

3.4 Answers

A-Level Pure Mathematics, Year 1
Additional Mathematics
The Algebra of Polynomials

3.4.1 Solution to 3.2 Example (Teaching Video)

$$f(x) = x^3 + 8x^2 + 5x - 50$$

(a) (i)
$$\begin{aligned} f(-1) &= (-1)^3 + 8(-1)^2 + 5(-1) - 50 \\ &= -1 + 8 - 5 - 50 \\ &= -48 \end{aligned}$$

That is, $f(-1) \neq 0$

$\therefore x + 1$ is not a factor of $f(x)$, by the factor theorem

[2 marks]

(ii)

$$\begin{aligned} f(2) &= (2)^3 + 8(2)^2 + 5(2) - 50 \\ &= 8 + 32 + 10 - 50 \\ &= 0 \end{aligned}$$

That is, $f(2) = 0$

$\therefore x - 2$ is a factor of $f(x)$, by the factor theorem

[2 marks]

(b)

$$\begin{array}{r} x^2 + 10x + 25 \\ x - 2 \overline{) x^3 + 8x^2 + 5x - 50} \\ \underline{x^3 - 2x^2} \\ 10x^2 + 5x \\ \underline{10x^2 - 20x} \\ 25x - 50 \\ \underline{25x - 50} \\ 0 \end{array} \leftarrow \text{As expected !}$$

$$\begin{aligned} f(x) &= x^3 + 8x^2 + 5x - 50 \\ &= (x - 2)(x^2 + 10x + 25) \\ &= (x - 2)(x + 5)(x + 5) \end{aligned}$$

[4 marks]

(c) If $f(x) = 0$

$$\text{then } (x - 2)(x + 5)(x + 5) = 0$$

in which case either $x = 2$ or $x = -5$

[1 mark]

3.4.2 Solution to 3.3 Exercise

Answers 1

$$\begin{aligned}f(x) &= 2x^3 + 3x^2 - 5x - 2 \\f(-2) &= 2 \times (-2)^3 + 3 \times (-2)^2 - 5 \times (-2) - 2 \\&= -16 + 12 + 6 - 2 \\&= 0\end{aligned}$$

That is, $f(-2) = 0$

$\therefore x + 2$ is a factor of $f(x)$, by the factor theorem

[2 marks]

Answer 2

$$\begin{aligned}g(x) &= x^3 + 6x^2 - 19x - 84 \\g(3) &= (3)^3 + 6 \times (3)^2 - 19 \times (3) - 84 \\&= 27 + 54 - 57 - 84 \\&= -60\end{aligned}$$

That is, $g(3) \neq 0$

$\therefore x - 3$ is not a factor of $g(x)$, by the factor theorem

[2 marks]

Answer 3

(a) (i)

$$\begin{aligned}h(x) &= x^3 - 3x^2 - 9x + 27 \\h(1) &= (1)^3 - 3 \times (1)^2 - 9 \times (1) + 27 \\&= 1 - 3 - 9 + 27 \\&= 16\end{aligned}$$

[1 mark]

(ii) As, $h(1) \neq 0$ it follows that $x - 1$ is not a factor of $h(x)$, by the factor theorem

[1 mark]

(b) (i)

$$\begin{aligned}h(x) &= x^3 - 3x^2 - 9x + 27 \\h(3) &= (3)^3 - 3 \times (3)^2 - 9 \times (3) + 27 \\&= 9 - 27 - 27 + 27 \\&= 0\end{aligned}$$

[1 mark]

(ii) As, $h(3) = 0$ it follows that $x - 3$ is a factor of $h(x)$, by the factor theorem

[1 mark]

Answer 4

The long division shows that $(x + 1)$ is not a factor of $k(x)$ because the remainder is not zero, it is -16

[2 marks]**Answer 5**

$$m(x) = x^3 + 12x^2 - 4x - 240$$

$$\begin{aligned} m(4) &= (4)^3 + 12 \times (4)^2 - 4 \times (4) - 240 \\ &= 64 + 192 - 16 - 240 \\ &= 0 \end{aligned}$$

$$\text{That is, } m(4) = 0$$

$\therefore x - 4$ is a factor of $m(x)$, by the factor theorem

[3 marks]**Answer 6**

The guessable possible factors are $(x \pm 1)$, $(x \pm 2)$

$$n(1) = -3, \quad n(-1) = -3, \quad n(2) = 0, \quad n(-2) = -12$$

Thus, by the factor theorem, $(x - 2)$ is a factor of $n(x)$

[3 marks]**Answer 7**

$$\begin{array}{r} \overline{x^2 + 8x + 16} \\ x - 3 \overline{) x^3 + 5x^2 - 8x - 48} \\ \underline{x^3 - 3x^2} \\ 8x^2 - 8x \\ \underline{8x^2 - 24x} \\ 16x - 48 \\ \underline{16x - 48} \\ 0 \leftarrow \text{Remainder} \end{array}$$

(i)

