A-Level

Pure Mathematics

~ Year 1 ~

Progress Test Revision

P R O G R E S S T E S T R E V I S I O N

Lesson 1

A-Level Pure Mathematics : Year 1

Progress Test Revision

1.1 Example #1

Given that, $(a + \sqrt{5})(4 + \sqrt{5}) = 29 + b\sqrt{5}$

find the value of a and the value of b both of which are integers.

[2 marks]

1.2 Example #2

Solve the equation, $3^{2x+1} - 28(3^x) + 3^2 = 0$

[3 marks]

1.3 Revision Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 70

Question 1

Given that, $\left(a + \sqrt{7}\right)\left(5 + \sqrt{7}\right) = 47 + b\sqrt{7}$

find the value of a and the value of b both of which are integers

[2 marks]

Question 2

Solve the equation, $2^{2x+2} - 33\left(2^{x}\right) + 2^{3} = 0$

[3 marks]

Question 3

Solve the inequality, $21x + 19 > 6x - 26$

[2 marks]

Question 4

Consider the inequality, $x^2 - 4x - 21 \geqslant 0$

(i) State the critical values of x.

[2 marks]

(ii) Sketch the graph of, $y = x^2 - 4x - 21$

[2 marks]

(iii) Solve the inequality, $x^2 - 4x - 21 \geqslant 0$

[2 marks]

(iv) Solve the related inequality, $x^2 - 4x - 21 < 0$

[2 marks]

Question 5

Points A and B have coordinates $(3, 5)$ and $(9, 17)$ respectively.

(a) Find the midpoint of the line segment AB.

[2 marks]

(b) Find the gradient of the line segment AB.

[2 marks]

(c) Find the equation of the line through A and B in the form $ax + by + c = 0$, where a, b and c are integers.

[3 marks]

(d) Find the equation of the perpendicular bisector of the line segment AB.
Give your answer in the form $ax + by + c = 0, \quad a, b, c \in \mathbb{Z}$

[3 marks]

Question 6

$$f(x) = x^3 + 4x^2 - 9x - 36$$

(i) Show that $f(-4) = 0$

[2 marks]

(ii) Hence, or otherwise, factorise the cubic polynomial completely.

[4 marks]

Question 7

Given that, $243\sqrt{3} = 3^a$, find the value of a

[3 marks]

Question 8

A-Level examination question from November 2002, (Edexcel)

(a) Solve the inequality, $3x - 8 > x + 13$

[2 marks]

(b) Solve the inequality, $x^2 - 5x - 14 > 0$

[2 marks]

Question 9

Rationalise the denominator of,

(i) $\dfrac{1}{2 + \sqrt{3}}$

[2 marks]

(ii) $\dfrac{1}{(3 - \sqrt{2})^2}$

[3 marks]

Question 10

Find the perpendicular bisector of the line segment between the points A and B where, A is (4, 11) and B is (16, 3)

[4 marks]

Question 11

Examination Question, Edexcel June 2003

Find the set of values for x for which

(a) $6x - 7 < 2x + 3$

[2 marks]

(b) $2x^2 - 11x + 5 < 0$

[3 marks]

(c) Both $6x - 7 < 2x + 3$ and $2x^2 - 11x + 5 < 0$

[1 mark]

Question 12

$$g\,(x) = x^3 - 3x^2 - 10x + 24$$

(i) Show that $g\,(4) = 0$

[1 mark]

(ii) Hence, or otherwise, factorise the cubic polynomial completely.

[4 marks]

Question 13

Solve the equation, $8 + x\sqrt{12} = \dfrac{8x}{\sqrt{3}}$

Give your answer in the form $a\sqrt{b}$ where a and b are integers

[4 marks]

Question 14

(i) Expand the brackets; $(x + 1)^2$

[1 mark]

(ii) Hence, or otherwise, solve simultaneously;

$$x^2 + y^2 = 25$$

$$y = x + 1$$

[4 marks]

Question 15

Given that $x + 2$ is a factor of $x^3 - 2x^2 + ax + 6$ find a

[3 marks]

1.4 Answers

1.4.1 Solution to the 1.1 Example

$$(a + \sqrt{5})(4 + \sqrt{5}) = 29 + b\sqrt{5}$$

$$4a + 4\sqrt{5} + a\sqrt{5} + 5 = 29 + b\sqrt{5}$$

$$4a + 5 + (4 + a)\sqrt{5} = 29 + b\sqrt{5}$$

$$\therefore 4a + 5 = 29 \quad \text{and} \quad 4 + a = b$$

$$4a = 24$$

$$a = 6 \quad \text{and so} \quad b = 10$$

[2 marks]

1.4.2 Solution to the 1.2 Example

$$3^{2x+1} - 28(3^x) + 3^2 = 0$$

$$3^{2x} \times 3^1 - 28(3^x) + 9 = 0$$

$$3(3^x)^2 - 28(3^x) + 9 = 0$$

Let $z = 3^x$ in which case,

$$3z^2 - 28z + 9 = 0$$

$$(3z - 1)(z - 9) = 0$$

Either $3z - 1 = 0$ or $z - 9 = 0$

$$z = \frac{1}{3} \qquad z = 9$$

$$3^x = \frac{1}{3} \qquad 3^x = 9$$

$$x = -1 \qquad x = 2$$

[3 marks]

1.4.3 Solution to the 1.3 Exercise

Answer 1

$$(a + \sqrt{7})(5 + \sqrt{7}) = 47 + b\sqrt{7}$$

$$5a + 5\sqrt{7} + a\sqrt{7} + 7 = 47 + b\sqrt{7}$$

$$5a + 7 + (5 + a)\sqrt{7} = 47 + b\sqrt{7}$$

$$\therefore 5a + 7 = 47 \qquad \text{and} \qquad 5 + a = b$$

$$5a = 40$$

$$a = 8 \quad \text{and so} \quad b = 13$$

[2 marks]

Answer 2

$$2^{2x+2} - 33\left(2^x\right) + 2^3 = 0$$

$$2^{2x} \times 2^2 - 33\left(2^x\right) + 8 = 0$$

$$4\left(2^x\right)^2 - 33\left(2^x\right) + 8 = 0$$

Let $z = 2^x$ in which case,

$$4z^2 - 33z + 8 = 0$$

$$(4z - 1)(z - 8) = 0$$

Either $4z - 1 = 0$ or $z - 8 = 0$

$$z = \frac{1}{4} \qquad z = 8$$

$$2^x = \frac{1}{4} \qquad 2^x = 8$$

$$x = -2 \qquad x = 3$$

[3 marks]

Answer 3

$$21x + 19 > 6x - 26$$

$$15x > -45$$

$$x > -3$$

[2 marks]

