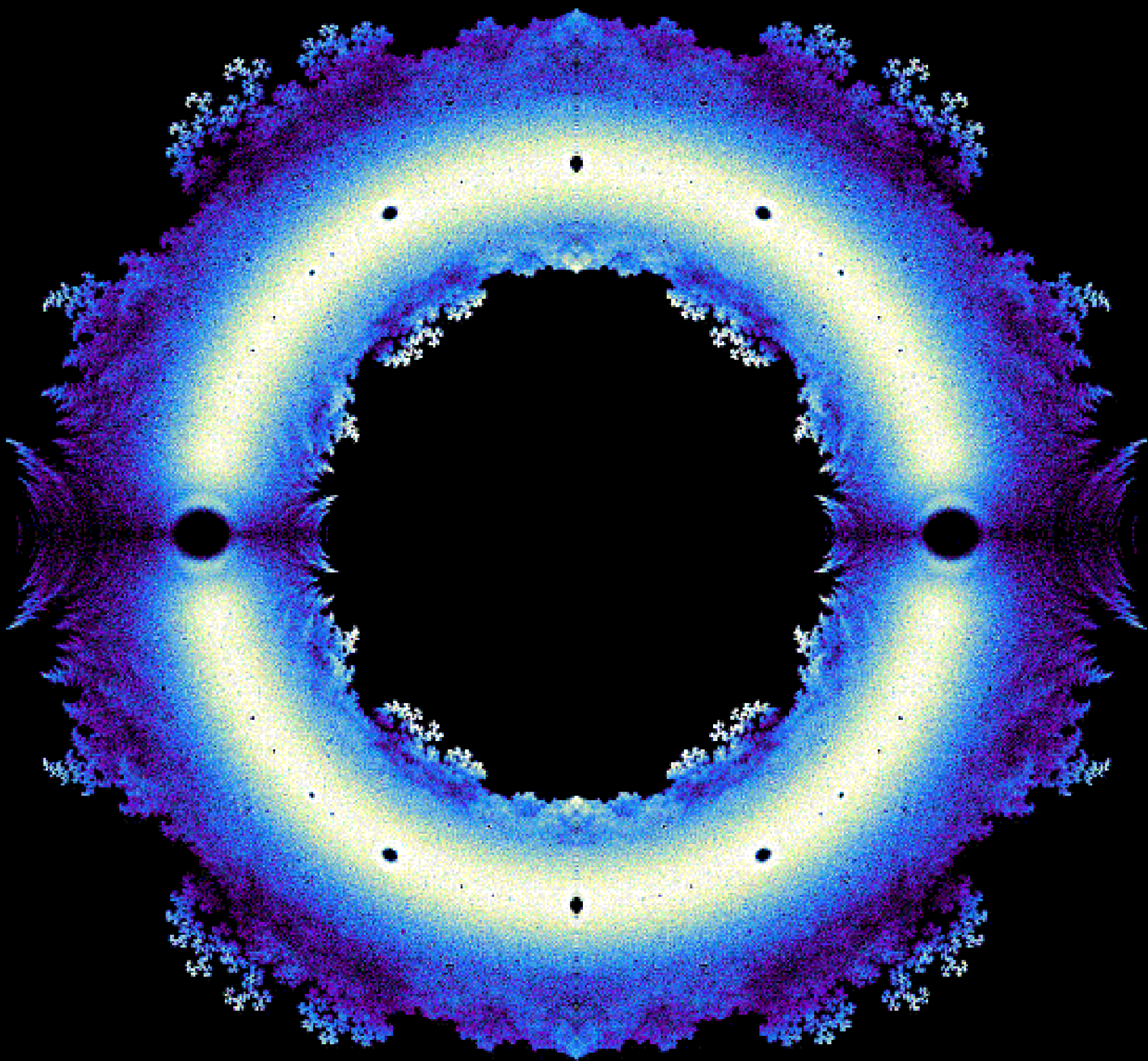
The Algebra Of Polynomials

The Factor and Remainder Theorems



Roots of all degree 18 polynomials with
coefficients +1 or −1 plotted in the complex plane
with roots on the real-axis removed
Image plotted Paul Nylander

The Algebra Of Polynomials

The Factor & The Remainder Theorems

Lesson 1

A-Level Pure Mathematics, Year 1
Additional Mathematics
The Algebra of Polynomials

1.1 What Are Polynomials ?

In this course, polynomials are algebraic expressions involving powers of x as the only function of x. The powers are integers that are non-negative. A typical polynomial is built by adding together multiples of the powers of x.

1.2 Exercise

Complete the following table,

Expression	Is the expression a polynomial ?	If the expression is not a polynomial, why not ?
$x^3 - x$		
$x^3 + 4.2x + 6.5$		
$x^2 + \sqrt{x}$		
$(x + 4)(x - 5)$		
$x^4 + \sqrt{3}\,x^2 + \pi^2$		
$x^{10} + x^1 + x^{0.1}$		
$x + \frac{1}{x}$		
$\sqrt{5}\,x^3 + x^{\sqrt{2}} - 41x + 20$		
$2\sin^2 x + 3\sin x + 12$		

[18 marks]

Polynomials are amongst the simplest form of mathematical function. In spite of this, they are remarkably useful. They are easy to integrate and differentiate and many other, more complicated functions, can be approximated by polynomials. They have a structure in the world of algebra that is similar to that of the integers in the world of numbers.

1.3 The Degree Of A Polynomial

The ***degree*** of a polynomial is the index of the highest power of x in the expression. A quadratic equation is an example of a polynomial of degree two. Polynomials of higher degree include;

Cubics (degree 3), Quartics (degree 4), Quintics (degree 5), Sextics (degree 6)

and....

... especially useful to medical students...

Septics (degree 7)

1.4 Exercise

Complete the following table,

Polynomial Expression	What is the degree and name of the polynomial ?
$x^5 + 4x^3 + 576x^2 + 1$	
$(2x + 7)(5x - 8)$	
$62 + 13x^4 - x^6$	
$(4 + x)(3x + 4)(1 - x)$	
$(x^2 + 1)(x^2 - 1)$	
$x^3(3x - 4 + x^4)$	
$3x - 5$	

[7 marks]

1.5 Answers

1.5.1 Solutions to 1.2 Exercise

Expression	Is the expression a polynomial ?	If the expression is not a polynomial, why not ?
$x^3 - x$	Yes	
$x^3 + 4.2x + 6.5$	Yes	
$x^2 + \sqrt{x}$	No	$\sqrt{x} = x^{0.5}$ which has a power that is not an integer
$(x + 4)(x - 5)$	Yes	
$x^4 + \sqrt{3}\,x^2 + \pi^2$	Yes	
$x^{10} + x^1 + x^{0.1}$	No	The power of $x^{0.1}$ is not an integer
$x + \frac{1}{x}$	No	$\frac{1}{x} = x^{-1}$ which has a power that is negative
$\sqrt{5}\,x^3 + x^{\sqrt{2}} - 41x + 20$	No	$x^{\sqrt{2}}$ has a power which is not an integer
$2\sin^2 x + 3\sin x + 12$	No	The only function allowed is "powers of x"

[18 marks]

1.5.2 Solutions to 1.4 Exercise

Polynomial Expression	What is the degree and name of the polynomial ?
$x^5 + 4x^3 + 576x^2 + 1$	5 : Quintic
$(2x + 7)(5x - 8)$	2 : Quadratic (whose graph is a Parabola)
$62 + 13x^4 - x^6$	6 : Sextic (or Hexic)
$(4 + x)(3x + 4)(1 - x)$	3 : Cubic
$\left(x^2 + 1\right)\left(x^2 - 1\right)$	4: Quartic
$x^3\left(3x - 4 + x^4\right)$	7 : Septic (or Heptic)
$3x - 5$	1: Linear (or Monic)

[7 marks]

Lesson 2

2.1 Algebraic Long Division Of Polynomials

Example

$$f(x) = x^3 + 12x^2 + 47x + 60$$

(i) Show by algebraic long division that $f(x)$ is divisible by $(x + 5)$

(ii) Hence factorise $f(x)$ completely

(iii) Hence sketch the graph of $f(x)$

Teaching Video : http://www.NumberWonder.co.uk/v9029/2a.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9029/2b.mp4 (Part 2)

<= Part 1 Part 2 =>

[4, 2, 2 marks]

2.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 46

Question 1

$$f(x) = x^3 + 7x^2 + 14x + 8$$

(i) Show by algebraic long division that $f(x)$ is divisible by $(x + 4)$
(ii) Hence factorise $f(x)$ completely
(iii) Hence sketch the graph of $f(x)$

[4, 2, 2 marks]

Question 2

$$f(x) = x^3 + 7x^2 - 9x - 63$$

(i) Show by algebraic long division that $f(x)$ is divisible by $(x + 3)$

(ii) Hence factorise $f(x)$ completely

(iii) Hence sketch the graph of $f(x)$

Be careful with minus signs

e.g. $(-9x) - (12x) = -21x$

[4, 2, 2 marks]

Question 3

$$f(x) = x^3 + 3x^2 - 4x - 12$$

(i) Show by algebraic long division that $f(x)$ is divisible by $(x - 2)$

(ii) Hence factorise $f(x)$ completely.

(iii) Hence sketch the graph of $f(x)$.

Be **VERY** careful with minus signs !

e.g. $(-4x) - (-10x) = 6x$

[4, 2, 2 marks]

Question 4

$$f(x) = x^3 + 6x^2 + 3x - 10$$

Notice that the function "ends" in – 10

As a result the likely factors are ($x \pm 1$), ($x \pm 2$), ($x \pm 5$) or ($x \pm 10$)

(i) Show by algebraic long division that $f(x)$ is NOT divisible by ($x + 1$)

(ii) Try other possibilities from the list, until you find a factor that divides $f(x)$

(iii) Hence factorise $f(x)$ completely

(iv) Hence sketch the graph of $f(x)$

[4, 4, 2, 2 marks]

Question 5

$$f(x) = x^3 + 4x^2 + x - 6$$

(i) Use the – 6 to list the likely factors of $f(x)$

(ii) By algebraic long division, find a factor of $f(x)$ of the form $(x + a)$ where a is an integer.

(iii) Hence factorise $f(x)$ completely

(iv) Hence sketch the graph of $f(x)$

[2, 4, 2, 2 marks]

2.3 Answers

2.3.1 Solution to 2.1 Example (Teaching Video)

(i) Setting out of the algebraic long division clearly is key to success,

$$
\begin{array}{r|rrrr}
 & x^2 & +\ 7x & +\ 12 & \\
\hline
x + 5 & x^3 + 12x^2 & +\ 47x & +\ 60 & \\
 & x^3 + 5x^2 & & & \\
\hline
 & 7x^2 & +\ 47x & & \\
 & 7x^2 & +\ 35x & & \\
\hline
 & & 12x & +\ 60 & \\
 & & 12x & +\ 60 & \\
\hline
 & & & 0 & \leftarrow \text{Remainder}
\end{array}
$$

$\therefore\ f(x)$ is divisible by $x + 5$ $\square$

[4 marks]

(ii) Part (i) reveals that,

$$\frac{x^3 + 12x^2 + 47x + 60}{x + 5} = x^2 + 7x + 12$$

$$\therefore\ x^3 + 12x^2 + 47x + 60 = \left(x^2 + 7x + 12\right)\left(x + 5\right)$$

$$= (x + 3)(x + 4)(x + 5)$$

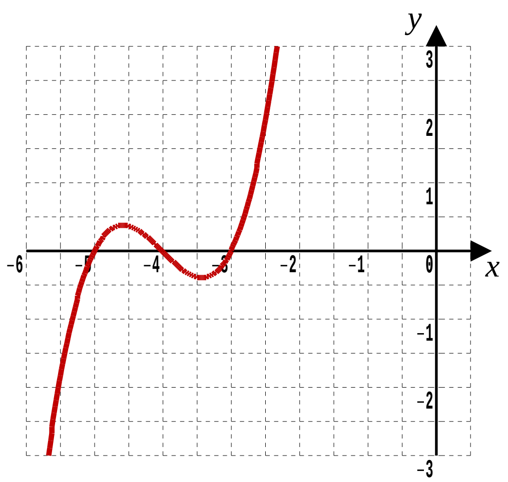
[2 marks]

