5.3.1 Solution to the 5.1 Example

(i) As the centre of the circle is (30, 21) the equation of the circle will be,

$(x - 30)^2 + (y - 21)^2 = r^2$ where the value of r needs to be found.

This is surprisingly tricky to determine and is what the question is about.

All lines perpendicular to the tangent with equation $y = 3x + 1$ will have a gradient of $-\frac{1}{3}$ (The sign changed reciprocal of 3)

We are after such a perpendicular line that passes through the centre of the circle, which we're told is (30, 21).

$$y = mx + c$$

$$21 = -\frac{1}{3} \times 30 + c$$

$$c = 31$$

$$\text{Equation sought : } y = -\frac{1}{3}x + 31$$

The point were the tangent touches the circle is now given by where this line intersects the tangent which we're told has equation $y = 3x + 1$

$$3x + 1 = -\frac{1}{3}x + 31$$

Multiply through by 3

$$9x + 3 = -x + 93$$

$$10x = 90$$

$$x = 9 \qquad \therefore \; y = 28$$

So the point of contact is (9, 28) and the radius will be the distance from this to the centre at (30, 21),

$$r = \sqrt{(\Delta y)^2 + (\Delta x)^2}$$

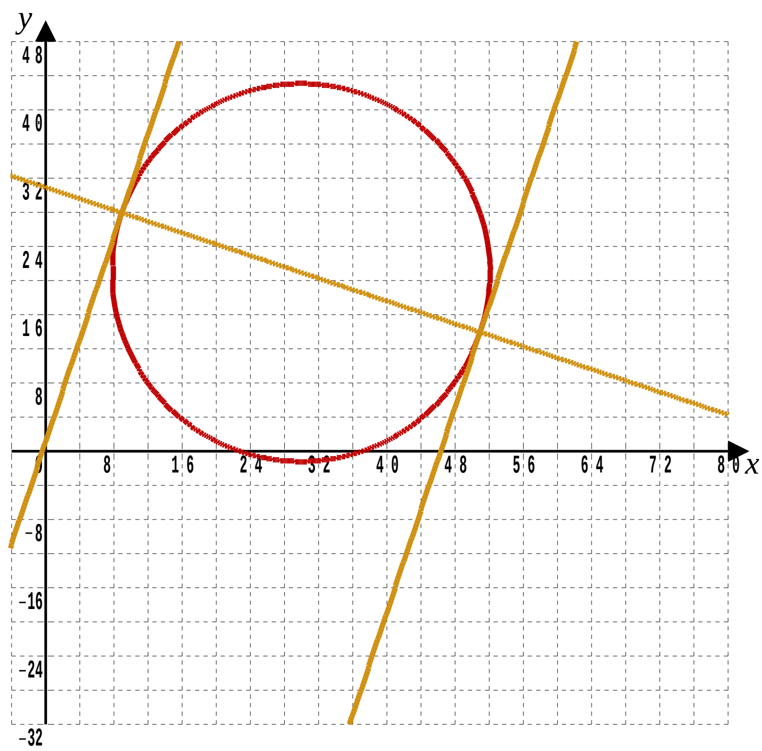
$$= \sqrt{(28 - 21)^2 + (9 - 30)^2}$$

$$= \sqrt{490}$$

So the equation of the circle is,

$$(x - 30)^2 + (y - 21)^2 = 490$$

[5 marks]



(ii) The vector translation that will move from (9, 28) to (30, 21) is,

$$\begin{pmatrix} 21 \\ -7 \end{pmatrix}$$

and so the point at the other end of the diameter will be given by,

$$\begin{pmatrix} 9 \\ 28 \end{pmatrix} + 2\begin{pmatrix} 21 \\ -7 \end{pmatrix} = \begin{pmatrix} 51 \\ 14 \end{pmatrix}$$

The line sought has gradient 3 and passes through (51, 14),

$$y = mx + k$$

$$14 = 3 \times 51 + k$$

$$k = -139$$

$$\therefore\ y = 3x - 139$$

[2 marks]

5.3.2 Solutions to 5.2 Exercise

Answer 1

(a) PA will have gradient $-\frac{1}{2}$ and pass through $A(7, 5)$

$$y = mx + c$$

$$5 = -\frac{1}{2} \times 7 + c \Rightarrow c = 8.5$$

$$\therefore \; y = -\frac{1}{2}x + 8.5$$

Multiply through by 2

$$2y = -x + 17$$

$$2y + x = 17 \qquad \square$$

[3 marks]

(b) To find P, $2x + 1 = -\frac{1}{2}x + 8.5$

$$4x + 2 = -x + 17$$

$$5x = 15$$

$$x = 3 \qquad \therefore \; y = 7$$

The get length of radius, PA,

$$|PA| = \sqrt{(7 - 5)^2 + (3 - 7)^2}$$

$$= \sqrt{20}$$

The equation for C is thus,

$$(x - 7)^2 + (y - 5)^2 = 20$$

[4 marks]

(c) The vector translation that will move from P (3, 7) to A (7, 5) is,

$$\begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

and so the point at the other end of the diameter will be given by,

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} + 2\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix}$$

The line sought has gradient 2 and passes through (11, 3),

$$y = mx + k$$

$$3 = 2 \times 11 + k$$

$$k = -19 \qquad \therefore \; y = 2x - 19$$

[3 marks]

