}{(1 - 3x)^3}$$

Show that;

$$f'(1) = \frac{9}{2}$$

[4 marks]

Question 4

A-Level Examination Question from January 2009, Paper C3 (Edexcel)

Find the equation of the tangent to the curve

$$x = \cos(2y + \pi) \quad \text{at} \quad \left(0, \frac{\pi}{4}\right)$$

Give your answer in the form $y = ax + b$, where a and b are constants to be found.

[6 marks]

Question 5

The curve

$$y = ln\left(x^2 - 3\right)$$

crosses the *x*-axis at *A* and *B*.

(i) Find the coordinates of *A* and *B*

[3 marks]

(ii) The normals at *A* and *B* meet at *P*.
Find the coordinates of *P*.

[5 marks]

Question 6

Show that the derivative of the inverse cotangent function

$$y = arccot\, x$$

is

$$\frac{dy}{dx} = -\frac{1}{1 + x^2}$$

The following trigonometry identity will be useful;

$$cot^2\, y + 1 = csc^2\, y$$

[6 marks]

Question 7
The curve

$$y = \frac{2x + 1}{2x - 1}$$

crosses the x-axis at A and the y-axis at B.

Find the point of intersection of the tangents to the curve at A and B.

[8 marks]

13.2 Answers

Answer 1

$$
\begin{aligned}
y &= \sec x \tan x \\
\frac{dy}{dx} &= \sec x \sec^2 x + \sec x \tan x \tan x \\
&= \sec x \left(\sec^2 x + \tan^2 x \right) \\
&= \sec x \left(\sec^2 x + \sec^2 x - 1 \right) \qquad \text{from identity } 1 + \tan^2 x = \sec^2 x \\
&= \sec x \left(2 \sec^2 x - 1 \right) \qquad \square
\end{aligned}
$$

[4 marks]

Answer 2

$$
\begin{aligned}
y &= \csc x \cot x \\
\frac{dy}{dx} &= \csc x \left(-\csc^2 x \right) + \left(-\csc x \cot x \right) \cot x \\
&= -\csc x \left(\csc^2 x + \cot^2 x \right) \\
&= -\csc x \left(\csc^2 x + \csc^2 x - 1 \right) \qquad \text{from identity } \cot^2 x + 1 = \csc^2 x \\
&= -\csc x \left(2 \csc^2 x - 1 \right) \\
&= \csc x \left(1 - 2 \csc^2 x \right)
\end{aligned}
$$

[4 marks]

Answer 3

$$
\begin{aligned}
f(x) &= \frac{8}{(1 - 3x)^3} \\
&= 8(1 - 3x)^{-3} \\
f'(x) &= -24\,(1 - 3x)^{-4} \times (-3) \\
&= \frac{72}{(1 - 3x)^4} \\
\therefore\ f'(1) &= \frac{72}{(-2)^4} \\
&= \frac{9}{2}
\end{aligned}
$$

[4 marks]

Answer 4

$$x = cos(2y + \pi)$$

$$\frac{dx}{dy} = -2\, sin(2y + \pi)$$

When $y = \frac{\pi}{4}, \quad \frac{dx}{dy} = -2\, sin\left(\frac{3\pi}{2}\right) = -2 \times -1 = 2$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \text{ at } \left(0, \frac{\pi}{4}\right)$$

For the tangent, $y = mx + c$

$$\frac{\pi}{4} = \frac{1}{2} \times 0 + c \Rightarrow c = \frac{\pi}{4}$$

$$\therefore \text{Tangent is } y = \frac{1}{2}x + \frac{\pi}{4}$$

[6 marks]

Answer 5

(i) The curve will cut the x-axis when $y = 0$

$$ln\left(x^2 - 3\right) = 0 \Leftrightarrow x^2 - 3 = e^0$$

$$x^2 = 4$$

$$x = \pm 2$$

$\therefore$ Crosses the x-axis at $(-2, 0)$, $(2, 0)$

[3 marks]

(ii) $\frac{dy}{dx} = \frac{2x}{x^2 - 3}$

When $x = 2, \ y = 0$ and $\frac{dy}{dx} = 4$ so this normal has gradient $\left(-\frac{1}{4}\right)$

It's equation will be, $y = mx + c$

$$0 = -\frac{1}{4} \times 2 + c \Rightarrow c = \frac{1}{2}$$

$$\therefore y = -\frac{1}{4}x + \frac{1}{2}$$

When $x = -2, \ y = 0$ and $\frac{dy}{dx} = -4$ so this normal has gradient $\left(\frac{1}{4}\right)$