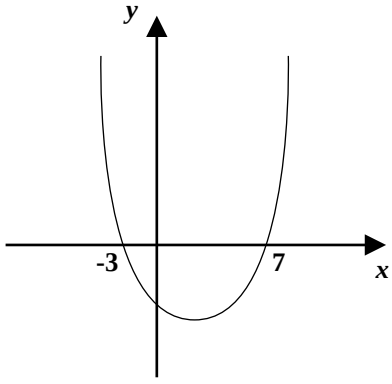
Answer 4

(i) $x^2 - 4x - 21 \geqslant 0$

$$(x + 3)(x - 7) \geqslant 0$$

$\therefore$ The critical values are $x = -3$ or $x = 7$

(ii)



(iii) $x \leqslant -3$ *or* $x \geqslant 7$

(iv) $-3 < x < 7$

[2, 2, 2, 2 marks]

Answer 5

(a) $(6, 11)$

(b) $m_{AB} = \dfrac{\Delta y}{\Delta x} = \dfrac{17 - 5}{9 - 3} = \dfrac{12}{6} = 2$

(c) $y = 2x - 1 \quad \Rightarrow \quad 2x - y - 1 = 0$

(d) $m_{PERP} = -\dfrac{1}{2}$ through $(6, 11)$

$$y = mx + c$$

$$11 = -\frac{1}{2} \times 6 + c \text{ so } c = 14$$

$$y = -\frac{1}{2}x + 14 \Rightarrow x + 2y - 28 = 0$$

[2, 2, 3, 3 marks]

Answer 6

(i)
$$f(x) = x^3 + 4x^2 - 9x - 36$$
$$f(-4) = (-4)^3 + 4 \times (-4)^2 - 9 \times (-4) - 36$$
$$= -64 + 64 + 36 - 36$$
$$= 0 \qquad \square$$

[2 marks]

(ii) The long division is unusual,

$$\begin{array}{r|l} & x^2 \qquad\qquad - 9 \\ \hline x + 4 & x^3 + 4x^2 - 9x - 36 \\ & \underline{x^3 + 4x^2} \\ & 0 \quad - 9x - 36 \\ & \underline{\quad - 9x - 36} \\ & 0 \end{array}$$

$$x^3 + 4x^2 - 9x - 36 = (x + 4)(x^2 - 9)$$
$$= (x + 4)(x + 3)(x - 3)$$

[4 marks]

Answer 7

$$243\sqrt{3} = 3^a$$
$$3^5 \times 3^{0.5} = 3^a \qquad \text{could use } FACT \text{ calculator button}$$
$$3^{5.5} = 3^a$$
$$\therefore \; a = 5.5$$

[3 marks]

Answer 8

(a) $3x - 8 > x + 13$

$2x > 21$

$x > 10.5$

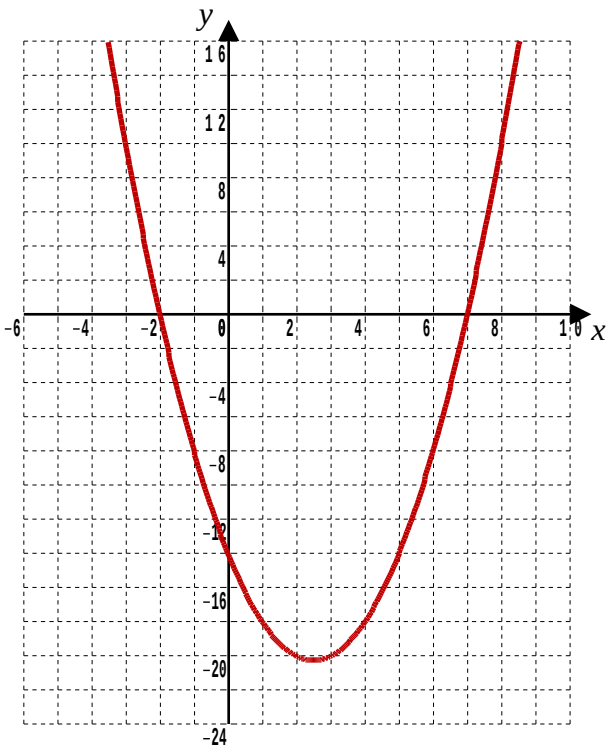
[2 marks]

(b) $x^2 - 5x - 14 > 0$

$(x + 2)(x - 7) > 0$

The critical values are $x = -2$ or $x = 7$

Sketch the graph to get the last bit correct...



$x < -2$ or $x > 7$

[2 marks]

Answer 9

(i)

$$\frac{1}{2+\sqrt{3}} = \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$$

$$= \frac{2-\sqrt{3}}{4-2\sqrt{3}+2\sqrt{3}-3}$$

$$= 2-\sqrt{3}$$

[2 marks]

(ii)

$$\frac{1}{(3-\sqrt{2})^2} = \frac{1}{9-6\sqrt{2}+2}$$

$$= \frac{1}{11-6\sqrt{2}} \times \frac{11+6\sqrt{2}}{11+6\sqrt{2}}$$

$$= \frac{11+6\sqrt{2}}{121+66\sqrt{2}-66\sqrt{2}-72}$$

$$= \frac{11+6\sqrt{2}}{49}$$

[3 marks]

Answer 10

$$Midpoint = \left(\frac{4+16}{2}, \frac{11+3}{2}\right) = (10,\ 7)$$

$$m_{AB} = \frac{\Delta y}{\Delta x} = \frac{3-11}{16-4} = -\frac{8}{12} = -\frac{2}{3}$$

$$m_{PERP} = \frac{3}{2}$$

$$y = mx + c$$

$$7 = \frac{3}{2} \times 10 + c$$

$$c = -8$$

$y = \frac{3}{2}x - 8$ is the perpendicular bisector

[4 marks]

Answer 11

(a) $6x - 7 < 2x + 3$

$4x < 10$

$x < 2.5$

[2 marks]

(b) $2x^2 - 11x + 5 < 0$

$(2x-1)(x-5) < 0$

The critical values are $x = \frac{1}{2}$ or $x = 5$

$\frac{1}{2} < x < 5$ (Draw a graph to see why)

[3 marks]

(c) $\frac{1}{2} < x < 2.5$

[1 mark]

Answer 12

(i)

$$g(x) = x^3 - 3x^2 - 10x + 24$$

$$g(4) = 4^3 - 3 \times 4^2 - 10 \times 4 + 24$$

$$= 64 - 48 - 40 + 24$$

$$= 0 \qquad \square$$

[1 mark]

(ii)

$$\begin{array}{rl|l} & & x^2 + x - 6 \\ \hline x - 4 & x^3 - 3x^2 - 10x + 24 & \\ & x^3 - 4x^2 & \\ \hline & + x^2 - 10x & \\ & x^2 - 4x & \\ \hline & - 6x + 24 & \\ & - 6x + 24 & \\ \hline & 0 & \end{array}$$

$$x^3 - 3x^2 - 10x + 24 = (x - 4)(x^2 + x - 6)$$

$$= (x - 4)(x + 3)(x - 2)$$

[4 marks]

Answer 13

$$8 + x\sqrt{12} = \frac{8x}{\sqrt{3}}$$

Multiply through by $\sqrt{3}$

$$8\sqrt{3} + x\sqrt{36} = 8x$$

$$8\sqrt{3} + 6x = 8x$$

$$2x = 8\sqrt{3}$$

$$x = 4\sqrt{3}$$

[4 marks]

