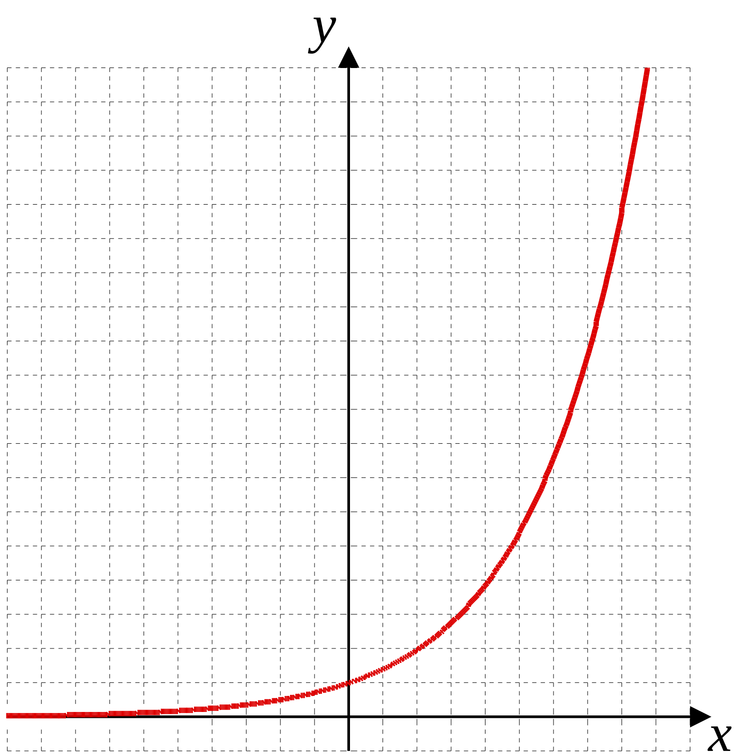
A-Level Mathematics

Year 1

~ Pure Mathematics ~

E X P O N E N T I A L S
a n d
L O G A R I T H M S



The graph of an exponential

E X P O N E N T I A L S
and
L O G A R I T H M S

Lesson 1

A-Level Pure Mathematics : Year 1
Exponentials and Logarithms

1.1 An Exponential Curve and its Inverse

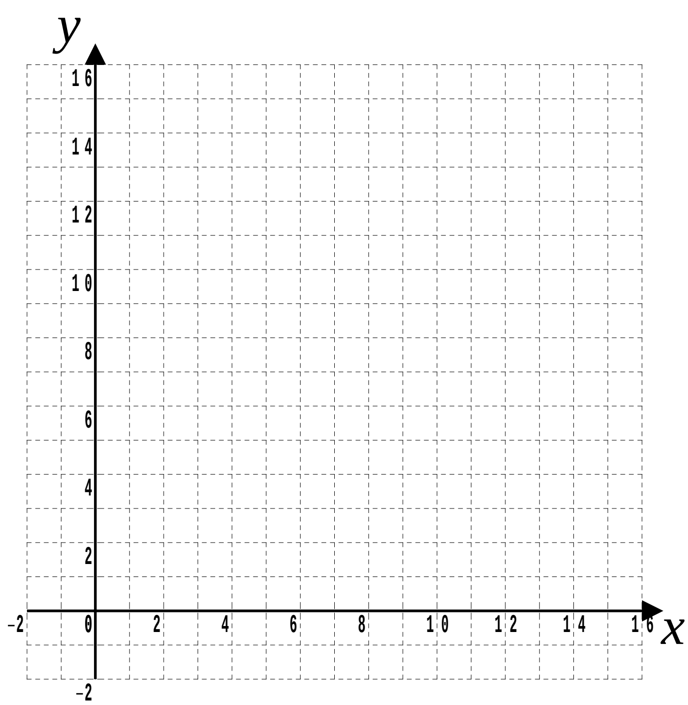
(i) Complete the table of values for the exponential function given by

$$y = 2^x$$

x	-2	-1	0	1	2	3	4
y				2	4		16

[2 marks]

(ii) Graph the function, using the values from the completed table.



[2 marks]

(iii) Use your graph, evidencing your method, to find $2^{2.5}$

[2 marks]

(iv) Use your graph, evidencing your method, to help you solve the equation

$$2^x = 14$$

[2 marks]

(v) Add the line $y = x$ to your graph.

[1 mark]

(vi) Reflect the exponential curve, plotted in part (ii), in the line $y = x$.

[2 marks]

The reflection curve is of the inverse function.
The inverse of an exponential function is a logarithm function.

In this case, the exponential function;

$$y = 2^x$$

has as an inverse, the logarithm function;

$$y = log_2 x$$

(vii) Use your reflection graph, evidencing your method, to find

$$log_2 9$$

[2 marks]

(viii) Use your reflection graph, evidencing your method, to find

$$log_2 16$$

Notice that this means, “2 to what power is 16 ?”

[2 marks]

(ix) This next question is 'off the graph'.
State the value of

$$log_2 64$$

by realising that this means, “2 to what power is 64 ?”

[1 mark]

1.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 40

Question 1

(i) State the value of,

$$log_3 81$$

by realising that this means, “3 to what power is 81 ?”

(ii) State the value of,

$$log_2 32$$

(iii) State the value of,

$$log_{10} 1000000$$

(iv) State the value of,

$$log_4 64$$

[4 marks]

Question 2

(i) State the value of,

$$log_3\left(\frac{1}{9}\right)$$

by realising that this means “3 to what power is $\frac{1}{9}$?”

(ii) State the value of,

$$log_2\left(\frac{1}{8}\right)$$

(iii) State the value of,

$$log_2 0.25$$

(iv) State the value of,

$$log_3\left(\frac{1}{27}\right)$$

[4 marks]

Question 3

(i) Find x given that,

$$log_x 125 = 3$$

by realising that this means, “what to the power 3 is 125 ?”

(ii) Find x given that,

$$log_x 81 = 4$$

(iii) Find x given that,

$$log_x 64 = 2$$

(iv) Find x given that,

$$log_x 9 = 0.5$$

[4 marks]

Question 4

(i) Find x given that,

$$log_2 x = 3$$

by realising that this means, “2 to the power 3 is what ?”

(ii) Find x given that,

$$log_{10} x = 5$$

(iii) Find x given that,

$$log_2 x = -3$$

(iv) Find x given that,

$$log_9 x = \frac{3}{2}$$

[4 marks]

Question 5

(i) Find x given that,

$$log_{10}\, 10000 \;=\; x$$

by realising that this means, "10 to what power is 10000 ?"

(ii) Find x given that,

$$log_{10}\, 100\,000\,000 \;=\; x$$

(iii) Find x given that,

$$log_{10}\, 10 \;=\; x$$

(iv) Find x given that,

$$log_{10} \left(\frac{1}{1000} \right) \;=\; x$$

(v) Find x given that,

$$log_{10}\; 0.01 \;=\; x$$

(vi) Find x given that,

$$log_{10}\; \sqrt{10} \;=\; x$$

(vii) Find x given that,

$$log_{10}\; 1 \;=\; x$$

(viii) Find x given that,

$$log_{8}\; 2 \;=\; x$$

(ix) Find x given that,

$$log_{5} \left(\frac{1}{125} \right) \;=\; x$$

[9 marks]

Question 6

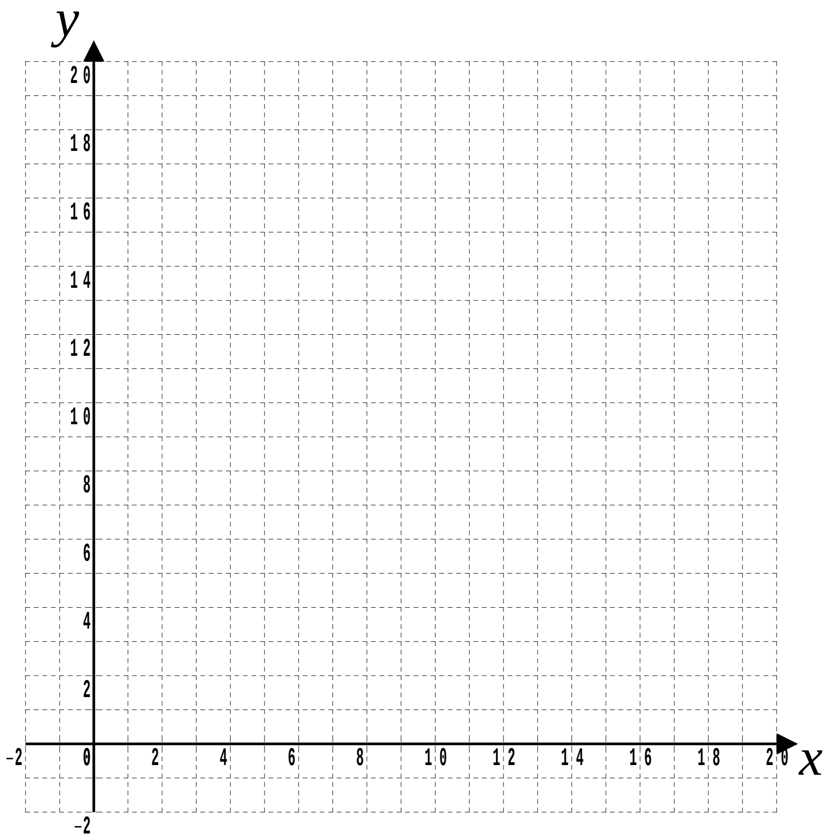
(i) Complete the table of values for the exponential function given by

$$y = 1.2^x$$

x	-2	-1	0	1	2	3	4	5	6	7	8	12	16
y				1.2	1.4		2.1						

[3 marks]

(ii) Graph the function, using the values from the completed table.



[2 marks]

(iii) Use your graph, evidencing your method, to find

$$1.2^{6.5}$$

[2 marks]

(iv) Use your graph, evidencing your method, to help you solve the equation

$$1.2^x = 12$$

[2 marks]

(v) Add the line $y = x$ to your graph.

[1 mark]

(vi) Reflect the exponential curve, plotted in part (ii), in the line $y = x$.

[1 mark]

This reflected curve is the inverse function.
The inverse of an exponential function is a logarithm function.

In this case, the exponential function;

$$y = 1.2^x$$

has as an inverse, the logarithm function;

$$y = log_{1.2}\, x$$

(vii) Use your reflection graph, evidencing your method, to find

$$log_{1.2}\, 8$$

[2 marks]

(viii) Use your reflection graph, evidencing your method, to find

$$log_{1.2}\, 18$$

[2 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

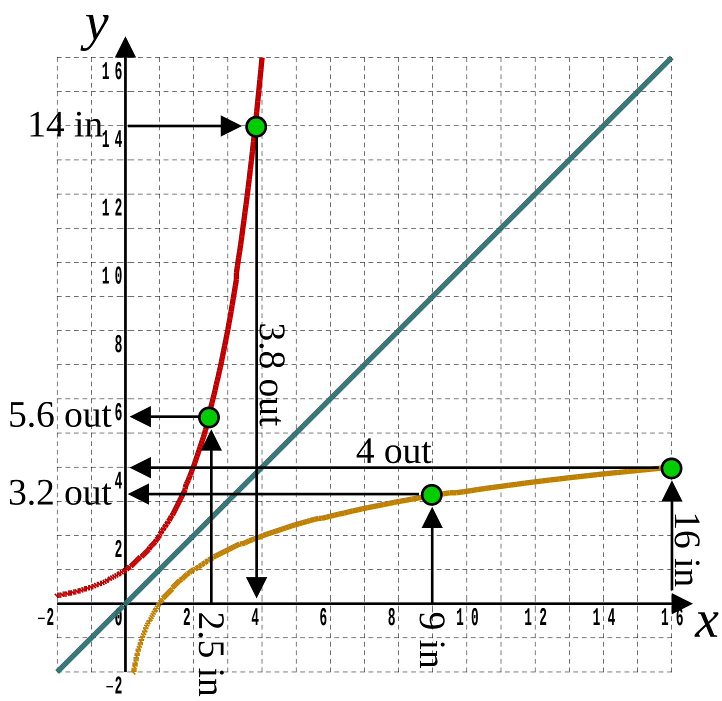
1.3 Answers

1.3.1 Solutions to 1.1 An Exponential Curve and its Inverse

(i)

x	-2	-1	0	1	2	3	4
y	0.25	0.5	1	2	4	8	16

[2 marks]



(ii) The red curve on the graph is plotted; $y = 2^x$

[2 marks]

(iii) 2.5 in gives 5.7 out on the red curve, $\therefore\ 2^{2.5} = 2.7$

[2 marks]

(iv) $2^x = 14$, 14 in gives 3.8 out on the red curve, $\therefore\ x = 3.8$

[2 marks]

(v) Line $y = x$ plotted accurately (in blue on the graph)

[1 mark]

(vi) Reflected curve plotted accurately (in gold on the graph)

[2 marks]

(vii) 9 in gives 3.2 out on the gold curve, $\therefore log_2 9 = 3.2$

[2 marks]

(viii) 16 in gives 4 out on the gold curve, $\therefore\ log_2 16 = 4$

[2 marks]

(ix) $log_2 64 = 6$

[1 mark]

1.3.2 Solutions to 1.2 Exercise

Answer 1

(i) 4 **(ii)** 5 **(iii)** 6 **(iv)** 3

[4 marks]

Answer 2

(i) -2 **(ii)** -3 **(iii)** -2 **(iv)** -3

[4 marks]

Answer 3

(i) 5 **(ii)** 3 **(iii)** 8 **(iv)** 81

[4 marks]

Answer 4

(i) 8 **(ii)** 100000

(iii) $\frac{1}{8}$ (or 0.125) **(iv)** 27

[4 marks]

Answer 5

(i) 4 **(ii)** 8 **(iii)** 1

(iv) -3 **(v)** -2 **(vi)** 0.5

(vii) 0 **(viii)** $\frac{1}{3}$ **(ix)** -3

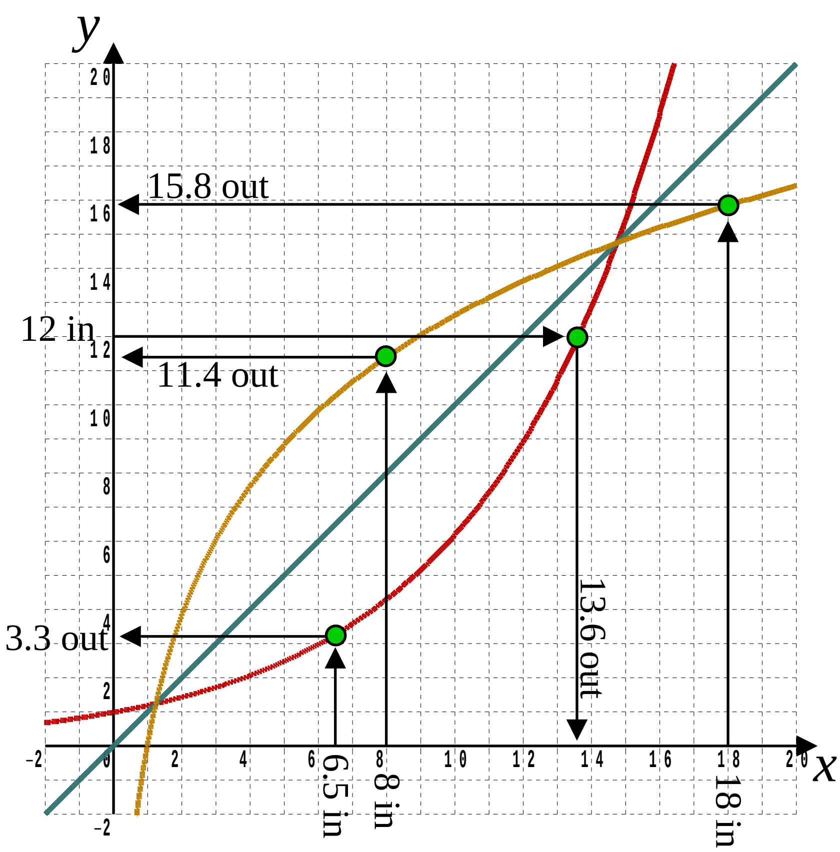
[9 marks]

Answer 6

(i)

x	-2	-1	0	1	2	3	4	5	6	7	8	12	16
y	0.7	0.8	1	1.2	1.4	1.7	2.1	2.5	3.0	3.6	4.3	8.9	18.5

[2 marks]



(ii) The red curve on the graph is plotted; $y = 1.2^x$

[2 marks]

(iii) 6.5 in gives 3. out on the red curve, $\therefore\ 1.2^{6.5} = 3.3$

[2 marks]

(iv) $1.2^x = 12$, 12 in gives 13.6 out on the red curve, $\therefore\ x = 13.6$

[2 marks]

(v) Line $y = x$ plotted accurately (in blue)

[1 mark]

(vi) Reflected curve plotted accurately (in gold)

[2 marks]

(vii) 8 in gives 11.4 out on the gold curve, $\therefore log_{1.2}\, 8 = 11.4$

[2 marks]

(viii) 18 in gives 15.8 out on the gold curve, $\therefore\ log_{1.2}\, 18 = 15.8$

