## 7.3 Answers to 7.2 Test

**A-Level Pure Mathematics, Year 1**
**Additional Mathematics**
**The Algebra of Polynomials**

**Answer 1**

$p(x)$ has degree 4 (It's a Quartic)

**[ 1 mark ]**

**Answer 2**

**( i )** A degree 2 polynomial is called a quadratic
(and its graph is termed a parabola)

**[ 1 mark ]**

**( ii )** A degree 5 polynomial is called a quintic

**[ 1 mark ]**

**Answer 3**

$$f(x) = 3x^3 - 5x^2 - 16x + 12$$

**( a )** The remainder when $f(x)$ is divided by $(x - 2)$ is given by $f(2)$ by the remainder theorem.

$$\begin{aligned} f(2) &= 3(2)^3 - 5(2)^2 - 16(2) + 12 \\ &= 24 - 20 - 32 + 12 \\ &= -16 \end{aligned}$$

**[ 2 marks ]**

**( b )**

$$\begin{array}{r|rrrr} & & 3x^2 & -\ 11x & +\ 6 \\ \hline x + 2 & 3x^3 & -\ 5x^2 & -\ 16x & +\ 12 \\ & 3x^3 & +\ 6x^2 & & \\ \hline & & -\ 11x^2 & -\ 16x & \\ & & -\ 11x^2 & -\ 22x & \\ \hline & & & 6x & +\ 12 \\ & & & 6x & +\ 12 \\ \hline & & & & 0 \end{array} \quad \leftarrow \text{ As expected}$$

$$\begin{aligned} f(x) &= 3x^3 - 5x^2 - 16x + 12 \\ &= \left(x + 2\right)\left(3x^2 - 11x + 6\right) \\ &= (x + 2)(3x - 2)(x - 3) \end{aligned}$$

**[ 4 marks ]**

**Answer 4**

$$f(x) = x^3 + 4x^2 + x - 6$$

**( a )**

$$\begin{aligned} f(-2) &= (-2)^3 + 4(-2)^2 + (-2) - 6 \\ &= -8 + 16 - 2 - 6 \\ &= 0 \end{aligned}$$

$\therefore$ $(x + 2)$ is a factor of $f(x)$ by the factor theorem

**[ 2 marks ]**

**( b )**

$$\begin{array}{r|l} & \quad\quad\quad\; x^2 + 2x - 3 \\ \cline{2-2} x + 2 & x^3 + 4x^2 + x - 6 \\ & x^3 + 2x^2 \\ \cline{2-2} & \quad\quad 2x^2 + x \\ & \quad\quad 2x^2 + 4x \\ \cline{2-2} & \quad\quad\quad\; -3x - 6 \\ & \quad\quad\quad\; -3x - 6 \\ \cline{2-2} & \quad\quad\quad\quad\quad\quad 0 \leftarrow \text{As expected} \end{array}$$

$$\begin{aligned} x^3 + 4x^2 + x - 6 &= \left(x + 2\right)\left(x^2 + 2x - 3\right) \\ &= (x + 2)(x - 1)(x + 3) \end{aligned}$$

**[ 4 marks ]**

**( c )** The solutions are,

$$x = -3,\ x = -2 \text{ or } x = 1$$

**[ 1 mark ]**

**Answer 5**

$$f(x) = ax^3 + bx^2 - 4x - 3$$

**( a )** The question is, in effect, saying that,

$$f(1) = 0 \quad \text{by the factor theorem}$$

$$a(1)^3 + b(1)^2 - 4(1) - 3 = 0$$

$$a + b - 4 - 3 = 0$$

$$a + b = 7 \qquad \square$$

**[ 2 marks ]**

**( b )** The question is, in effect, saying that,

$$f(-2) = 9 \quad \text{by the remainder theorem}$$

$$a(-2)^3 + b(-2)^2 - 4(-2) - 3 = 9$$

$$-8a + 4b + 8 - 3 = 9$$

$$-8a + 4b = 4$$

Divide through by $-4$

$$2a - b = -1$$

Combining this equation with that of part (a) simultaneously,

$$a + b = 7$$

ADD $\quad 2a - b = -1$

$$3a = 6$$

$$a = 2 \quad \Rightarrow \quad b = 5$$

**[ 4 marks ]**

**Answer 6**

$$f(x) = x^3 - x^2 - 7x + c$$

**( a )**

$$f(4) = 0$$
$$(4)^3 - (4)^2 - 7(4) + c = 0$$
$$64 - 16 - 28 + c = 0$$
$$c = -20$$

**[ 2 marks ]**

**( b )** Part (a) indirectly revealed that $( x - 4 )$ is a factor of $f(x)$ by the factor theorem

$$\begin{array}{r|llllll}
 & & x^2 & + & 3x & + & 5 \\ \hline
x - 4 & x^3 & - x^2 & - & 7x & - & 20 \\
 & x^3 & - 4x^2 & & & & \\ \hline
 & & 3x^2 & - & 7x & & \\
 & & 3x^2 & - & 12x & & \\ \hline
 & & & & 5x & - & 20 \\
 & & & & 5x & - & 20 \\ \hline
 & & & & & & 0 \leftarrow \text{As expected}
\end{array}$$

$$f(x) = x^3 - x^2 - 7x - 20$$
$$= \left( x - 4 \right)\left( x^2 + 3x + 5 \right)$$