Answer 14

(i)

$$(x + 1)^2 = x^2 + 2x + 1$$

[1 mark]

(ii)

$$x^2 + y^2 = 25$$

$$x^2 + (x + 1)^2 = 25$$

$$x^2 + x^2 + 2x + 1 = 25$$

$$2x^2 + 2x - 24 = 0$$

Divide through by 2

$$x^2 + x - 12 = 0$$

$$(x + 4)(x - 3) = 0$$

Either $x + 4 = 0$ or $x - 3 = 0$

$$x = -4 \qquad x = 3$$

The solutions are $(-4, -3)$ or $(3, 4)$

[4 marks]

Answer 15

Told, effectively, that,

$$f(-2) = 0$$

$$x^3 - 2x^2 + ax + 6 = 0$$

$$(-2)^3 - 2 \times (-2)^2 + a(-2) + 6 = 0$$

$$-8 - 8 - 2a + 6 = 0$$

$$2a = -10$$

$$a = -5$$

[3 marks]

2.1 A Useful "Flip"

Power Inversion Rule

$$\left(\frac{p}{q}\right)^{-m} = \left(\frac{q}{p}\right)^{m} \qquad p,\ q \neq 0$$

2.2 Examples

Simplify fully each of the following,

(i) $\left(\dfrac{3a^6}{5a^2\sqrt{b}}\right)^{-2}$ **(ii)** $\left(\dfrac{4x^2}{9\sqrt[3]{y}}\right)^{-\frac{3}{2}}$

[3, 3 marks]

2.3 Revision Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Simplify fully each of the following

(i) $\left(\dfrac{36x^6}{25}\right)^{-\frac{1}{2}}$ **(ii)** $\left(\dfrac{4xy^2}{3x^2y^3}\right)^{-1}$

[3, 3 marks]

Question 2

Expand the brackets and simplify; $(x + 3)(x - 3)(x + 4)$

[3 marks]

Question 3

Factorise completely, $4x^2 - x^4$

[3 marks]

Question 4

Showing your method, rationalise the denominator of, $\frac{8}{3 - \sqrt{5}}$

[3 marks]

Question 5

(i) Simplify $\sqrt{75} + \sqrt{12}$, giving your answer in the form $a\sqrt{3}$, where a is an integer to be found.

[2 marks]

(ii) Simplify $\dfrac{\sqrt{75} + \sqrt{12}}{2 - \sqrt{3}}$ giving your answer in the form $b + c\sqrt{3}$ where b and c are integers to be found.

[4 marks]

Question 6

Here is a FALSE PROOF that, $\dfrac{a+b}{a-\sqrt{b}} = \dfrac{a+\sqrt{b}}{a-b}$ where $a \neq b$, $a \neq \sqrt{b}$

$$\text{LHS} = \frac{a+b}{a-\sqrt{b}}$$

$$= \frac{a+b}{a-\sqrt{b}} \times \frac{a+\sqrt{b}}{a+\sqrt{b}}$$

$$= \frac{(a+b)(a+\sqrt{b})}{a^2-b^2}$$

$$= \frac{(a+b)(a+\sqrt{b})}{(a+b)(a-b)} \quad \text{(Difference of two squares)}$$

$$= \frac{a+\sqrt{b}}{a-b}$$

$$= \text{RHS} \qquad \square$$

Where is the error ?

[3 marks]

Question 7

A-Level Examination Question from November 2021, Paper 1, Q1 (Edexcel)

Using algebra, solve the inequality

$$x^2 - x > 20$$

writing your answer in set notation.

[3 marks]

Question 8

Given that $y = 3^x$

(i) Express 9^x in terms of y

[1 mark]

(ii) Hence, or otherwise, solve $9\left(9^x\right) - 10\left(3^x\right) + 1 = 0$

[6 marks]

Question 9

AS-Level Examination Question from October 2020, Paper 1, Q3 (Edexcel)

(i) Solve the equation,

$$x\sqrt{2} - \sqrt{18} = x$$

writing the answer as a surd in simplest form.

[3 marks]

(ii) Solve the equation,

$$4^{3x-2} = \frac{1}{2\sqrt{2}}$$

[3 marks]

2.4 Answers

A-Level Pure Mathematics : Year 1
Progress Test Revision

2.4.1 Solutions to 2.2 Examples

(i) $\left(\dfrac{3a^6}{5a^2\sqrt{b}}\right)^{-2} = \left(\dfrac{5a^2\sqrt{b}}{3a^6}\right)^{2} = \left(\dfrac{5\sqrt{b}}{3a^4}\right)^{2} = \dfrac{25b}{9a^8}$

[3 marks]

(ii) $\left(\dfrac{4x^2}{9\sqrt[3]{y}}\right)^{-\frac{3}{2}} = \left(\dfrac{9\sqrt[3]{y}}{4x^2}\right)^{\frac{3}{2}} = \dfrac{27\sqrt{y}}{8x^3}$

[3 marks]

2.4.2 Solutions to 2.3 Exercise

Answer 1

(i) $\left(\dfrac{36x^6}{25}\right)^{-\frac{1}{2}} = \left(\dfrac{25}{36x^6}\right)^{\frac{1}{2}} = \dfrac{5}{6x^3}$

[3 marks]

(ii) $\left(\dfrac{4xy^2}{3x^2y^3}\right)^{-1} = \dfrac{3x^2y^3}{4xy^2} = \dfrac{3xy}{4}$

[3 marks]

Answer 2

$(x+3)(x-3)(x+4) = (x^2-9)(x+4)$ Diff of two squares

$= x^3 + 4x^2 - 9x - 36$

[3 marks]

Answer 3

$4x^2 - x^4 = x^2(4 - x^2)$

$= x^2(2+x)(2-x)$

[3 marks]

Answer 4

$$\frac{8}{3-\sqrt{5}} = \frac{8}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$$

$$= \frac{8(3+\sqrt{5})}{9-5}$$

$$= 2(3+\sqrt{5})$$

$$= 6 + 2\sqrt{5}$$

[3 marks]

Answer 5

(i) $\sqrt{75} + \sqrt{12} = \sqrt{25 \times 3} + \sqrt{4 \times 3}$

$= 5\sqrt{3} + 2\sqrt{3}$

$= 7\sqrt{3}$ That is, $a = 7$

[2 marks]

(ii) $\dfrac{\sqrt{75} + \sqrt{12}}{2 - \sqrt{3}} = \dfrac{7\sqrt{3}}{2 - \sqrt{3}}$ from part (i)

$= \dfrac{7\sqrt{3}}{2 - \sqrt{3}} \times \dfrac{2 + \sqrt{3}}{2 + \sqrt{3}}$ rationalising the denominator

$= \dfrac{7\sqrt{3}\,(2 + \sqrt{3})}{4 - 3}$

$= 14\sqrt{3} + 21$

$= 21 + 14\sqrt{3}$ that is, $b = 21$ and $c = 14$

[4 marks]