[2 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

2.1 The Rules of Logs

The value of many "nice" logarithms can be determined immediately and without the use of a calculator by making use of the following rule;

The "Jump out of *logs*" Rule

$$\log_c a = b \quad \Leftrightarrow \quad c^b = a$$

Proof

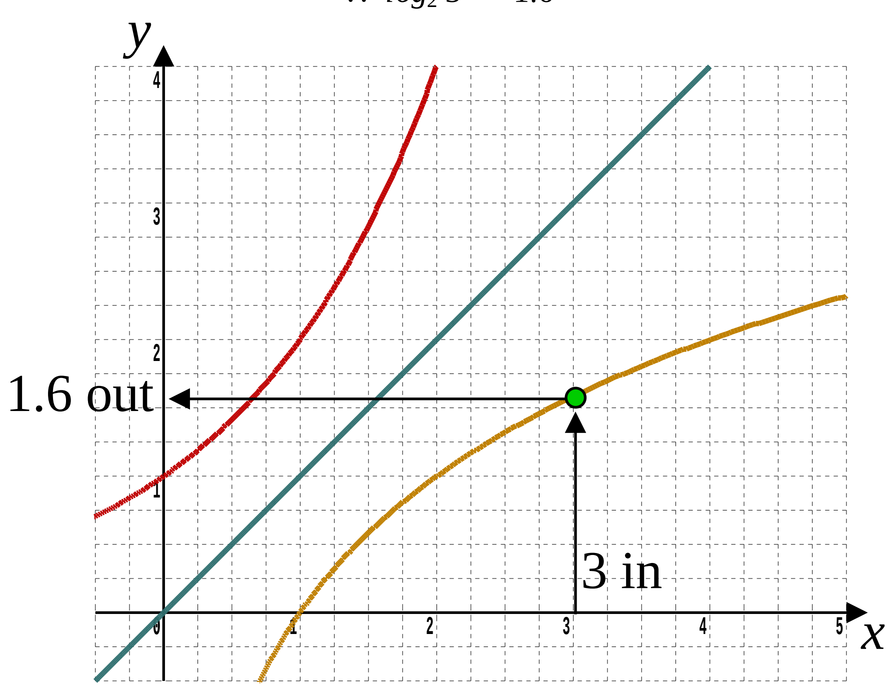
$$\log_c a = b$$
$$\therefore \quad c^{\log_c a} = c^b$$
$$a = c^b \qquad \square$$

For example, to determine the value of $\log_2 32$ the question translates into "2 to what power is 32 ?" which yields the answer 5 because $2^5 = 32$

For more "awkward" logarithms the use of a calculator can still be avoided and a value for the logarithm found by plotting a graph and reading off the required value. For example, to find the value of $\log_2 3$, plot $y = 2^x$ (in red) and then reflect in the line $y = x$ to obtain the graph of the inverse function $y = \log_2 x$.

Then go up from $x = 2$ to the reflection graph and across to 1.6

$$\therefore \log_2 3 = 1.6$$



Whilst drawing a graph aids understanding rather too long winded.
Using a calculator, you can determine a better approximate value of any logarithm.
For example, my Casio Classwiz fx-991ex calculator tells me that,

$$log_2 3 \ = \ 1.584962501...$$

This is an irrational number, like π and $\sqrt{2}$

With such numbers, if an exact answer is requested, do NOT move into decimals.

The exact value of $log_2 3$ is $log_2 3$ in the same sense that the exact value of π is π.

This in turn necessitates a need to be able to simplify expressions such as,

$log_2 3 \ + \ 2\ log_2 5$ without resorting to decimals.

(Decimals can only yield an approximation to the exact answer)

In this case $log_2 3 \ + \ 2\ log_2 5 \ = \ log_2 75$

To efficiently perform such calculations mathematicians have established a collection rules; “The Rules of Logs”

2.2.1 The First Rule

$$log_c (ab) \ = \ log_c a \ + \ log_c b$$

Proof : Let $log_c a \ = \ x$ and $log_c b \ = \ y$ in which case

$$c^x \ = \ a \quad \text{and} \quad c^y \ = \ b$$

by using the “Jump out of *logs* rule”

By the law of indices,

$$ab \ = \ c^x \times c^y \ \Leftrightarrow \ ab \ = \ c^{x+y}$$

Now, using the “Jump out of *logs* Rule” backwards,

$$log_c (ab) \ = \ x \ + \ y$$

Giving, $$log_c (ab) \ = \ log_c a \ + \ log_c b \qquad \square$$

Illustration: Verify that; $log_2 4 \ + \ log_2 16 \ = \ log_2 64$ where $4 \times 16 \ = \ 64$

2.2.2 The Second Rule

$$log_c\left(\frac{a}{b}\right) = log_c\, a - log_c\, b$$

Proof : Let $log_c\, a = x$ and $log_c\, b = y$ in which case

$$c^x = a \quad \text{and} \quad c^y = b$$

by using the "Jump out of *logs* rule"

By the laws of indices,

$$\frac{a}{b} = \frac{c^x}{c^y} \Leftrightarrow \frac{a}{b} = c^{x-y}$$

Now, using the "Jump out of *logs* Rule" backwards,

$$log_c\left(\frac{a}{b}\right) = x - y$$

Giving, $$log_c\left(\frac{a}{b}\right) = log_c\, a - log_c\, b \qquad \square$$

Illustration: Verify that, $log_3 81 - log_3 27 = log_3 3$ where $\frac{81}{27} = 3$

2.2.3 The Third Rule

$$log_c\, a^n = n\, log_c\, a$$

Proof : Let $log_c\, a = x$ in which case $c^x = a$ ("Jump out of *logs* rule")

By the laws of indices,

$$a^n = (c^x)^n \quad \Leftrightarrow \quad a^n = c^{nx}$$

And so, $log_c\, a^n = n\,x$ ("Jump out of logs rule" backwards)

Giving, $log_c\, a^n = n\, log_c\, a$ $\quad \square$

Illustration: Verify that, $log_2 8^2 = 2\, log_2 8$

2.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available: 25

Question 1

Use the rules of logs to prove that,

$$log_2 3 + 2\, log_2 5 = log_2 75$$

[3 marks]

Question 2

Use the rules of logs to prove that,

$$log_7 45 - 2\, log_7 3 = log_7 5$$

[3 marks]

Question 3

Use the rules of logs to prove that,

$$3\, log_5 3 - 2\, log_5 6 + 2\, log_5 2 = log_5 3$$

[4 marks]

Question 4

A-Level Examination Question from May 2006, paper C2, Q3 (Edexcel)

(i) Write down the value of

$$log_6 36$$

[1 mark]

(ii) Express $2\, log_a 3 + log_a 11$ as a single logarithm to base a.

[3 marks]

Question 5

A-Level Examination Question from January 2008, paper C2, Q5 (Edexcel)

Given that a and b are positive constants, solve the simultaneous equations

$$a = 3b$$

$$log_3 a + log_3 b = 2$$

Give your answers as exact numbers.

[6 marks]

Question 6

(i) Use the rules of logs to prove that,

$$3\,log_3 x - log_3 7x + log_3 28 = 2\,log_3 2x$$

[3 marks]

(ii) Hence, or otherwise, prove that,

$$3\,log_2 x - log_2 7x + log_2 28 = 2\,(\,1 + log_2 x\,)$$

[2 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

2.4 Answers

2.4.1 Solutions to 2.4 Exercise

Answer 1

Statement: $log_2 3 + 2\, log_2 5 = log_2 75$

Proof

$$\begin{aligned} \text{LHS} &= log_2 3 + 2\, log_2 5 \\ &= log_2 3 + log_2 5^2 \\ &= log_2 \left(3 \times 5^2\right) \\ &= log_2 75 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

Answer 2

Statement: $log_7 45 - 2\, log_7 3 = log_7 5$

Proof

$$\begin{aligned} \text{LHS} &= log_7 45 - 2\, log_7 3 \\ &= log_7 45 - log_7 3^2 \\ &= log_7 \left(\frac{45}{3^2} \right) \\ &= log_7 5 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

Answer 3

Statement: $3\, log_5 3 - 2\, log_5 6 + 2\, log_5 2 = log_5 3$

Proof

$$\begin{aligned} \text{LHS} &= 3\, log_5 3 - 2\, log_5 6 + 2\, log_5 2 \\ &= log_5 3^3 - log_5 6^2 + log_5 2^2 \\ &= log_5 \left(\frac{3^3 \times 2^2}{6^2} \right) \\ &= log_5 3 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[4 marks]

Answer 4

(i) 2

[1 mark]

(ii)

$$2\,log_a 3 + log_a 11 = log_a 3^2 + log_a 11$$
$$= log_a \left(3^2 \times 11 \right)$$
$$= log_a 99$$

[3 marks]

Answer 5

$$log_3 a + log_3 b = 2$$
$$log_3 (3b) + log_3 b = 2 \qquad \text{as } a = 3b$$
$$log_3 \left(3b \times b \right) = 2$$
$$log_3 \left(3\,b^2 \right) = 2$$
$$3^2 = 3\,b^2 \qquad \text{Jump out of } \textit{logs} \text{ rule}$$
$$3\,b^2 = 9$$
$$b^2 = 3$$
$$b = \pm\sqrt{3}$$
$$b = \sqrt{3} \qquad \text{as told } b \text{ is positive}$$
$$\therefore\ a = 3\sqrt{3} \text{ and } b = \sqrt{3}$$

[6 marks]

Answer 6

(i) Statement: $3\,log_3\,x - log_3\,7x + log_3\,28 = 2\,log_3\,2x$

Proof

$$
\begin{aligned}
\text{LHS} &= 3\,log_3\,x - log_3\,7x + log_3\,28 \\
&= log_3\,x^3 - log_3(7x) + log_3\,28 \\
&= log_3\left(\frac{28\,x^3}{7x}\right) \\
&= log_3\left(4\,x^2\right) \\
&= log_3\,(2x)^2 \\
&= 2\,log_3\,2x \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[3 marks]

(ii) Statement: $3\,log_2\,x - log_2\,7x + log_2\,28 = 2\,(1 + log_2\,x)$

Proof

$$
\begin{aligned}
\text{LHS} &= 3\,log_2\,x - log_2\,7x + log_2\,28 \\
&= 2\,log_2\,2x \qquad \text{from part (i)} \\
&= 2\left(log_2\,2x\right) \\
&= 2\left(log_2\,2 + log_2\,x\right) \\
&= 2\left(1 + log_2\,x\right) \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[2 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

Lesson 3

3.1 Consolidation Exercise (Non Calculator)

Any solution based entirely on graphical or numerical methods is not acceptable
Marks Available : 70

Question 1
Solve these equations;

(i) $5^x = 125$

(ii) $2^x = \dfrac{1}{8}$

(iii) $9^x = 3$

(iv) $17^x = 1$

(v) $(\sqrt{2})^x = 8$

(vi) $27^x = \dfrac{1}{9}$

[6 marks]

Question 2
Determine the value of;

(i) 5^{-2}

(ii) $8^{\frac{2}{3}}$

(iii) 13^0

(iv) $9^{-\frac{3}{2}}$

(v) $(\sqrt{6})^4$

(vi) $100000000^{0.25}$

[6 marks]

Question 3
Solve these equations;

(i) $log_4 64 = x$ **(ii)** $log_3 \left(\frac{1}{81}\right) = x$

(iii) $log_9 27 = x$ **(iv)** $log_7 7 = x$

(v) $log_8 1 = x$ **(vi)** $log_{100} 10 = x$

[6 marks]

Question 4
Solve these equations;

(i) $log_x 8 = 3$ **(ii)** $log_x 7 = 0.5$

(iii) $log_x \left(\frac{1}{9}\right) = -1$ **(iv)** $log_x 1000 = 3$

(v) $log_x 25 = \frac{2}{3}$ **(vi)** $log_x 81 = 4$

[6 marks]

Question 5

Solve these equations;

(i) $log_2 x = 5$

(ii) $log_5 x = -1$

(iii) $log_8 x = \frac{5}{3}$

(iv) $log_3 x = -3$

(v) $log_{0.5} x = 4$

(vi) $log_{\sqrt{10}} x = -4$

[6 marks]

Question 6

(i) Use the rules of logs to prove that,

$$3\, log_x 8 - 2\, log_x 4 = log_x 32$$

[3 marks]

(ii) Hence, or otherwise, determine the integer value of,

$$3\, log_2 8 - 2\, log_2 4$$

[2 marks]

Question 7

Use the rules of logs to prove that,

$$2\,log_7 2 \;+\; 3\,log_7 5 \;=\; log_7 500$$

[4 marks]

Question 8

Given that *a* and *b* are positive constants, solve the simultaneous equations

$$a \;=\; 8b$$

$$log_2 a \;+\; 2\,log_2 b \;=\; 4$$

Give your answers as exact numbers.

[6 marks]

Question 9

AS-Level Examination Question from May 2018, Paper 1, Q5 (Edexcel)

A student's attempt to solve the equation $2\,log_2 x - log_2 \sqrt{x} = 3$ is as follows;

$$2\,log_2 x - log_2 \sqrt{x} = 3$$

$$2\,log_2 \left(\frac{x}{\sqrt{x}} \right) = 3 \qquad \text{using the subtraction law for logs}$$

$$2\,log_2 \left(\sqrt{x} \right) = 3 \qquad \text{simplifying}$$

$$log_2 x = 3 \qquad \text{using the power law for logs}$$

$$x = 3^2 \qquad \text{using the definition of a log}$$

$$x = 9$$

(a) Identify two errors made by this student, giving a brief explanation of each.

[2 marks]

(b) Write out the correct solution.

[3 marks]

Question 10

(i) Write down the value of $log_{36} 6$

[1 mark]

(ii) Express $3\,log_a 2 + log_a 13$ as a single logarithm to base a.

[3 marks]

Question 11

A-Level Examination Question from 2018, Specimen Paper 1, Q5 (Edexcel)

$$f(x) = x^3 + ax^2 - ax + 48, \text{ where } a \text{ is a constant}$$

Given that $f(-6) = 0$,

(a) **(i)** show that $a = 4$

[2 marks]

(ii) express $f(x)$ as a product of two algebraic factors.

[2 marks]

Given that $2\log_2(x+2) + \log_2 x - \log_2(x-6) = 3$

(b) show that $x^3 + 4x^2 - 4x + 48 = 0$

[4 marks]

(c) hence explain why

$$2\log_2(x+2) + \log_2 x - \log_2(x-6) = 3$$

has no real roots.

[2 marks]

Question 12

(i) Use the rules of logs to prove that,

$$5\,log_3\,x\ -\ 2\,log_3\,3x\ +\ log_3\,72\ =\ 3\,log_3\,2x$$

[4 marks]

(ii) Hence, or otherwise, prove that,

$$5\,log_2\,x\ -\ 2\,log_2\,3x\ +\ log_2\,72\ =\ 3\ (1\ +\ log_2\,x)$$

[2 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

3.2 Answers

3.2.1 Solutions (3.1 Exercise)

Answer 1

(i)	3	**(ii)**	-3
(iii)	0.5	**(iv)**	0
(v)	6	**(vi)**	$-\dfrac{2}{3}$

[6 marks]

Answer 2

(i)	$\dfrac{1}{25}$	**(ii)**	4
(iii)	1	**(iv)**	$\dfrac{1}{27}$
(v)	36	**(vi)**	100

[6 marks]

Answer 3

(i)	3	**(ii)**	-4
(iii)	$\dfrac{3}{2}$	**(iv)**	1
(v)	0	**(vi)**	$\dfrac{1}{2}$

[6 marks]

Answer 4

(i)	2	**(ii)**	49
(iii)	9	**(iv)**	10
(v)	125	**(vi)**	3

[6 marks]

Answer 5

(i)	32	**(ii)**	$\dfrac{1}{5}$
(iii)	32	**(iv)**	$\dfrac{1}{27}$
(v)	$\dfrac{1}{16}$	**(vi)**	$\dfrac{1}{100}$

[6 marks]

Answer 6

(i) $$3\,log_x 8 - 2\,log_x 4 = log_x 32$$

METHOD 1 :
$$\begin{aligned} \text{LHS} &= 3\,log_x 8 - 2\,log_x 4 \\ &= log_x 8^3 - log_x 4^2 \\ &= log_x\left(\frac{8^3}{4^2}\right) \\ &= log_x\left(\frac{2^9}{2^4}\right) \\ &= log_x 2^5 \\ &= log_x 32 \\ &= \text{RHS} \qquad \square \end{aligned}$$

METHOD 2 :
$$\begin{aligned} \text{LHS} &= 3\,log_x 8 - 2\,log_x 4 \\ &= 3\,log_x 2^3 - 2\,log_x 2^2 \\ &= 9\,log_x 2 - 4\,log_x 2 \\ &= 5\,log_x 2 \\ &= log_x 2^5 \\ &= log_x 32 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

(ii) $3\,log_2 8 - 2\,log_2 4 = log_2 32$ using the part (i) result

$$\Leftrightarrow 2^x = 32$$

$$\therefore\ x = 5$$

[2 marks]

Answer 7

$$2\,log_7 2 + 3\,log_7 5 = log_7 500$$

$$\begin{aligned} \text{LHS} &= 2\,log_7 2 + 3\,log_7 5 \\ &= log_7 2^2 + log_7 5^3 \\ &= log_7(4 \times 125) \\ &= log_7 500 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[4 marks]

Answer 8

Substitute $a = 8b$ into $log_2 a + 2\,log_2 b = 4$

$$\therefore \; log_2\,(8b) + 2\,log_2 b = 4$$
$$log_2 8 + log_2 b + 2\,log_2 b = 4$$
$$3 + 3\,log_2 b = 4$$
$$log_2 b = \frac{1}{3}$$
$$\Leftrightarrow \; 2^{\frac{1}{3}} = b$$

That is, $b = \sqrt[3]{2}$ and $a = 8\sqrt[3]{2}$

[6 marks]

Answer 9

(a) Whilst it is true that $log\,A + log\,B = log(AB)$,

it does not follow that $2\,log\,A + log\,B = 2\,log(AB)$ which is the incorrect deduction made by the student in the first step in their solution.

