4.5 Answers

4.5.1 Solution to 4.3 Example (Teaching Video)

$$\begin{aligned} f(x) &= x^4 + x^2 + kx \\ &= x\left(x^3 + x + k\right) \end{aligned}$$

As $x = 1$ is a root, $f(1) = 0$ by the factor theorem

$$\begin{aligned} \therefore \left(1\right)\left(1^3 + 1 + k\right) &= 0 \\ k &= -2 \end{aligned}$$

Thus, $f(x) = x\left(x^3 + x - 2\right)$

As $x = 1$ is a root, $\left(x - 1\right)$ is a factor

$$\begin{array}{rrrrrrrr}
 & & & x^2 & + & x & + & 2 \\ \cline{2-8}
x - 1 & \multicolumn{1}{|r}{x^3} & + & 0x^2 & + & x & - & 2 \\
 & x^3 & - & x^2 & & & & \\ \cline{2-4}
 & & & x^2 & + & x & & \\
 & & & x^2 & - & x & & \\ \cline{4-6}
 & & & & & 2x & - & 2 \\
 & & & & & 2x & - & 2 \\ \cline{6-8}
 & & & & & & & 0 \quad \leftarrow \text{ As expected !}
\end{array}$$

$$f(x) = x\left(x - 1\right)\left(x^2 + x + 2\right)$$

This is, in fact, fully factorised, as the quadratic does not factorise further.
The function has two linear factors and one quadratic factor.

[5 marks]

4.5.2 Solutions to 4.4 Exercise

Answer 1

$$x^4 - 3x^2 + 2x = 0$$

$$x\left(x^3 - 3x + 2\right) = 0$$

Let $m(x) = x^3 - 3x + 2$, and notice that $m(1) = 0$

So $(x - 1)$ is a factor of $m(x)$ by the factor theorem

$$\begin{array}{r|l}
 & x^2 + x - 2 \\
\hline
x - 1 & x^3 + 0x^2 - 3x + 2 \\
 & x^3 - x^2 \\
\hline
 & \quad x^2 - 3x \\
 & \quad x^2 - x \\
\hline
 & \qquad - 2x + 2 \\
 & \qquad - 2x + 2 \\
\hline
 & \qquad\qquad 0 \leftarrow \text{As expected !}
\end{array}$$

$$x^4 - 3x^2 + 2x = 0$$

$$x\left(x^3 - 3x + 2\right) = 0$$

$$x\left(x - 1\right)\left(x^2 + x - 2\right) = 0$$

$$x\left(x - 1\right)\left(x - 1\right)\left(x + 2\right) = 0$$

Either $x = 0$ or $x = 1$ or $x = -2$

Note : This is an old Additional Mathematics Examination Question
From 21st June 1995, Paper 2
I don't know which examination board

[5 marks]

Answer 2

As $x = 2$ is a root, it will satisfy the given equation

$$x^3 - 3x + k = 0$$
$$2^3 - 3 \times 2 + k = 0$$
$$k = -2$$

The equation to be solved is thus,

$$x^3 - 3x - 2 = 0$$

and, as $x = 2$ is a root, $(x - 2)$ is a factor

$$\begin{array}{r|rrrr}
 & x^2 & +\ 2x & +\ 1 & \\
\hline
x - 2 & x^3 + 0x^2 & -\ 3x & -\ 2 & \\
 & x^3 - 2x^2 & & & \\
\hline
 & 2x^2 & -\ 3x & & \\
 & 2x^2 & -\ 4x & & \\
\hline
 & & x & -\ 2 & \\
 & & x & -\ 2 & \\
\hline
 & & & 0 & \leftarrow \text{As expected !}
\end{array}$$

$$x^3 - 3x^2 - 2 = 0$$
$$(x - 2)(x^2 + 2x + 1) = 0$$
$$(x - 2)(x + 1)(x + 1) = 0$$

Either $x = 2$ or $x = -1$

$\therefore$ The “other root” are thus $x = -1$

(Other than the root given in the question)