(iii) To sketch the graph, work out that it crosses the x-axis when $y = 0$

That is, when $(x + 3)(x + 4)(x + 5) = 0$

Either $x = -3$, or $x = -4$, or $x = -5$

It crosses the y-axis when $y = 60$



[2 marks]

2.3.2 Solution to 2.2 Exercise

Answer 1

(i)

$$
\begin{array}{r|l}
 & x^2 + 3x + 2 \\
\cline{2-2}
x + 4 & x^3 + 7x^2 + 14x + 8 \\
 & x^3 + 4x^2 \\
\cline{2-2}
 & \quad 3x^2 + 14x \\
 & \quad 3x^2 + 12x \\
\cline{2-2}
 & \qquad 2x + 8 \\
 & \qquad 2x + 8 \\
\cline{2-2}
 & \qquad\quad 0 \leftarrow \text{Remainder}
\end{array}
$$

$\therefore\ f(x)$ is divisible by $x + 4$ □

[4 marks]

(ii)

$$\frac{x^3 + 7x^2 + 14x + 8}{x + 4} = x^2 + 3x + 2$$

$$\therefore\ x^3 + 7x^2 + 14x + 8 = \left(x^2 + 3x + 2\right)\left(x + 4\right)$$

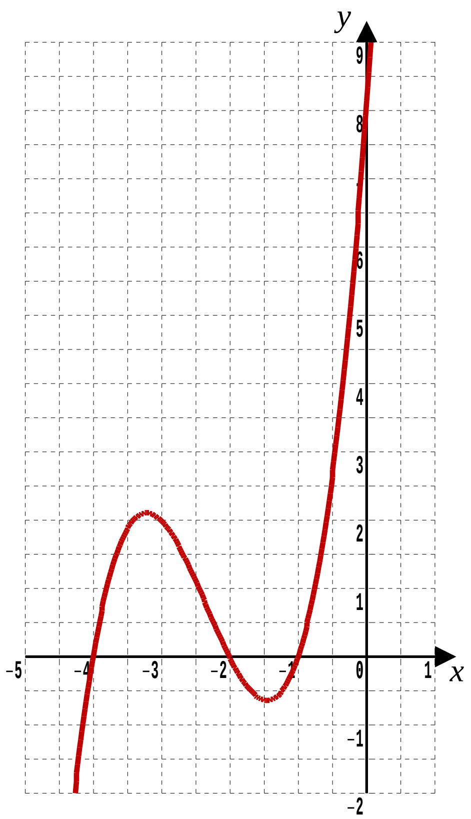
$$= (x + 1)(x + 2)(x + 4)$$

[2 marks]

(iii) When $y = 0$, $(x + 1)(x + 2)(x + 4) = 0$

$\therefore$ Either $x = -1$, or $x = -2$, or $x = -4$

Also the graph of $f(x)$ crosses the y-axis when $y = 8$



[2 marks]

Answer 2

(i)

$$
\begin{array}{r|l}
 & x^2 + 4x - 21 \\
\hline
x + 3 & x^3 + 7x^2 - 9x - 63 \\
 & \underline{x^3 + 3x^2} \\
 & \qquad 4x^2 - 9x \\
 & \qquad \underline{4x^2 + 12x} \\
 & \qquad\qquad -21x - 63 \\
 & \qquad\qquad \underline{-21x - 63} \\
 & \qquad\qquad\qquad 0 \leftarrow \text{Remainder}
\end{array}
$$

$\therefore\ f(x)$ is divisible by $x + 3$ ☐

[4 marks]

(ii)

$$\frac{x^3 + 7x^2 - 9x - 63}{x + 3} = x^2 + 4x - 21$$

$$\therefore\ x^3 + 7x^2 - 9x - 63 = \left(x^2 + 4x - 21\right)\left(x + 3\right)$$

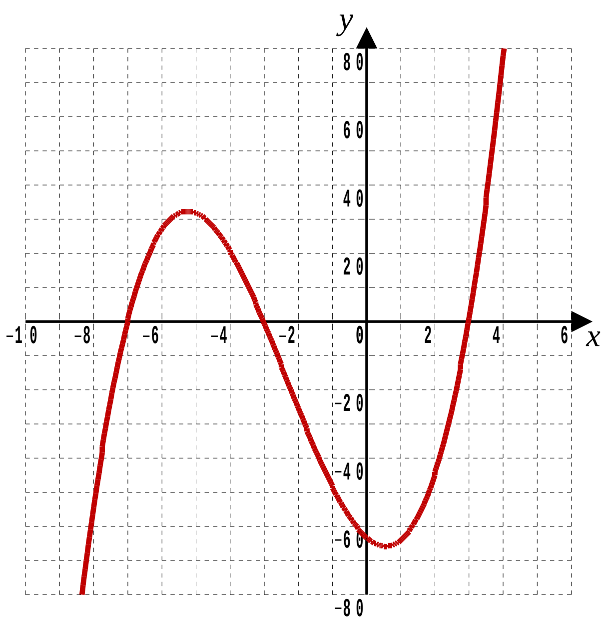
$$= (x - 3)(x + 7)(x + 3)$$

[2 marks]

(iii) When $y = 0$, $(x - 3)(x + 7)(x + 3) = 0$

$\therefore$ Either $x = 3$, or $x = -7$, or $x = -3$

Also the graph of $f(x)$ crosses the y-axis when $y = -63$



[2 marks]

Answer 3

(i)

$$
\begin{array}{r|rrrr}
 & x^2 & +\ 5x & +\ 6 & \\
\cline{2-5}
x - 2 & x^3 + 3x^2 & -\ 4x & -\ 12 & \\
 & x^3 - 2x^2 & & & \\
\cline{2-2}
 & 5x^2 & -\ 4x & & \\
 & 5x^2 & -\ 10x & & \\
\cline{2-3}
 & & 6x & -\ 12 & \\
 & & 6x & -\ 12 & \\
\cline{3-4}
 & & & 0 & \leftarrow \text{Remainder}
\end{array}
$$

$\therefore\ f(x)$ is divisible by $x - 2$ ☐

[4 marks]

(ii) $$\frac{x^3 + 3x^2 - 4x - 12}{x - 2} = x^2 + 5x + 6$$

$$\therefore\ x^3 + 7x^2 - 9x - 63 = \left(x^2 + 5x + 6\right)\left(x - 2\right)$$

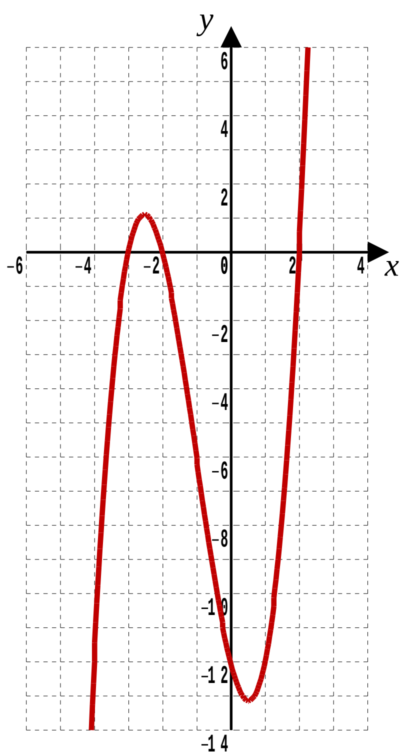
$$= (x + 3)(x + 2)(x - 2)$$

[2 marks]

(iii) When $y = 0,\ (x + 3)(x + 2)(x - 2) = 0$

$\therefore$ Either $x = -3$, or $x = -2$, or $x = 2$

Also the graph of $f(x)$ crosses the y-axis when $y = -12$



[2 marks]

Answer 4

(i)

$$
\begin{array}{r|rrrrrrr}
 & & & x^2 & + & 5x & - & 2 \\ \cline{2-8}
x + 1 & x^3 & + & 6x^2 & + & 3x & - & 10 \\
 & x^3 & + & x^2 & & & & \\ \cline{2-4}
 & & & 5x^2 & + & 3x & & \\
 & & & 5x^2 & + & 5x & & \\ \cline{4-6}
 & & & & - & 2x & - & 10 \\
 & & & & - & 2x & - & 2 \\ \cline{5-8}
 & & & & & & - & 8 \leftarrow \text{Remainder}
\end{array}
$$

$\therefore\ f(x)$ is not divisible by $x + 1$ □

[4 marks]

(ii) The factors that will give an exact division are, $(x - 1)$, $(x + 2)$, $(x + 5)$
Looking for a successful long division with one of those factors.

[4 marks]

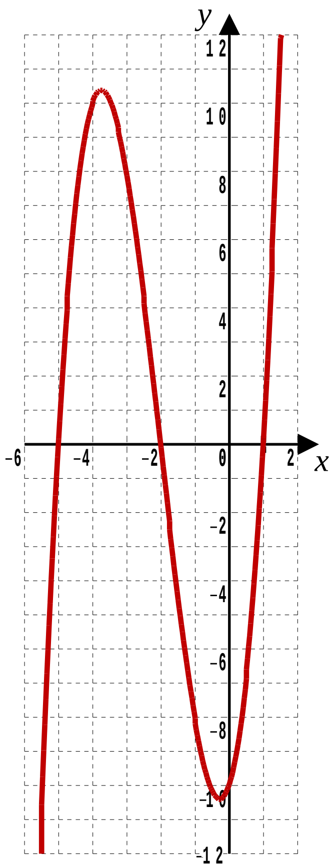
(iii) $\therefore\ x^3 + 6x^2 + 3x - 10 = (x + 5)(x + 2)(x - 1)$

[2 marks]

(iv) When $y = 0$, $(x + 5)(x + 2)(x - 1) = 0$

$\therefore$ Either $x = -5$, or $x = -2$, or $x = 1$

Also the graph of $f(x)$ crosses the y-axis when $y = -10$



[2 marks]

Answer 5

(i) The likely factors are $(x \pm 1)$, $(x \pm 2)$, $(x \pm 3)$, or $(x \pm 6)$

[2 marks]

(ii) The factors that will give an exact division are $(x - 1)$, $(x + 2)$, $(x + 3)$
Look for a successful long division with one of those factors.

[4 marks]

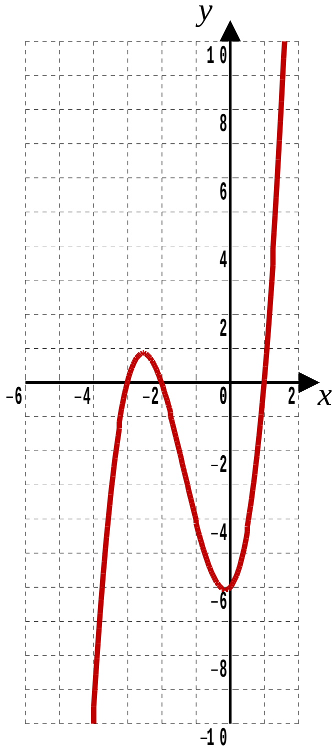
(iii) $\therefore\ x^3 + 4x^2 + x - 6 = (x - 1)(x + 2)(x + 3)$

[2 marks]

(iv) When $y = 0$, $(x + 3)(x + 2)(x - 1) = 0$

$\therefore$ Either $x = -3$, or $x = -2$, or $x = 1$

Also the graph of $f(x)$ crosses the y-axis when $y = -6$



[2 marks]

3.1 Less Long Division

Faced with a cubic equation, an ability to do algebraic long division gave a method of factorising the cubic. The main effort involved was in initially finding one of the factors. Once a first factor was found then, via an algebraic long division, the other two factors were easy to extract from the quotient quadratic. Having to try to find that initial factor is best done, not by algebraic long division, but by making use of a The Factor Theorem.

The Factor Theorem

If, for a given polynomial function $p(x)$, $p(a) = 0$ (for some constant, a)

then $(x - a)$ is a factor of $p(x)$

3.2 Example

(a) Use the factor theorem to determine which, if any, of the following are factors of $f(x) = x^3 + 8x^2 + 5x - 50$

(i) $(x + 1)$

(ii) $(x - 2)$

(b) Factorise $f(x)$ fully

(c) Solve the equation $f(x) = 0$

Teaching Video : http://www.NumberWonder.co.uk/v9029/3a.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9029/3b.mp4 (Part 2)

<= Part 1 Part 2 =>

[2, 2, 4, 1 marks]

3.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

$$f(x) = 2x^3 + 3x^2 - 3x - 2$$

Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$

[2 marks]

Question 2

$$g(x) = x^3 + 6x^2 - 19x - 84$$

Use the factor theorem to show that $(x - 3)$ is not a factor of $g(x)$

[2 marks]

Question 3

$$h(x) = x^3 - 3x^2 - 9x + 27$$

(a) **(i)** Determine the value of $h(1)$

[1 mark]

(ii) What does this tell you about $(x - 1)$ with regards to $h(x)$?