Whilst it is true that $log_a b = c \Leftrightarrow a^c = b$ the student has become muddled and claimed incorrectly that $log_a b = c \Leftrightarrow c^a = b$ in the last step in their solution.

(b)

$$2\,log_2 x - log_2 \sqrt{x} = 3$$
$$log_2 x^2 - log_2 \sqrt{x} = 3$$
$$log_2\left(\frac{x^2}{x^{\frac{1}{2}}}\right) = 3$$
$$log_2 x^{\frac{3}{2}} = 3$$
$$\frac{3}{2}\,log_2 x = 3$$
$$log_2 x = 2$$
$$x = 2^2 = 4$$

[3 marks]

Answer 10

(i) 0.5

[1 mark]

(ii)

$$3\,log_a 2 + log_a 13 = log_a 2^3 + log_a 13$$
$$= log_a(8 \times 13)$$
$$= log_a 104$$

[3 marks]

Answer 11

$$f(x) = x^3 + ax^2 - ax + 48$$

(a) **(i)**

$$f(-6) = 0$$

$$(-6)^3 + a(-6)^2 - a(-6) + 48 = 0$$

$$-216 + 36a + 6a + 48 = 0$$

$$42a = 168$$

$$a = 4 \quad \square$$

[2 marks]

(ii)

$$\begin{array}{r|l} & x^2 - 2x + 8 \\ \hline x + 6 & x^3 + 4x^2 - 4x + 48 \\ & \underline{x^3 + 6x^2} \\ & \quad -2x^2 - 4x \\ & \quad \underline{-2x^2 - 12x} \\ & \qquad\qquad 8x + 48 \\ & \qquad\qquad \underline{8x + 48} \\ & \qquad\qquad\qquad 0 \end{array}$$

$0 \leftarrow$ As expected !

$$x^3 + 4x^2 - 4x + 48 = (x + 6)(x^2 - 2x + 8)$$

[2 marks]

(b)

$$2\, log_2(x + 2) + log_2 x - log_2(x - 6) = 3$$

$$log_2 (x + 2)^2 + log_2 x - log_2 (x - 6) = 3$$

$$log_2 \left(\frac{(x + 2)^2 x}{(x - 6)} \right) = 3$$

$$\frac{(x + 2)^2 x}{x - 6} = 2^3$$

$$x(x + 2)^2 = 8(x - 6)$$

$$x^3 + 4x^2 + 4x - 8x + 48 = 0$$

$$x^3 + 4x^2 - 4x + 48 = 0 \quad \square$$

[4 marks]

(c) From parts (a) and (b),

$$2\, log_2 (\, x + 2\,) + log_2\, x - log_2 (\, x - 6\,) = 3$$

$$\Leftrightarrow\; x^3 + 4x^2 - 4x + 48 = 0$$

$$\Leftrightarrow\; (\, x + 6\,)\left(x^2 - 2x + 8 \right) = 0$$

As the discriminant of $x^2 - 2x + 8$ is negative...

$$\ldots\; (-2)^2 - 4 \times 1 \times 8 = -28 \ldots$$

... the only possible root is $x = -6$ but this is immediately rejected

because the logarithms of negative numbers do not exist.

That is, $2\, log_2 (\, x + 2\,) + log_2\, x - log_2 (\, x - 6\,) = 3$ has no real roots.

□

[2 marks]

Answer 12

$$5\, log_3\, x - 2\, log_3\, 3x + log_3\, 72 = 3\, log_3\, 2x$$

(i)

$$\begin{aligned} \text{LHS} &= 5\, log_3\, x - 2\, log_3\, 3x + log_3\, 72 \\ &= log_3\, x^5 - log_3\, (3x)^2 + log_3 (\, 9 \times 8\,) \\ &= log_3 \left(\frac{x^5 \times 9 \times 8}{3^2\, x^2} \right) \\ &= log_3 \left(8x^3 \right) \\ &= log_3\, (2x)^3 \\ &= 3\, log_3\, 2x \\ &= \text{RHS} \qquad \square \end{aligned}$$

[4 marks]

(ii) From part (i),

$$\begin{aligned} 5\, log_2\, x - 2\, log_2\, 3x + log_2\, 72 &= 3\, log_2\, 2x \\ &= 3 \left(log_2\, 2 + log_2\, x \right) \\ &= 3 \left(1 + log_2\, x \right) \end{aligned}$$

[2 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

4.1 Taking the *log* of both sides

Given an equation, the "golden rule" of algebra is to "do the same to both sides". Taking the *log* (of any base) of both sides is sometimes the key to solving an equation that has x as an index.

4.2 A Worked Example

Find the solution to the equation $2^{x+1} = 5^x$

Give your answer accurate to three decimal places.

[4 marks]

Solution :

$$2^{x+1} = 5^x$$

Take the natural logarithm of both sides

$$\ln 2^{x+1} = \ln 5^x$$

$$(x + 1) \ln 2 = x \ln 5$$

$$x \ln 2 + \ln 2 = x \ln 5$$

$$x \ln 5 - x \ln 2 = \ln 2$$

$$x(\ln 5 - \ln 2) = \ln 2$$

$$x = \frac{\ln 2}{\ln 5 - \ln 2}$$

$$x = 0.756 \quad \text{to three decimal places}$$

[4 marks]

Note : It would be fine to take either log_2 or log_5 of both sides.
Indeed, that would save a little working.
The standard approach is to use *ln* as there is a handy special calculator button that avoids having to type in a base.

Note : $ln = log_e$ where $e = 2.718...$ which is an irrational number like π or $\sqrt{2}$

Note : On some calculators there may be a button marked simply *log*

This button will actually be log_{10}

Note : In some textbooks, including older A-Level texts, log_{10} may be written *lg*

4.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1
Solve this equation, giving your answer to three decimal places,

$$2^{x+5} = 3^x$$

[4 marks]

Question 2
By first using a Law of Indices, or otherwise, solve the equation,

$$3^{4x} \times 3^{5-x} = 2^x$$

Give your answer to three significant figures.

[4 marks]

Question 3

Find the solution to the equation,

$$\frac{1}{3^{2x-1}} = 4^x$$

Give your answer accurate to three decimal places.

[4 marks]

Question 4

Solve, correct to 3 decimal places, the equation

$$3^x - 5 = 0$$

[3 marks]

Question 5

(i) Sketch the curve

$$y = 2^x$$

[2 marks]

(ii) Hence or otherwise, discuss trying to solve the equation

$$2^x = -0.5$$

[2 marks]

Question 6

Use the fact that, $ln\left(5 \times 3^x\right) = ln\,5 + ln\,3^x$ to assist in solving the equation,

$$5 \times 3^x = 4^{3x}$$

Give your answer accurate to three decimal places.

[4 marks]

Question 7
Solve the equation,

$$3 \times 5^{x+2} = 8^x$$

Give your answer accurate to three decimal places.

[4 marks]

Question 8
(i) Solve,

$$log_2 x = 6$$

[1 mark]

(ii) Solve,

$$log_5 x = 0.5$$

Give your answer to three decimal places.

[1 mark]

(iii) Solve,

$$log_2 x = -4$$

Give the exact answer.

[1 mark]

Question 9
Solve this equation, giving your answer to three decimal places,

$$3^{2x+5} = 4^{x-2}$$

[5 marks]

Question 10
Solve the equation,

$$4 \times 3^{2x+1} = 5^{x}$$

Give your answer accurate to three decimal places.

[5 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

4.4 Answers

4.4.1 Solutions to 4.3 Exercise

Answer 1

$$\begin{aligned}
2^{x+5} &= 3^x \\
\ln 2^{x+5} &= \ln 3^x \\
(x+5)\ln 2 &= x\ln 3 \\
x\ln 2 + 5\ln 2 &= x\ln 3 \\
x\ln 3 - x\ln 2 &= 5\ln 2 \\
x(\ln 3 - \ln 2) &= 5\ln 2 \\
x &= \frac{5\ln 2}{\ln 3 - \ln 2} \\
x &= 8.547
\end{aligned}$$

[4 marks]

Answer 2

$$\begin{aligned}
3^{4x} \times 3^{5-x} &= 2^x \\
3^{3x+5} &= 2^x \\
\ln 3^{3x+5} &= \ln 2^x \\
(3x+5)\ln 3 &= x\ln 2 \\
3x\ln 3 + 5\ln 3 &= x\ln 2 \\
x\ln 2 - 3x\ln 3 &= 5\ln 3 \\
x(\ln 2 - 3\ln 3) &= 5\ln 3 \\
x &= \frac{5\ln 3}{\ln 2 - 3\ln 3} \\
x &= -2.11
\end{aligned}$$

[4 marks]

Answer 3

$$\frac{1}{3^{2x-1}} = 4^x$$

$$3^{1-2x} = 4^x$$

$$ln\, 3^{1-2x} = ln\, 4^x$$

$$(1 - 2x)\, ln\, 3 = x\, ln\, 4$$

$$ln\, 3 - 2x\, ln\, 3 = x\, ln\, 4$$

$$x\, ln\, 4 + 2x\, ln\, 3 = ln\, 3$$

$$x\,(ln\, 4 + 2\, ln\, 3) = ln\, 3$$

$$x = \frac{ln\, 3}{ln\, 4 + 2\, ln\, 3}$$

$$x = 0.307$$

[4 marks]

Answer 4

$$3^x - 5 = 0$$

$$3^x = 5$$

$$ln\, 3^x = ln\, 5$$

$$x\, ln\, 3 = ln\, 5$$

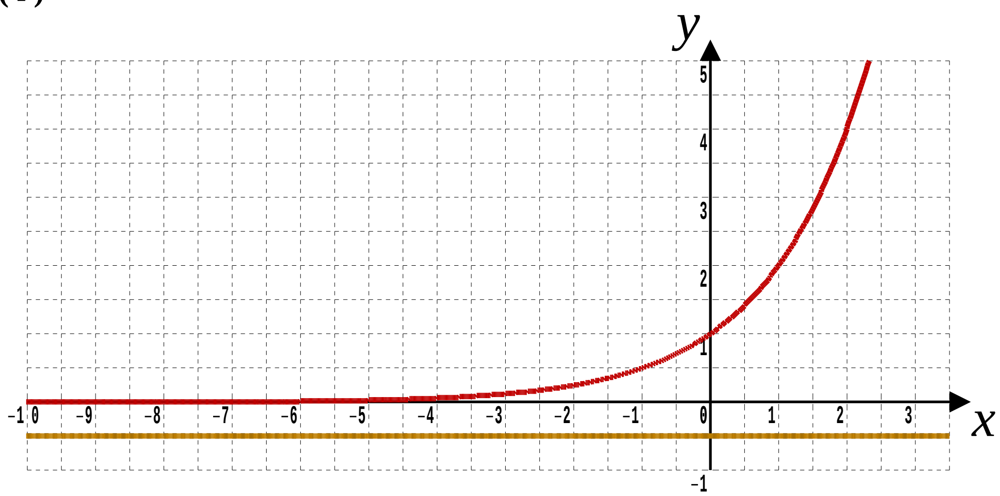
$$x = \frac{ln\, 5}{ln\, 3}$$

$$= 1.465$$

[3 marks]

Answer 5

(i)



Red: $y = 2^x$ Gold: $y = -0.5$

(ii) As the two graphs never intersect the equation $2^x = -0.5$ has no solutions.

[2, 2 marks]

Answer 6

$$\begin{aligned} 5 \times 3^x &= 4^{3x} \\ \ln(5 \times 3^x) &= \ln 4^{3x} \\ \ln 5 + \ln 3^x &= \ln 4^{3x} \\ \ln 5 + x \ln 3 &= 3x \ln 4 \\ 3x \ln 4 - x \ln 3 &= \ln 5 \\ x(3 \ln 4 - \ln 3) &= \ln 5 \\ x &= \frac{\ln 5}{3 \ln 4 - \ln 3} \\ &= 0.526 \end{aligned}$$

[4 marks]

Answer 7

$$\begin{aligned} 3 \times 5^{x+2} &= 8^x \\ \ln(3 \times 5^{x+2}) &= \ln 8^x \\ \ln 3 + \ln 5^{x+2} &= \ln 8^x \\ \ln 3 + (x + 2) \ln 5 &= x \ln 8 \\ \ln 3 + x \ln 5 + 2 \ln 5 &= x \ln 8 \\ x \ln 8 - x \ln 5 &= \ln 3 + 2 \ln 5 \\ x(\ln 8 - \ln 5) &= \ln 3 + 2 \ln 5 \\ x &= \frac{\ln 3 + 2 \ln 5}{\ln 8 - \ln 5} \\ &= 9.186 \end{aligned}$$

[4 marks]

Answer 8

(i) $\log_2 x = 6 \Leftrightarrow x = 2^6 \Leftrightarrow x = 64$

[1 mark]

(ii) $\log_5 x = 0.5 \Leftrightarrow x = 5^{0.5} \Leftrightarrow x = 2.236$

[1 mark]

(iii) $\log_2 x = -4 \Leftrightarrow x = 2^{-4} \Leftrightarrow x = \frac{1}{16}$

[1 mark]

Answer 9

$$
\begin{aligned}
3^{2x+5} &= 4^{x-2} \\
\ln 3^{2x+5} &= \ln 4^{x-2} \\
(2x + 5) \ln 3 &= (x - 2) \ln 4 \\
2x \ln 3 + 5 \ln 3 &= x \ln 4 - 2 \ln 4 \\
x \ln 4 - 2x \ln 3 &= 5 \ln 3 + 2 \ln 4 \\
x(\ln 4 - 2 \ln 3) &= 5 \ln 3 + 2 \ln 4 \\
x &= \frac{5 \ln 3 + 2 \ln 4}{\ln 4 - 2 \ln 3} \\
&= -10.193
\end{aligned}
$$

[5 marks]

Answer 10

$$
\begin{aligned}
4 \times 3^{2x+1} &= 5^x \\
\ln\left(4 \times 3^{2x+1}\right) &= \ln 5^x \\
\ln 4 + \ln 3^{2x+1} &= \ln 5^x \\
\ln 4 + (2x + 1) \ln 3 &= x \ln 5 \\
\ln 4 + 2x \ln 3 + \ln 3 &= x \ln 5 \\
x \ln 5 - 2x \ln 3 &= \ln 4 + \ln 3 \\
x(\ln 5 - 2 \ln 3) &= \ln 4 + \ln 3 \\
x &= \frac{\ln 4 + \ln 3}{\ln 5 - 2 \ln 3} \\
&= -4.228
\end{aligned}
$$

[5 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

4.5 Homework

A-Level Pure Mathematics : Year 1
Exponentials and Logarithms

Any solution based entirely on graphical or numerical methods is not acceptable
Marks Available : 40

Question 1
Solve this equation, giving your answer to three decimal places,

$$3^{x-7} = 2^x$$

[4 marks]

Question 2
By first using a Law of Indices, or otherwise, solve the equation,

$$2^{5x} \times 2^{2x-3} = 3^x$$

Give your answer to three significant figures.

[4 marks]

Question 3

Find the solution to the equation,

$$\frac{1}{5^{3x-1}} = 7^x$$

Give your answer accurate to three decimal places.

[4 marks]

Question 4

Solve, correct to 3 decimal places, the equation

$$5^x - 3 = 0$$

[3 marks]

Question 5

(i) Sketch the curve

$$y = 3^x$$

[2 marks]

(ii) Hence or otherwise, discuss trying to solve the equation

$$3^x = -1$$

[2 marks]

Question 6

Use the fact that $ln\left(4 \times 3^x\right) = ln\,4 + ln\,3^x$ to assist in finding the solution to,

$$4 \times 3^x = 5^{3x}$$

Give your answer accurate to three decimal places.

[4 marks]

Question 7

Solve the equation,

$$3 \times 7^{x+2} = 4^x$$

Give your answer accurate to three decimal places.

[4 marks]

Question 8

(i) Solve,

$$log_3 x = 4$$

[1 mark]

(ii) Solve,

$$log_7 x = 0.25$$

Give your answer to three decimal places.

[1 mark]

(iii) Solve,

$$log_2 x = -5$$

Give the exact answer.