Answer 6

LHS $= \dfrac{a + b}{a - \sqrt{b}}$

$= \dfrac{a + b}{a - \sqrt{b}} \times \dfrac{a + \sqrt{b}}{a + \sqrt{b}}$

$= \dfrac{(a + b)(a + \sqrt{b})}{a^2 - b^2}$ <== This denominator is wrong

$= \dfrac{(a + b)(a + \sqrt{b})}{(a + b)(a - b)}$ (Difference of two squares)

$= \dfrac{a + \sqrt{b}}{a - b}$

= RHS $\square$

[3 marks]

Answer 7

$x^2 - x > 20$

$x^2 - x - 20 > 0$

$(x + 4)(x - 5) > 0$

The critical values are, $x = -4$ or $x = 5$

$x < -4$ or $x > 5$ (Draw a graph to see why)

$\{x : x < -4\} \cup \{x : x > 5\}$

[3 marks]

Answer 8

(i) $9^x = (3^2)^x$

$= (3^x)^2$

$= y^2$

[1 mark]

(ii) $9(9^x) - 10(3^x) + 1 = 0$

$9y^2 - 10y + 1 = 0 \quad$ with $y = 3^x$

$(9y - 1)(y - 1) = 0$

Either $9y - 1 = 0$ in which case $y = \frac{1}{9}$

Or $y - 1 = 0$ in which case $y = 1$

But $y = 3^x$ so either $3^x = \frac{1}{9}$ giving $x = -2$

or $3^x = 1$ giving $x = 0$

$\therefore x \in \{-2, 0\}$

[5 marks]

Answer 9

(i) $x\sqrt{2} - \sqrt{18} = x$

$x(\sqrt{2} - 1) = \sqrt{18}$

$x = \frac{\sqrt{18}}{(\sqrt{2} - 1)} \times \frac{(\sqrt{2} + 1)}{(\sqrt{2} + 1)}$

$= 6 + 3\sqrt{2}$

[3 marks]

(ii) $4^{3x-2} = \frac{1}{2\sqrt{2}}$

$(2^2)^{3x-2} = 2^{-\frac{3}{2}}$

$6x - 4 = -\frac{3}{2}$

$6x = \frac{5}{2}$

$x = \frac{5}{12}$

[3 marks]

3.1 Example #1

Solve the equation, $9^{5x-3} = 27^{x+5}$

[3 marks]

3.2 Example #2

The equation $x^2 + 3mx + (4m + 1) = 0$, where m is a constant, has equal roots.

(i) Find the possible values of m

[4 marks]

(ii) For these values of m, solve the equation

$$x^2 + 3mx + (4m + 1) = 0$$

[4 marks]

3.3 Revision Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 52

Question 1

Solve the equation, $8^{7x-6} = 16^{8-x}$

[3 marks]

Question 2

The equation $x^2 + 2mx + (3m + 4) = 0$, where m is a constant, has equal roots.

(i) Find the possible values of m

[4 marks]

(ii) For these values of m, solve the equation

$$x^2 + 2mx + (3m + 4) = 0$$

[4 marks]

Question 3

Simplify the following algebraic expressions by first factorising the quadratics:

$$\frac{x^2 + 2x - 24}{x^2 - 3x - 54}$$

[2 marks]

Question 4

Solve simultaneously,

$$5x^2 - 4xy + y^2 = 13$$

$$y = x + 1$$

[5 marks]

Question 5

$$m(x) = x^3 + 13x^2 + ax + 63$$

(i) Given that $x + 7$ is a factor of $m(x)$ determine the integer value of a

[2 marks]

(ii) Hence, factorise $m(x)$ completely.

[4 marks]

Question 6

Determine the coordinates of the centre and the radius of the circle with equation,

$$x^2 + y^2 + 96x - 110y = 0$$

[4 marks]

Question 7

Determine the value of a

$$\frac{\sqrt{3}}{81} = 3^a$$

[3 marks]

Question 8

A circle, C, has equation,

$$(x - 3)^2 + (y - 15)^2 = 17^2$$

(i) Write down the coordinates of the point X, the centre of the circle.

[2 marks]

(ii) What is the diameter of the circle ?

[1 mark]

(iii) Give the coordinates of the points M and N, where the circle crosses the x-axis.

[3 marks]

(iv) What is the perimeter of triangle MXN ?

[2 marks]

Question 9

It is wished to solve the inequality,

$$\frac{5}{x-3} < 2 \qquad x \neq 3$$

(i) Explain the reason for the restriction, $x \neq 3$

[1 mark]

(ii) Explain why this inequality can not be solved by multiplying both sides by $(x-3)$

[1 mark]

(iii) Solve the inequality by first multiplying both sides by $(x-3)^2$

[3 marks]

(iv) Explain why it was OK to multiply both sides by $(x-3)^2$

[1 mark]

Question 10

The following equation, where k is a constant, has no real roots

$$k\,x^2 - 2kx + 3 = 0$$

Prove that k satisfies the inequality, $0 \leqslant k < 3$

[4 marks]

Question 11

AS-Level Examination Question, May 2018, Q4 (Edexcel)

The line L_1 has equation $4y - 3x = 10$

The line L_2 passes through the points $(5, -1)$ and $(-1, 8)$

Determine, giving full reasons for your answer, whether lines L_1 and L_2 are parallel, perpendicular or neither.

[3 marks]

3.4 Answers

3.4.1 Solution to the 3.1 Example #1

$$9^{5x-3} = 27^{x+5}$$
$$\left(3^2\right)^{5x-3} = \left(3^3\right)^{x+5}$$
$$3^{10x-6} = 3^{3x+15}$$
$$10x - 6 = 3x + 15$$
$$7x = 21$$
$$x = 3$$

[3 marks]

3.4.2 Solution to the 3.1 Example #2

$$x^2 + 3mx + (4m + 1) = 0$$

(i) For "equal roots" the discriminant must equal zero,

$$b^2 - 4ac = 0 \qquad \text{For } ax^2 + bx + c = 0$$
$$(3m)^2 - 4 \times 1 \times (4m + 1) = 0$$
$$9m^2 - 16m - 4 = 0$$
$$(9m + 2)(m - 2) = 0$$
$$\text{Either } 9m + 2 = 0 \text{ or } m - 2 = 0$$
$$m = -\frac{2}{9} \qquad m = 2$$

[4 marks]

(ii) When $m = 2$, $x^2 + 3mx + (4m + 1) = 0$

$$x^2 + 6x + 9 = 0$$
$$(x + 3)^2 = 0$$
$$x + 3 = 0$$
$$x = -3$$

When $m = -\frac{2}{9}$, $x^2 + 3mx + (4m + 1) = 0$

$$x^2 - \frac{2}{3}x + \frac{1}{9} = 0$$
$$\left(x - \frac{1}{3}\right)^2 = 0$$
$$x = \frac{1}{3}$$

[4 marks]