[1 mark]

(b) **(i)** Determine the value of $h(3)$

[1 mark]

(ii) What does this tell you about $(x - 3)$ with regards to $h(x)$?

[1 mark]

Question 4

$$k(x) = x^3 - 3x^2 - 36x - 48$$

Here is an algebraic long division of $k(x)$ by $(x + 1)$,

$$\begin{array}{r|l}
 & \quad\;\; x^2 - 4x - 32 \\ \hline
x + 1 & x^3 - 3x^2 - 36x - 48 \\
 & x^3 + x^2 \\ \hline
 & \quad - 4x^2 - 36x \\
 & \quad - 4x^2 - 4x \\ \hline
 & \qquad\quad - 32x - 48 \\
 & \qquad\quad - 32x - 32 \\ \hline
 & \qquad\qquad\quad - 16
\end{array}$$

From studying this long division, what do you deduce about $k(x)$ and $(x + 1)$?
Give a reason for your answer

[2 marks]

Question 5

$$m(x) = x^3 + 12x^2 - 4x - 240$$

Use the factor theorem to decide if $(x - 4)$ is a factor of $m(x)$

[3 marks]

Question 6

$$n(x) = x^3 - x^2 - x - 2$$

Use the factor theorem to find a factor of $n(x)$

[3 marks]

Question 7

$$f(x) = x^3 + 5x^2 - 8x - 48$$

(i) Show by algebraic long division that $(x - 3)$ is a factor of $f(x)$

[4 marks]

(ii) Hence factorise $f(x)$ completely

[2 marks]

(iii) Solve the equation, $f(x) = 0$

[1 mark]

Question 8

$$g(x) = x^3 + 6x^2 - 15x - 100$$

(a) Use the factor theorem to determine which, if any, of the following are factors of $g(x)$;

(i) $(x - 4)$

[2 marks]

(ii) $(x + 2)$

[2 marks]

(b) Factorise $g(x)$ fully

[4 marks]

(c) Solve the equation $g(x) = 0$

[1 mark]

Question 9

$$h(x) = x^3 + 18x^2 + 108x + 216$$

(a) Determine

(i) $h(-4)$

[2 marks]

(ii) $h(-6)$

[2 marks]

(b) Factorise $h(x)$ fully

[4 marks]

(c) Solve the equation $h(x) = 0$

[1 mark]

Question 10

$$f(x) = x^3 + ax^2 + bx - 12, \quad \text{for some constants } a \text{ and } b$$

(i) Given that $(x - 1)$ is a factor of $f(x)$ write down an equation which must be satisfied by a and b

[2 marks]

(ii) Given that $(x + 2)$ is a factor of $f(x)$ write down another equation which must be satisfied by a and b

[2 marks]

(iii) Solve your part (i) and part (ii) equations simultaneously to find the value of a and the value of b

[3 marks]

(iv) Hence, or otherwise, factorise $f(x)$ completely

[2 marks]

3.4 Answers

3.4.1 Solution to 3.2 Example (Teaching Video)

$$f(x) = x^3 + 8x^2 + 5x - 50$$

(a) **(i)**

$$\begin{aligned} f(-1) &= (-1)^3 + 8(-1)^2 + 5(-1) - 50 \\ &= -1 + 8 - 5 - 50 \\ &= -48 \end{aligned}$$

That is, $f(-1) \neq 0$

$\therefore\ x + 1$ is not a factor of $f(x)$, by the factor theorem

[2 marks]

(ii)

$$\begin{aligned} f(2) &= (2)^3 + 8(2)^2 + 5(2) - 50 \\ &= 8 + 32 + 10 - 50 \\ &= 0 \end{aligned}$$

That is, $f(2) = 0$

$\therefore\ x - 2$ is a factor of $f(x)$, by the factor theorem

[2 marks]

(b)

$$\begin{array}{r|rrrrrrr} & & & x^2 & + & 10x & + & 25 \\ \cline{2-8} x - 2 & x^3 & + & 8x^2 & + & 5x & - & 50 \\ & x^3 & - & 2x^2 & & & & \\ \cline{2-4} & & & 10x^2 & + & 5x & & \\ & & & 10x^2 & - & 20x & & \\ \cline{4-6} & & & & & 25x & - & 50 \\ & & & & & 25x & - & 50 \\ \cline{6-8} & & & & & & & 0 \end{array} \quad \leftarrow \text{ As expected !}$$

$$\begin{aligned} f(x) &= x^3 + 8x^2 + 5x - 50 \\ &= \left(x - 2\right)\left(x^2 + 10x + 25\right) \\ &= (x - 2)(x + 5)(x + 5) \end{aligned}$$

[4 marks]

(c) If $f(x) = 0$

then $(x - 2)(x + 5)(x + 5) = 0$

in which case either $x = 2$ or $x = -5$

[1 mark]

3.4.2 Solution to 3.3 Exercise

Answers 1

$$f(x) = 2x^3 + 3x^2 - 5x - 2$$

$$f(-2) = 2 \times (-2)^3 + 3 \times (-2)^2 - 5 \times (-2) - 2$$

$$= -16 + 12 + 6 - 2$$

$$= 0$$

That is, $f(-2) = 0$

$\therefore\ x + 2$ is a factor of $f(x)$, by the factor theorem

[2 marks]

Answer 2

$$g(x) = x^3 + 6x^2 - 19x - 84$$

$$g(3) = (3)^3 + 6 \times (3)^2 - 19 \times (3) - 84$$

$$= 27 + 54 - 57 - 84$$

$$= -60$$

That is, $g(3) \neq 0$

$\therefore\ x - 3$ is not a factor of $g(x)$, by the factor theorem

[2 marks]

Answer 3

(a) **(i)**

$$h(x) = x^3 - 3x^2 - 9x + 27$$

$$h(1) = (1)^3 - 3 \times (1)^2 - 9 \times (1) + 27$$

$$= 1 - 3 - 9 + 27$$

$$= 16$$

[1 mark]

(ii) As, $h(1) \neq 0$ it follows that $x - 1$ is not a factor of $h(x)$, by the factor theorem

[1 mark]

(b) **(i)**

$$h(x) = x^3 - 3x^2 - 9x + 27$$

$$h(3) = (3)^3 - 3 \times (3)^2 - 9 \times (3) + 27$$

$$= 9 - 27 - 27 + 27$$

$$= 0$$

[1 mark]

(ii) As, $h(3) = 0$ it follows that $x - 3$ is a factor of $h(x)$, by the factor theorem

[1 mark]

Answer 4

The long division shows that $(x+1)$ is not a factor of $k(x)$ because the remainder is not zero, it is -16

[2 marks]

Answer 5

$$m(x) = x^3 + 12x^2 - 4x - 240$$

$$m(4) = (4)^3 + 12 \times (4)^2 - 4 \times (4) - 240$$

$$= 64 + 192 - 16 - 240$$

$$= 0$$

That is, $m(4) = 0$

$\therefore\ x - 4$ is a factor of $m(x)$, by the factor theorem

[3 marks]

Answer 6

The guessable possible factors are $(x \pm 1)$, $(x \pm 2)$

$n(1) = -3,\quad n(-1) = -3,\quad n(2) = 0,\quad n(-2) = -12$

Thus, by the factor theorem, $(x-2)$ is a factor of $n(x)$

[3 marks]

Answer 7

(i)

$$\begin{array}{r|l}
 & x^2 + 8x + 16 \\
\hline
x - 3 & x^3 + 5x^2 - 8x - 48 \\
 & \underline{x^3 - 3x^2} \\
 & \quad 8x^2 - 8x \\
 & \quad \underline{8x^2 - 24x} \\
 & \qquad 16x - 48 \\
 & \qquad \underline{16x - 48} \\
 & \qquad\qquad 0 \leftarrow \text{Remainder}
\end{array}$$

The remainder of zero shows that $(x-3)$ is a factor of $f(x)$

[4 marks]

(ii)

$$f(x) = x^3 + 5x^2 - 8x - 48$$

$$= (x-3)(x^2 + 8x + 16)$$

$$= (x-3)(x+4)(x+4)$$

[2 marks]

(iii)

If $f(x) = 0$

then $(x-3)(x+4)(x+4) = 0$

in which case either $x = 3$ or $x = -4$

[1 mark]

Answer 8

$$g(x) = x^3 + 6x^2 - 15x - 100$$

(a) **(i)**

$$\begin{aligned} g(4) &= (4)^3 + 6(4)^2 - 15(4) - 100 \\ &= 64 + 96 - 60 - 100 \\ &= 0 \end{aligned}$$

That is, $g(4) = 0$

$\therefore\ x - 4$ is a factor of $g(x)$, by the factor theorem

[2 marks]

(ii)

$$\begin{aligned} g(-2) &= (-2)^3 + 6(-2)^2 - 15(-2) - 100 \\ &= -8 + 24 + 30 - 100 \\ &= -54 \end{aligned}$$

That is, $g(-2) \neq 0$

$\therefore\ x + 2$ is not a factor of $g(x)$, by the factor theorem

[2 marks]

(b)

$$\begin{array}{r|lllllll} & & & x^2 & + & 10x & + & 25 \\ \hline x - 4 & x^3 & + & 6x^2 & - & 15x & - & 100 \\ & x^3 & - & 4x^2 & & & & \\ \cline{2-4} & & & 10x^2 & - & 15x & & \\ & & & 10x^2 & - & 40x & & \\ \cline{4-6} & & & & & 25x & - & 100 \\ & & & & & 25x & - & 100 \\ \cline{6-8} & & & & & & & 0 \end{array} \quad \leftarrow \text{ As expected !}$$

$$\begin{aligned} g(x) &= x^3 + 6x^2 - 15x - 100 \\ &= \left(x - 4\right)\left(x^2 + 10x + 25\right) \\ &= (x - 4)(x + 5)(x + 5) \end{aligned}$$

[4 marks]

(c)

If $g(x) = 0$

then $(x - 4)(x + 5)(x + 5) = 0$

in which case either $x = 4$ or $x = -5$

[1 mark]

Answer 9

$$h(x) = x^3 + 18x^2 + 108x + 216$$

(a) **(i)**

$$\begin{aligned} h(-4) &= (-4)^3 + 18(-4)^2 + 108(-4) + 216 \\ &= -64 + 288 - 432 + 216 \\ &= 8 \end{aligned}$$

That is, $h(-4) = 8$

$\therefore\ x + 4$ is not a factor of $h(x)$, by the factor theorem

[2 marks]

(ii)

$$\begin{aligned} h(-6) &= (-6)^3 + 18(-6)^2 + 108(-6) + 216 \\ &= -216 + 648 - 648 + 216 \\ &= 0 \end{aligned}$$

That is, $h(-6) = 0$

$\therefore\ x + 6$ is a factor of $h(x)$, by the factor theorem

[2 marks]

(b)

$$\begin{array}{r|rrrrrrr} & & & x^2 & + & 12x & + & 36 \\ \hline x + 6 & x^3 & + & 18x^2 & + & 108x & + & 216 \\ & x^3 & + & 6x^2 & & & & \\ \cline{2-4} & & & 12x^2 & + & 108x & & \\ & & & 12x^2 & + & 72x & & \\ \cline{4-6} & & & & & 36x & + & 216 \\ & & & & & 36x & + & 216 \\ \cline{6-8} & & & & & & & 0 \end{array} \quad \leftarrow \text{ As expected !}$$

$$\begin{aligned} h(x) &= x^3 + 18x^2 + 108x + 216 \\ &= \left(x + 6\right)\left(x^2 + 12x + 36\right) \\ &= (x + 6)(x + 6)(x + 6) \end{aligned}$$

[4 marks]

(c)

If $h(x) = 0$

then $(x + 6)(x + 6)(x + 6) = 0$

in which case $x = -6$

[1 mark]

Answer 10

(i) As $(x-1)$ is a factor of $f(x)$, $f(1) = 0$, by the factor theorem

$$f(1) = 0$$
$$(1)^3 + a(1)^2 + b(1) - 12 = 0$$
$$1 + a + b - 12 = 0$$
$$a + b = 11$$

[2 marks]

(ii) As $(x+2)$ is a factor of $f(x)$, $f(-2) = 0$, by the factor theorem

$$f(-2) = 0$$
$$(-2)^3 + a(-2)^2 + b(-2) - 12 = 0$$
$$-8 + 4a - 2b - 12 = 0$$
$$4a - 2b = 20$$

Divide through by 2

$$2a - b = 10$$

[2 marks]

(iii)

$$a + b = 11$$

ADD $\quad 2a - b = 10$

$$3a \quad = 21$$
$$a = 7 \qquad \Rightarrow \qquad b = 4$$

[3 marks]

(iv) Making use of the fact that two factors are known,

$$(x-1)(x+2)(x+\lambda) = x^3 + 7x^2 + 4x - 12$$
$$\therefore \ \lambda = 6$$

Thus,

$$f(x) = x^3 + 7x^2 + 4x - 12$$
$$= (x-1)(x+2)(x+6)$$

[2 marks]

4.1 Deeper Understanding

Suppose that $f(x) = x^2 + 7x + 10$
To find the factors of this using The Factor Theorem would involve testing from the list $(x \pm 1)$, $(x \pm 2)$, $(x \pm 5)$ and $(x \pm 10)$
An astute observer would notice that substituting in any positive value for x could not result in $f(x) = 0$ as there are no minus signs in the function.
So the list is halved in size !