[1 mark]

Question 9

Solve this equation, giving your answer to three decimal places,

$$5^{4x+7} = 2^{x-3}$$

[5 marks]

Question 10

Solve the equation,

$$6 \times 5^{2x+1} = 7^{x}$$

Give your answer accurate to three decimal places.

[5 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

4.6 Answers

4.6.1 Solutions to 4.5 Exercise

Answer 1

$$\begin{aligned} 3^{x-7} &= 2^{x} \\ \ln 3^{x-7} &= \ln 2^{x} \\ (x - 7)\ln 3 &= x \ln 2 \\ x \ln 3 - 7 \ln 3 &= x \ln 2 \\ x \ln 3 - x \ln 2 &= 7 \ln 3 \\ x(\ln 3 - \ln 2) &= 7 \ln 3 \\ x &= \frac{7 \ln 3}{\ln 3 - \ln 2} \\ &= 18.967 \end{aligned}$$

[4 marks]

Answer 2

$$\begin{aligned} 2^{5x} \times 2^{2x-3} &= 3^{x} \\ 2^{7x-3} &= 3^{x} \\ \ln 2^{7x-3} &= \ln 3^{x} \\ (7x - 3)\ln 2 &= x \ln 3 \\ 7x \ln 2 - 3 \ln 2 &= x \ln 3 \\ 7x \ln 2 - x \ln 3 &= 3 \ln 2 \\ x(7 \ln 2 - \ln 3) &= 3 \ln 2 \\ x &= \frac{3 \ln 2}{7 \ln 2 - \ln 3} \\ &= 0.554 \end{aligned}$$

[4 marks]

Answer 3

$$\frac{1}{5^{3x-1}} = 7^x \quad \Leftrightarrow \quad 5^{1-3x} = 7^x$$

$$\ln 5^{1-3x} = \ln 7^x$$

$$(1 - 3x) \ln 5 = x \ln 7$$

$$\ln 5 - 3x \ln 5 = x \ln 7$$

$$x \ln 7 + 3x \ln 5 = \ln 5$$

$$x(\ln 7 + 3 \ln 5) = \ln 5$$

$$x = \frac{\ln 5}{\ln 7 + 3 \ln 5}$$

$$= 0.238$$

[4 marks]

Answer 4

$$5^x - 3 = 0 \quad \Leftrightarrow \quad 5^x = 3$$

$$\ln 5^x = \ln 3$$

$$x \ln 5 = \ln 3$$

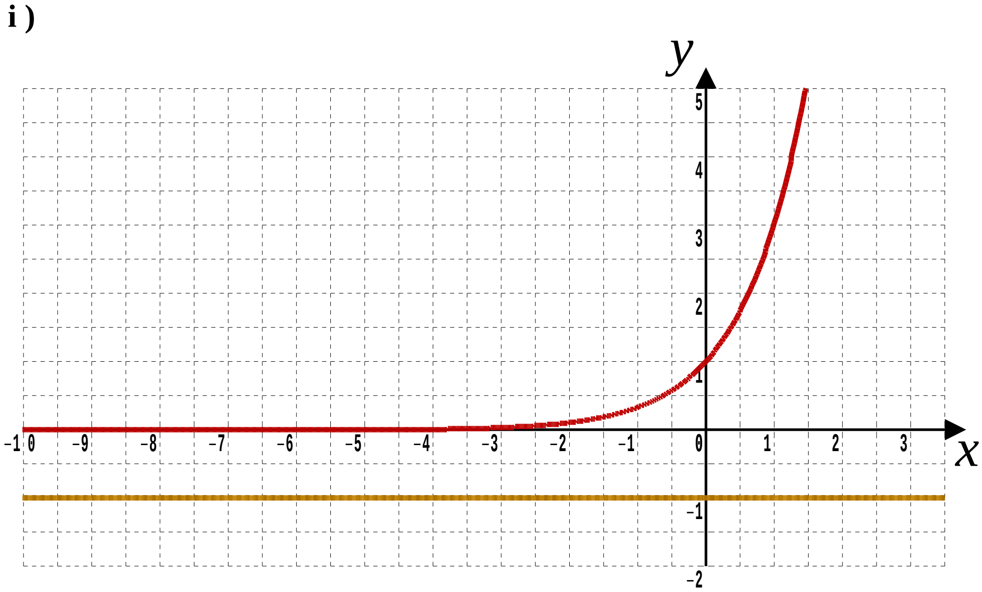
$$x = \frac{\ln 3}{\ln 5}$$

$$= 0.683$$

[3 marks]

Answer 5

(i)



Red: $y = 3^x$ Gold: $y = -1$

(ii) As the two graphs never intersect the equation $3^x = -1$ has no solutions.

[2, 2 marks]

Answer 6

$$\begin{aligned} 4 \times 3^{x} &= 5^{3x} \\ ln\left(4 \times 3^{x} \right) &= ln\, 5^{3x} \\ ln\, 4 + ln\, 3^{x} &= ln\, 5^{3x} \\ ln\, 4 + x\, ln\, 3 &= 3x\, ln\, 5 \\ 3x\, ln\, 5 - x\, ln\, 3 &= ln\, 4 \\ x\, (3\, ln\, 5 - ln\, 3) &= ln\, 4 \\ x &= \frac{ln\, 4}{3\, ln\, 5 - ln\, 3} \\ &= 0.372 \end{aligned}$$

[4 marks]

Answer 7

$$\begin{aligned} 3 \times 7^{x+2} &= 4^{x} \\ ln\left(3 \times 7^{x+2} \right) &= ln\, 4^{x} \\ ln\, 3 + ln\, 7^{x+2} &= ln\, 4^{x} \\ ln\, 3 + (x + 2)\, ln\, 7 &= x\, ln\, 4 \\ ln\, 3 + x\, ln\, 7 + 2\, ln\, 7 &= x\, ln\, 4 \\ x\, ln\, 4 - x\, ln\, 7 &= ln\, 3 + 2\, ln\, 7 \\ x\, (ln\, 4 - ln\, 7) &= ln\, 3 + 2\, ln\, 7 \\ x &= \frac{ln\, 3 + 2\, ln\, 7}{ln\, 4 - ln\, 7} \\ &= -8.918 \end{aligned}$$

[4 marks]

Answer 8

(i) $log_3\, x = 4 \Leftrightarrow x = 3^4 \Leftrightarrow x = 81$

[1 mark]

(ii) $log_7\, x = 0.25 \Leftrightarrow x = 7^{0.25} \Leftrightarrow x = 1.627$

[1 mark]

(iii) $log_2\, x = -5 \Leftrightarrow x = 2^{-5} \Leftrightarrow x = \dfrac{1}{32}$

[1 mark]

Answer 9

$$\begin{aligned}
5^{4x+7} &= 2^{x-3} \\
ln\, 5^{4x+7} &= ln\, 2^{x-3} \\
(4x + 7)\, ln\, 5 &= (x - 3)\, ln\, 2 \\
4x\, ln\, 5 + 7\, ln\, 5 &= x\, ln\, 2 - 3\, ln\, 2 \\
x\, ln\, 2 - 4x\, ln\, 5 &= 7\, ln 5 + 3\, ln\, 2 \\
x(\, ln\, 2 - 4\, ln\, 5\,) &= 7\, ln\, 5 + 3\, ln\, 2 \\
x &= \frac{7\, ln\, 5 + 3\, ln\, 2}{ln\, 2 - 4\, ln\, 5} \\
&= -2.323
\end{aligned}$$

[5 marks]

Answer 10

$$\begin{aligned}
6 \times 5^{2x+1} &= 7^x \\
ln\left(6 \times 5^{2x+1}\right) &= ln\, 7^x \\
ln\, 6 + ln\, 5^{2x+1} &= ln\, 7^x \\
ln\, 6 + (2x + 1)\, ln\, 5 &= x\, ln\, 7 \\
ln\, 6 + 2x\, ln\, 5 + ln\, 5 &= x\, ln\, 7 \\
x\, ln\, 7 - 2x\, ln\, 5 &= ln\, 6 + ln\, 5 \\
x(\, ln\, 7 - 2\, ln\, 5\,) &= ln\, 6 + ln\, 5 \\
x &= \frac{ln\, 6 + ln\, 5}{ln\, 7 - 2\, ln\, 5} \\
&= -2.672
\end{aligned}$$

[5 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

5.1 Index Equations

Example 1

Solve the equation, $3^x = 81$

[1 mark]

The solution is obviously 4 but the real question is "How could the answer be obtained in a systematic manner ?"

$$\begin{aligned} 3^x &= 81 \\ \ln 3^x &= \ln 81 \\ x \ln 3 &= \ln 81 \\ x &= \frac{\ln 81}{\ln 3} \\ x &= 4 \end{aligned}$$

[1 mark]

Example 2

Solve the equation, $9^x - 3^{x+1} + 2 = 0$

After some thought you may spot one value of x that makes this true.
However, there is a second solution, much more elusive.
It's an irrational number and so needs a methodical method to track down.

$$\begin{aligned} 9^x - 3^{x+1} + 2 &= 0 \\ (3^2)^x - 3^{x+1} + 2 &= 0 \\ 3^{2x} - 3^x \times 3^1 + 2 &= 0 \\ (3^x)^2 - 3(3^x) + 2 &= 0 \\ z^2 - 3z + 2 &= 0 \qquad \text{by letting } z = 3^x \\ (z - 2)(z - 1) &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \text{ Either } z &= 2 && \text{or} \quad z = 1 \\ 3^x &= 2 && \text{or} \quad 3^x = 1 \\ \ln 3^x &= \ln 2 && \text{or} \quad x = 0 \\ x &= \frac{\ln 2}{\ln 3} \\ x &= 0.631 \end{aligned}$$

[5 marks]

5.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Solve the equation

$$2^x = 3$$

Give your answer accurate to 3 decimal places.

[2 marks]

Question 2

A-Level Examination Question from June 2008, paper C2, Q4 (Edexcel)

(a) Find, to 3 significant figures, the value of x for which

$$5^x = 7$$

[2 marks]

(b) Solve the equation,

$$5^{2x} - 12\left(5^x\right) + 35 = 0$$

[3 marks]

Question 3

Show that the equation

$$25^x - 5^{x+1} + 4 = 0$$

can be written in the form

$$(5^x - 4)(5^x - 1) = 0$$

and hence solve the equation.

[5 marks]

Question 4

Without using logarithms, solve the equation, $4^{2x+2} = 8^{x-5}$

[3 marks]

Question 5

Without using logarithms, solve the equation, $9^x = 27^{2-x}$

[3 marks]

Question 6

Solve the equation, $9^x - 3^{x+2} + 20 = 0$

Give your answers correct to 3 decimal places.

[5 marks]

Question 7

Solve the equation, $3^x \times 3^{2x+1} = 9^x$

[3 marks]

Question 8

Solve the equation, $16^x \times 8^{4x+3} = 4^x$

Give your answer as an exact fraction.

[3 marks]

Question 9

A-Level Examination Question from January 2007, paper C2, Q4 (Edexcel)

Solve the equation, $5^x = 17$

Give your answer to 3 significant figures.

[2 marks]

Question 10

Solve for x

$$9^x - 2 \times 3^{x+1} + 8 = 0$$

[5 marks]

Question 11

Solve for x

$$4^x - 6\left(2^x\right) + 5 = 0$$

[5 marks]

Question 12

A-Level Examination Question from May 2015, paper C1, Q7 (Edexcel)

Given that, $y = 2^x$

(a) express 4^x in terms of y

[1 mark]

(b) Hence, or otherwise, solve

$$8(4^x) - 9(2^x) + 1 = 0$$

[4 marks]

Question 13

Show that the equation

$$8^{x} - 2^{x+6} = 0$$

can be written in the form

$$2^{x}\left(2^{2x} - 64\right) = 0$$

and hence solve the equation.

[4 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

5.3 Answers

Answer 1

$$
\begin{aligned}
2^x &= 3 \\
\ln 2^x &= 3 \\
x \ln 2 &= \ln 3 \\
x &= \frac{\ln 3}{\ln 2} \\
&= 1.585
\end{aligned}
$$

[2 marks]

Answer 2

(a)

$$
\begin{aligned}
5^x &= 7 \\
\ln 5^x &= \ln 7 \\
x \ln 5 &= \ln 7 \\
x &= \frac{\ln 7}{\ln 5} \\
&= 1.21
\end{aligned}
$$

[2 marks]

(b)

$$
\begin{aligned}
5^{2x} - 12\left(5^x\right) + 35 &= 0 \\
\left(5^x\right)^2 - 12\left(5^x\right) + 35 &= 0 \\
z^2 - 12z + 35 &= 0 \qquad \text{by letting } z = 5^x \\
(z - 5)(z - 7) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\therefore \text{ Either } z &= 5 & \text{or} \quad z &= 7 \\
5^x &= 5 & \text{or} \quad 5^x &= 7 \\
x &= 1 & \text{or} \quad \ln 5^x &= \ln 7 \\
& & x &= 1.21 \text{ (from part(a))}
\end{aligned}
$$

[3 marks]

Answer 3

(a)

$$25^x - 5^{x+1} + 4 = 0$$

$$\left(5^2\right)^x - 5^1\, 5^x + 4 = 0$$

$$\left(5^x\right)^2 - 5\left(5^x\right) + 4 = 0$$

$$z^2 - 5z + 4 = 0$$

$$(z - 4)(z - 1) = 0$$

$$\left(5^x - 4\right)\left(5^x - 1\right) = 0 \qquad \square$$

[3 marks]

(b)

$$\text{Either } 5^x = 4 \qquad \text{or } 5^x = 1$$

$$ln\, 5^x = ln\, 4 \qquad \text{or } x = 0$$

$$x\, ln\, 5 = ln\, 4$$

$$x = \frac{ln\, 4}{ln\, 5}$$

$$x = 0.861$$

[2 marks]

Answer 4

$$4^{2x+2} = 8^{x-5}$$

$$\left(2^2\right)^{2x+2} = \left(2^3\right)^{x-5}$$

$$2^{4x+4} = 2^{3x-15}$$

$$4x + 4 = 3x - 15$$

$$x = -19$$

[3 marks]

Answer 5

$$9^x = 27^{2-x}$$

$$\left(3^2\right)^x = \left(3^3\right)^{2-x}$$

$$3^{2x} = 3^{6-3x}$$

$$2x = 6 - 3x$$

$$5x = 6$$

$$x = \frac{6}{5}$$

[3 marks]

Answer 6

$$9^x - 3^{x+2} + 20 = 0$$
$$(3^2)^x - 3^x 3^2 + 20 = 0$$
$$(3^x)^2 - 9(3^x) + 20 = 0$$
$$z^2 - 9z + 20 = 0 \qquad \text{by letting } z = 3^x$$
$$(z-4)(z-5) = 0$$
$$\therefore \text{ Either } z = 4 \qquad \text{or} \qquad z = 5$$
$$3^x = 4 \qquad \text{or} \qquad 3^x = 5$$
$$ln\, 3^x = ln\, 4 \qquad ln\, 3^x = ln\, 5$$
$$x\, ln\, 3 = ln\, 4 \qquad x\, ln\, 3 = ln\, 5$$
$$x = \frac{ln\, 4}{ln\, 3} \qquad x = \frac{ln\, 5}{ln\, 3}$$
$$x = 1.262 \qquad \text{or} \qquad x = 1.465$$

[5 marks]

Answer 7

$$3^x \times 3^{2x+1} = 9^x$$
$$3^{3x+1} = (3^2)^x$$
$$3^{3x+1} = 3^{2x}$$
$$3x + 1 = 2x$$
$$x = -1$$

[3 marks]

Answer 8

$$16^x \times 8^{4x+3} = 4^x$$
$$(2^4)^x \times (2^3)^{4x+3} = (2^2)^x$$
$$2^{4x} \times 2^{12x+9} = 2^{2x}$$
$$2^{16x+9} = 2^{2x}$$
$$16x + 9 = 2x$$
$$14x = -9$$
$$x = -\frac{9}{14}$$

[3 marks]

Answer 9

$$5^x = 17$$
$$\ln 5^x = \ln 17$$
$$x \ln 5 = \ln 17$$
$$x = \frac{\ln 17}{\ln 5}$$
$$= 1.76$$

[2 marks]

Answer 10

$$9^x - 2 \times 3^{x+1} + 8 = 0$$
$$\left(3^2\right)^x - 2\,3^x\,3^1 + 8 = 0$$
$$\left(3^x\right)^2 - 6\left(3^x\right) + 8 = 0$$
$$z^2 - 6z + 8 = 0 \qquad \text{by letting } z = 3^x$$
$$(z - 2)(z - 4) = 0$$
$$\therefore \text{ Either } z = 2 \qquad \text{or} \qquad z = 4$$
$$3^x = 2 \qquad \text{or} \qquad 3^x = 4$$
$$\ln 3^x = \ln 2 \qquad \ln 3^x = \ln 4$$
$$x \ln 3 = \ln 2 \qquad x \ln 3 = \ln 4$$
$$x = \frac{\ln 2}{\ln 3} \qquad x = \frac{\ln 4}{\ln 3}$$
$$x = 0.631 \qquad \text{or} \qquad x = 1.262$$

[5 marks]

Answer 11

$$4^x - 6\left(2^x\right) + 5 = 0$$
$$\left(2^2\right)^x - 6\left(2^x\right) + 5 = 0$$
$$\left(2^x\right)^2 - 6\left(2^x\right) + 5 = 0$$
$$z^2 - 6z + 5 = 0 \qquad \text{by letting } z = 2^x$$
$$(z - 1)(z - 5) = 0$$
$$\therefore \text{ Either } z = 1 \qquad \text{or} \qquad z = 5$$
$$2^x = 1 \qquad \text{or} \qquad 2^x = 5$$
$$x = 0 \qquad x = \frac{\ln 5}{\ln 2}$$
$$x = 2.322$$

[5 marks]

Answer 12

(a) $4^x = \left(2^x\right)^2 = y^2$

[1 mark]

(b)

$$8\left(4^x\right) - 9\left(2^x\right) + 1 = 0$$

$$8y^2 - 9y + 1 = 0 \quad \text{using the part (i) suggestion}$$

$$(8y - 1)(y - 1) = 0$$

$$\therefore \text{ Either } y = \frac{1}{8} \quad \text{or} \quad y = 1$$

$$2^x = \frac{1}{8} \quad \text{or} \quad 2^x = 1$$

$$x = -3 \quad \text{or} \quad x = 0$$

[4 marks]

Answer 13

$$8^x - 2^{x+6} = 0$$

$$\left(2^3\right)^x - 2^x 2^6 = 0$$

$$\left(2^x\right)^3 - 64\left(2^x\right) = 0$$

$$2^x\left(2^{2x} - 64\right) = 0$$

Either $2^x = 0$ which has no solutions

or $2^{2x} = 64$ which gives $x = 3$

[4 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted “**Independent School of the Year 2020**”
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

6.1 Extraneous Solutions

When solving equations involving the logarithm function, it's important to verify that the answers obtained after a sequence of algebraic manipulations do, in fact, satisfy the original equation.
This stems from the fact that certain manipulations can turn a false statement into a true statement. As an example, consider squaring,

$$3 = -3 \qquad \text{(False)}$$
$$3^2 = (-3)^2 \qquad \text{Square both sides}$$
$$9 = 9 \qquad \text{(True)}$$

Here is an illustration of what can happen when a variable is involved,

$$x + 3 = -5 \qquad \left(\text{This has the solution set } \{-8\}\right)$$
$$(x + 3)^2 = 25 \qquad \left(\text{This has the solution set } \{-8,\ 2\}\right)$$

Squaring both sides caused an ***extraneous solution*** to be added.