3.4.3 Solution to the 3.3 Revision Exercise

Answer 1

$$8^{7x-6} = 16^{8-x}$$
$$\left(2^3\right)^{7x-6} = \left(2^4\right)^{8-x}$$
$$2^{21x-18} = 2^{32-4x}$$
$$21x - 18 = 32 - 4x$$
$$25x = 50$$
$$x = 2$$

[3 marks]

Answer 2

$$x^2 + 2mx + (3m + 4) = 0$$

(i) For "equal roots" the discriminant must equal zero,

$$b^2 - 4ac = 0 \qquad \text{for } ax^2 + bx + c = 0$$
$$(2m)^2 - 4 \times 1 \times (3m + 4) = 0$$
$$4m^2 - 12m - 16 = 0$$

divide through by 4

$$m^2 - 3m - 4 = 0$$
$$(m + 1)(m - 4) = 0$$

Either $m + 1 = 0$ or $m - 4 = 0$

$$m = -1 \qquad m = 4$$

[4 marks]

(ii)

When $m = -1$,
$$x^2 + 2mx + (3m + 4) = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x - 1)^2 = 0$$
$$x = 1$$

When $m = 4$,
$$x^2 + 2mx + (3m + 4) = 0$$
$$x^2 + 8x + 16 = 0$$
$$(x + 4)^2 = 0$$
$$x = -4$$

[4 marks]

Answer 3

$$\frac{x^2 + 2x - 24}{x^2 - 3x - 54} = \frac{(x + 6)(x - 4)}{(x - 9)(x + 6)}$$

$$= \frac{x - 4}{x - 9}$$

[2 marks]

Answer 4

$$5x^2 - 4xy + y^2 = 13$$

$$5x^2 - 4x(x + 1) + (x + 1)^2 = 13$$

$$5x^2 - 4x^2 - 4x + x^2 + 2x + 1 = 13$$

$$2x^2 - 2x - 12 = 0$$

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

$$x = 3 \text{ or } x = -2$$

The (x, y) solutions are $(3, 4)$ or $(-2, -1)$

[5 marks]

Answer 5

(i)

$$m(x) = x^3 + 13x^2 + ax + 63$$

Told that, $m(-7) = 0$

$$(-7)^3 + 13(-7)^2 + a(-7) + 63 = 0$$

$$-343 + 637 - 7a + 63 = 0$$

$$357 - 7a = 0$$

$$a = 51$$

[2 marks]

(ii)

$$\begin{array}{rl|l} & & x^2 + 6x + 9 \\ \hline x + 7 & x^3 + 13x^2 + 51x + 63 & \\ & x^3 + 7x^2 & \\ \hline & + 6x^2 + 51x & \\ & 6x^2 + 42x & \\ \hline & 9x + 63 & \\ & 9x + 63 & \\ \hline & 0 & \end{array}$$

$$x^3 + 13x^2 + 51x + 63 = (x + 7)(x^2 + 6x + 9)$$

$$= (x + 7)(x + 3)(x + 3)$$

[4 marks]

Answer 6

$$x^2 + y^2 + 96x - 110y = 0$$
$$\left[x^2 + 96x\right] + \left[y^2 - 110y\right] = 0$$
$$\left[(x + 48)^2 - 2304\right] + \left[(y - 55)^2 - 3025\right] = 0$$
$$(x + 48)^2 + (y - 55)^2 = 5329$$
$$(x + 48)^2 + (y - 55)^2 = 73^2$$

Circle has centre $(-48,\ 55)$ and radius 73

[4 marks]

Answer 7

$$\frac{\sqrt{3}}{81} = 3^a$$
$$\frac{3^{0.5}}{3^4} = 3^a$$
$$3^{-3.5} = 3^a$$
$$\therefore\ a = -3.5$$

[3 marks]

Answer 8

(i) $X\,(3, 15)$

[2 marks]

(ii) Diameter = 34

[1 mark]

(iii) Crosses the x-axis when $y = 0$

$$(x - 3)^2 + (-15)^2 = 17^2$$
$$x^2 - 6x + 9 + 225 - 289 = 0$$
$$x^2 - 6x - 55 = 0$$
$$(x - 11)(x + 5) = 0$$

Either $x - 11 = 0$ or $x + 5 = 0$

$x = 11$ $\qquad$ $x = -5$

[3 marks]

(iv) $Perimeter = 17 + 17 + 16$

$= 50$

[2 marks]

Answer 9

(i) If x were allowed to equal 3 a division by zero would occur which has no meaning in mathematics.

[1 mark]

(ii) Multiplying both sides by $(x - 3)$ is not a good move because it is not known if $(x - 3)$ is positive or negative so it's then not known if the inequality sign should be reversed or not.

[1 mark]

(iii)

$$5 (x - 3) < 2 (x - 3)^2$$

$$5x - 15 < 2x^2 - 12x + 18$$

$$2x^2 - 12x + 18 > 5x - 15$$

$$2x^2 - 17x + 33 > 0$$

$$(2x - 11)(x - 3) > 0$$

$$x < 3 \quad \text{or} \quad x > \frac{11}{2} \qquad \text{(Sketch a graph to see why)}$$

[3 marks]

(iv) It's valid to multiply both sides by a quantity that is certain to be greater than or equal to zero, as it's then known to leave the inequality sign as it is.

$$(x - 3)^2 \geqslant 0 \quad \text{for all real values of } x$$

[1 mark]

Answer 10

To have no real roots the discriminant must be negative.

$$D < 0$$

$$(-2k)^2 - 4 \times k \times 3 < 0$$

$$4k^2 - 12k < 0$$

divide through by 4

$$k^2 - 3k < 0$$

$$k (k - 3) < 0$$

Critical values are $k = 0$ and $k = 3$

$$0 < k < 3$$

Observe that $k = 0$ does not give a solvable equation either,

$$\therefore \; 0 \leqslant k < 3$$

[4 marks]

Answer 11

$$L_1 : 4y - 3x = 10$$

$$y = \frac{3}{4}x + \frac{10}{4} \text{ which has gradient } \frac{3}{4}$$

$$\begin{aligned} L_2 : m &= \frac{\Delta y}{\Delta x} \\ &= \frac{8 - (-1)}{-1 - 5} \\ &= \frac{9}{-6} \\ &= -\frac{3}{2} \end{aligned}$$

If the two gradients were the same the two lines would be parallel, but they are not.
If each gradient were the sign changed reciprocal of the other they would be mutually perpendicular, but they are not.

$\therefore$ The two lines are neither parallel nor perpendicular.

[3 marks]

4.1 Example

Given that $y = 5^x$, express each of the following in terms of y.
Write each expression in its simplest form.

(i) 5^{4x}

[1 mark]

(ii) 5^{x-2}

[1 mark]

(iii) $5^{-\frac{x}{2}}$

[2 marks]

(iv) 125^x

[1 mark]

(v) $\dfrac{5^3}{25^{x+2}}$

[3 marks]

4.2 Revision Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 35

Question 1

Given that $y = 3^x$, express each of the following in terms of y.
Write each expression in its simplest form.