It's equation will be, $y = mx + c$

$$0 = \frac{1}{4} \times -2 + c \Rightarrow c = \frac{1}{2}$$

$$\therefore y = \frac{1}{4}x + \frac{1}{2}$$

Point of intersection is thus $\left(0, \frac{1}{2}\right)$

[5 marks]

Answer 6

$$y = arccot\, x$$
$$\therefore\ x = cot\, y$$
$$\frac{dx}{dy} = -csc^2 y$$
$$= -\left(cot^2 y + 1 \right)$$
$$= -\left(x^2 + 1 \right)$$
$$\frac{dy}{dx} = -\frac{1}{1 + x^2}$$

[6 marks]

Answer 7

Cuts x-axis when $y = 0$, which is when $2x + 1 = 0$ $\quad \therefore\ A\left(-\frac{1}{2},\ 0\right)$

Cuts y-axis when $x = 0$. which is when $y = -1$ $\quad \therefore\ B\ (\ 0.\ -\ 1\ )$

$$y = \frac{2x + 1}{2x - 1} \Rightarrow \frac{dy}{dx} = \frac{(2x - 1)\ 2 - 2\,(\ 2x + 1\ )}{(\ 2x - 1\ )^2}$$
$$= \frac{4x - 2 - 4x - 2}{(\ 2x - 1\ )^2}$$
$$= -\frac{4}{(\ 2x - 1\ )^2}$$

For A, $\frac{dy}{dx} = -1$ giving a tangent, $\quad y = mx + c$

$$0 = -1 \times -\frac{1}{2} + c \Rightarrow c = -\frac{1}{2}$$
$$y = -x - \frac{1}{2}$$

For B, $\frac{dy}{dx} = -4$ giving a tangent, $\quad y = mx + c$

$$-\ 1 = -4 \times 0 + c \Rightarrow c = -1$$
$$y = -4x - 1$$

These tangents intersect when, $\quad -\ 4x - 1 = -x - \frac{1}{2}$

$$x = -\frac{1}{6} \text{ with } y = -\frac{1}{3} \text{ i.e. } \left(-\frac{1}{6}, -\frac{1}{3}\right)$$

[8 marks]

Lesson 14

14.1 Extension Material

Marks Available : 40

Question 1

Differentiate each of the following with respect to *x*,

(i) $y = e^x$ **(ii)** $y = x^e$ **(iii)** $y = e^e$

[6 marks]

Question 2

Write down the derivative of each of the following with respect to *x*,

(i) $y = \ln x$ **(ii)** $y = \ln x^2$ **(iii)** $y = \ln\left(\frac{1}{x}\right)$

[6 marks]

Question 3

Find the equation of the tangent to the curve

$$y = e^{\frac{1}{2}x}$$

at the point where it intercepts the *y*-axis.

Write your answer in the form $ay + bx + c = 0$, where $a, b, c \in \mathbb{Z}$

[6 marks]

Question 4

Find the points on the following curve where the gradient is 3

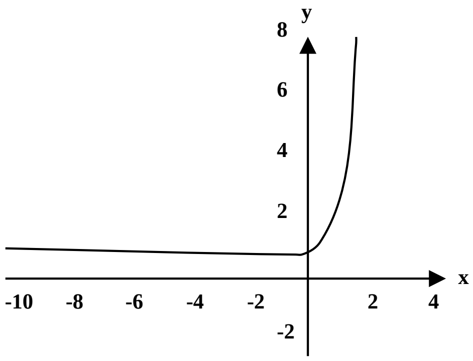
$$y = 5\sqrt{x} - \frac{1}{2} ln\ x \qquad x \in \mathbb{R},\ x > 0$$

Give your answers correct to three significant figures.

[10 marks]

Question 5

A sketch graph of the function, $f\,(x) \;=\; e^{2x} \;-\; e^{x} \;+\; 1$ is given below.



(i) Find $f'(x)$

(ii) Explain, briefly, why the equation $e^{x} = 0$ has no solution for $x \in \mathbb{R}$

(iii) Find the value of x such that $f'(x) = 0$

(iv) The graph of the $f\,(x)$ has a turning point of the form ($\ln a$, b)
Determine the value of a and the value of b

(v) Find $f''(x)$ at the turning point.

(vi) Explain what your part (v) answer tells you about the turning point.