6.2 Example

Solve for x, carefully checking for, and discarding, any extraneous solutions,

$$log_6(x + 5) + log_6(x - 4) = 2$$

[4 marks]

6.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 88

Question 1

A-Level Examination Question from January 2007, paper C2, Q4 (Edexcel)

Solve the equation

$$5^x = 17$$

giving your answer to 3 significant figures.

[3 marks]

Question 2

A-Level Examination Question from June 2005, paper C2, Q2 (Edexcel)

(a) Solve

$$5^x = 8$$

giving your answer to 3 significant figures.

[3 marks]

(b) Solve

$$log_2 (x + 1) - log_2 x = log_2 7$$

[3 marks]

Question 3

A-Level Examination Question from May 2011, paper C2, Q3 (Edexcel)

Find, giving your answers to 3 significant figures where appropriate, the value of *x* for which;

(a) $$5^{x} = 10$$

[2 marks]

(b) $$log_3 (x - 2) = -1$$

[2 marks]

Question 4

A-Level Examination Question from January 2005, paper C2, Q3 (Edexcel)

Find, giving your answers to 3 significant figures where appropriate, the value of *x* for which;

(a)

$$3^{x} = 5$$

[3 marks]

(b)

$$log_2 (2x + 1) - log_2 x = 2$$

[2 marks]

Question 5

A-Level Examination Question from May 2007, paper C2, Q6 (Edexcel)

(a) Find, to 3 significant figures, the value of x for which,

$$8^x = 0.8$$

[2 marks]

(b) Solve the equation

$$2\,log_3 x - log_3 7x = 1$$

[4 marks]

Question 6

A-Level Examination Question from June 2003, paper P2, Q1 (Edexcel)

(a) Simplify, $\dfrac{x^2 + 4x + 3}{x^2 + x}$

[2 marks]

(b) Find the value of x for which

$$log_2\left(x^2 + 4x + 3\right) - log_2\left(x^2 + x\right) = 4$$

[4 marks]

Question 7

A-Level Examination Question from January 2010, paper C2, Q5 (Edexcel)

(a) Find the positive value of x such that,

$$log_x 64 = 2$$

[2 marks]

(b) Solve for x

$$log_2 (11 - 6x) = 2\, log_2 (x - 1) + 3$$

[6 marks]

Question 8

A-Level Examination Question from January 2009, paper C2, Q4 (Edexcel)

Given that $0 < x < 4$ and

$$log_5 (4 - x) - 2\, log_5 x = 1$$

find the value of x.

[6 marks]

Question 9

A-Level Examination Question from June 2008, paper C2, Q4 (Edexcel)

(a) Find, to 3 significant figures, the value of x for which

$$5^x = 7$$

[2 marks]

(b) Solve the equation

$$5^{2x} - 12(5^x) + 35 = 0$$

[4 marks]

Question 10

A-Level Examination Question from January 2012, paper C2, Q4 (Edexcel)

Given that $y = 3x^2$

(a) show that $log_3 y = 1 + 2\,log_3 x$

[3 marks]

(b) Hence, or otherwise, solve the equation

$$1 + 2\,log_3 x = log_3(28x - 9)$$

[3 marks]

Question 11

A-Level Examination Question from January 2011, paper C2, Q8 (Edexcel)

Sketch the graph of $y = 7^x$ showing the coordinates of any points at which the graph crosses the axes.

[2 marks]

(b) Solve the equation, $7^{2x} - 4(7^x) + 3 = 0$
giving your answers to 2 decimal places where appropriate.

[6 marks]

Question 12

A-Level Examination question from June 2010, paper C2, Q7 (Edexcel)

(a) Given that

$$2\,log_3(x - 5) - log_3(2x - 13) = 1$$

show that

$$x^2 - 16x + 64 = 0$$

[5 marks]

(b) Hence, or otherwise, solve

$$2\,log_3(x - 5) - log_3(2x - 13) = 1$$

[2 marks]

Question 13

A-Level Examination Question from June 2002, paper P2, Q5 (Edexcel)

(a) Given that

$$3 + 2\log_2 x = \log_2 y$$

show that

$$y = 8x^2$$

[3 marks]

(b) Hence, or otherwise, find the roots α and β, where $\alpha < \beta$ of the equation,

$$3 + 2\log_2 x = \log_2 (14x - 3)$$

[3 marks]

(c) Show that $\log_2 \alpha = -2$

[1 mark]

(d) Calculate $\log_2 \beta$ giving your answer to 3 significant figures.

[3 marks]

Question 14

A-Level Examination Question from June 2009, paper C2, Q8 (Edexcel)

(a) Find the value of y such that

$$log_2 y = -3$$

[2 marks]

(b) Find the values of x such that

$$\frac{log_2 32 + log_2 16}{log_2 x} = log_2 x$$

[5 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

6.4 Answers

6.4.1 Solution to 6.2 Example

$$log_6(x+5) + log_6(x-4) = 2$$

$$log_6\left((x+5)(x-4) \right) = 2$$

$$(x+5)\ (x-4) = 6^2$$

$$x^2 + x - 20 = 36$$

$$x^2 + x - 56 = 0$$

$$(x-7)(x+8) = 0$$

Either $x = 7$ or $x = -8$

However, $x = -8$ is an extraneous solution

$\therefore$ The only valid solution is $x = 7$

[4 marks]

It can be seen that it was on the very first manipulation that the extraneous solution was generated for,

$$(x+5)\ (x-4)$$

with $x = -8$ becomes $(-3) \times (-12)$

and with $x = 7$ becomes $(12) \times (3)$

Only one of these products has terms that can have their logarithms taken.

6.4.2 Solutions to 6.3 Exercise

In these solutions when it is required to take a logarithm of both sides the natural logarithm, *ln*, has been used simply because on the Casio ClassWiz calculator this involves a single button press rather than two or three.

Answer 1

$$5^x = 17$$

Take the *ln* of both sides

$$ln\, 5^x = ln\, 17$$

$$x\, ln\, 5 = ln\, 17$$

$$x = \frac{ln\, 17}{ln\, 5}$$

$$= 1.76$$ (Three significant figures asked for)

[3 marks]

Answer 2

(a)

$$5^x = 8$$

Take the *ln* of both sides

$$ln\,5^x = ln\,8$$

$$x\,ln\,5 = ln\,8$$

$$x = \frac{ln\,8}{ln\,5}$$

$$= 1.29$$ (Three significant figures asked for)

[3 marks]

(b)

$$log_2(x + 1) - log_2 x = log_2 7$$

$$log_2\left(\frac{x + 1}{x}\right) = log_2 7$$

$$\therefore \frac{x + 1}{x} = 7$$

$$x + 1 = 7x$$

$$x = \frac{1}{6}$$ (Which is a valid solution)

[3 marks]

Answer 3

(a)

$$5^x = 10$$

Take the log_{10} of both sides

$$log_{10} 5^x = log_{10} 10$$

$$x\,log_{10} 5 = 1$$

$$x = \frac{1}{log_{10} 5}$$

$$= 1.43$$ (Three significant figures asked for)

[2 marks]

(b)

$$log_3(x - 2) = -1$$

$$3^{-1} = x - 2$$

$$x = 2 + \frac{1}{3}$$

$$x = 2.33$$ (to three significant figures)

[2 marks]

Answer 4

(a)

$$3^x = 5$$

Take the *ln* of both sides

$$ln\, 3^x = ln\, 5$$

$$x\, ln\, 3 = ln\, 5$$

$$x = \frac{ln\, 5}{ln\, 3}$$

$$= 1.46 \qquad \text{(Three significant figures asked for)}$$

[3 marks]

(b)

$$log_2 (2x + 1) - log_2 x = 2$$

$$log_2 \left(\frac{2x + 1}{x} \right) = 2$$

$$2^2 = \frac{2x + 1}{x}$$

$$2x = 1$$

$$x = 0.5 \qquad \text{(Which is a valid solution)}$$

[2 marks]

Answer 5

(a)

$$8^x = 0.8$$

Take the *ln* of both sides

$$ln\, 8^x = ln\, 0.8$$

$$x\, ln\, 8 = ln\, 0.8$$

$$x = \frac{ln\, 0.8}{ln\, 8}$$

$$= -0.107 \qquad \text{(Three significant figures asked for)}$$

[2 marks]

(b)

$$2\, log_3 x - log_3 7x = 1$$

$$log_3 x^2 - log_3 7x = 1$$

$$log_3 \left(\frac{x^2}{7x} \right) = 1 \qquad \text{(As } x = 0 \text{ is clearly not a solution)}$$

$$3^1 = \frac{x}{7}$$

$$x = 21$$

[4 marks]

Answer 6

(a)

$$\frac{x^2 + 4x + 3}{x^2 + x} = \frac{(x+3)(x+1)}{x(x+1)}$$

$$= \frac{x+3}{x}$$

[2 marks]

(b)

$$log_2\left(x^2 + 4x + 3\right) - log_2\left(x^2 + x\right) = 4$$

$$log_2\left(\frac{x^2 + 4x + 3}{x^2 + x}\right) = 4$$

(As $x = 0$ or $x = -1$ are clearly not solutions)

$$log_2\left(\frac{x+3}{x}\right) = 4$$

$$2^4 = \frac{x+3}{x}$$

$$16x = x + 3$$

$x = 0.2$ (Which is a valid solution)

[4 marks]

Answer 7

(a)

$$log_x 64 = 2$$

$$x^2 = 64$$

$\therefore\ x = 8$ (The base must be a positive real number not equal to 1)

[2 marks]

(b)

$$log_2(11 - 6x) = 2\,log_2(x - 1) + 3$$

$$log_2(11 - 6x) = log_2(x - 1)^2 + 3$$

$$log_2(11 - 6x) - log_2(x - 1)^2 = 3$$

$$log_2\left(\frac{11 - 6x}{(x-1)^2}\right) = 3$$

$$2^3 = \left(\frac{11 - 6x}{(x-1)^2}\right)$$

$$8\left(x^2 - 2x + 1\right) = 11 - 6x$$

$$8x^2 - 16x + 8 = 11 - 6x$$

$$8x^2 - 10x - 3 = 0$$

$$(4x + 1)(2x - 3) = 0$$

Either $x = -0.25$ or $x = 1.5$

Only $x = 1.5$ is a valid solution

[6 marks]

Answer 8

$$log_5(4 - x) - 2\,log_5 x = 1$$

$$log_5(4 - x) - log_5 x^2 = 1$$

$$log_5\left(\frac{4 - x}{x^2}\right) = 1$$

$$5^1 = \left(\frac{4 - x}{x^2}\right)$$

$$5x^2 + x - 4 = 0$$

$$(5x - 4)(x + 1) = 0$$

Either $x = \frac{4}{5}$ or $x = -1$

Only $x = \frac{4}{5}$ is a valid solution

[6 marks]

Answer 9

(a)

$$5^x = 7$$

Take the *ln* of both sides

$$ln\,5^x = ln\,7$$

$$x\,ln\,5 = ln\,7$$

$$x = \frac{ln\,7}{ln\,5}$$

$= 1.21$ (Three significant figures asked for)

[2 marks]

(b)

$$5^{2x} - 12(5^x) + 35 = 0$$

$$(5^x)^2 - 12(5^x) + 35 = 0$$

$$(5^x - 5)(5^x - 7) = 0$$

Either $5^x = 5$ or $5^x = 7$

$x = 1$ or $x = 1.21$

[4 marks]

Answer 10

(a)

$$\begin{aligned} \text{LHS} &= \log_3 y \\ &= \log_3 \left(3x^2\right) \\ &= \log_3 3 + \log_3 x^2 \\ &= 1 + 2\log_3 x \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

(b)

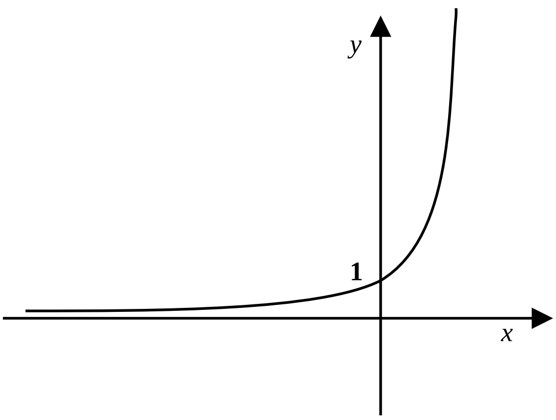
$$\begin{aligned} 1 + 2\log_3 x &= \log_3 (28x - 9) \\ \log_3 \left(3x^2\right) &= \log_3 (28x - 9) \\ 3x^2 &= 28x - 9 \\ 3x^2 - 28x + 9 &= 0 \\ (3x - 1)(x - 9) &= 0 \end{aligned}$$

Either $x = \dfrac{1}{3}$ or $x = 9$ (Both valid solutions)

[3 marks]

Answer 11

(a)



[2 marks]

(b)

$$\begin{aligned} 7^{2x} - 4\left(7^x\right) + 3 &= 0 \\ \left(7^x\right)^2 - 4\left(7^x\right) + 3 &= 0 \\ \left(7^x - 3\right)\left(7^x - 1\right) &= 0 \end{aligned}$$

Either $7^x = 3$ or $7^x = 1$

$\ln 7^x = \ln 3$ or $x = 0$

$$x = \frac{\ln 3}{\ln 7}$$

$$x = 0.565$$

[6 marks]

Answer 12

(a)

$$2\,log_3(x-5) - log_3(2x-13) = 1$$

$$log_3(x-5)^2 - log_3(2x-13) = 1$$

$$log_3\left(\frac{(x-5)^2}{(2x-13)}\right) = 1$$

$$3^1 = \left(\frac{(x-5)^2}{(2x-13)}\right)$$

$$6x - 39 = x^2 - 10x + 25$$

$$x^2 - 16x + 64 = 0 \quad \square$$

[5 marks]

(b)

$$(x-8)^2 = 0$$

$$x - 8 = 0$$

$x = 8$ (Which is a valid solution)

[2 marks]

Answer 13

(a)

$$\begin{aligned}
\text{LHS} &= 3 + 2\,log_2 x \\
&= 3 \times 1 + log_2 x^2 \\
&= 3 \times log_2 2 + log_2 x^2 \\
&= log_2 2^3 + log_2 x^2 \\
&= log_2 8 + log_2 x^2 \\
&= log_2 8x^2 \\
&= log_2 y \quad \text{where } y = 8x^2 \quad \square
\end{aligned}$$

[3 marks]

(b)

$$3 + 2\,log_2 x = log_2(14x-3)$$

$$log_2 8x^2 = log_2(14x-3)$$

$$8x^2 = 14x - 3$$

$$8x^2 - 14x + 3 = 0$$

$$(4x-1)(2x-3) = 0$$

Either $x = 0.25$ or $x = 1.5$ (Both valid solutions)