(i) $3^{0.5x}$

[1 mark]

(ii) 3^{1+2x}

[1 mark]

(iii) 3^{-3x}

[2 marks]

(iv) $\dfrac{1}{9^{x+1}}$

[2 marks]

(v) $\dfrac{1^x}{27^{2x}}$

[2 marks]

Question 2

$$f(x) = x^2 + 6x - 41$$

(i) Express $f(x)$ in the form $(x + a)^2 + b$, where a and b are constants to be found.

[2 marks]

(ii) Hence or otherwise, find the exact solutions to the equation,

$$x^2 + 6x - 41 = 0$$

Write your answer in the form $p \pm q\sqrt{r}$, where p, q and r are integers

[3 marks]

Question 3

Given that $\dfrac{3x^2 + 12x + 9}{108x - 12x^3} = \dfrac{x + a}{bx(x + c)}$ where a, b and c are constants, work out the values of a, b and c.

[4 marks]

Question 4

A-Level Examination Question from June 2022, Paper 1, Q2 (Edexcel)

$$f(x) = (x + 4)(x^2 - 3x + k) - 42$$

Given that $(x + 2)$ is a factor of $f(x)$, find the value of k

[3 marks]

Question 5

AS-Level Examination Question from November 2021, Paper 1, Q2 (Edexcel)

Given,

$$\frac{9^{x-1}}{3^{y+2}} = 81$$

express y in terms of x, writing your answer in simplest form.

[3 marks]

Question 6

AS-Level Examination Question from May 2019, Paper 1, Q2

Find, using algebra, all real solutions to the equation,

(i) $16\,a^2 = 2\sqrt{a}$

[4 marks]

(ii) $b^4 + 7\,b^2 - 18 = 0$

[4 marks]

Question 7

The curve with equation $y = 3 \times 2^x$ meets the curve with equation $y = 28 - 2^{x-1}$ at the point P. Find, using algebra, the coordinates of P.

[4 marks]

4.3 Answers

4.3.1 Solution to 4.1 Example

(i) $5^{4x} = (5^x)^4 = y^4$

[1 mark]

(ii) $5^{x-2} = 5^x \times 5^{-2} = \frac{y}{25}$

[1 mark]

(iii) $5^{-\frac{x}{2}} = \left(\frac{5}{1}\right)^{-\frac{x}{2}} = \left(\frac{1}{5}\right)^{\frac{x}{2}} = \left(\frac{1^x}{5^x}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{y}}$

[2 marks]

(iv) $125^x = (5^3)^x = (5^x)^3 = y^3$

[1 mark]

(v) $\frac{5^3}{25^{x+2}} = \frac{5^3}{(5^2)^x \times (5^2)^2} = \frac{5^3}{(5^x)^2 \times 5^4} = \frac{1}{5y^2}$

[3 marks]

4.3.2 Solutions to 4.2 Exercise

Answer 1

(i) $3^{0.5x} = (3^x)^{0.5} = \sqrt{y}$

[1 mark]

(ii) $3^{1+2x} = 3^1 \times (3^x)^2 = 3y^2$

[1 mark]

(iii) $3^{-3x} = \left(\frac{3}{1}\right)^{-3x} = \left(\frac{1}{3}\right)^{3x} = \left(\frac{1^x}{3^x}\right)^3 = \frac{1}{y^3}$

[2 marks]

(iv) $\frac{1}{9^{x+1}} = \frac{1}{9^x \times 9^1} = \frac{1}{(3^2)^x \times 9} = \frac{1}{(3^x)^2 \times 9} = \frac{1}{9y^2}$

[2 marks]

(v) $\frac{1^x}{27^{2x}} = \frac{1}{(3^3)^{2x}} = \frac{1}{(3^x)^6} = \frac{1}{y^6}$

[2 marks]

Answer 2

(i)

$$f(x) = x^2 + 6x - 41$$
$$= \left[x^2 + 6x \right] - 41$$
$$= \left[(x+3)^2 - 9 \right] - 41$$
$$= (x+3)^2 - 50$$
$$a = 3, \quad b = -50$$

[2 marks]

(ii)

$$x^2 + 6x - 41 = 0$$
$$(x+3)^2 - 50 = 0 \qquad \text{from part (i)}$$
$$(x+3)^2 = 50$$
$$x + 3 = \pm\sqrt{50}$$
$$x = -3 \pm \sqrt{25 \times 2}$$
$$= -3 \pm 5\sqrt{2}$$
$$p = -3, \quad q = 5, \quad r = 2$$

[3 marks]

Answer 3

$$\frac{3x^2 + 12x + 9}{108x - 12x^3} = \frac{3\left(x^2 + 4x + 3\right)}{12x\left(9 - x^2\right)}$$
$$= \frac{(x+3)(x+1)}{4x(3-x)(3+x)}$$
$$= \frac{x+1}{-4x(x-3)}$$
$$a = 1, \; b = -4 \text{ and } c = -3$$

[3 marks]

Answer 4

By the factor theorem, $f(-2) = 0$

$$\left(-2 + 4\right)\left((-2)^2 - 3(-2) + k \right) - 42 = 0$$
$$2(4 + 6 + k) = 42$$
$$k + 10 = 21$$
$$k = 11$$

[3 marks]

Answer 5

$$\frac{9^{x-1}}{3^{y+2}} = 81$$

$$\frac{(3^2)^{x-1}}{3^{y+2}} = 3^4$$

$$3^{2x-2-y-2} = 3^4$$

$$2x - 2 - y - 2 = 4$$

$$2x - y = 8$$

$$y = 2x - 8$$

[3 marks]

Answer 6

(i)

$16a^2 = 2\sqrt{a} \quad \Rightarrow$ ... alternatively ...

$8a^2 = \sqrt{a}$ $\qquad 256a^4 = 4a$

$8a^2 - \sqrt{a} = 0$ $\qquad 256a^4 - 4a = 0$

$\sqrt{a}\left(8a^{\frac{3}{2}} - 1\right) = 0$ $\qquad 4a\left(64a^3 - 1\right) = 0$

Either $a = 0$ or $a^{\frac{3}{2}} = \frac{1}{8}$ $\qquad a = 0$ or $64a^3 = 1$

$a^3 = \frac{1}{64} \quad \Leftarrow$

$a = \frac{1}{4}$

[4 marks]

(ii)

$$b^4 + 7b^2 - 18 = 0$$

Let $z = b^2$ in which case,

$$z^2 + 7z - 18 = 0$$

$$(z + 9)(z - 2) = 0$$

Either $z = -9$ or $z = 2$

$b^2 = -9$ or $b^2 = 2$

$b = \pm\sqrt{2}$ are the only real solutions

[4 marks]

Answer 7

$$3 \times 2^x = 28 - 2^{x-1}$$

$$3 \times 2^x = 28 - \frac{2^x}{2}$$

Multiply through by 2

$$6 \times 2^x = 56 - 2^x$$

$$6 \times 2^x + 2^x = 56$$

$$7 \times 2^x = 56$$

$$2^x = 8$$

$$x = 3$$

[4 marks]

5.1 Example

The line with equation $y = 3x + 1$ is a tangent to a circle with centre $(30, 21)$

(i) Find the equation of the circle.