The remainder of zero shows that $(x - 3)$ is a factor of $f(x)$

[4 marks]

$$\begin{aligned} \text{(ii)} \quad f(x) &= x^3 + 5x^2 - 8x - 48 \\ &= (x - 3)(x^2 + 8x + 16) \\ &= (x - 3)(x + 4)(x + 4) \end{aligned}$$

[2 marks]

$$\begin{aligned} \text{(iii)} \quad &\text{If } f(x) = 0 \\ &\text{then } (x - 3)(x + 4)(x + 4) = 0 \\ &\text{in which case either } x = 3 \text{ or } x = -4 \end{aligned}$$

[1 mark]

Answer 8

$$g(x) = x^3 + 6x^2 - 15x - 100$$

(a) (i)

$$\begin{aligned} g(4) &= (4)^3 + 6(4)^2 - 15(4) - 100 \\ &= 64 + 96 - 60 - 100 \\ &= 0 \end{aligned}$$

That is, $g(4) = 0$ $\therefore x - 4$ is a factor of $g(x)$, by the factor theorem**[2 marks]****(ii)**

$$\begin{aligned} g(-2) &= (-2)^3 + 6(-2)^2 - 15(-2) - 100 \\ &= -8 + 24 + 30 - 100 \\ &= -54 \end{aligned}$$

That is, $g(-2) \neq 0$ $\therefore x + 2$ is not a factor of $g(x)$, by the factor theorem**[2 marks]****(b)**

$$\begin{array}{r} \overline{x^2 + 10x + 25} \\ x - 4 \overline{\left| \begin{array}{r} x^3 + 6x^2 - 15x - 100 \\ x^3 - 4x^2 \\ \hline 10x^2 - 15x \\ 10x^2 - 40x \\ \hline 25x - 100 \\ 25x - 100 \\ \hline 0 \end{array} \right.} \end{array}$$

0 $\leftarrow$ As expected !

$$\begin{aligned} g(x) &= x^3 + 6x^2 - 15x - 100 \\ &= (x - 4)(x^2 + 10x + 25) \\ &= (x - 4)(x + 5)(x + 5) \end{aligned}$$

[4 marks]**(c)**If $g(x) = 0$ then $(x - 4)(x + 5)(x + 5) = 0$ in which case either $x = 4$ or $x = -5$ **[1 mark]**

Answer 9

$$h(x) = x^3 + 18x^2 + 108x + 216$$

(a) (i)

$$\begin{aligned} h(-4) &= (-4)^3 + 18(-4)^2 + 108(-4) + 216 \\ &= -64 + 288 - 432 + 216 \\ &= 8 \end{aligned}$$

That is, $h(-4) = 8$ $\therefore x + 4$ is not a factor of $h(x)$, by the factor theorem**[2 marks]****(ii)**

$$\begin{aligned} h(-6) &= (-6)^3 + 18(-6)^2 + 108(-6) + 216 \\ &= -216 + 648 - 648 + 216 \\ &= 0 \end{aligned}$$

That is, $h(-6) = 0$ $\therefore x + 6$ is a factor of $h(x)$, by the factor theorem**[2 marks]****(b)**

$$\begin{array}{r} \quad \quad \quad x^2 + 12x + 36 \\ x + 6 \overline{) x^3 + 18x^2 + 108x + 216} \\ \underline{x^3 + 6x^2} \\ 12x^2 + 108x \\ \underline{12x^2 + 72x} \\ 36x + 216 \\ \underline{36x + 216} \\ 0 \end{array} \leftarrow \text{As expected !}$$

$$\begin{aligned} h(x) &= x^3 + 18x^2 + 108x + 216 \\ &= (x + 6)(x^2 + 12x + 36) \\ &= (x + 6)(x + 6)(x + 6) \end{aligned}$$

[4 marks]**(c)**If $h(x) = 0$ then $(x + 6)(x + 6)(x + 6) = 0$ in which case $x = -6$ **[1 mark]**

Answer 10**(i)** As $(x - 1)$ is a factor of $f(x)$, $f(1) = 0$, by the factor theorem

$$f(1) = 0$$

$$(1)^3 + a(1)^2 + b(1) - 12 = 0$$

$$1 + a + b - 12 = 0$$

$$a + b = 11$$

[2 marks]**(ii)** As $(x + 2)$ is a factor of $f(x)$, $f(-2) = 0$, by the factor theorem

$$f(-2) = 0$$

$$(-2)^3 + a(-2)^2 + b(-2) - 12 = 0$$

$$-8 + 4a - 2b - 12 = 0$$

$$4a - 2b = 20$$

Divide through by 2

$$2a - b = 10$$

[2 marks]**(iii)**

$$a + b = 11$$

ADD

$$2a - b = 10$$

$$3a = 21$$

$$a = 7 \Rightarrow b = 4$$

[3 marks]**(iv)** Making use of the fact that two factors are known,

$$(x - 1)(x + 2)(x + \lambda) = x^3 + 7x^2 + 4x - 12$$

$$\therefore \lambda = 6$$

Thus,

$$f(x) = x^3 + 7x^2 + 4x - 12$$

$$= (x - 1)(x + 2)(x + 6)$$

[2 marks]