$\therefore\ \alpha = 0.25$ and $\beta = 1.5$

[3 marks]

(c)

$$\begin{aligned} \text{LHS} &= log_2\, \alpha \\ &= log_2\, 0.25 \\ &= log_2\, \frac{1}{4} \\ &= log_2\, \frac{1}{2^2} \\ &= log_2\, 2^{-2} \\ &= (-2)\ log_2\, 2 \\ &= (-2) \times 1 \\ &= -2 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[1 mark]

(d)

$$\begin{aligned} \text{Let } x &= log_2\, \beta \\ x &= log_2\, 1.5 \\ \therefore\ 2^x &= 1.5 \end{aligned}$$

Take the *ln* of both sides

$$\begin{aligned} ln\, 2^x &= ln\, 1.5 \\ x &= \frac{ln\, 1.5}{ln\, 2} \\ &= 0.585 \quad \text{(Three significant figures requested)} \end{aligned}$$

[3 marks]

Answer 14

(a)

$$\begin{aligned} log_2\, y &= -3 \\ 2^{-3} &= y \\ y &= \frac{1}{8} \quad \text{(Which is a valid solution)} \end{aligned}$$

[2 marks]

(b)

$$\begin{aligned} \frac{log_2\, 32 \,+\, log_2\, 16}{log_2\, x} &= log_2\, x \\ log_2\, 2^5 \,+\, log_2\, 2^4 &= (\, log_2\, x\,)^2 \\ 9\, log_2\, 2 &= (\, log_2\, x\,)^2 \\ 9 \times 1 &= (\, log_2\, x\,)^2 \\ log_2\, x &= \pm 3 \\ \text{Either } x &= 2^3 \text{ or } x = 2^{-3} \\ &= 8 \quad \text{or} \quad = \frac{1}{8} \qquad \text{(Both valid solutions)} \end{aligned}$$

[5 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

7.1 Logarithms And Non-Linear Data

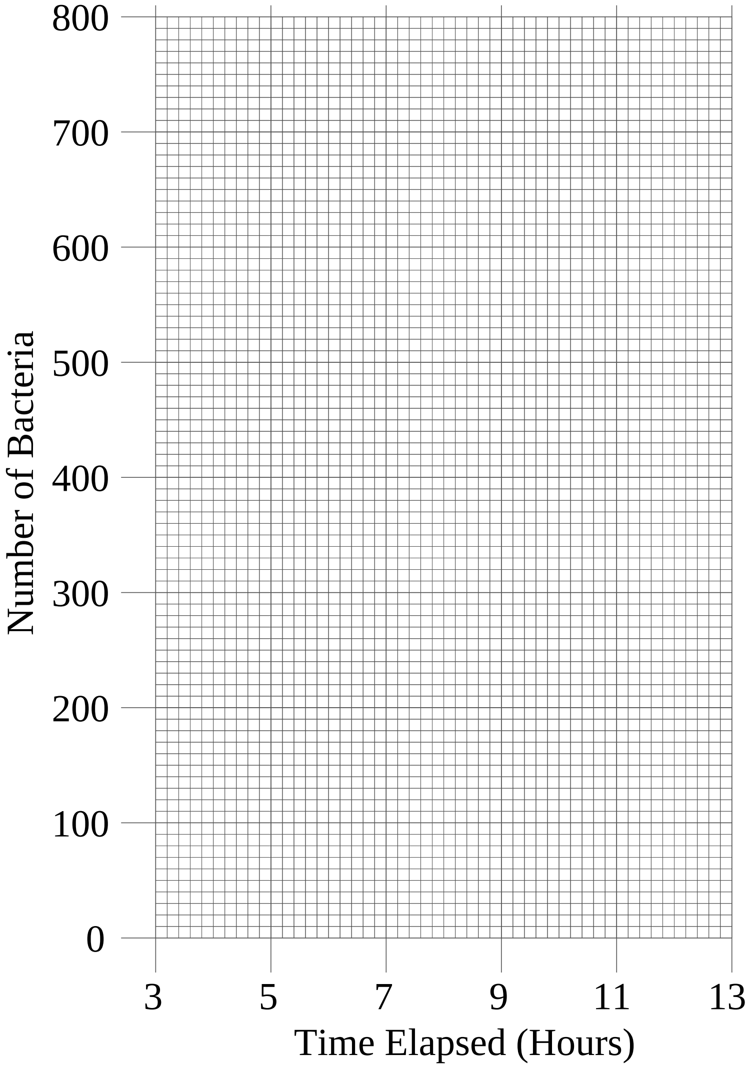
Logarithms are a surprisingly useful tool when it comes to looking at the data generated by experiments and observations in physics, chemistry, biology and economics.

7.2 Example

The table shows the number a bacteria, y, in a petri-dish after x hours have elapsed.

x hours	3	5	6	8	9	11	12	13
y bacteria	105	150	180	260	310	450	580	740

(i) Plot a graph of the data;



[4 marks]

(ii) By eye, draw in a line of best fit.
It should go through the point (Mean x, Mean y)

[2 marks]

(iii) Determine the equation of your line of best fit.

[2 marks]

(iv) Veronica has read in her biology textbook that bacteria, when unconstrained by food or space, exhibit exponential growth.
She reads that to test for exponential growth the data is first coded using;

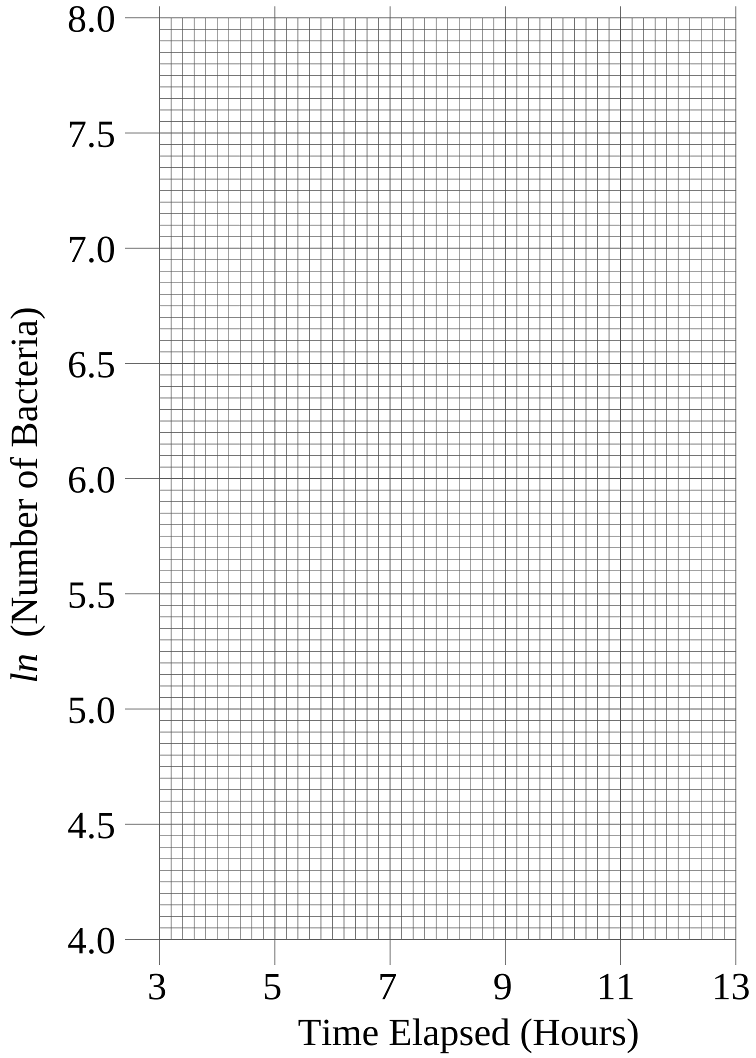
$$Y = \ln y \qquad \text{and} \qquad X = x$$

and then a plot of Y against X is made.
If a straight line relationship is then evident, the original (uncoded) growth data is indeed better modelled by an exponential curve.
Use this coding to complete the following table;

X	3			8				
$Y = \ln y$	4.65			5.56				

[2 marks]

(v) Plot a graph of the coded bivarate data below;



[4 marks]

(vi) Is a straight line relationship evident in the graph of the coded data ?

[1 mark]

(vii) By eye, draw in the line of best fit.

[2 marks]

(viii) Determine the equation of your line of best fit.
it should go through the point (Mean X, Mean Y)

[2 marks]

(ix) How does your part (vii) answer confirm that an exponential curve is indeed a better fit for the uncoded data than a straight line ?

[1 mark]

(x) The general equation of an exponential curve is

$$y = a\, b^x$$

And the equation of the coded data straight line is

$$Y = A + B X$$

The values of a and b are found using the decoding

$$a = e^A \quad \text{and} \quad b = e^B$$

Write out the equation of the exponential curve of best fit in the form

$$y = a\, b^x$$

where a and b are numbers, correct to three significant figures that you have determined.

[3 marks]

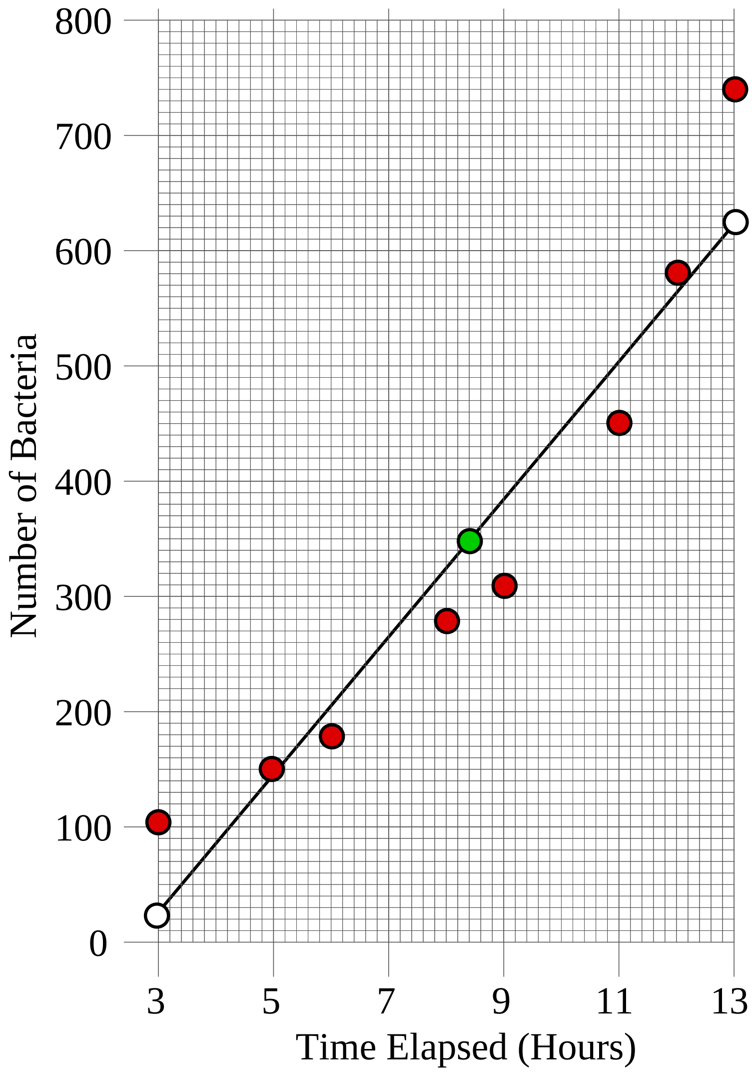
(xi) Carefully plot the exponential curve of best fit onto your part (i) graph.

[1 mark]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

7.3 Solution to 7.2 Example

(i) Here are the raw data points are plotted as red spots on a scattergraph,



[4 marks]

(ii) (Mean x, Mean y) = (8.375, 346.875)
This point is plotted on the above graph (in green).
The line of best fit has been added by eye.

[2 marks]

(iii) Obtaining the equation of the straight line of best fit from a calculator is part of the Year 2 applied mathematics (statistics) course: $y = -158.7 + 60.36x$ Here, having drawn the line by eye, making sure it goes through the point (Mean x, Mean y), the idea is to pick two far apart points on the line, say those plotted in white on the above graph, and so obtain the equation,

$$
\begin{aligned}
b &= \frac{\Delta y}{\Delta x} \\
&= \frac{626 - 22.4}{13 - 3} \\
&= 60.36 \\
y &= a + bx \\
346.875 &= a + 60.36 \times 8.375 \\
a &= -158.7 \\
\therefore\ y &= -158.7 + 60.36x
\end{aligned}
$$

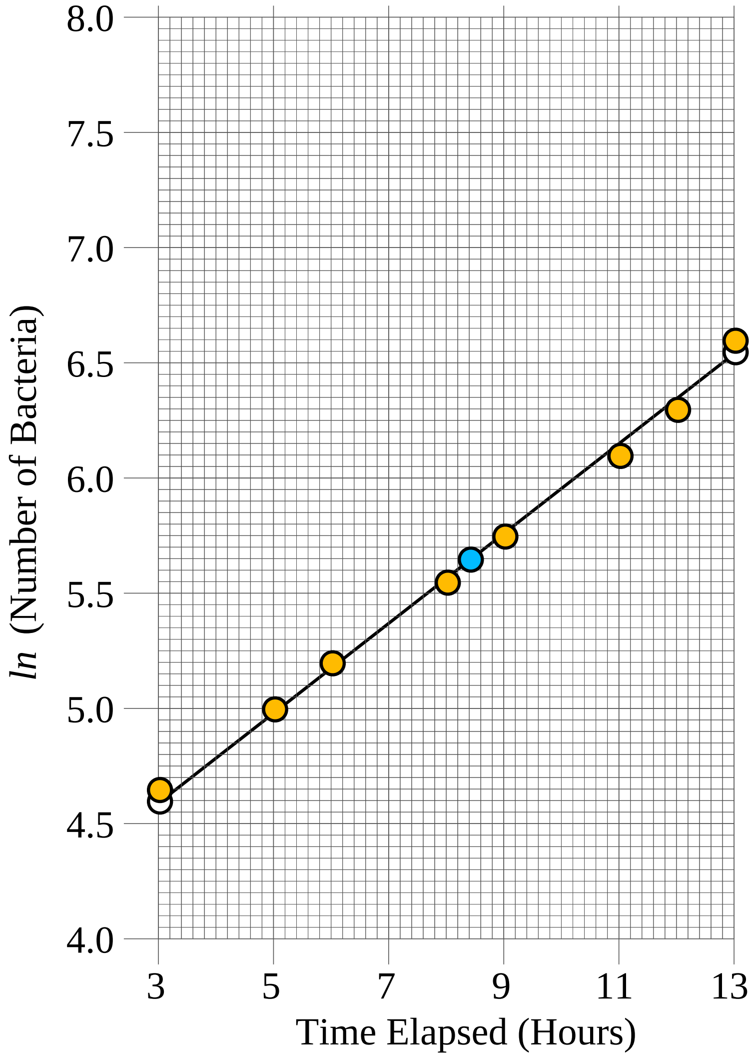
[2 marks]

(iv)

X	3	5	6	8	9	11	12	13
$Y = \ln y$	4.65	5.01	5.19	5.56	5.74	6.11	6.36	6.61

[2 marks]

(v) Here are the coded data points are plotted as orange spots on a scattergraph,



[4 marks]

(vi) Yes, a straight line relationship has been found in the coded data.

[1 mark]

(vii) (Mean x, Mean $ln\ y$) = (8.375, 5.654)
This point is plotted on the above graph (in blue).
The line of best fit has been added by eye.

[2 marks]

(viii) Obtaining the equation of the straight line of best fit from a calculator is part of the Year 2 applied mathematics (statistics) course: $y = 4.04 + 0.193x$
Here, having drawn the line by eye, making sure it goes through the point (Mean *x*, Mean *ln y*), the idea is to pick two far apart points on the line, say those plotted in white on the above graph, and so obtain the equation,

$$\begin{aligned} b &= \frac{\Delta Y}{\Delta X} \\ &= \frac{6.55 - 4.62}{13 - 3} \\ &= 0.193 \\ Y &= a + bX \\ 5.654 &= a + 0.193 \times 8.375 \\ a &= 4.04 \\ \therefore\ Y &= 4.04 + 0.193X \end{aligned}$$

[2 marks]

(ix) The coded scatter graph's straight line of best fit has the points much closer (on average) to it than was the case with the raw data scattergraph and its straight line of best fit.

[1 mark]

(x) Using the claimed decoding procedure,

$$\begin{aligned} a &= e^{4.04} & \qquad b &= e^{0.193} \\ &= 56.8 & &= 1.21 \end{aligned}$$

$$\therefore\ y = 56.8 \times 1.21^{x}$$

[3 marks]

(xi) This curve is so good that when plotted on the raw data scatter graph it very nearly passes through all of the original raw data points.

[1 mark]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted “**Independent School of the Year 2020**”
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

8.1 The General Exponential Non-Linear Relationship Model

In Lesson 7 we investigated a situation where bivarate data, when plotted on a graph, had a relationship that was better captured by an exponential curve of best fit, rather than a straight line of best fit.