To see if $(x + 1)$ is a factor test if $f(-1)$ is equal to zero or not,

$$f(-1) = (-1)^2 + 7(-1) + 10$$
$$= 4 \qquad \text{As } f(-1) \neq 0,\ (x+1) \text{ is not a factor, by the factor theorem}$$

To see if $(x + 2)$ is a factor test if $f(-2)$ is equal to zero or not,

$$f(-2) = (-2)^2 + 7(-2) + 10$$
$$= 0 \qquad \text{As } f(-2) = 0,\ (x+2) \text{ is a factor, by the factor theorem}$$

4.2 Why does the factor theorem work ?

Putting $x = -1$ into $x^2 + 7x + 10$ is the same as putting it into $(x + 2)(x + 5)$

$$\begin{aligned} f(x) &= x^2 + 7x + 10 \\ &= (x+2)(x+5) \\ \therefore f(-1) &= (-1+2)(-1+5) \\ &= 1 \times 4 \leftarrow \text{No factor "hit" so no zero} \\ &= 4 \end{aligned}$$

Putting $x = -2$ into $x^2 + 7x + 10$ is the same as putting it into $(x + 2)(x + 5)$

$$\begin{aligned} f(x) &= x^2 + 7x + 10 \\ &= (x+2)(x+5) \\ \therefore f(-2) &= (-2+2)(-2+5) \\ &= 0 \times 3 \leftarrow \text{Factor "hit" gives a zero} \\ &= 0 \end{aligned}$$

This shows that whenever $(x - a)$ is a factor, substituting a into the function $f(a) = 0$, which is The Factor Theorem ! □

Here is a reminder of exactly what the factor theorem says, which may make more sense having gained experience in using it and an understanding why it works.

The Factor Theorem

If, for a given polynomial function $p(x)$, $p(a) = 0$ (for some constant, a)

then $(x - a)$ is a factor of $p(x)$

4.3 Example

The function, $f(x) = x^4 + x^2 + kx$, where k is a constant, has a root $x = 1$

Fully factorise $f(x)$

Teaching Video : http://www.NumberWonder.co.uk/v9029/4.mp4

[5 marks]

4.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

By using the factor theorem, solve the equation

$$x^4 - 3x^2 + 2x = 0$$

[5 marks]

Question 2

Additional Mathematics Examination Question from June 2019, Q7 (OCR)

In this question you must show detailed reasoning

The equation $x^3 - 3x + k = 0$, where k is a constant, has a root at $x = 2$

Find the numerical value(s) of the other roots of this equation

[5 marks]

Question 3

Additional Mathematics Examination Question from June 2011, Q7 (OCR)

(a) Determine whether or not each of the following is a factor of the expression

$$x^3 - 7x + 6$$

You must show working

(i) $(x - 2)$

[2 marks]

(ii) $(x + 1)$

[1 mark]

(b) **(i)** Factorise the function $f(x) = x^3 - 7x + 6$

[3 marks]

(ii) Solve the equation $f(x) = 0$

[1 mark]

Question 4

Additional Mathematics Examination Question from June 2017, Q6 (OCR)

You are given that the equation $x^3 - x^2 - 10x + 6 = 0$ has two non-integer positive roots and one negative integer root.

(i) Using the factor theorem, find the negative root

[2 marks]

(ii) Hence solve the equation

[4 marks]

Question 5

Additional Mathematics Examination Question from June 2016, Q4 (OCR)

You are given that $f(x) = x^3 - x^2 + x - 6$

Show that

(i) $(x - 2)$ is a factor of $f(x)$

[1 mark]

(ii) the equation $f(x) = 0$ has only one real root

[4 marks]

Question 6

If the equations

$$x^3 + x^2 - ax - 2b = 0$$

$$3x^3 - 2x^2 - 3ax + 4b = 0$$

both have a solution $x = 2$, find a and b.

With these values of a and b show that the equation

$$x^3 + x^2 - ax - 2b = 0$$

also has a solution $x = -2$, and find the third solution.

[6 marks]

Question 7

Given that

$$f(x) = x^4 - 2x^3 - 11x^2 + 12x + 36$$

find the values of $f(1)$, $f(-2)$ and $f(-3)$

It is given that $f(x)$ has two pairs of repeated factors, so that $f(x)$ may be expressed in the form $(x - a)^2(x - b)^2$ where *a* and *b* are integers

Find an expression for $f(x)$ in the form $\left(px^2 + qx + r\right)^2$ where *p*, *q* and *r* are integers to be found.

[6 marks]

4.5 Answers

4.5.1 Solution to 4.3 Example (Teaching Video)

$$\begin{aligned} f(x) &= x^4 + x^2 + kx \\ &= x\left(x^3 + x + k\right) \end{aligned}$$

$$\text{As } x = 1 \text{ is a root, } f(1) = 0 \text{ by the factor theorem}$$

$$\begin{aligned} \therefore (1)\left(1^3 + 1 + k\right) &= 0 \\ k &= -2 \end{aligned}$$

$$\text{Thus, } f(x) = x\left(x^3 + x - 2\right)$$

$$\text{As } x = 1 \text{ is a root, } (x - 1) \text{ is a factor}$$

$$\begin{array}{r|rrrr} & & x^2 & +\; x & +\; 2 \\ \hline x - 1 & x^3 & +\; 0x^2 & +\; x & -\; 2 \\ & x^3 & -\; x^2 & & \\ \cline{2-3} & & x^2 & +\; x & \\ & & x^2 & -\; x & \\ \cline{3-4} & & & 2x & -\; 2 \\ & & & 2x & -\; 2 \\ \cline{4-5} & & & & 0 \end{array} \quad \leftarrow \text{ As expected !}$$

$$f(x) = x\left(x - 1\right)\left(x^2 + x + 2\right)$$

This is, in fact, fully factorised, as the quadratic does not factorise further.
The function has two linear factors and one quadratic factor.

[5 marks]

4.5.2 Solutions to 4.4 Exercise

Answer 1

$$x^4 - 3x^2 + 2x = 0$$

$$x\left(x^3 - 3x + 2\right) = 0$$

Let $m(x) = x^3 - 3x + 2$, and notice that $m(1) = 0$

So $(x - 1)$ is a factor of $m(x)$ by the factor theorem

$$\begin{array}{r|rrrr}
 & x^2 & +\ x & -\ 2 & \\ \cline{2-5}
x - 1 & x^3 + 0x^2 & -\ 3x & +\ 2 & \\
 & x^3 - x^2 & & & \\ \cline{2-2}
 & x^2 & -\ 3x & & \\
 & x^2 & -\ x & & \\ \cline{2-3}
 & & -\ 2x & +\ 2 & \\
 & & -\ 2x & +\ 2 & \\ \cline{3-4}
 & & & 0 & \leftarrow \text{ As expected !}
\end{array}$$

$$x^4 - 3x^2 + 2x = 0$$

$$x\left(x^3 - 3x + 2\right) = 0$$

$$x\left(x - 1\right)\left(x^2 + x - 2\right) = 0$$

$$x(x - 1)(x - 1)(x + 2) = 0$$

Either $x = 0$ or $x = 1$ or $x = -2$

Note : This is an old Additional Mathematics Examination Question
From 21st June 1995, Paper 2
I don't know which examination board

[5 marks]

Answer 2

As $x = 2$ is a root, it will satisfy the given equation

$$x^3 - 3x + k = 0$$

$$2^3 - 3 \times 2 + k = 0$$

$$k = -2$$

The equation to be solved is thus,

$$x^3 - 3x - 2 = 0$$

and, as $x = 2$ is a root, $(x - 2)$ is a factor

$$
\begin{array}{r|l}
 & x^2 + 2x + 1 \\
\cline{2-2}
x - 2 & x^3 + 0x^2 - 3x - 2 \\
 & \underline{x^3 - 2x^2} \\
 & \quad 2x^2 - 3x \\
 & \quad \underline{2x^2 - 4x} \\
 & \qquad\quad x - 2 \\
 & \qquad\quad \underline{x - 2} \\
 & \qquad\qquad 0 \leftarrow \text{As expected !}
\end{array}
$$

$$x^3 - 3x^2 - 2 = 0$$

$$(x - 2)(x^2 + 2x + 1) = 0$$

$$(x - 2)(x + 1)(x + 1) = 0$$

Either $x = 2$ or $x = -1$

$\therefore$ The "other root" are thus $x = -1$

(Other than the root given in the question)

[5 marks]

Answer 3

(a) **(i)** If $(x - 2)$ is a factor then replacing x with 2 would make the given expression have a value of zero, by the factor theorem.

$$x^3 - 7x + 6 \Rightarrow 2^3 - 7 \times 2 + 6 = 0$$

$$\therefore \ (x - 2) \text{ is a factor}$$

[2 marks]

(ii) If $(x + 1)$ is a factor then replacing x with -1 would make the given expression have a value of zero, by the factor theorem.

$$x^3 - 7x + 6 \Rightarrow (-1)^3 - 7 \times (-1) + 6 = 12$$

$$\therefore \ (x + 1) \text{ is not a factor}$$

[1 mark]

(b) **(i)** As $(x - 2)$ is a factor...

$$\begin{array}{r|l}
 & x^2 + 2x - 3 \\
\hline
x - 2 & x^3 + 0x^2 - 7x + 6 \\
 & x^3 - 2x^2 \\
\hline
 & 2x^2 - 7x \\
 & 2x^2 - 4x \\
\hline
 & -3x + 6 \\
 & -3x + 6 \\
\hline
 & 0 \leftarrow \text{As expected !}
\end{array}$$

$$x^3 - 7x + 6 = \left(x - 2 \right)\left(x^2 + 2x - 3 \right)$$

$$= (x - 2)(x - 1)(x + 3)$$

[3 marks]

(ii) The solutions are $x \in \{-3, \ 1, \ 2 \}$

[1 mark]

Answer 4

(i) The possible negative integer roots are $(x+1)$, $(x+2)$, $(x+3)$ and $(x+6)$

Try $x = -1$: $(-1)^3 - (-1)^2 - 10(-1) + 6 = 14$

Try $x = -2$: $(-2)^3 - (-2)^2 - 10(-2) + 6 = 14$

Try $x = -3$: $(-3)^3 - (-3)^2 - 10(-3) + 6 = 0$

Try $x = -6$: $(-6)^3 - (-6)^2 - 10(-6) + 6 = -186$

$\therefore$ The negative root is $x = -3$, by the factor theorem

[2 marks]

(ii) As $x = -3$ is a root, $(x+3)$ is a factor.