[5 marks]

Answer 3

(a) **(i)** If $(x - 2)$ is a factor then replacing x with 2 would make the given expression have a value of zero, by the factor theorem.

$$x^3 - 7x + 6 \Rightarrow 2^3 - 7 \times 2 + 6 = 0$$

$$\therefore \ (x - 2) \text{ is a factor}$$

[2 marks]

(ii) If $(x + 1)$ is a factor then replacing x with -1 would make the given expression have a value of zero, by the factor theorem.

$$x^3 - 7x + 6 \Rightarrow (-1)^3 - 7 \times (-1) + 6 = 12$$

$$\therefore \ (x + 1) \text{ is not a factor}$$

[1 mark]

(b) **(i)** As $(x - 2)$ is a factor...

$$\begin{array}{r|l}
 & x^2 + 2x - 3 \\
\hline
x - 2 & x^3 + 0x^2 - 7x + 6 \\
 & x^3 - 2x^2 \\
\cline{2-2}
 & \quad 2x^2 - 7x \\
 & \quad 2x^2 - 4x \\
\cline{2-2}
 & \qquad -3x + 6 \\
 & \qquad -3x + 6 \\
\cline{2-2}
 & \qquad\qquad 0 \leftarrow \text{As expected !}
\end{array}$$

$$x^3 - 7x + 6 = \left(x - 2\right)\left(x^2 + 2x - 3\right)$$
$$= (x - 2)(x - 1)(x + 3)$$

[3 marks]

(ii) The solutions are $x \in \{-3, \ 1, \ 2\}$

[1 mark]

Answer 4

(i) The possible negative integer roots are $(x + 1)$, $(x + 2)$, $(x + 3)$ and $(x + 6)$

Try $x = -1$: $(-1)^3 - (-1)^2 - 10(-1) + 6 = 14$

Try $x = -2$: $(-2)^3 - (-2)^2 - 10(-2) + 6 = 14$

Try $x = -3$: $(-3)^3 - (-3)^2 - 10(-3) + 6 = 0$

Try $x = -6$: $(-6)^3 - (-6)^2 - 10(-6) + 6 = -186$

$\therefore$ The negative root is $x = -3$, by the factor theorem

[2 marks]

(ii) As $x = -3$ is a root, $(x + 3)$ is a factor.

$$
\begin{array}{r|rrrr}
 & x^2 & -\ 4x & +\ 2 & \\
\cline{2-5}
x + 3 & x^3 - & x^2 & -\ 10x & +\ 6 \\
 & x^3 + & 3x^2 & & \\
\cline{2-3}
 & - & 4x^2 & -\ 10x & \\
 & - & 4x^2 & -\ 12x & \\
\cline{3-4}
 & & & 2x & +\ 6 \\
 & & & 2x & +\ 6 \\
\cline{4-5}
 & & & & 0
\end{array}
\leftarrow \text{ As expected !}
$$

Thus, $x^3 - 3x - 2 = \left(x + 3\right)\left(x^2 - 4x + 2\right)$

One solution is $x = -3$ and the other two are given by,

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 2}}{2 \times 1}$$

$$= 2 \pm \sqrt{2}$$

[4 marks]

Answer 5

(i)

$$f(x) = x^3 - x^2 + x - 6$$

$$f(2) = 2^3 - 2^2 + 2 - 6$$

$$= 8 - 4 + 2 - 6$$

$$= 0$$

$\therefore$ $(x - 2)$ is a factor of $f(x)$ by the factor theorem

[1 mark]

(ii)

$$\begin{array}{r|l}
 & x^2 + x + 3 \\ \hline
x - 2 & x^3 - x^2 + x - 6 \\
 & \underline{x^3 - 2x^2} \\
 & \quad x^2 + x \\
 & \quad \underline{x^2 - 2x} \\
 & \qquad 3x - 6 \\
 & \qquad \underline{3x - 6} \\
 & \qquad\qquad 0 \leftarrow \text{As expected !}
\end{array}$$

$$f(x) = x^3 - x^2 + x - 6$$

$$= \left(x - 2\right)\left(x^2 + x + 3\right)$$

The quadratic root does not factorise further to give any more real roots other than the one already known about at $x = 2$