When an exponential relationship is suspected the original data should be coded using;

$$Y = \ln y \qquad \text{and} \qquad X = x$$

and a plot of Y against X made.

If the graph of the coded data is still curved, then the exponential model is NOT appropriate. Other codings could be tried to test for other types of non-linear relationship.

The jackpot has been hit, however, if the graph of the coded data is clearly linear for this then this confirms that the original data can indeed be effectively modelled by an exponential curve of the form

$$y = a\,b^x$$

and the task now becomes one of determining the values of the constants a and b. This is done by obtaining the equation of the straight line of best fit for the coded data, in the form

$$Y = A + B\,X$$

and using the decoding

$$a = e^A \qquad \textit{and} \qquad b = e^B$$

8.2 Why Does It Work ?

If

$$y = a\,b^x$$

then

$$\ln y = \ln\left(a\,b^x\right)$$

$$\ln y = \ln a + \ln b^x$$

$$\ln y = \ln a + x\ln b$$

$$\ln y = \ln a + (\ln b)\,x$$

which is in the form of the straight line

$$Y = A + B\,X$$

where

$$Y = \ln y \qquad A = \ln a \qquad B = \ln b \qquad \text{and} \qquad X = x$$

the first and last give the coding, reversing the middle two gives the decoding ☐

8.3 Use of log to the base 10

It does not have to be the natural log, *ln*, that is used to code the data.
A log to any suitable base could be used.
Often, examination questions use $\log_{10}$ in which case the decoding would be

$$a = 10^A \qquad \text{and} \qquad b = 10^B$$

When log is written without a base in this course you should assume the base is 10.

8.4 Example

On 1st February 2020, Nick began gathering data on the relationship between the size of a population of rabbits, n, on the small Outer Hebridean Island of *Ensay*, over time, t, measured in months.
He codes his data using;

$$Y = \log n \qquad \text{and} \qquad X = t$$

The resulting relationship between Y and X appears to be linear with line of best fit

$$Y = 1.380 + 0.0969\,X$$

Consequently an exponential curve of best fit is the most appropriate model for the relationship between n and t
That is,

$$n = a\,b^t$$

(i) Determine the values of the constants a and b

[2 marks]

(ii) How many rabbits were there when Nick started gathering data ?

[1 mark]

(iii) How many rabbits were there one month later ?

[1 mark]

(iv) What does this model predict the rabbit population will be on the small Island of *Ensay* in 20 years time ?

[2 marks]

(v) Explain why this prediction for the number of rabbits in 20 years time should be treated with caution.

[2 marks]

8.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available: 30

Question 1

Use your calculator to find, correct to 3 significant figures;

(i) $10^{2.3}$ **(ii)** e^4 **(iii)** $ln\ 0.4$

(iv) $log\ 3.8$ **(v)** $10^{-0.2}$ **(vi)*** $log_3 11$

[6 marks]

* HELP for Graphic Calculator fx-CG50 users

In Comp Mode (MENU 1) press EXIT then F4 (MATH) then $(log_a b)$

Question 2

The line of best fit for some coded data passes through the points (0, 4.8) and (1.4, 7.3)
Given that the data has been coded using

$$Y = log\, y \quad \text{and} \quad X = x$$

determine the relationship between y and x, writing your answer in the form

$$y = a\, b^x$$

where a and b are constants, the values of which you should find.

[3 marks]

Question 3

Data are coded using

$$Y = \log y \qquad \text{and} \qquad X = x$$

to give a linear relationship.

The equation of the line of best fit for the coded data is

$$Y = 0.8 - 0.05\,X$$

(i) Determine the relationship between y and x, writing your answer in the form

$$y = a\,b^{x}$$

where a and b are constants, the values of which you should find.

[2 marks]

(ii) Complete the following table

x	0	1	2	3
y				

[2 marks]

(iii) Sketch the curve of best fit, marking on the value of any points where the curve passes through the coordinate axes.

[2 marks]

Question 4

The table shows some data collected on the temperature, t °C, of a colony of insect larvae and the growth rate, g, of the population.

The reading for the growth rate when t is 17°C has been lost.

Temp, t °C	13	17	21	25	26	28
Growth rate, g	5.37		13.29	20.91	23.42	29.38

The data are coded using the changes of variable

$$Y = \ln g \qquad \text{and} \qquad X = t$$

The line of best fit of y on x is found to be

$$Y = 0.208 + 0.113\,X$$

(i) Given that the data can be modelled by an equation of the form

$$g = a\,b^{t}$$

where a and b are constants, find the values of a and b

[2 marks]

(ii) Give an interpretation of the constant b in this equation.

[1 mark]

(iii) What does the model predict the growth rate will be when the temperature is 17 °C

[1 mark]

(iv) Give an interpretation of the constant a in this equation and also explain why this constant can not be reliably checked by direct measurement.

[2 marks]

Question 5

400 identical six sided biased dice are to be rolled repeatedly.

The probability of rolling a '6' is unknown.

Every time the dice are rolled all those showing a '6' are considered ***dead*** and removed.
Those dice not yet ***dead*** are considered to be ***live***.

The table below shows the number of dice still ***live*** after the *n* th roll.

Roll number, n	0	1	2	3	4	5	6	7	8
N° of ***live*** dice, A	400	272	186	130	95	69	51	34	25

Theory predicts exponential decay and so the data are coded using

$$Y = \log L \qquad and \qquad X = n$$

The equation of the line of best fit for the coded data is

$$Y = 2.58 - 0.148\,X$$

(i) Determine the relationship between *n* and *A*, writing your answer in the form

$$L = a\,b^n$$

where *a* and *b* are constants, the values of which you should find

[3 marks]

(ii) Give an interpretation of the constant *b* in this equation.

[1 mark]

(iii) Complete the third row of the following table to show the N° of ***live*** dice that are predicted after each roll by your part **(i)** exponential curve of best fit formula.

Roll number, n	0	1	2	3	4	5	6	7	8
N° of ***live*** dice, L	400	272	186	130	95	69	51	34	25
Prediction of L									

[3 marks]

(iv) What is the probability of rolling a '6' with these biased dice.
Justify your answer.

[2 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

8.6 Answers

8.6.1 Solution to 8.4 Example

(i)

$$a = 10^{1.380} \qquad b = 10^{0.0969}$$
$$= 24 \qquad\qquad = 1.25$$
$$\therefore\ n = 24 \times 1.25^{t}$$

[2 marks]

(ii) When $t = 0$, $n = 24$ rabbits.

[1 mark]

(iii) When $t = 1$, $n = 24 \times 1.25^{1}$

$= 30$ rabbits

[1 mark]

(iv) When $t = 240$, $n = 24 \times 1.25^{240}$

$= 4.35 \times 10^{24}$

$= 4\,350\,000\,000\,000\,000\,000\,000\,000$ rabbits

[2 marks]

(v) The rabbits are likely to run out of food and space well before their population reaches such a substantive size on Ensay which was described as being a small island.

[2 marks]

8.6.2 Solution to 8.5 Exercise

Answer 1

(i)	200	**(ii)**	54.6	**(iii)**	– 0.916
(iv)	0.580	**(v)**	0.631	**(vi)**	2.18

[6 marks]

Answer 2

$$B = \frac{\Delta Y}{\Delta X}$$
$$= \frac{7.3 - 4.8}{1.4 - 0}$$
$$= 1.786$$
$$\therefore\ Y = 4.8 + 1.786X$$

$$a = 10^{4.8} \qquad b = 10^{1.786}$$
$$= 63000 \qquad\qquad = 61.1$$
$$\therefore\ y = 63000 \times 61.1^{x}$$

[3 marks]

Answer 3

(i) $a = 10^{0.8}$ $b = 10^{-0.05}$

$= 6.31$ $= 0.891$

$\therefore\ y = 6.31 \times 0.891^x$

[2 marks]

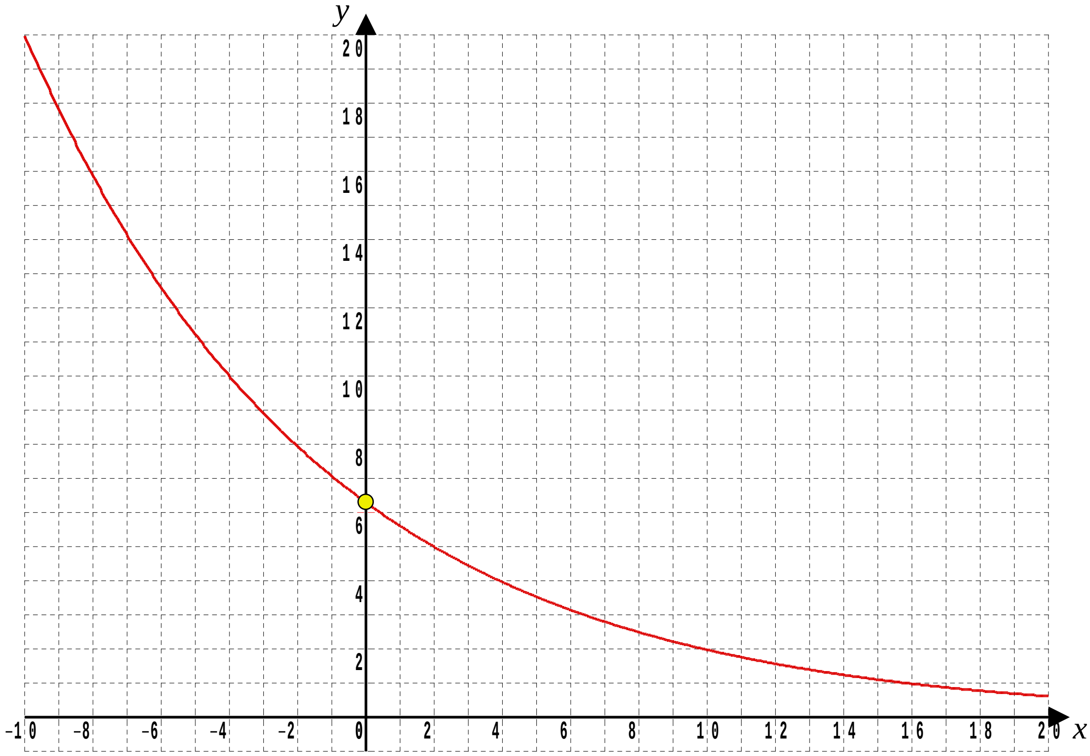
(ii)

x	0	1	2	3
y	6.31	5.62	5.01	4.46

[2 marks]

(iii) Here is the graph of $y = 6.31 \times 0.891^x$

The only axis crossed is the y-axis at (0, 6.31)



[2 marks]

Answer 4

(i) $a = e^{0.208}$ $b = e^{0.113}$

$= 1.23$ $= 1.12$

$\therefore\ g = 1.23 \times 1.12^{t}$

[2 marks]

(ii) b is the rate of change of the growth rate.

[1 mark]

(iii) 8.45

[1 mark]

(iv) a is the predicted rate of growth when $t = 0°$ C
This temperature is below the known data domain of $13 \leqslant t \leqslant 28$
Possibly, if the larvae become frozen they might die, for example.

[2 marks]

Answer 5

(i) $a = 10^{2.58}$ $b = 10^{-0.148}$

$= 380$ $= 0.711$

$\therefore\ L = 380 \times 0.711^{n}$

[3 marks]

(ii) b is the probability of NOT rolling a '6'

[1 mark]

(iii)

Roll number, n	0	1	2	3	4	5	6	7	8
N° of ***live*** dice, L	400	272	186	130	95	69	51	34	25
Prediction of L	380	270	192	137	97	69	49	35	25

[3 marks]

(iv) 0.289 because $1 - 0.711 = 0.289$

[2 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

9.1 The Power Function Model

There are many non-linear relationship models that are in common usage.
Here are six that are well known;

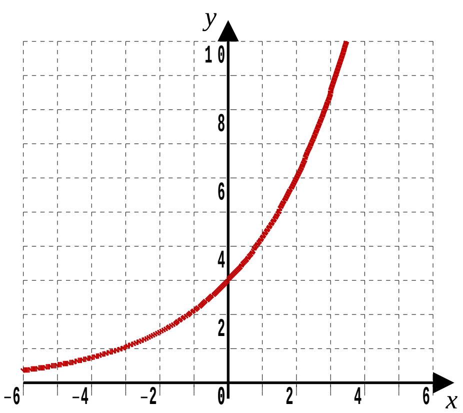
- ◇ The best ***quadratic*** curve: $y = a + bx + cx^2$
- ◇ The best ***logarithmic*** curve: $y = a + b \ln x$
- ◇ The best ***e powered*** curve: $y = a\,e^{bx}$
- ◇ The best (general) ***exponential*** curve: $y = a\,b^x$ (Lesson 8)
- ◇ The best ***power function*** curve: $y = a\,x^b$ (Lesson 9)
- ◇ The best ***simple reciprocal*** curve: $y = a + \dfrac{b}{x}$

There are many more and the adventurous may like to try building their own.

In other subjects, especially physics, chemistry, biology, economics and geography, you may want to explore some of these non-linear models if you are doing a project or investigation and need to fit a curve to your data.

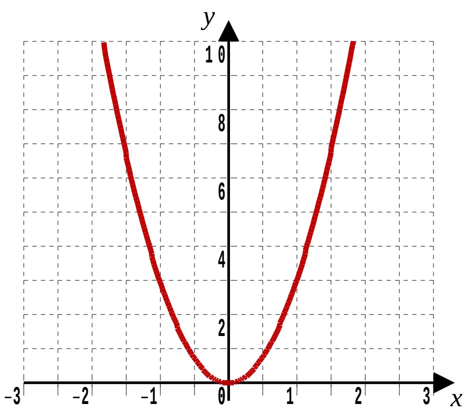
In this course one other Non-Linear Model is studied; the ***power function*** curve.

9.2 Spot The Difference



$y = 3 \times 2^x$

An ***exponential*** curve



$y = 3x^2$

A ***power function*** curve

9.3 The Theory Behind The Power Function Model

If

$$y = a\,x^b$$

then

$$\ln y = \ln\left(a\,x^b\right)$$

$$\ln y = \ln a + \ln x^b$$

$$\ln y = \ln a + b \ln x$$

which is in the form of the straight line

$$Y = A + B\,X$$

where $Y = \ln y$ $A = \ln a$ $B = b$ and $X = \ln x$

the first and last give the coding, reversing the middle two gives the decoding as;

$$a = e^A \quad \text{and} \quad b = B \qquad \square$$

Notice that the more complicated power function coding, compared with last lesson's exponential model, has yielded a simpler decoding.

9.4 Example

(i) Data are coded using

$$Y = \log y \quad \text{and} \quad X = x$$

and a curved graph results when the coded data is plotted as *Y* against *X*
What does this tell you about the relationship between the uncoded data, *y* and *x* ?

[1 mark]

(ii) The data are now coded using

$$Y = \log y \quad \text{and} \quad X = \log x$$

and a straight line graph results when the coded date is plotted, *Y* against *X*
What does this tell you about the relationship between the uncoded data, *y* and *x* ?

[2 marks]

(iii) Given that the line of best fit for the appropriately coded data is

$$Y = 1.2 + 0.4\,X$$

write down the relationship between *y* and *x* and find the values of the constants.

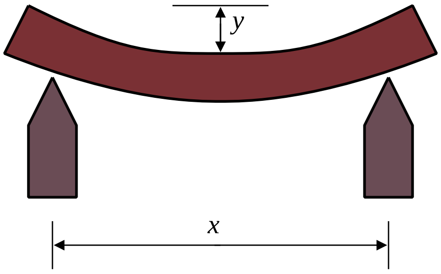
[3 marks]

9.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available: 35

Question 1

An experiment is conducted to determine how the sag of a steel beam, y mm, varies with the distance between its supports, x m.



The data is coded using

$$Y = \log y \qquad \text{and} \qquad X = \log x$$

to give a linear relationship with line of best fit

$$Y = -0.947 + 2.84\,X$$

(i) Determine the power function curve of best fit.

It will be of the form, $y = a\,x^{b}$

where a and b are constants that you have to determine.

[2 marks]

(ii) Complete the following table,

x metres	2	4	6	8
y millimetres				

[2 marks]

(iii) Sketch the curve of best fit, marking on the value of any points where the curve passes through the coordinate axes.

[2 marks]

Question 2

A pendulum-swinging experiment is conducted to investigate how the average time of a swing, T, (measured in seconds) relates to the length, L, (measured in metres) of the pendulum.

The data is coded using

$$Y = \log T \qquad \text{and} \qquad X = \log L$$

to give a linear relationship with line of best fit

$$Y = 0.30 + 0.50\,X$$

(i) Determine the power function curve of best fit which will be of the form

$$T = a\,L^{b}$$

where a and b are constants that you have determined.

[2 marks]

(ii) Complete the following table,

L metres	0.1	0.2	0.5	1
T seconds				

[2 marks]

(iii) Sketch the curve of best fit, marking on the value of any points where the curve passes through the coordinate axes.

[2 marks]

Question 3

A scientist is modelling the number of people, N, who have fallen sick with a virus after t days. He codes his data using;

$$Y = log\, N \qquad \text{and} \qquad X = t$$

and is pleased to see a linear relationship in the coded data.