Answer 2

$$(7 - \sqrt{3})(5 - \sqrt{3}) = a + b\sqrt{3}$$

$$35 - 7\sqrt{3} - 5\sqrt{3} + 3 = a + b\sqrt{3}$$

$$38 - 12\sqrt{3} = a + b\sqrt{3}$$

Thus $a = 38$ and $b = -12$

[2 marks]

Answer 3

(i)

$$\begin{aligned} f(x) &= x^2 - 8x + 23 \\ &= [\, x^2 - 8x \,] + 23 \\ &= [\, (x - 4)^2 - 16 \,] + 23 \\ &= (x - 4)^2 + 7 \end{aligned}$$

[1 mark]

(ii) To cross the x-axis the function must have a value of zero which corresponds to having zero height. However, $(x - 4)^2$ has a least value of zero and, even then, 7 is added on.

So the function never has zero height and so does not cross the x-axis.

[2 marks]

(iii) Negative

[1 mark]

Answer 4

$$\begin{aligned} \frac{\sqrt{3}}{3 + 2\sqrt{3}} \times \frac{3 - 2\sqrt{3}}{3 - 2\sqrt{3}} &= \frac{3\sqrt{3} - 6}{9 - 6\sqrt{3} + 6\sqrt{3} - 12} \\ &= \frac{3(\sqrt{3} - 2)}{-3} \\ &= 2 - \sqrt{3} \end{aligned}$$

$\therefore\ m = 2$ and $n = -1$

[3 marks]

Answer 5

$$\begin{aligned} 5x^2 + 10x - 2 &= 5[\, x^2 + 2x \,] - 2 \\ &= 5[\, (x + 1)^2 - 1 \,] - 2 \\ &= 5(x + 1)^2 - 5 - 2 \\ &= 5(x + 1)^2 - 7 \end{aligned}$$

giving $a = 5$, $b = 1$ and $c = -7$

[3 marks]

Answer 6

(i)

$$3\,(2x + 1) > 5 - 2x$$

$$6x + 3 > 5 - 2x$$

$$x > \frac{1}{4}$$

[2 marks]

(ii)

$$2\,x^2 - 7x + 3 > 0$$

$$(2x - 1)(x - 3) > 0$$

The critical values are $x = \frac{1}{2}$ or $x = 3$

$x < \frac{1}{2}$ or $x > 3$ (Sketch the graph to see why)

[3 marks]

(iii) $\frac{1}{4} < x < \frac{1}{2}$ or $x > 3$

[1 mark]

Answer 7

(a)

$$x^2 + y^2 - 6x + 10y + 9 = 0$$

$$\left[x^2 - 6x\right] + \left[y^2 + 10y\right] + 9 = 0$$

$$\left[(x - 3)^2 - 9\right] + \left[(y + 5)^2 - 25\right] + 9 = 0$$

$$(x - 3)^2 + (y + 5)^2 = 25$$

(i) Centre is (3, – 5) **(ii)** Radius is 5

[3 marks]

(b) As no part of the circle is in the first quadrant, k cannot be positive. As the circle is tangential to the x-axis, and because the line has to cut the circle at two distinct points (not one) k cannot equal zero. Thus $k < 0$

$$x^2 + y^2 - 6x + 10y + 9 = 0$$

$$x^2 + (kx)^2 - 6x + 10\,(kx) + 9 = 0$$

$$(1 + k^2)\,x^2 + (10k - 6)\,x + 9 = 0$$

Discriminant: : $(10k - 6)^2 - 4\,(1 + k^2) \times 9 > 0$ (for two distinct roots)

$$100\,k^2 - 120k + 36 - 36 - 36\,k^2 > 0$$

$$64\,k^2 - 120k > 0$$

$$8\,k^2 - 15k > 0$$

$$k\,(8k - 15) > 0$$

Overall, $k < 0$ or $k > \frac{15}{8}$

[6 marks]

Answer 8

(a)
$$f(x) = x^2 + (k+3)x + k$$
$$D = (k+3)^2 - 4 \times 1 \times k$$
$$= k^2 + 6k + 9 - 4k$$
$$= k^2 + 2k + 9$$

[2 marks]

(b)
$$D = k^2 + 2k + 9$$
$$= \left[k^2 + 2k \right] + 9$$
$$= \left[(k+1)^2 - 1 \right] + 9$$
$$= (k+1)^2 + 8$$

That is, $a = 1$ and $b = 8$

[2 marks]

(c) This smallest that any real number squared can be is zero.
Even then, 8 is added on.
So D is always positive.
Thus $f(x)$ always has two distinct real roots □

[2 marks]

Answer 9

$$8^{2x} - 10\left(8^x\right) + 16 = 0$$
$$\left(8^x\right)^2 - 10\left(8^x\right) + 16 = 0$$
$$\left(\left(8^x - 2\right)\left(8^x - 8\right)\right) = 0$$

Either $8^x - 2 = 0$ or $8^x - 8 = 0$

$8^x = 2$ $\quad$ $8^x = 8$

$x = \dfrac{1}{3}$ $\quad$ $x = 1$

[3 marks]

Answer 10

Various ways through are possible, but all should end up with,

$$m(x) = x^4 + 2x^3 - 3x^2 - 8x - 4$$
$$= (x+1)^2(x-2)(x+2)$$

[6 marks]