For $x^2 + x + 3 = 0$, $\quad x = \dfrac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 3}}{2 \times 1}$

$$= \frac{-1 \pm \sqrt{-11}}{2} \text{ which is not real}$$

[4 marks]

Answer 6

$$\text{With } x = 2 \Rightarrow x^3 + x^2 - ax - 2b = 0$$
$$8 + 4 - 2a - 2b = 0$$
$$2a + 2b = 12$$

$$\text{With } x = 2 \Rightarrow 3x^3 - 2x^2 - 3ax + 4b = 0$$
$$24 - 8 - 6a + 4b = 0$$
$$6a - 4b = 16$$

Divide through by 2

$$3a - 2b = 8$$

Solving these two equations simultaneously,

$$\begin{array}{rl} & 2a + 2b = 12 \\ \text{ADD} & 3a - 2b = 8 \\ \hline & 5a \quad\; = 20 \end{array}$$

$$a = 4 \quad \Rightarrow \quad b = 2$$

and $x^3 + x^2 - ax - 2b = 0$ becomes, $x^3 + x^2 - 4x - 4 = 0$.

Remember this has a solution $x = 2$

That is, has $(x - 2)$ as a factor,

$$\begin{array}{r|l} & \quad x^2 + 3x + 2 \\ \cline{2-2} x - 2 & x^3 + x^2 - 4x - 4 \\ & \underline{x^3 - 2x^2} \\ & \qquad 3x^2 - 4x \\ & \qquad \underline{3x^2 - 6x} \\ & \qquad\qquad 2x - 4 \\ & \qquad\qquad \underline{2x - 4} \\ & \qquad\qquad\quad 0 \leftarrow \text{As expected !} \end{array}$$

$$x^3 + x^2 - 4x - 4 = 0$$
$$(x - 2)(x^2 + 3x + 2) = 0$$
$$(x - 2)(x + 2)(x + 1) = 0$$

It has thus been shown that the equation also has a solution $x = -2$

and the third solution is $x = -1$

[6 marks]

Note : This is an old Additional Mathematics Examination Question
From 22nd June 1994, Paper 2

Answer 7

$$f(x) = x^4 - 2x^3 - 11x^2 + 12x + 36$$

$$\begin{aligned} f(1) &= (1)^4 - 2(1)^3 - 11(1)^2 + 12(1) + 36 \\ &= 1 - 2 - 11 + 12 + 36 \\ &= 36 \end{aligned}$$

$$\begin{aligned} f(-2) &= (-2)^4 - 2(-2)^3 - 11(-2)^2 + 12(-2) + 36 \\ &= 16 + 16 - 44 - 24 + 36 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(-3) &= (-3)^4 - 2(-3)^3 - 11(-3)^2 + 12(-3) + 36 \\ &= 81 + 54 - 99 - 36 + 36 \\ &= 36 \end{aligned}$$

By the factor theorem it's now known that $(x + 2)$ is a factor so let $a = -2$

When $(x - a)^2(x - b)^2$ is expanded the term independent of x is $a^2 b^2$

$$\begin{aligned} a^2 b^2 &= 36 \\ b^2 &= \frac{36}{4} \\ b &= \pm 3 \end{aligned}$$

But it has already been observed that $(x + 3)$ is not a factor, so $b = 3$

$$\begin{aligned} (x + 2)^2(x - 3)^2 &= ((x + 2)(x - 3))^2 \\ &= \left(x^2 - x - 6\right)^2 \end{aligned}$$

That is, $p = 1$, $q = -1$ and $r = -6$

[6 marks]

Note : This is an old Additional Mathematics Examination Question
From 13th June 1997, Paper 1