$$
\begin{array}{r|l}
 & \phantom{x^3 -{}} x^2 - 4x + 2 \\
\cline{2-2}
x + 3 & x^3 - x^2 - 10x + 6 \\
 & \underline{x^3 + 3x^2} \\
 & - 4x^2 - 10x \\
 & \underline{- 4x^2 - 12x} \\
 & \phantom{x^3 - 4x^2 -{}} 2x + 6 \\
 & \phantom{x^3 - 4x^2 -{}} \underline{2x + 6} \\
 & \phantom{x^3 - 4x^2 - 2x + {}} 0 \leftarrow \text{As expected !}
\end{array}
$$

Thus, $x^3 - 3x - 2 = \left(x+3\right)\left(x^2 - 4x + 2\right)$

One solution is $x = -3$ and the other two are given by,

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 2}}{2 \times 1}$$

$$= 2 \pm \sqrt{2}$$

[4 marks]

Answer 5

(i)

$$f(x) = x^3 - x^2 + x - 6$$

$$\begin{aligned} f(2) &= 2^3 - 2^2 + 2 - 6 \\ &= 8 - 4 + 2 - 6 \\ &= 0 \end{aligned}$$

$\therefore$ $(x - 2)$ is a factor of $f(x)$ by the factor theorem

[1 mark]

(ii)

$$\begin{array}{r|l} & x^2 + x + 3 \\ \hline x - 2 & x^3 - x^2 + x - 6 \\ & \underline{x^3 - 2x^2} \\ & \quad x^2 + x \\ & \quad \underline{x^2 - 2x} \\ & \qquad 3x - 6 \\ & \qquad \underline{3x - 6} \\ & \qquad\quad 0 \end{array}$$

$\leftarrow$ As expected !

$$\begin{aligned} f(x) &= x^3 - x^2 + x - 6 \\ &= (x - 2)(x^2 + x + 3) \end{aligned}$$

The quadratic root does not factorise further to give any more real roots other than the one already known about at $x = 2$

For $x^2 + x + 3 = 0$, $\quad x = \dfrac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 3}}{2 \times 1}$

$= \dfrac{-1 \pm \sqrt{-11}}{2}$ which is not real

[4 marks]

Answer 6

$$\text{With } x = 2 \Rightarrow x^3 + x^2 - ax - 2b = 0$$
$$8 + 4 - 2a - 2b = 0$$
$$2a + 2b = 12$$

$$\text{With } x = 2 \Rightarrow 3x^3 - 2x^2 - 3ax + 4b = 0$$
$$24 - 8 - 6a + 4b = 0$$
$$6a - 4b = 16$$

Divide through by 2

$$3a - 2b = 8$$

Solving these two equations simultaneously,

$$\begin{array}{lrcl} & 2a + 2b & = & 12 \\ \text{ADD} & 3a - 2b & = & 8 \\ \hline & 5a & = & 20 \end{array}$$

$$a = 4 \quad \Rightarrow \quad b = 2$$

and $x^3 + x^2 - ax - 2b = 0$ becomes, $x^3 + x^2 - 4x - 4 = 0$.

Remember this has a solution $x = 2$

That is, has $(x - 2)$ as a factor,

$$\begin{array}{r|l} & \quad\; x^2 + 3x + 2 \\ \cline{2-2} x - 2 & x^3 + x^2 - 4x - 4 \\ & x^3 - 2x^2 \\ \cline{2-2} & \quad 3x^2 - 4x \\ & \quad 3x^2 - 6x \\ \cline{2-2} & \qquad\quad 2x - 4 \\ & \qquad\quad 2x - 4 \\ \cline{2-2} & \qquad\qquad\; 0 \end{array}$$

$0 \leftarrow$ As expected !

$$x^3 + x^2 - 4x - 4 = 0$$
$$(x - 2)(x^2 + 3x + 2) = 0$$
$$(x - 2)(x + 2)(x + 1) = 0$$

It has thus been shown that the equation also has a solution $x = -2$

and the third solution is $x = -1$

[6 marks]

Note : This is an old Additional Mathematics Examination Question From 22nd June 1994, Paper 2

Answer 7

$$f(x) = x^4 - 2x^3 - 11x^2 + 12x + 36$$

$$\begin{aligned} f(1) &= (1)^4 - 2(1)^3 - 11(1)^2 + 12(1) + 36 \\ &= 1 - 2 - 11 + 12 + 36 \\ &= 36 \end{aligned}$$

$$\begin{aligned} f(-2) &= (-2)^4 - 2(-2)^3 - 11(-2)^2 + 12(-2) + 36 \\ &= 16 + 16 - 44 - 24 + 36 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(-3) &= (-3)^4 - 2(-3)^3 - 11(-3)^2 + 12(-3) + 36 \\ &= 81 + 54 - 99 - 36 + 36 \\ &= 36 \end{aligned}$$

By the factor theorem it's now known that $(x + 2)$ is a factor so let $a = -2$

When $(x - a)^2(x - b)^2$ is expanded the term independent of x is $a^2 b^2$

$$\begin{aligned} a^2 b^2 &= 36 \\ b^2 &= \frac{36}{4} \\ b &= \pm 3 \end{aligned}$$

But it has already been observed that $(x + 3)$ is not a factor, so $b = 3$

$$\begin{aligned} (x + 2)^2(x - 3)^2 &= ((x + 2)(x - 3))^2 \\ &= \left(x^2 - x - 6\right)^2 \end{aligned}$$

That is, $p = 1$, $q = -1$ and $r = -6$

[6 marks]

Note : This is an old Additional Mathematics Examination Question
From 13th June 1997, Paper 1

5.1 The Remainder Theorem

Consider the following algebraic long division,

$$
\begin{array}{r|rrrr}
 & x^2 & -\;4x & -\;32 & \\
\cline{2-5}
x+1 & x^3 \;-\; 3x^2 & -\;36x & -\;48 & \\
 & x^3 \;+\; x^2 & & & \\
\cline{2-2}
 & -\;4x^2 & -\;36x & & \\
 & -\;4x^2 & -\;4x & & \\
\cline{2-3}
 & & -\;32x & -\;48 & \\
 & & -\;32x & -\;32 & \\
\cline{3-4}
 & & & -\;16 & \leftarrow \text{Remainder}
\end{array}
$$

It shows the cubic $p(x) = x^3 - 3x^2 - 36x - 48$ being divided by $(x + 1)$ and that the remainder is -16

The calculation is telling us is that,

$$\frac{x^3 - 3x^2 - 36x - 48}{x + 1} = x^2 - 4x - 32 + \frac{-16}{x + 1}$$

The remainder theorem gives a quick method of finding out what the remainder at the end of the algebraic long division will be without having to do the division.

It says,
"To find the remainder when $p(x)$ is divided by $(x + 1)$ calculate $p(-1)$"

$$
\begin{aligned}
p(-1) &= (-1)^3 - 3 \times (-1)^2 - 36 \times (-1) - 48 \\
&= -1 - 3 + 36 - 48 \\
&= -16
\end{aligned}
$$

The Remainder Theorem

When a polynomial $p(x)$ is divided by $(x - a)$, where a is a constant,

the remainder is $p(a)$

Notice that, in the example, saying the remainder is – 16 meant $\dfrac{-16}{x + 1}$

5.2 Why Does The Remainder Theorem Work ?

As observed previously, from the long division,

$$\frac{x^3 - 3x^2 - 36x - 48}{x + 1} = x^2 - 4x - 32 + \frac{-16}{x + 1}$$

If this is multiplied throughout by $(x + 1)$ it becomes,

$$x^3 - 3x^2 - 36x - 48 = \left(x^2 - 4x - 32\right)\left(x + 1\right) - 16$$

Now, putting the number 1 into the LHS is the same as putting it into the RHS.

On the RHS $(x + 1)$ then becomes zero and all that's left is the -16

5.3 Example

$f(x) = x^3 - a x^2 + bx - 4$, where a and b are constants to be found

- The remainder when $f(x)$ is divided by $(x - 1)$ is 2
- The remainder when $f(x)$ is divided by $(x - 2)$ is also 2

Teaching Video : http://www.NumberWonder.co.uk/v9029/5.mp4

[5 marks]

5.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

$f(x) = 4x^3 + ax^2 + bx - 27$, where a and b are constants to be found

- The remainder when $f(x)$ is divided by $(x - 1)$ is -2
- The remainder when $f(x)$ is divided by $(x + 1)$ is -100

[5 marks]

Question 2

Find the values of the constants *a* and *b* if the function

$$f(x) = ax^3 + bx^2 - 28x + 15$$

- is exactly divisible by $(x + 3)$
- leaves a remainder of – 60 when divided by $(x - 3)$

[5 marks]

Question 3

Additional Mathematics Examination Question from June 2014, Q6 (OCR)

The function $f(x) = x^3 - 4x^2 + ax + b$ is such that

- $x = 3$ is a root of the equation $f(x) = 0$
- when $f(x)$ is divided by $(x - 1)$ there is a remainder of 4

(i) Find the value of a and the value of b

[4 marks]

(ii) Solve the equation $f(x) = 0$

[3 marks]

Question 4

Additional Mathematics Examination question from June 2007, Q11 (OCR)

(a) You are given that

$$f(x) = x^3 - 3x^2 - 4x$$

(i) Find the three points where the curve $y = f(x)$ cuts the x-axis

[4 marks]

(ii) Sketch the graph of $y = f(x)$

[1 mark]

(b) You are given that

$$g(x) = x^3 - 3x^2 - 4x + 12$$

(i) Find the remainder when $g(x)$ is divided by $(x + 1)$

[2 marks]

(ii) Show that $(x - 2)$ is a factor of $g(x)$

[1 mark]

(iii) Hence solve the equation $g(x) = 0$

[4 marks]

Question 5

$$p(x) = x^4 + ax^3 + bx^2 + cx + d$$

The polynomial $p(x)$ has factors x and $(x + 1)$

(i) Prove that $a - b + c = 1$

[2 marks]

(ii) If the remainder when $p(x)$ is divided by $(x - 1)$ is 5, find the value of b

[3 marks]

Question 6

$$p(x) = 9x^2 + ax + b$$

$p(x)$ has remainder 12 when divided by either $(x - 1)$ or $(x + 2)$

(a) **(i)** Find the values of a and b

[2 marks]

(ii) Hence obtain the remainder when $p(x)$ is divided by $(x + 3)$

[2 marks]

(b) Prove that $9x^3 + ax^2 + bx - 6$ is divisible by $(x + 1)$

[2 marks]

5.5 Answers

5.5.1 Solution to 5.3 Example (Teaching Video)

$$f(x) = x^3 - ax^2 + bx - 4$$

The question is, in effect, saying that,

$$\begin{aligned} f(1) &= 2 \\ (1)^3 - a(1)^2 + b(1) - 4 &= 2 \\ 1 - a + b - 4 &= 2 \\ -a + b &= 5 \\ a - b &= -5 \end{aligned}$$

The question is also, in effect, saying that,

$$\begin{aligned} f(2) &= 2 \\ (2)^3 - a(2)^2 + b(2) - 4 &= 2 \\ 8 - 4a + 2b - 4 &= 2 \\ -4a + 2b &= -2 \\ &\text{Divide through by 2} \\ -2a + b &= -1 \\ 2a - b &= 1 \end{aligned}$$

Combining these two results,

$$\begin{array}{lrcl} & 2a - b &=& 1 \\ \text{SUBTRACT} & a - b &=& -5 \\ \hline & a &=& 6 \qquad \Rightarrow \qquad b = 11 \end{array}$$

[5 marks]

5.5.2 Solutions to 5.4 Exercise

Answer 1

$$f(x) = 4x^3 + ax^2 + bx - 27$$

The question is, in effect, saying that,

$$\begin{aligned} f(1) &= -2 \\ 4(1)^3 + a(1)^2 + b(1) - 27 &= -2 \\ 4 + a + b - 27 &= -2 \\ a + b &= 21 \end{aligned}$$

The question is also, in effect, saying that,

$$\begin{aligned} f(-1) &= -100 \\ 4(-1)^3 + a(-1)^2 + b(-1) - 27 &= -100 \\ -4 + a - b - 27 &= -100 \\ a - b &= -69 \end{aligned}$$

Combining these two results,

$$\begin{array}{lrcr} & a + b &=& 21 \\ \text{ADD} & a - b &=& -69 \\ \hline & 2a & = & -48 \\ & a &=& -24 \end{array} \quad \Rightarrow \; b = 45$$

[5 marks]