[5 marks]

The line with equation $y = 3x + k$, $k \neq 1$, is also a tangent to the circle

(ii) Find the value of the constant k

[2 marks]

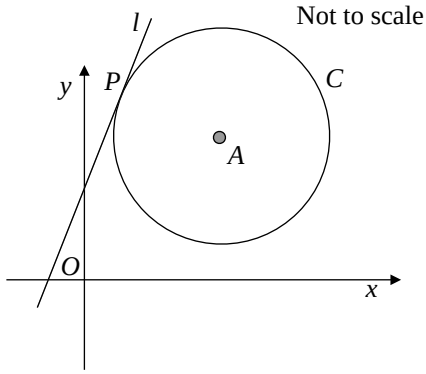
5.2 Revision Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 52

Question 1

A-Level Examination Question, June 2018, Paper 1, Q6 (Edexcel)



The circle C has centre A with coordinates (7, 5)

The line l, with equation $y = 2x + 1$, is the tangent to C at the point P

(a) Show that an equation of the line PA is $2y + x = 17$

[3 marks]

(b) Find an equation for C

[4 marks]

The line with equation $y = 2x + k$, $k \neq 1$, is also a tangent to C

(c) Find the value of the constant k

[3 marks]

Question 2

Given that, $\left(7 - \sqrt{3}\right)\left(5 - \sqrt{3}\right) = a + b\sqrt{3}$, where a and b are integers, find the value of a and the value of b

[2 marks]

Question 3

(i) Complete the square for the following function,

$$f(x) = x^2 - 8x + 23$$

[1 marks]

(ii) Use your part **(i)** answer to explain why the graph of the function will not have any x-axis crossing points

[2 marks]

(iii) Given your previous answers, would you expect the discriminant of the function to be positive, negative or zero ?

[1 mark]

Question 4

Given that, $\dfrac{\sqrt{3}}{3 + 2\sqrt{3}} = m + n\sqrt{3}$, where m and n are integers, find the value of m and the value of n

[3 marks]

Question 5

Given that,

$$5x^2 + 10x - 2 = a(x + b)^2 + c$$

where a, b and c are integers, find the value of a, the value of b and the value of c

[3 marks]

Question 6

Find the set of values of x for which

(i) $3(2x + 1) > 5 - 2x$

[2 marks]

(ii) $2x^2 - 7x + 3 > 0$

[3 marks]

(iii) **both** $3(2x + 1) > 5 - 2x$

and $2x^2 - 7x + 3 > 0$

[1 mark]

Question 7

AS-Level Examination Question, June 2018, Q14 (Edexcel)

The circle C has equation

$$x^2 + y^2 - 6x + 10y + 9 = 0$$

(a) Find

(i) the coordinates of the centre of C

(ii) the radius of C

[3 marks]

The line with equation $y = kx$, where k is a constant, cuts C at two distinct points

(b) Find the range of values of k

[6 marks]

Question 8

$f(x) = x^2 + (k + 3)x + k$ where k is a real constant

(a) Find the discriminant of $f(x)$ in terms of k

[2 marks]

(b) Show that the discriminant of $f(x)$ can be expressed in the form $(k + a)^2 + b$ where a and b are integers to be found.

[2 marks]

(c) Show that, for all values of k, the equation $f(x) = 0$ has real roots

[2 marks]

Question 9

Solve the equation, $8^{2x} - 10\left(8^x\right) + 16 = 0$

[3 marks]

Question 10

$$m(x) = x^4 + 2x^3 - 3x^2 - 8x - 4$$

Factorise the quartic polynomial completely.

[6 marks]

5.3 Answers

A-Level Pure Mathematics : Year 1
Progress Test Revision

5.3.1 Solution to the 5.1 Example

(i) As the centre of the circle is (30, 21) the equation of the circle will be,

$(x - 30)^2 + (y - 21)^2 = r^2$ where the value of r needs to be found.
This is surprisingly tricky to determine and is what the question is about.

All lines perpendicular to the tangent with equation $y = 3x + 1$ will have a gradient of $-\frac{1}{3}$ (The sign changed reciprocal of 3)

We are after such a perpendicular line that passes through the centre of the circle, which we're told is (30, 21).

$$y = mx + c$$

$$21 = -\frac{1}{3} \times 30 + c$$

$$c = 31$$

$$\text{Equation sought : } y = -\frac{1}{3}x + 31$$

The point were the tangent touches the circle is now given by where this line intersects the tangent which we're told has equation $y = 3x + 1$

$$3x + 1 = -\frac{1}{3}x + 31$$

Multiply through by 3

$$9x + 3 = -x + 93$$

$$10x = 90$$

$$x = 9 \qquad \therefore\ y = 28$$

So the point of contact is (9, 28) and the radius will be the distance from this to the centre at (30, 21),

$$r = \sqrt{(\Delta y)^2 + (\Delta x)^2}$$

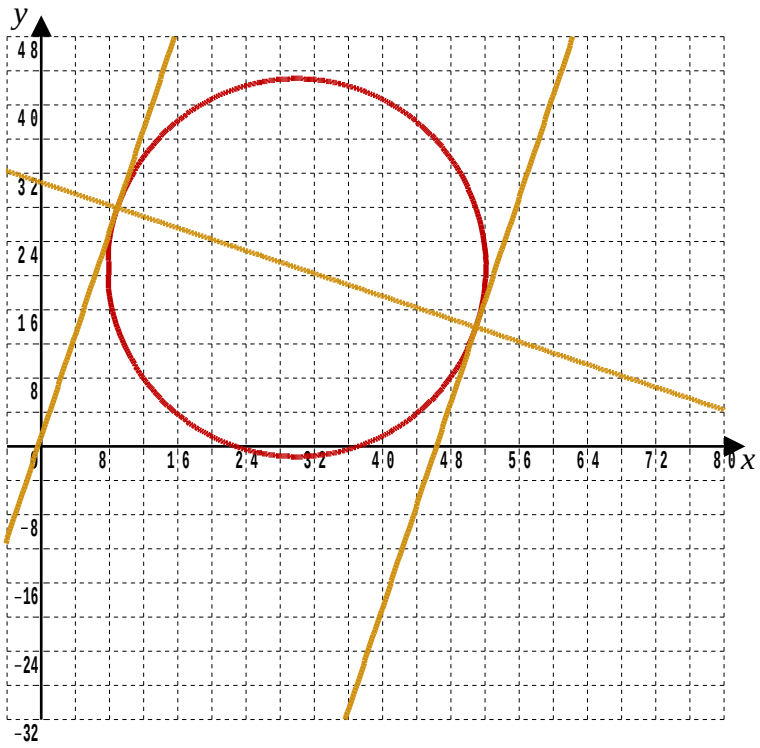
$$= \sqrt{(28 - 21)^2 + (9 - 30)^2}$$

$$= \sqrt{490}$$

So the equation of the circle is,

$$(x - 30)^2 + (y - 21)^2 = 490$$

[5 marks]