[12 marks]

14.2 Answers

Answer 1

(i) $y = e^x \quad \Rightarrow \quad \dfrac{dy}{dx} = e^x$

(ii) $y = x^e \quad \Rightarrow \quad \dfrac{dy}{dx} = e\,x^{e-1}$

(iii) $y = e^e \quad \Rightarrow \quad \dfrac{dy}{dx} = 0$

[6 marks]

Answer 2

(i) $y = \ln x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{x}$

(ii) $y = \ln x^2$

$= 2\ln x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{2}{x}$

(iii) $y = \ln\left(\dfrac{1}{x}\right)$

$= \ln x^{-1}$

$= -\ln x \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{1}{x}$

[6 marks]

Answer 3

$$y = e^{\frac{1}{2}x}$$

$$\frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x}$$

The curve intercepts the y-axis when $x = 0$

so $(0,\ 1)$ is on the tangent which has a gradient of $\dfrac{1}{2}$

$$\therefore y = \frac{1}{2}x + 1$$

$$\Leftrightarrow 2y - x - 2 = 0$$

[6 marks]

Answer 4

$$y = 5\sqrt{x} - \frac{1}{2}\ln x$$

$$= 5x^{\frac{1}{2}} - \frac{1}{2}\ln x$$

$$\frac{dy}{dx} = \frac{5}{2}x^{-\frac{1}{2}} - \frac{1}{2} \times \frac{1}{x}$$

$$= \frac{5}{2\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}} - \frac{1}{2x}$$

$$= \frac{5\sqrt{x} - 1}{2x}$$

To find where the gradient is 3

$$\frac{5\sqrt{x} - 1}{2x} = 3$$

$$5\sqrt{x} - 1 = 6x$$

$$6x - 5\sqrt{x} + 1 = 0$$

Let $z = \sqrt{x} \Rightarrow z^2 = x$ and the equation becomes,

$$6z^2 - 5z + 1 = 0$$

$$(3z - 1)(2z - 1) = 0$$

$\therefore$ Either $\sqrt{x} = \frac{1}{3}$ or $\sqrt{x} = \frac{1}{2}$

Either $x = \frac{1}{9}$ or $x = \frac{1}{4}$

For $x = \frac{1}{9}$, $y = 5\sqrt{\frac{1}{9}} - \frac{1}{2}\ln\left(\frac{1}{9}\right)$

$$= \frac{5}{3} - \frac{1}{2}\ln 3^{-2}$$

$$= \frac{5}{3} + \ln 3$$

For $x = \frac{1}{4}$, $y = 5\sqrt{\frac{1}{4}} - \frac{1}{2}\ln\left(\frac{1}{4}\right)$

$$= \frac{5}{2} - \frac{1}{2}\ln 2^{-2}$$

$$= \frac{5}{2} + \ln 2$$

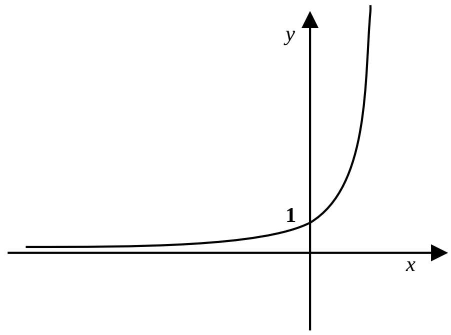
The points, correct to three significant figures, are $(0.111,\ 2.77)$, $(0.250,\ 3.19)$

[10 marks]

Answer 5

(i) $f(x) = e^{2x} - e^x + 1 \Rightarrow f'(x) = 2e^{2x} - e^x$

(ii) The graph of $y = e^x$ has the x-axis as an asymptote so the curve never has an intersection with the x-axis. $\therefore\ e^x = 0$ has no solution. ☐



(iii)

$$2e^{2x} - e^x = 0$$

$$e^x(2e^x - 1) = 0$$

Either $e^x = 0$ which has no solution

Or $2e^x - 1 = 0$

$$e^x = \frac{1}{2}$$

$$x = ln\left(\frac{1}{2}\right)$$

(iv)

$$y = e^{2x} - e^x + 1$$

When $x = ln\left(\frac{1}{2}\right)$, $y = e^{2\,ln\left(\frac{1}{2}\right)} - e^{ln\left(\frac{1}{2}\right)} + 1$

$$= e^{ln\left(\frac{1}{2}\right)^2} - e^{ln\left(\frac{1}{2}\right)} + 1$$

$$= \frac{1}{4} - \frac{1}{2} + 1$$

$= \frac{3}{4}$ That is, $a = \frac{1}{2}$ and $b = \frac{3}{4}$

(v) $f''(x) = 4e^{2x} - e^x \Rightarrow f''\left(ln\left(\frac{1}{2}\right)\right) = 4e^{2ln\left(\frac{1}{2}\right)} - e^{ln\left(\frac{1}{2}\right)} = 4 \times \frac{1}{4} - \frac{1}{2}$

so $f''(x) = \frac{1}{2}$ at the turning point

(vi) As $f''(x)$ is positive the turning point is a minimum

[12 marks]