Answer 2

$$f(x) = ax^3 + bx^2 - 28x + 15$$

The question is, in effect, saying that,

$$f(-3) = 0$$

$$a(-3)^3 + b(-3)^2 - 28(-3) + 15 = 0$$

$$-27a + 9b + 84 + 15 = 0$$

$$-27a + 9b = -99$$

Divide through by 9

$$-3a + b = -11$$

The question is also, in effect, saying that,

$$f(3) = -60$$

$$a(3)^3 + b(3)^2 - 28(3) + 15 = -60$$

$$27a + 9b - 84 + 15 = -60$$

$$27a + 9b = 9$$

Divide through by 9

$$3a + b = 1$$

Combining these two results,

$$\begin{array}{lr} & -3a + b = -11 \\ \text{ADD} & 3a + b = \quad 1 \\ \hline & 2b = -10 \\ & b = -5 \end{array} \qquad \Rightarrow \quad a = 2$$

[5 marks]

Note : The example used for the teaching video and also this question are from a single question in an old Additional Mathematics examination paper. The date I have for that is before "Tuesday 24th March 1992"

Answer 3

$$f(x) = x^3 - 4x^2 + ax + b$$

(i) The question is, in effect, saying that,

$$f(3) = 0$$

$$(3)^3 - 4(3)^2 + a(3) + b = 0$$

$$27 - 36 + 3a + b = 0$$

$$3a + b = 9$$

The question is also, in effect, saying that,

$$f(1) = 4$$

$$(1)^3 - 4(1)^2 + a(1) + b = 4$$

$$1 - 4 + a + b = 4$$

$$a + b = 7$$

Combining these two results,

$$\begin{array}{lrcl} & 3a + b & = & 9 \\ \text{SUBTRACT} & a + b & = & 7 \\ \hline & 2a \quad & = & 2 \end{array}$$

$$a = 1 \qquad \Rightarrow b = 6$$

[4 marks]

(ii) From part (i), $f(x) = x^3 - 4x^2 + x + 6$ has $(x - 3)$ as a factor

$$\begin{array}{r|l} & x^2 - x - 2 \\ \hline x - 3 & x^3 - 4x^2 + x + 6 \\ & x^3 - 3x^2 \\ \hline & \quad - x^2 + x \\ & \quad - x^2 + 3x \\ \hline & \qquad - 2x + 6 \\ & \qquad - 2x + 6 \\ \hline & \qquad\qquad 0 \end{array} \leftarrow \text{ As expected}$$

$$f(x) = 0$$

$$x^3 - 4x^2 + x + 6 = 0$$

$$(x - 3)(x^2 - x - 2) = 0$$

$$(x - 3)(x - 2)(x + 1) = 0$$

Either $x = 3$ or $x = 2$ or $x = -1$ are the solutions

[3 marks]

Answer 4

(a) **(i)** The function will cut the x-axis when $f(x) = 0$

$$f(x) = 0$$

$$x^3 - 3x^2 - 4x = 0$$

$$x\left(x^2 - 3x - 4\right) = 0$$

$$x(x - 4)(x + 1) = 0$$

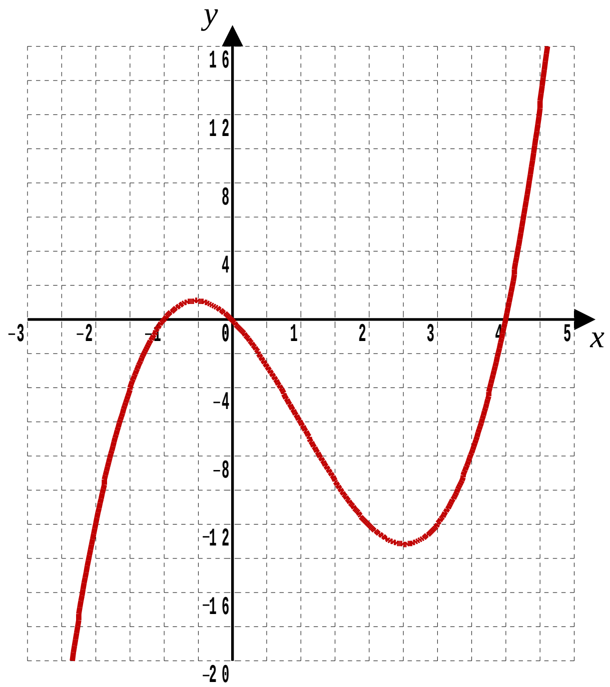
Either $x = 0$, $x = 4$ or $x = -1$

That is, the curve $y = f(x)$ cuts the x-axis at

$(-1,\ 0)$, $(0,\ 0)$ and $(4,\ 0)$

[4 marks]

(ii)



[1 mark]

(b) **(i)** $g(x) = x^3 - 3x^2 - 4x + 12$

By the remainder theorem, the remainder is given by, $g(-1)$

$$\begin{aligned} g(-1) &= (-1)^3 - 3(-1)^2 - 4(-1) + 12 \\ &= -1 - 3 + 4 + 12 \\ &= 12 \end{aligned}$$

[2 marks]

(ii)

$$\begin{aligned} g(2) &= (2)^3 - 3(2)^2 - 4(2) + 12 \\ &= 8 - 12 - 8 + 12 \\ &= 0 \end{aligned}$$

$\therefore$ By the factor theorem, $(x - 2)$ is a factor of $g(x)$

[1 mark]

(iii)

$$\begin{array}{r|l} & \quad\quad\quad x^2 - x - 6 \\ \cline{2-2} x - 2 & x^3 - 3x^2 - 4x + 12 \\ & \underline{x^3 - 2x^2} \\ & \quad -x^2 - 4x \\ & \quad \underline{-x^2 + 2x} \\ & \quad\quad -6x + 12 \\ & \quad\quad \underline{-6x + 12} \\ & \quad\quad\quad\quad 0 \leftarrow \text{As expected} \end{array}$$

$$\begin{aligned} g(x) &= 0 \\ x^3 - 3x^2 - 4x + 12 &= 0 \\ (x - 2)(x^2 - x - 6) &= 0 \\ (x - 2)(x - 3)(x + 2) &= 0 \end{aligned}$$

Either $x = 2$ or $x = 3$ or $x = -2$ are the solutions

[4 marks]

Answer 5

(i) As x is a factor, which could be thought of as the factor $(x - 0)$,

putting $x = 0$ into $x^4 + a x^3 + b x^2 + cx + d$ will cause it to equal zero, by the factor theorem.

$$\therefore \; d = 0$$

As $(x + 1)$ is a factor, putting $x = -1$ into $x^4 + a x^3 + b x^2 + cx + d$ cause it to equal zero, by the factor theorem.

$$\therefore \; (-1)^4 + a(-1)^3 + b(-1)^2 + c(-1) + d = 0$$

but it was just shown that $d = 0$ and so,

$$1 - a + b - c + 0 = 0$$

$$\therefore \; a - b + c = 1 \qquad \square$$

[2 marks]

(ii) The question is, in effect, saying that putting $x = 1$ into $x^4 + a x^3 + b x^2 + cx + d$ will cause it to equal 5, by the remainder theorem

$$(1)^4 + a(1)^3 + b(1)^2 + c(1) + d = 0$$

again, using the established fact that $d = 0$,

$$a + b + c = 4$$

Combining equations allows the extraction of a value for b

$$\begin{array}{lr} & a + b + c = 4 \\ \text{SUBTRACT} & a - b + c = 1 \\ \hline & 2b \quad = 3 \\ & b = \dfrac{3}{2} \end{array}$$

[3 marks]

Answer 6

$$p(x) = 9x^2 + ax + b$$

(a) **(i)**

$$p(1) = 12 \qquad \text{by the remainder theorem}$$

$$9\,(1)^2 + a(1) + b = 12$$

$$a + b = 3$$

$$p(-2) = 12 \qquad \text{by the remainder theorem}$$

$$9\,(-2)^2 + a(-2) + b = 12$$

$$-2a + b = -24$$

$$2a - b = 24$$

Combining these two results simultaneously,

$$a + b = 3$$

$$\text{ADD} \qquad 2a - b = 24$$

$$3a \quad = 27$$

$$a = 9 \Rightarrow \qquad b = -6$$

[2 marks]

(ii) By the remainder theorem, the remainder is given by,

$$\begin{aligned} p(-3) &= 9\,(-3)^2 + a(-3) + b \\ &= 9 \times 9 + 9 \times (-3) + (-6) \\ &= 81 - 27 - 6 \\ &= 48 \end{aligned}$$

[2 marks]

(b) If $9x^3 + ax^2 + bx - 6$ is divisible by $(x + 1)$ then substituting in $x = -1$ will cause it to equal zero, by the factor theorem.

$$\begin{aligned} 9(-1)^3 + (9)(-1)^2 + (-6)(-1) - 6 &= -9 + 9 + 6 - 6 \\ &= 0 \end{aligned}$$

$\therefore$ $(x + 1)$ is indeed a factor of $9x^3 + ax^2 + bx - 6$ ☐

[2 marks]

Lesson 6

6.1 Revision

The Factor Theorem

If, for a given polynomial function $p(x)$, $p(a) = 0$ (for some constant, a)

then $(x - a)$ is a factor of $p(x)$

The Remainder Theorem

When a polynomial $p(x)$ is divided by $(x - a)$, where a is a constant,

the remainder is $p(a)$

6.2 The Revision

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

$$p(x) = 2x^3 + 4x^2 - 6x + 14$$

What is the degree of polynomial $p(x)$?

[1 mark]

Question 2

A polynomial of degree five is called a Quintic.

(i) What is a polynomial of degree 2 called ?

[1 mark]

(ii) What is a polynomial of degree 4 called ?

[1 mark]

Question 3

A-Level Examination Question from May 2011, Paper C2, Q1 (Edexcel)

$$f(x) = 2x^3 - 7x^2 - 5x + 4$$

(a) Find the remainder when $f(x)$ is divided by $(x - 1)$

[2 marks]

(b) Use the factor theorem to show that $(x + 1)$ is a factor of $f(x)$

[2 marks]

(c) Factorise $f(x)$ completely

[4 marks]

Question 4

A-Level Examination Question from January 2011, Paper C2, Q1 (Edexcel)

$$f(x) = x^4 + x^3 + 2x^2 + ax + b \quad \text{where } a \text{ and } b \text{ are constants}$$

When $f(x)$ is divided by $(x - 1)$ the remainder is 7

(a) Show that $a + b = 3$

[2 marks]

When $f(x)$ is divided by $(x + 2)$ the remainder is -8

(b) Find the value of a and the value of b

[5 marks]

Question 5

A-Level Examination Question from January 2008, Paper C2, Q1 (Edexcel)

(a) Find the remainder when

$$x^3 - 2x^2 - 4x + 8$$

is divided by

(i) $x - 3$

(ii) $x + 2$

[3 marks]

(b) Hence, or otherwise, find all solutions to the equation

$$x^3 - 2x^2 - 4x + 8 = 0$$

[4 marks]

Question 6

Additional Mathematics Examination Question from June 2012, Q3 (OCR)

The function $f(x) = x^3 + ax + 6$ is such that when $f(x)$ is divided by $(x - 3)$ the remainder is 12

(i) Show that the value of a is -7

[2 marks]

(ii) Factorise $f(x)$

[3 marks]

Question 7

A-Level Examination Question from January 2012, Paper C2, Q5 (Edexcel)

$$f(x) = x^3 + ax^2 + bx + 3 \quad \text{where } a \text{ and } b \text{ are constants}$$

Given that when $f(x)$ is divided by $(x + 2)$ the remainder is 7,

(a) Show that $2a - b = 6$

[2 marks]

Given also that when $f(x)$ is divided by $(x - 1)$ the remainder is 4

(b) Find the value of a and the value of b

[4 marks]

Question 8

A-Level Examination question from June 2008, Paper C2, Q1 (Edexcel)

$$f(x) = 2x^3 - 3x^2 - 39x + 20$$

(a) Use the factor theorem to show that $(x + 4)$ is a factor of $f(x)$

[2 marks]

(b) Factorise $f(x)$ completely

[4 marks]

Question 9

A-Level Examination Question from January 2005, Paper C2, Q5 (Edexcel)

$$f(x) = x^3 - 2x^2 + ax + b \quad \text{where } a \text{ and } b \text{ are constants}$$