(ii) The vector translation that will move from (9, 28) to (30, 21) is,

$$\begin{pmatrix} 21 \\ -7 \end{pmatrix}$$

and so the point at the other end of the diameter will be given by,

$$\begin{pmatrix} 9 \\ 28 \end{pmatrix} + 2\begin{pmatrix} 21 \\ -7 \end{pmatrix} = \begin{pmatrix} 51 \\ 14 \end{pmatrix}$$

The line sought has gradient 3 and passes through (51, 14),

$$y = mx + k$$

$$14 = 3 \times 51 + k$$

$$k = -139$$

$$\therefore\ y = 3x - 139$$

[2 marks]

5.3.2 Solutions to 5.2 Exercise

Answer 1

(a) PA will have gradient $-\frac{1}{2}$ and pass through $A(7, 5)$

$$y = mx + c$$

$$5 = -\frac{1}{2} \times 7 + c \Rightarrow c = 8.5$$

$$\therefore \; y = -\frac{1}{2}x + 8.5$$

Multiply through by 2

$$2y = -x + 17$$

$$2y + x = 17 \qquad \square$$

[3 marks]

(b) To find P, $2x + 1 = -\frac{1}{2}x + 8.5$

$$4x + 2 = -x + 17$$

$$5x = 15$$

$$x = 3 \qquad \therefore \; y = 7$$

The get length of radius, PA,

$$|PA| = \sqrt{(7 - 5)^2 + (3 - 7)^2}$$

$$= \sqrt{20}$$

The equation for C is thus,

$$(x - 7)^2 + (y - 5)^2 = 20$$

[4 marks]

(c) The vector translation that will move from $P(3, 7)$ to $A(7, 5)$ is,

$$\begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

and so the point at the other end of the diameter will be given by,

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} + 2\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix}$$

The line sought has gradient 2 and passes through $(11, 3)$,

$$y = mx + k$$

$$3 = 2 \times 11 + k$$

$$k = -19 \qquad \therefore \; y = 2x - 19$$

[3 marks]

Answer 2

$$(7 - \sqrt{3})(5 - \sqrt{3}) = a + b\sqrt{3}$$
$$35 - 7\sqrt{3} - 5\sqrt{3} + 3 = a + b\sqrt{3}$$
$$38 - 12\sqrt{3} = a + b\sqrt{3}$$
$$\text{Thus } a = 38 \text{ and } b = -12$$

[2 marks]

Answer 3

(i)
$$f(x) = x^2 - 8x + 23$$
$$= [x^2 - 8x] + 23$$
$$= [(x - 4)^2 - 16] + 23$$
$$= (x - 4)^2 + 7$$

[1 mark]

(ii) To cross the x-axis the function must have a value of zero which corresponds to having zero height. However, $(x - 4)^2$ has a least value of zero and, even then, 7 is added on.
So the function never has zero height and so does not cross the x-axis.

[2 marks]

(iii) Negative

[1 mark]

Answer 4

$$\frac{\sqrt{3}}{3 + 2\sqrt{3}} \times \frac{3 - 2\sqrt{3}}{3 - 2\sqrt{3}} = \frac{3\sqrt{3} - 6}{9 - 6\sqrt{3} + 6\sqrt{3} - 12}$$
$$= \frac{3(\sqrt{3} - 2)}{-3}$$
$$= 2 - \sqrt{3}$$
$$\therefore m = 2 \text{ and } n = -1$$

[3 marks]

Answer 5

$$5x^2 + 10x - 2 = 5[x^2 + 2x] - 2$$
$$= 5[(x + 1)^2 - 1] - 2$$
$$= 5(x + 1)^2 - 5 - 2$$
$$= 5(x + 1)^2 - 7$$
$$\text{giving } a = 5, \quad b = 1 \text{ and } c = -7$$

[3 marks]

Answer 6

(i) $3(2x+1) > 5-2x$

$6x+3 > 5-2x$

$x > \frac{1}{4}$

[2 marks]

(ii) $2x^2 - 7x + 3 > 0$

$(2x-1)(x-3) > 0$

The critical values are $x = \frac{1}{2}$ or $x = 3$

$x < \frac{1}{2}$ or $x > 3$ (Sketch the graph to see why)

[3 marks]

(iii) $\frac{1}{4} < x < \frac{1}{2}$ or $x > 3$

[1 mark]

Answer 7

(a) $x^2 + y^2 - 6x + 10y + 9 = 0$

$[x^2 - 6x] + [y^2 + 10y] + 9 = 0$

$[(x-3)^2 - 9] + [(y+5)^2 - 25] + 9 = 0$

$(x-3)^2 + (y+5)^2 = 25$

(i) Centre is $(3, -5)$ **(ii)** Radius is 5

[3 marks]

(b) As no part of the circle is in the first quadrant, k cannot be positive.
As the circle is tangential to the x-axis, and because the line has to cut the circle at two distinct points (not one) k cannot equal zero. Thus $k < 0$

$x^2 + y^2 - 6x + 10y + 9 = 0$

$x^2 + (kx)^2 - 6x + 10(kx) + 9 = 0$

$(1+k^2)x^2 + (10k-6)x + 9 = 0$

Discriminant: : $(10k-6)^2 - 4(1+k^2) \times 9 > 0$ (for two distinct roots)

$100k^2 - 120k + 36 - 36 - 36k^2 > 0$

$64k^2 - 120k > 0$

$8k^2 - 15k > 0$

$k(8k-15) > 0$

Overall, $k < 0$ or $k > \frac{15}{8}$

[6 marks]

Answer 8

(a)
$$f(x) = x^2 + (k+3)x + k$$
$$D = (k+3)^2 - 4 \times 1 \times k$$
$$= k^2 + 6k + 9 - 4k$$
$$= k^2 + 2k + 9$$

[2 marks]

(b)
$$D = k^2 + 2k + 9$$
$$= \left[k^2 + 2k \right] + 9$$
$$= \left[(k+1)^2 - 1 \right] + 9$$
$$= (k+1)^2 + 8$$

That is, $a = 1$ and $b = 8$

[2 marks]

(c) This smallest that any real number squared can be is zero.
Even then, 8 is added on.
So D is always positive.
Thus $f(x)$ always has two distinct real roots □

[2 marks]

Answer 9

$$8^{2x} - 10\left(8^x\right) + 16 = 0$$
$$\left(8^x\right)^2 - 10\left(8^x\right) + 16 = 0$$
$$\left(\left(8^x - 2\right)\left(8^x - 8\right)\right) = 0$$

Either $8^x - 2 = 0$ or $8^x - 8 = 0$

$8^x = 2$ $\qquad$ $8^x = 8$

$x = \frac{1}{3}$ $\qquad$ $x = 1$

[3 marks]

Answer 10

Various ways through are possible, but all should end up with,

$$m(x) = x^4 + 2x^3 - 3x^2 - 8x - 4$$
$$= (x+1)^2(x-2)(x+2)$$

[6 marks]