**[ 3 marks ]**

**( c )** It is required that the following equation be solved,

$$f(x) = 0$$
$$x^3 - x^2 - 7x - 20 = 0$$
$$\left( x - 4 \right)\left( x^2 + 3x + 5 \right) = 0$$

Either $\left( x - 4 \right) = 0$ in which case $x = 4$

Or $\left( x^2 + 3x + 5 \right) = 0$ in which case $x = \dfrac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 5}}{2 \times 1}$

$$x = \frac{-3 \pm \sqrt{-11}}{2}$$

but this has no real solutions

$\therefore\ x = 4$ is the only solution □

**[ 2 marks ]**

**Answer 7**

$$p(x) = 4x^3 + ax^2 + bx + 15$$

By the remainder theorem, $p(1) = 90$

$$4(1)^3 + a(1)^2 + b(1) + 15 = 90$$
$$4 + a + b + 15 = 90$$
$$a + b = 71$$

By the remainder theorem, $f(-1) = -4$

$$4(-1)^3 + a(-1)^2 + b(-1) + 15 = -4$$
$$-4 + a - b + 15 = -4$$
$$a - b = -15$$

Combining the two results as simultaneous equations,

$$\begin{array}{lrcl} & a + b & = & 71 \\ \text{ADD} & a - b & = & -15 \\ \hline & 2a & = & 56 \end{array}$$

$$a = 28 \qquad \Rightarrow \qquad b = 43$$

**[ 7 marks ]**

**Answer 8**

$$f(x) = x^4 - 3x^2 - ax + b \qquad g(x) = x^3 - 3ax + 5b$$

By the factor theorem, if a polynomial is divisible by $(x - 2)$ then letting $x = 2$ causes the polynomial to equal zero.

$$f(2) = 0 \qquad\qquad g(2) = 0$$

$$(2)^4 - 3(2)^2 - a(2) + b = 0 \qquad\qquad (2)^3 - 3a(2) + 5b = 0$$
$$16 - 12 - 2a + b = 0 \qquad\qquad 8 - 6a + 5b = 0$$
$$-2a + b = -4 \qquad\qquad -6a + 5b = -8$$
$$2a - b = 4 \qquad\qquad 6a - 5b = 8$$

Thus, $2(2a - b) = 6a - 5b$

$$4a - 2b = 6a - 5b$$
$$3b = 2a \qquad \square$$

**[ 5 marks ]**

Alternatively solve simultaneous equation to get $a = 3$ and $b = 2$ from which the requested result follows.

**Answer 9**

$$f(x) = 2x^3 - 5x^2 + ax + 18$$

**( a )** As $(x - 3)$ is a factor, putting $x = 3$ into $f(x)$ will cause $f(x)$ to equal zero, by the factor theorem,

$$\therefore\ 2(3)^3 - 5(3)^2 + a(3) + 18 = 0$$

$$54 - 45 + 3a + 18 = 0$$

$$3a = -27 \Leftrightarrow a = -9 \qquad \square$$

**[ 2 marks ]**

**( b )**

$$\begin{array}{r|rrrr}
 & 2x^2 & + x & - 6 & \\
\hline
x - 3 & 2x^3 - 5x^2 & - 9x & + 18 & \\
 & 2x^3 - 6x^2 & & & \\
\hline
 & x^2 & - 9x & & \\
 & x^2 & - 3x & & \\
\hline
 & & - 6x & + 18 & \\
 & & - 6x & + 18 & \\
\hline
 & & & 0 & \leftarrow \text{ As expected}
\end{array}$$

$$\begin{aligned} f(x) &= 2x^3 - 5x^2 - 9x + 18 \\ &= (x - 3)(2x^2 + x - 6) \\ &= (x - 3)(2x - 3)(x + 2) \end{aligned}$$

**[ 4 marks ]**

**( c )** It's helpful to solve the equation $f(x) = 0$

That is, $(x - 3)(2x - 3)(x + 2) = 0$

Either $x = 3$ or $x = \dfrac{3}{2}$ or $x = -2$

$$\begin{aligned} g(y) &= 2\left(3^{3y}\right) - 5\left(3^{2y}\right) - 9\left(3^{y}\right) + 18 \\ &= 2\left(3^{y}\right)^3 - 5\left(3^{y}\right)^2 - 9\left(3^{y}\right) + 18 \end{aligned}$$

Solving $g(x) = 0$ is the same as solving $f(x) = 0$ with $x = 3^y$

$$\therefore\ \text{Either } 3^y = 3 \Rightarrow y = 1$$

$$\text{Or } 3^y = \frac{3}{2} \Rightarrow y = \frac{\ln 3 - \ln 2}{\ln 3} = 0.3690702464\ldots \text{ (accept T+I)}$$

$$\text{Or } 3^y = -2 \Rightarrow \text{ No solutions}$$

**[ 3 marks ]**