- When $f(x)$ is divided by $(x - 2)$ the remainder is 1
- When $f(x)$ is divided the $(x + 1)$ the remainder is 28

(a) Find the value of a and the value of b

[6 marks]

(b) Show that $(x - 3)$ is a factor of $f(x)$

[2 marks]

6.3 Answers to 6.2 Exercise

Answer 1

$p(x)$ has degree 3 (It's a cubic)

[1 mark]

Answer 2

(i) A degree 2 polynomial is called a quadratic (and its graph is termed a parabola)

[1 mark]

(ii) A degree 4 polynomial is called a quartic

[1 mark]

Answer 3

$$f(x) = 2x^3 - 7x^2 - 5x + 4$$

(a) The remainder when $f(x)$ is divided by $(x - 1)$ is given by $f(1)$ by the remainder theorem.

$$\begin{aligned} f(1) &= 2(1)^3 - 7(1)^2 - 5(1) + 4 \\ &= 2 - 7 - 5 + 4 \\ &= -6 \end{aligned}$$

[2 marks]

(b)

$$\begin{aligned} f(-1) &= 2(-1)^3 - 7(-1)^2 - 5(-1) + 4 \\ &= -2 - 7 + 5 + 4 \\ &= 0 \quad \therefore \; (x + 1) \text{ is a factor of } f(x) \text{ by the factor theorem} \end{aligned}$$

[2 marks]

(c)

$$\begin{array}{r|rrrr} & 2x^2 & -\;9x & +\;4 & \\ \hline x + 1 & 2x^3 - 7x^2 & -\;5x & +\;4 & \\ & 2x^3 + 2x^2 & & & \\ \hline & -\;9x^2 & -\;5x & & \\ & -\;9x^2 & -\;9x & & \\ \hline & & 4x & +\;4 & \\ & & 4x & +\;4 & \\ \hline & & & 0 & \leftarrow \text{ As expected} \end{array}$$

$$\begin{aligned} f(x) &= 0 \\ 2x^3 - 7x^2 - 5x + 4 &= 0 \\ (x + 1)(2x^2 - 9x + 4) &= 0 \\ (x + 1)(2x - 1)(x - 4) &= 0 \end{aligned}$$

[4 marks]

Answer 4

$$f(x) = x^4 + x^3 + 2x^2 + ax + b$$

(a) $f(1) = 7$ by the remainder theorem

$$(1)^4 + (1)^3 + 2(1)^2 + a(1) + b = 7$$

$$1 + 1 + 2 + a + b = 7$$

$$a + b = 3 \qquad \square$$

[2 marks]

(b) $f(-2) = -8$ by the remainder theorem

$$(-2)^4 + (-2)^3 + 2(-2)^2 + a(-2) + b = -8$$

$$16 - 8 + 8 - 2a + b = -8$$

$$-2a + b = -24$$

Solving simultaneous equations,

$$\begin{array}{lrcl} & a + b & = & 3 \\ \text{SUBTRACT} & -2a + b & = & -24 \\ \hline & 3a & = & 27 \end{array}$$

$$a = 9 \quad \Rightarrow \quad b = -6$$

[5 marks]

Answer 5

(a)

(i) With $x = 3$, $x^3 - 2x^2 - 4x + 8 = (3)^3 - 2(3)^2 - 4(3) + 8$

$$= 27 - 18 - 12 + 8$$

$$= 5$$

(ii) With $x = -2$, $x^3 - 2x^2 - 4x + 8 = (-2)^3 - 2(-2)^2 - 4(-2) + 8$

$$= -8 - 8 + 8 + 8$$

$$= 0$$

[3 marks]

(b) From part (a), $(x + 2)$ is a factor, by the factor theorem

$$
\begin{array}{r|rrrr}
 & x^2 & -4x & +4 & \\
\cline{2-5}
x + 2 & x^3 - 2x^2 & -4x & +8 & \\
 & x^3 + 2x^2 & & & \\
\cline{2-2}
 & -4x^2 & -4x & & \\
 & -4x^2 & -8x & & \\
\cline{2-3}
 & & 4x & +8 & \\
 & & 4x & +8 & \\
\cline{3-4}
 & & & 0 & \leftarrow \text{As expected}
\end{array}
$$

$$x^3 - 2x^2 - 4x + 8 = 0$$

$$\left(x + 2\right)\left(x^2 - 4x + 4\right) = 0$$

$$(x + 2)(x - 2)(x - 2) = 0$$

Either $x = -2$ or $x = 2$

The solution set is $x \in \{\pm 2\}$

[4 marks]

Answer 6

$$f(x) = x^3 + ax + 6$$

(i) By the remainder theorem, $f(3) = 12$

$$(3)^3 + a(3) + 6 = 12$$

$$27 + 3a + 6 = 12$$

$$3a = -21$$

$$a = -7 \qquad \square$$

[2 marks]

(ii) The function can now be written with -7 in place of a

$$f(x) = x^3 - 7x + 6$$

Looking at this, it's obvious that $x = 1$ is a solution

That is, $(x - 1)$ is a factor

$$\begin{array}{r|rrrr}
 & & x^2 & +\ x & -\ 6 \\ \hline
x - 1 & x^3 & +\ 0x^2 & -\ 7x & +\ 6 \\
 & x^3 & -\ x^2 & & \\ \cline{2-3}
 & & x^2 & -\ 7x & \\
 & & x^2 & -\ x & \\ \cline{3-4}
 & & & -\ 6x & +\ 6 \\
 & & & -\ 6x & +\ 6 \\ \cline{4-5}
 & & & & 0
\end{array} \leftarrow \text{ As expected}$$

$$x^3 - 7x + 6 = 0$$

$$\left(x - 1\right)\left(x^2 + x - 6\right) = 0$$

$$(x - 1)(x + 3)(x - 2) = 0$$

[3 marks]

Answer 7

$$f(x) = x^3 + ax^2 + bx + 3$$

(a) By the remainder theorem, $f(-2) = 7$

$$(-2)^3 + a(-2)^2 + b(-2) + 3 = 7$$

$$-8 + 4a - 2b + 3 = 7$$

$$4a - 2b = 12$$

Divide through by 2

$$2a - b = 6 \qquad \square$$

[2 marks]

(b) By the remainder theorem, $f(1) = 4$

$$(1)^3 + a(1)^2 + b(1) + 3 = 4$$

$$1 + a + b + 3 = 4$$

$$a + b = 0$$

Combining the two results as simultaneous equations,

$$\begin{array}{lrcl} & 2a - b & = & 6 \\ \text{ADD} & a + b & = & 0 \\ \hline & 3a & = & 6 \end{array}$$

$$a = 2 \qquad \Rightarrow \qquad b = -2$$

[4 marks]

Answer 8

$$f(x) = 2x^3 - 3x^2 - 39x + 20$$

(a)

$$\begin{aligned} f(-4) &= 2(-4)^3 - 3(-4)^2 - 39(-4) + 20 \\ &= -128 - 48 + 156 + 20 \\ &= 0 \end{aligned}$$

$\therefore$ $(x + 4)$ is a factor of $f(x)$ by the factor theorem

[2 marks]

(b)

$$\begin{array}{r|rrrr} & & 2x^2 & -\ 11x & +\ 5 \\ \hline x + 4 & 2x^3 & -\ 3x^2 & -\ 39x & +\ 20 \\ & 2x^3 & +\ 8x^2 & & \\ \hline & & -\ 11x^2 & -\ 39x & \\ & & -\ 11x^2 & -\ 44x & \\ \hline & & & 5x & +\ 20 \\ & & & 5x & +\ 20 \\ \hline & & & & 0 \end{array} \quad \leftarrow \text{ As expected}$$

$$2x^3 - 3x^2 - 39x + 20 = 0$$

$$(x + 4)(2x^2 - 11x + 5) = 0$$

$$(x + 4)(2x - 1)(x - 5) = 0$$

[4 marks]

Answer 9

$$f(x) = x^3 - 2x^2 + ax + b$$

(a) By the remainder theorem, $f(2) = 1$

$$(2)^3 - 2(2)^2 + a(2) + b = 1$$

$$8 - 8 + 2a + b = 1$$

$$2a + b = 1$$

By the remainder theorem, $f(-1) = 28$

$$(-1)^3 - 2(-1)^2 + a(-1) + b = 28$$

$$-1 - 2 - a + b = 28$$

$$-a + b = 31$$

Combining the two results as simultaneous equations,

$$2a + b = 1$$

SUBTRACT $\quad -a + b = 31$

$$3a = -30$$

$$a = -10 \qquad \Rightarrow \qquad b = 21$$

[6 marks]

(b) The function is now known to be,

$$f(x) = x^3 - 2x^2 - 10x + 21$$

$$f(3) = (3)^3 - 2(3)^2 - 10(3) + 21$$

$$= 27 - 18 - 30 + 21$$

$$= 0$$

$\therefore$ $(x - 3)$ is a factor of $f(x)$, by the factor theorem

[2 marks]

Lesson 7

7.1 Test : Provided Materials

The Factor Theorem

If, for a given polynomial function $p(x)$, $p(a) = 0$ (for some constant, a)

then $(x - a)$ is a factor of $p(x)$

The Remainder Theorem

When a polynomial $p(x)$ is divided by $(x - a)$, where a is a constant,

the remainder is $p(a)$

7.2 The Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

$$p(x) = 3x^4 + 5x^2 - 23x + 31$$

What is the degree of polynomial $p(x)$?

[1 mark]

Question 2

A polynomial of degree seven is called a Septic.

(i) What is a polynomial of degree 2 called ?

[1 mark]

(ii) What is a polynomial of degree 5 called ?

[1 mark]

Question 3

A-Level Examination Question from May 2007, Paper C2, Q2 (Edexcel)

$$f(x) = 3x^3 - 5x^2 - 16x + 12$$

(a) Find the remainder when $f(x)$ is divided by $(x - 2)$

[2 marks]

Given that $(x + 2)$ is a factor of $f(x)$

(b) factorise $f(x)$ completely

[4 marks]

Question 4

A-Level Examination Question from January 2007, Paper C2, Q5 (Edexcel)

$$f(x) = x^3 + 4x^2 + x - 6$$

(a) Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$.

[2 marks]

(b) Factorise $f(x)$ completely.

[4 marks]

(c) Write down all the solutions to the equation

$$x^3 + 4x^2 + x - 6 = 0$$

[1 mark]

Question 5

A-Level Examination Question from January 2013, Paper C2, Q2 (Edexcel)

$f(x) = ax^3 + bx^2 - 4x - 3,$ where a and b are constants

Given that $(x - 1)$ is a factor of $f(x)$

(a) Show that $a + b = 7$

[2 marks]

Given also that, when $f(x)$ is divided by $(x + 2)$, the remainder is 9,

(b) find the value of a and the value of b, showing each step in your working.

[4 marks]

Question 6

$$f(x) = x^3 - x^2 - 7x + c$$

Given that $f(4) = 0$

(a) find the value of c

[2 marks]

(b) factorise $f(x)$ as the product of a linear factor and a quadratic factor.

[3 marks]

(c) Explain why, apart from at $x = 4$, there are no other real values of x for which $f(x) = 0$

[2 marks]

Question 7

$$p(x) = 4x^3 + ax^2 + bx + 15$$

- When $p(x)$ is divided by $(x - 1)$ the remainder is 90
- When $p(x)$ is divided by $(x + 1)$ the remainder is -4

Find the values of *a* and *b*

[7 marks]

Question 8

$$f(x) = x^4 - 3x^2 - ax + b$$

$$g(x) = x^3 - 3ax + 5b$$

Both $f(x)$ and $g(x)$ are divisible by $(x - 2)$

Prove that $3b = 2a$

[5 marks]

Question 9

A-Level Examination Question from May 2013, Paper C2, Q3 (Edexcel)

$$f(x) = 2x^3 - 5x^2 + ax + 18 \quad \text{where } a \text{ is a constant}$$

Given that $(x - 3)$ is a factor of $f(x)$

(a) Show that $a = -9$

[2 marks]

(b) Factorise $f(x)$ completely

[4 marks]

Given that $g(y) = 2\left(3^{3y}\right) - 5\left(3^{2y}\right) - 9\left(3^{y}\right) + 18$

(c) By considering your earlier answers, find the values of y that satisfy $g(y) = 0$, giving your answers to 2 decimal places where appropriate. You may use "trial and improvement" if you wish.

