5.5 Answers

5.5.1 Solution to 5.3 Example (Teaching Video)

$$f(x) = x^3 - ax^2 + bx - 4$$

The question is, in effect, saying that,

$$\begin{aligned} f(1) &= 2 \\ (1)^3 - a(1)^2 + b(1) - 4 &= 2 \\ 1 - a + b - 4 &= 2 \\ -a + b &= 5 \\ a - b &= -5 \end{aligned}$$

The question is also, in effect, saying that,

$$\begin{aligned} f(2) &= 2 \\ (2)^3 - a(2)^2 + b(2) - 4 &= 2 \\ 8 - 4a + 2b - 4 &= 2 \\ -4a + 2b &= -2 \end{aligned}$$

Divide through by 2

$$\begin{aligned} -2a + b &= -1 \\ 2a - b &= 1 \end{aligned}$$

Combining these two results,

$$\begin{aligned} 2a - b &= 1 \\ \text{SUBTRACT} \qquad a - b &= -5 \\ \hline a \qquad &= 6 \qquad \Rightarrow \qquad b = 11 \end{aligned}$$

[5 marks]

5.5.2 Solutions to 5.4 Exercise

Answer 1

$$f(x) = 4x^3 + ax^2 + bx - 27$$

The question is, in effect, saying that,

$$\begin{aligned} f(1) &= -2 \\ 4(1)^3 + a(1)^2 + b(1) - 27 &= -2 \\ 4 + a + b - 27 &= -2 \\ a + b &= 21 \end{aligned}$$

The question is also, in effect, saying that,

$$\begin{aligned} f(-1) &= -100 \\ 4(-1)^3 + a(-1)^2 + b(-1) - 27 &= -100 \\ -4 + a - b - 27 &= -100 \\ a - b &= -69 \end{aligned}$$

Combining these two results,

$$\begin{array}{lrcl} & a + b &=& 21 \\ \text{ADD} & a - b &=& -69 \\ \hline & 2a &=& -48 \\ & a &=& -24 \qquad \Rightarrow b = 45 \end{array}$$

[5 marks]

Answer 2

$$f(x) = a x^3 + b x^2 - 28x + 15$$

The question is, in effect, saying that,

$$\begin{aligned} f(-3) &= 0 \\ a(-3)^3 + b(-3)^2 - 28(-3) + 15 &= 0 \\ -27a + 9b + 84 + 15 &= 0 \\ -27a + 9b &= -99 \\ \text{Divide through by } 9 & \\ -3a + b &= -11 \end{aligned}$$

The question is also, in effect, saying that,

$$\begin{aligned} f(3) &= -60 \\ a(3)^3 + b(3)^2 - 28(3) + 15 &= -60 \\ 27a + 9b - 84 + 15 &= -60 \\ 27a + 9b &= 9 \\ \text{Divide through by } 9 & \\ 3a + b &= 1 \end{aligned}$$

Combining these two results,

$$\begin{aligned} -3a + b &= -11 \\ \text{ADD} \quad 3a + b &= 1 \\ \hline 2b &= -10 \\ b &= -5 \qquad \Rightarrow \quad a = 2 \end{aligned}$$

[5 marks]

Note : The example used for the teaching video and also this question are from a single question in an old Additional Mathematics examination paper. The date I have for that is before “Tuesday 24th March 1992”

Answer 3

$$f(x) = x^3 - 4x^2 + ax + b$$

(i) The question is, in effect, saying that,

$$f(3) = 0$$

$$(3)^3 - 4(3)^2 + a(3) + b = 0$$

$$27 - 36 + 3a + b = 0$$

$$3a + b = 9$$

The question is also, in effect, saying that,

$$f(1) = 4$$

$$(1)^3 - 4(1)^2 + a(1) + b = 4$$

$$1 - 4 + a + b = 4$$

$$a + b = 7$$

Combining these two results,

$$\begin{array}{rrcl} & 3a + b & = & 9 \\ \text{SUBTRACT} & a + b & = & 7 \\ \hline & 2a \quad & = & 2 \end{array}$$

$$a = 1 \qquad \Rightarrow \; b = 6$$

[4 marks]

(ii) From part (i), $f(x) = x^3 - 4x^2 + x + 6$ has $(x - 3)$ as a factor

$$\begin{array}{r|l} & x^2 - x - 2 \\ \hline x - 3 & x^3 - 4x^2 + x + 6 \\ & \underline{x^3 - 3x^2} \\ & \quad - x^2 + x \\ & \quad \underline{- x^2 + 3x} \\ & \qquad - 2x + 6 \\ & \qquad \underline{- 2x + 6} \\ & \qquad\qquad 0 \leftarrow \text{As expected} \end{array}$$

$$f(x) = 0$$

$$x^3 - 4x^2 + x + 6 = 0$$

$$\left(x - 3\right)\left(x^2 - x - 2\right) = 0$$

$$(x - 3)(x - 2)(x + 1) = 0$$

Either $x = 3$ or $x = 2$ or $x = -1$ are the solutions

[3 marks]

Answer 4

(a) **(i)** The function will cut the x-axis when $f(x) = 0$

$$f(x) = 0$$

$$x^3 - 3x^2 - 4x = 0$$

$$x\left(x^2 - 3x - 4\right) = 0$$

$$x(x - 4)(x + 1) = 0$$

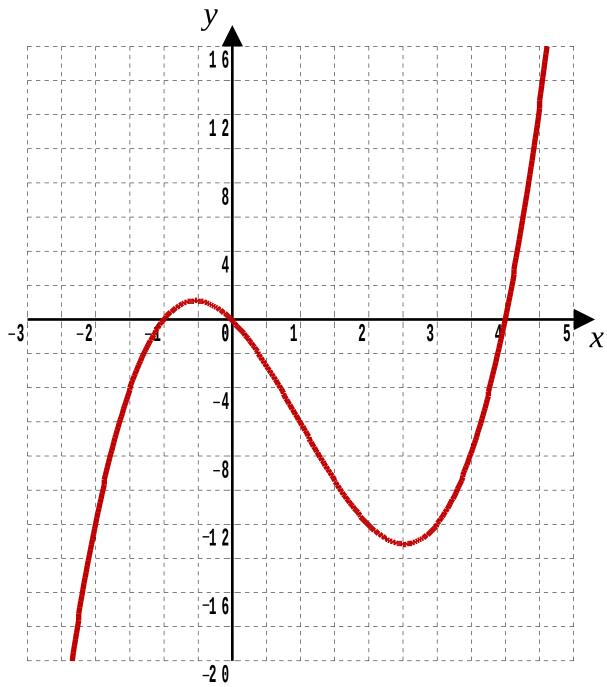
Either $x = 0$, $x = 4$ or $x = -1$

That is, the curve $y = f(x)$ cuts the x-axis at

$(-1,\ 0)$, $(0,\ 0)$ and $(4,\ 0)$

[4 marks]

(ii)



[1 mark]

(b) **(i)** $g(x) = x^3 - 3x^2 - 4x + 12$

By the remainder theorem, the remainder is given by, $g(-1)$

$$\begin{aligned} g(-1) &= (-1)^3 - 3(-1)^2 - 4(-1) + 12 \\ &= -1 - 3 + 4 + 12 \\ &= 12 \end{aligned}$$

[2 marks]

(ii)

$$\begin{aligned} g(2) &= (2)^3 - 3(2)^2 - 4(2) + 12 \\ &= 8 - 12 - 8 + 12 \\ &= 0 \end{aligned}$$

$\therefore$ By the factor theorem, $(x - 2)$ is a factor of $g(x)$

[1 mark]

(iii)

$$\begin{array}{r|l} & \quad x^2 - x - 6 \\ \cline{2-2} x - 2 & x^3 - 3x^2 - 4x + 12 \\ & \underline{x^3 - 2x^2} \\ & \quad -x^2 - 4x \\ & \quad \underline{-x^2 + 2x} \\ & \qquad -6x + 12 \\ & \qquad \underline{-6x + 12} \\ & \qquad\qquad 0 \leftarrow \text{As expected} \end{array}$$

$$\begin{aligned} g(x) &= 0 \\ x^3 - 3x^2 - 4x + 12 &= 0 \\ (x - 2)(x^2 - x - 6) &= 0 \\ (x - 2)(x - 3)(x + 2) &= 0 \end{aligned}$$

Either $x = 2$ or $x = 3$ or $x = -2$ are the solutions

[4 marks]

Answer 5

(i) As x is a factor, which could be thought of as the factor $(x - 0)$,

putting $x = 0$ into $x^4 + a x^3 + b x^2 + cx + d$ will cause it to equal zero, by the factor theorem.

$$\therefore \ d = 0$$

As $(x + 1)$ is a factor, putting $x = -1$ into $x^4 + a x^3 + b x^2 + cx + d$ cause it to equal zero, by the factor theorem.

$$\therefore \ (-1)^4 + a(-1)^3 + b(-1)^2 + c(-1) + d = 0$$

but it was just shown that $d = 0$ and so,

$$1 - a + b - c + 0 = 0$$

$$\therefore \ a - b + c = 1 \qquad \square$$

[2 marks]

(ii) The question is, in effect, saying that putting $x = 1$ into $x^4 + a x^3 + b x^2 + cx + d$ will cause it to equal 5, by the remainder theorem

$$(1)^4 + a(1)^3 + b(1)^2 + c(1) + d = 0$$

again, using the established fact that $d = 0$,

$$a + b + c = 4$$

Combining equations allows the extraction of a value for b

$$\begin{array}{rl} & a + b + c = 4 \\ \text{SUBTRACT} & a - b + c = 1 \\ \hline & 2b \qquad = 3 \\ & \qquad b = \dfrac{3}{2} \end{array}$$

[3 marks]

Answer 6

$$p(x) = 9x^2 + ax + b$$

(a) **(i)**

$$p(1) = 12 \quad \text{by the remainder theorem}$$

$$9\,(1)^2 + a(1) + b = 12$$

$$a + b = 3$$

$$p(-2) = 12 \quad \text{by the remainder theorem}$$

$$9\,(-2)^2 + a(-2) + b = 12$$

$$-2a + b = -24$$

$$2a - b = 24$$

Combining these two results simultaneously,

$$a + b = 3$$

$$\text{ADD} \qquad 2a - b = 24$$

$$3a \qquad = 27$$

$$a = 9 \;\Rightarrow\; \qquad b = -6$$

[2 marks]

(ii) By the remainder theorem, the remainder is given by,

$$p(-3) = 9\,(-3)^2 + a(-3) + b$$

$$= 9 \times 9 + 9 \times (-3) + (-6)$$

$$= 81 - 27 - 6$$

$$= 48$$

[2 marks]

(b) If $9x^3 + ax^2 + bx - 6$ is divisible by $(x + 1)$ then substituting in $x = -1$ will cause it to equal zero, by the factor theorem.

$$9(-1)^3 + (9)(-1)^2 + (-6)(-1) - 6 = -9 + 9 + 6 - 6$$

$$= 0$$

$\therefore$ $(x + 1)$ is indeed a factor of $9x^3 + ax^2 + bx - 6$ ☐

[2 marks]