[3 marks]

7.3 Answers to 7.2 Test

A-Level Pure Mathematics, Year 1
Additional Mathematics
The Algebra of Polynomials

Answer 1

$p(x)$ has degree 4 (It's a Quartic)

[1 mark]

Answer 2

(i) A degree 2 polynomial is called a quadratic
(and its graph is termed a parabola)

[1 mark]

(ii) A degree 5 polynomial is called a quintic

[1 mark]

Answer 3

$$f(x) = 3x^3 - 5x^2 - 16x + 12$$

(a) The remainder when $f(x)$ is divided by $(x - 2)$ is given by $f(2)$ by the remainder theorem.

$$\begin{aligned} f(2) &= 3(2)^3 - 5(2)^2 - 16(2) + 12 \\ &= 24 - 20 - 32 + 12 \\ &= -16 \end{aligned}$$

[2 marks]

(b)

$$\begin{array}{r|rrrr}
 & & 3x^2 & -\ 11x & +\ 6 \\ \hline
x + 2 & 3x^3 & -\ 5x^2 & -\ 16x & +\ 12 \\
 & 3x^3 & +\ 6x^2 & & \\ \hline
 & & -\ 11x^2 & -\ 16x & \\
 & & -\ 11x^2 & -\ 22x & \\ \hline
 & & & 6x & +\ 12 \\
 & & & 6x & +\ 12 \\ \hline
 & & & & 0
\end{array} \leftarrow \text{As expected}$$

$$\begin{aligned} f(x) &= 3x^3 - 5x^2 - 16x + 12 \\ &= (x + 2)(3x^2 - 11x + 6) \\ &= (x + 2)(3x - 2)(x - 3) \end{aligned}$$

[4 marks]

Answer 4

$$f(x) = x^3 + 4x^2 + x - 6$$

(a) $$f(-2) = (-2)^3 + 4(-2)^2 + (-2) - 6$$
$$= -8 + 16 - 2 - 6$$
$$= 0$$

$\therefore$ $(x + 2)$ is a factor of $f(x)$ by the factor theorem

[2 marks]

(b)

$$\begin{array}{r|rrrr} & x^2 & +\ 2x & -\ 3 & \\ \hline x + 2 & x^3 + 4x^2 & +\ x & -\ 6 & \\ & x^3 + 2x^2 & & & \\ \hline & 2x^2 & +\ x & & \\ & 2x^2 & +\ 4x & & \\ \hline & & -\ 3x & -\ 6 & \\ & & -\ 3x & -\ 6 & \\ \hline & & & 0 & \leftarrow \text{As expected} \end{array}$$

$$x^3 + 4x^2 + x - 6 = \left(x + 2\right)\left(x^2 + 2x - 3\right)$$
$$= (x + 2)(x - 1)(x + 3)$$

[4 marks]

(c) The solutions are,

$$x = -3,\ x = -2 \text{ or } x = 1$$

[1 mark]

Answer 5

$$f(x) = ax^3 + bx^2 - 4x - 3$$

(a) The question is, in effect, saying that,

$$f(1) = 0 \quad \text{by the factor theorem}$$

$$a(1)^3 + b(1)^2 - 4(1) - 3 = 0$$

$$a + b - 4 - 3 = 0$$

$$a + b = 7 \qquad \square$$

[2 marks]

(b) The question is, in effect, saying that,

$$f(-2) = 9 \quad \text{by the remainder theorem}$$

$$a(-2)^3 + b(-2)^2 - 4(-2) - 3 = 9$$

$$-8a + 4b + 8 - 3 = 9$$

$$-8a + 4b = 4$$

Divide through by – 4

$$2a - b = -1$$

Combining this equation with that of part (a) simultaneously,

$$\begin{array}{lrcl} & a + b & = & 7 \\ \text{ADD} & 2a - b & = & -1 \\ \hline & 3a & = & 6 \end{array}$$

$$a = 2 \qquad \Rightarrow \qquad b = 5$$

[4 marks]

Answer 6

$$f(x) = x^3 - x^2 - 7x + c$$

(a)

$$f(4) = 0$$

$$(4)^3 - (4)^2 - 7(4) + c = 0$$

$$64 - 16 - 28 + c = 0$$

$$c = -20$$

[2 marks]

(b) Part (a) indirectly revealed that $(x - 4)$ is a factor of $f(x)$ by the factor theorem

$$\begin{array}{r|l}
 & \quad\quad\quad x^2 + 3x + 5 \\
\hline
x - 4 & x^3 - x^2 - 7x - 20 \\
 & \underline{x^3 - 4x^2} \\
 & \quad\quad 3x^2 - 7x \\
 & \quad\quad \underline{3x^2 - 12x} \\
 & \quad\quad\quad\quad 5x - 20 \\
 & \quad\quad\quad\quad \underline{5x - 20} \\
 & \quad\quad\quad\quad\quad\quad 0 \leftarrow \text{As expected}
\end{array}$$

$$f(x) = x^3 - x^2 - 7x - 20$$

$$= \left(x - 4\right)\left(x^2 + 3x + 5\right)$$

[3 marks]

(c) It is required that the following equation be solved,

$$f(x) = 0$$

$$x^3 - x^2 - 7x - 20 = 0$$

$$\left(x - 4\right)\left(x^2 + 3x + 5\right) = 0$$

Either $\left(x - 4\right) = 0$ in which case $x = 4$

Or $\left(x^2 + 3x + 5\right) = 0$ in which case $x = \dfrac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 5}}{2 \times 1}$

$$x = \frac{-3 \pm \sqrt{-11}}{2}$$

but this has no real solutions

$\therefore\ x = 4$ is the only solution □

[2 marks]

Answer 7

$$p(x) = 4x^3 + ax^2 + bx + 15$$

$$\text{By the remainder theorem, } p(1) = 90$$
$$4(1)^3 + a(1)^2 + b(1) + 15 = 90$$
$$4 + a + b + 15 = 90$$
$$a + b = 71$$

$$\text{By the remainder theorem, } f(-1) = -4$$
$$4(-1)^3 + a(-1)^2 + b(-1) + 15 = -4$$
$$-4 + a - b + 15 = -4$$
$$a - b = -15$$

Combining the two results as simultaneous equations,

$$\begin{array}{lrcl} & a + b & = & 71 \\ \text{ADD} & a - b & = & -15 \\ \hline & 2a & = & 56 \end{array}$$

$$a = 28 \quad \Rightarrow \quad b = 43$$

[7 marks]

Answer 8

$$f(x) = x^4 - 3x^2 - ax + b \qquad g(x) = x^3 - 3ax + 5b$$

By the factor theorem, if a polynomial is divisible by $(x - 2)$ then letting $x = 2$ causes the polynomial to equal zero.

$$f(2) = 0 \qquad g(2) = 0$$
$$(2)^4 - 3(2)^2 - a(2) + b = 0 \qquad (2)^3 - 3a(2) + 5b = 0$$
$$16 - 12 - 2a + b = 0 \qquad 8 - 6a + 5b = 0$$
$$-2a + b = -4 \qquad -6a + 5b = -8$$
$$2a - b = 4 \qquad 6a - 5b = 8$$

$$\text{Thus, } 2(2a - b) = 6a - 5b$$
$$4a - 2b = 6a - 5b$$
$$3b = 2a \qquad \square$$

[5 marks]

Alternatively solve simultaneous equation to get $a = 3$ and $b = 2$ from which the requested result follows.

Answer 9

$$f(x) = 2x^3 - 5x^2 + ax + 18$$

(a) As $(x - 3)$ is a factor, putting $x = 3$ into $f(x)$ will cause $f(x)$ to equal zero, by the factor theorem,

$$\therefore\ 2(3)^3 - 5(3)^2 + a(3) + 18 = 0$$

$$54 - 45 + 3a + 18 = 0$$

$$3a = -27 \Leftrightarrow a = -9 \qquad \square$$

[2 marks]

(b)

$$\begin{array}{r|l}
 & \quad 2x^2 + x - 6 \\
x - 3 & \overline{2x^3 - 5x^2 - 9x + 18} \\
 & \underline{2x^3 - 6x^2} \\
 & \qquad x^2 - 9x \\
 & \qquad \underline{x^2 - 3x} \\
 & \qquad\quad -6x + 18 \\
 & \qquad\quad \underline{-6x + 18} \\
 & \qquad\qquad 0 \leftarrow \text{ As expected}
\end{array}$$

$$\begin{aligned} f(x) &= 2x^3 - 5x^2 - 9x + 18 \\ &= \left(x - 3\right)\left(2x^2 + x - 6\right) \\ &= (x - 3)(2x - 3)(x + 2) \end{aligned}$$

[4 marks]

(c) It's helpful to solve the equation $f(x) = 0$

That is, $(x - 3)(2x - 3)(x + 2) = 0$

Either $x = 3$ or $x = \dfrac{3}{2}$ or $x = -2$

$$\begin{aligned} g(y) &= 2\left(3^{3y}\right) - 5\left(3^{2y}\right) - 9\left(3^y\right) + 18 \\ &= 2\left(3^y\right)^3 - 5\left(3^y\right)^2 - 9\left(3^y\right) + 18 \end{aligned}$$

Solving $g(x) = 0$ is the same as solving $f(x) = 0$ with $x = 3^y$

$$\therefore \text{ Either } 3^y = 3 \Rightarrow y = 1$$

Or $3^y = \dfrac{3}{2} \Rightarrow y = \dfrac{\ln 3 - \ln 2}{\ln 3} = 0.3690702464\ldots$ (accept T+I)

Or $3^y = -2 \Rightarrow$ No solutions

[3 marks]

8.1 Later Date Revision

The Factor Theorem
If for a given polynomial function $p(x)$, $p(a) = 0$ then $(x - a)$ is a factor of $p(x)$.

The Remainder Theorem
When $p(x)$ is divided by $(x - a)$ the remainder is $p(a)$

8.2 Exercise

Marks Available : 47

Question 1

A-Level Examination question from June 2005, Paper C2, Q3 (Edexcel)

(a) Use the factor theorem to show that $(x + 4)$ is a factor of $2x^3 + x^2 - 25x + 12$

[2 marks]

(b) Factorise $2x^3 + x^2 - 25x + 12$ completely.

[4 marks]

Question 2

A-Level Examination question from January 2006, Paper C2, Q1 (Edexcel)

$$f(x) = 2x^3 + x^2 - 5x + c$$

where c is a constant.

Given that $f(1) = 0$,

(a) Find the value of c

[2 marks]

(b) Factorise $f(x)$ completely

[4 marks]

(c) Find the remainder when $f(x)$ is divided by $(2x - 3)$

[2 marks]

Question 3

A-Level Examination question from June 2010, Paper C2, Q2 (Edexcel)

$$f(x) = 3x^3 - 5x^2 - 58x + 40$$

(a) Find the remainder when $f(x)$ is divided by $(x - 3)$

[2 marks]

Given that $(x - 5)$ is a factor of $f(x)$

(b) Find all the solutions of $f(x) = 0$.

[5 marks]

Question 4

A-Level Examination question from June 2009, Paper C2, Q3 (Edexcel)

$$f(x) = (3x - 2)(x - k) - 8$$

where k is a constant.

(a) Write down the value of $f(k)$

[1 mark]

When $f(x)$ is divided by $(x - 2)$ the remainder is 4

(b) Find the value of k

[2 marks]

(c) Factorise $f(x)$ completely

[3 marks]

Question 5

A-Level Examination question from January 2010, Paper C2, Q3 (Edexcel)

$$f(x) = 2x^3 + ax^2 + bx - 6$$

where a and b are constants.

When $f(x)$ is divided by $(2x - 1)$ the remainder is -5

When $f(x)$ is divided by $(x + 2)$ there is no remainder.

(a) Find the value of a and the value of b

[6 marks]

(b) Factorise $f(x)$ completely.

[3 marks]

Question 6

A-Level Examination question from January 2009, Paper C2, Q6 (Edexcel)

$$f(x) = x^4 + 5x^3 + ax + b$$

where a and b are constants.

The remainder when $f(x)$ is divided by $(x - 2)$ is equal to the remainder when $f(x)$ is divided by $(x + 1)$

(a) Find the value of a

[5 marks]

Given that $(x + 3)$ is a factor of $f(x)$

(b) find the value of b

[3 marks]