His coded data's line of best fit passes through the end points (0, 2.4) and (6, 5.1)

(i) Determine the equation of the line of best fit for the coded data.

[2 marks]

(ii) Hence, write down the relationship between N and t that models the uncoded data. Include the values of any constants in the relationship.

[3 marks]

(iii) State the number of people that the model predicts will be sick after,

(a) 1 day **(b)** 4 days **(c)** 8 days

[3 marks]

(iv) Use your model to predict the number of sick people after 30 days.
Give one reason why this might be an overestimate.

[2 marks]

Question 4

The period of oscillation, P seconds, for bars of uniform material is thought to be proportional to some power of their length, L metres.

One set of measurements is given;

L	2	3	4	5	6
P	4.5	5.5	6.4	7.2	8.7

One result, however, has been incorrectly copied.

(i) Complete the following table;

$\ln L$		1.099			
$\ln P$				1.974	

[2 marks]

(ii) On the graph paper on the next page, plot the five points on a graph of $\ln P$ against $\ln L$

[2 marks]

(iii) Add an approximate straight line graph that goes through four of the five points; throw away the fifth error point.

[2 marks]

(iv) Determine the equation of your part (iii) line of best fit.

[3 marks]

(v) Now give the corresponding formula connecting P and L which should be of the form

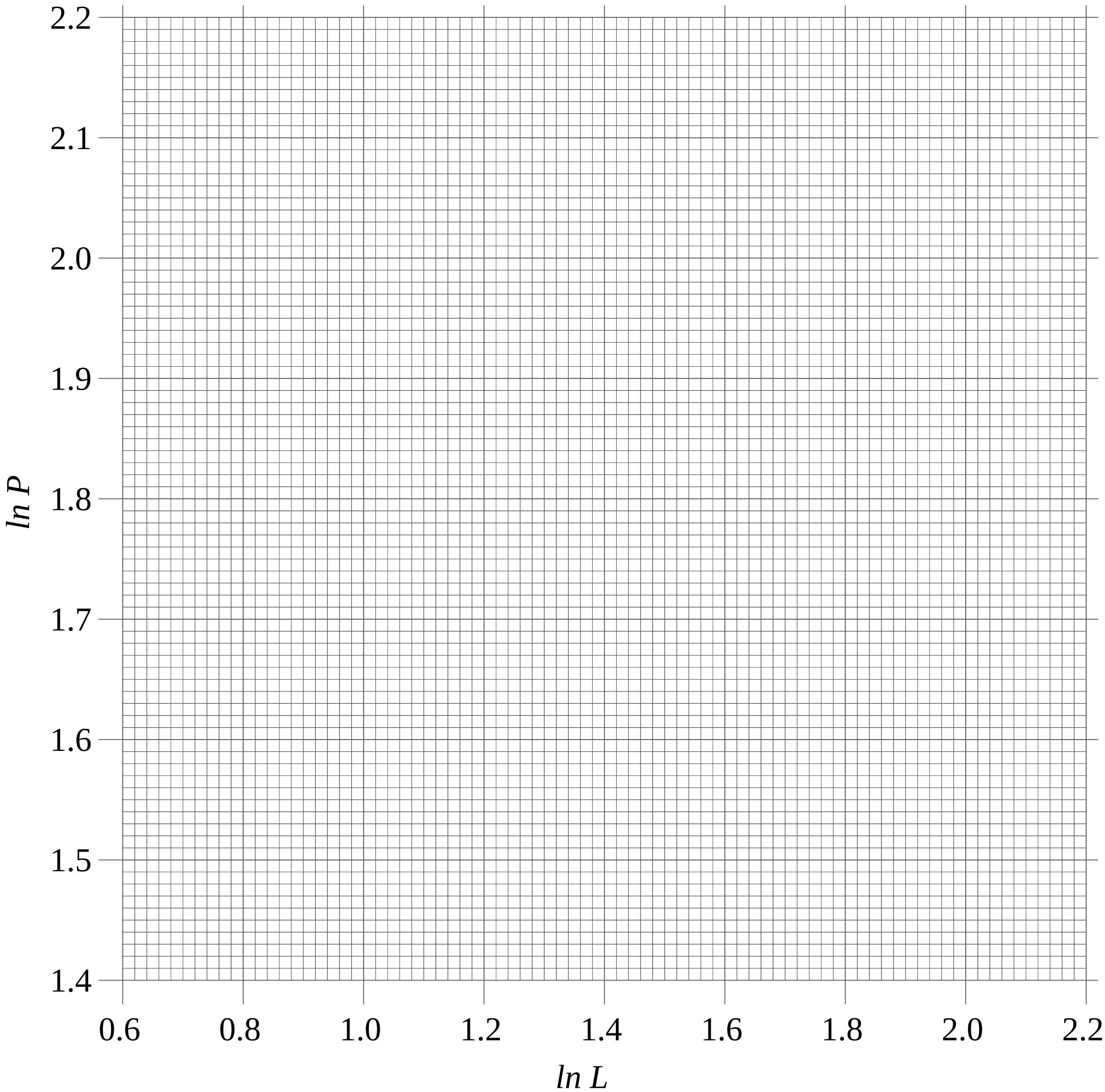
$$P = a L^b$$

where a and b are constants the values of which you have found.

[3 marks]

(vi) State what the incorrect result should have been.

[1 mark]



This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

9.6 Answers

9.6.1 Solution to 9.4 Example

(i) This is the coding to test for an exponential relationship of the form

$$y = a\,b^x$$

The coded curve tells us that this is NOT the form of a suitable equation with which to model the data.

[1 mark]

(ii) This is the coding to test for a power function relationship of the form

$$y = a\,x^b$$

The coded straight line tells us that this is the form of a suitable equation with which to model the data.

[2 marks]

(iii) Notice that the data has been coded using a *log* to the base 10

$$a = 10^{1.2} \qquad b = 0.4$$

$$= 15.8$$

$$\therefore\ y = 15.8 \times x^{0.4}$$

[3 marks]

9.6.2 Solution to 9.5 Exercise

Answer 1

(i) Notice that the data has been coded using a *log* to the base 10

$$a = 10^{-0.947} \qquad b = 2.84$$

$$= 0.113$$

$$\therefore\ y = 0.113 \times x^{2.84}$$

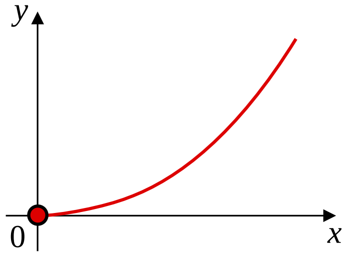
[2 marks]

(ii)

x metres	2	4	6	8
y millimetres	0.81	5.79	18.32	41.5

[2 marks]

(iii)



Like all power function curves, this passes through the origin.

[2 marks]

Answer 2

(i) Notice that the data has been coded using a *log* to the base 10

$$a = 10^{0.3} \qquad b = 0.5$$
$$= 2$$
$$\therefore\ T = 2\sqrt{L}$$

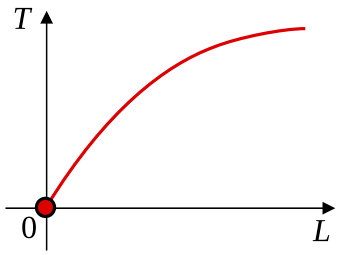
[2 marks]

(ii)

L metres	0.1	0.2	0.5	1
T seconds	0.63	0.89	1.41	2

[2 marks]

(iii)



Like all power function curves, this passes through the origin.

[2 marks]

Answer 3

(i) $m = \dfrac{\Delta y}{\Delta x} = \dfrac{5.1 - 2.4}{6} = 0.45 \qquad \therefore$ Straight line is $Y = 2.4 + 0.45X$

[2 marks]

(ii) The coding is for an exponential curve and not for a power function !

$$N = a\,b^t$$
$$a = 10^{2.4} \qquad b = 10^{0.45}$$
$$= 251 \qquad = 2.82$$
$$\therefore\ N = 251 \times 2.82^t$$

[3 marks]

(iii) **(a)** 708 **(b)** 15 873 **(c)** 1 003 845

[3 marks]

(iv) 8.075×10^{15}

After 30 days some people may have recovered so the disease may stop to spread as quickly as the immune proportion of the population increases The give away is the mention of 'end points' in the question which reveals that the known data was gathered over 6 days and so 30 days is well beyond the data domain that gave rise to the exponential curve being used to model the disease.

[2 marks]

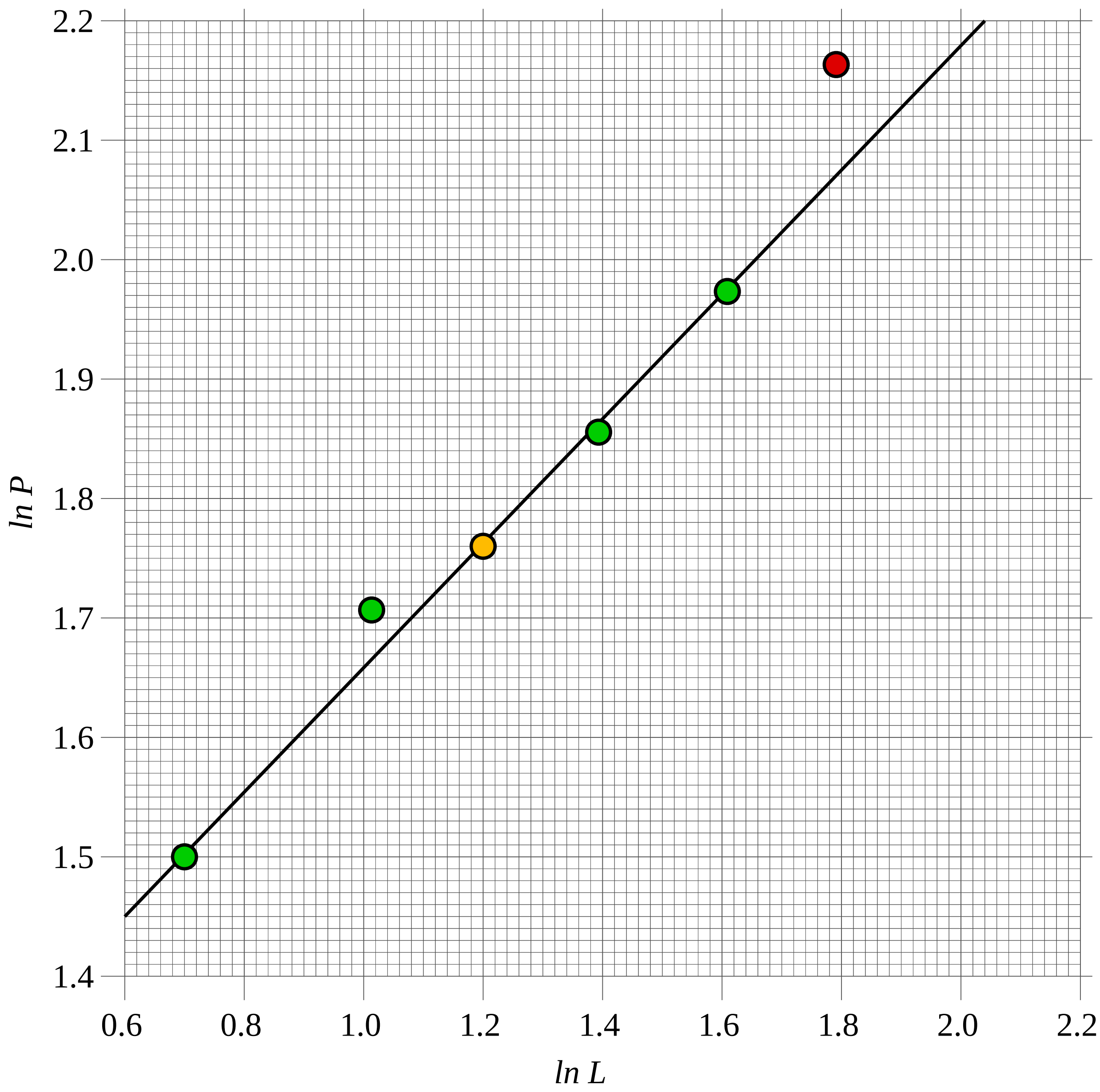
Answer 4

(i)

ln L	0.693	1.099	1.386	1.609	1.792
ln P	1.504	1.705	1.856	1.974	2.163

[2 marks]

(ii) The points need to be plotted accurately to avoid rejecting the wrong point !



[2 marks]

(iii) For the four green points,

$$(\textit{Mean } \ln L, \ \textit{Mean } \ln P) = (1.196, \ 1.760)$$

This point has been added in orange.

The line of best fit should go through this point.

[2 marks]

(iv) The exact answer is,

$$Y = 1.146 + 0.513\,X$$

For 'by eye' solutions, some variation on this should be allowed.

[3 marks]

(v)

$$P = a\,L^b$$

$$a = e^{1.146} \qquad b = 0.513$$

$$= 3.15$$

$$P = 3.15\ L^{0.513}$$

[3 marks]

(vi)

$$P = 3.15 \times 6^{0.513}$$

$$= 7.9$$

$\therefore$ The 8.7 should have been 7.9

[1 mark]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted “**Independent School of the Year 2020**”
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

10.1 Miscellaneous Old Examination Questions

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available: 42

Question 1

Solve the equation, $log_5 (3t + 8) = 2.1$

Round your answer to 3 significant figures.

[3 marks]

Question 2

Consider the formula

$$r = 0.1\, e^{0.5\, t \,+\, 1}$$

By first multiplying through by 10, rearrange the formula to make *t* the subject.

[3 marks]

Question 3

The temperature, T °C, of a cup of tea is given by

$$T = 55e^{-0.125t} + 20 \qquad t \geqslant 0$$

where t is the time in minutes since measurements began

(a) Briefly explain why $t \geqslant 0$

[1 mark]

(b) State the starting temperature of the cup of tea.

[1 mark]

(c) Find the time at which the temperature of the tea is 50 °C, giving your answer in minutes and seconds to the nearest second.

[3 marks]

(d) By sketching a graph or otherwise, explain why the temperature of the tea will never fall below 20 °C

[2 marks]

Question 4

The amount of an antibiotic in the bloodstream, from a given dose, is modelled by the formula

$$x = De^{-0.2t}$$

where x is the amount of the antibiotic in the bloodstream in milligrams, D is the dose given in milligrams and t is the time in hours after the antibiotic has been given.

A first dose of 15 mg of the antibiotic is given.

(a) Use the model to find the amount of the antibiotic in the bloodstream 4 hours after the dose is given. Give your answer in mg to 3 decimal places.

[2 marks]

A second dose of 15 mg is given 5 hours after the first dose has been given.
Using the same model for the second dose,

(b) show that the total amount of the antibiotic in the bloodstream 2 hours after the second dose is given is between 13.5 mg and 14 mg.
Give your answer to 3 decimal places.

[2 marks]

Question 5

The equation of a curve is

$$y = 20x^n$$

where n is a constant.

The point (15, 590) is on the curve.
Calculate the value of n

[5 marks]

Question 6

The population, P, of a colony of endangered Caledonian owlet-nightjars can be modelled by the equation $P = a\,b^t$ where a and b are constants and t is the time, in months, since the population was first recorded.

A graph of $log_{10} P$ against t is linear and has a line of best fit that passes through the point (0, 2) on the y-axis and also the point, twenty months later, (20, 2.2)

(a) Determine the equation of the straight line through the two points.

[3 marks]

(b) Work out the value of a and interpret this value in the context of the model

[3 marks]

(c) Work out the value of b, giving your answer correct to 3 decimal places.

[2 marks]

(d) Find the population predicted by the model when $t = 30$

[1 mark]

Question 7

The following table gives estimates of the world population in different years.

Date	1400	1500	1600	1700	1800
Population (millions)	375	460	540	640	945

Sam is trying to model the population by an equation of the form

$$y = A e^{kt}$$

where y is the population in millions and t is the number of centuries after 1400.

Sam decides to find values of A and k that gives the exact values for 1400 and 1800

(a) Show that Sam's value for A is 375

[1 mark]

(b) Find Sam's value for k

[4 marks]

(c) Does Sam's formula give reasonable value for the population in 1600 ?
Justify your answer.

[2 marks]

Question 8

Solve the equation, $2^{2x} + 2^3 = 2^{x+3}$

[4 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted “**Independent School of the Year 2020**”
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

10.2 Answers

Answer 1

$$\begin{aligned} log_5(3t + 8) &= 2.1 \\ 5^{2.1} &= 3t + 8 \qquad \text{Using } \; log_c a = b \Leftrightarrow c^b = a \\ 29.36 &= 3t + 8 \\ t &= 7.12 \end{aligned}$$

[3 marks]

Answer 2

$$\begin{aligned} r &= 0.1\, e^{0.5t + 1} \\ 10r &= e^{0.5t+1} \\ ln\,(10r) &= 0.5t + 1 \\ ln\,(10r) - 1 &= 0.5t \\ t &= 2\, ln(10r) - 2 \end{aligned}$$

[3 marks]

Answer 3

(a) Accept "Time cannot go backwards"
The model has only been set up for positive values of t

[1 mark]

(b) When $t = 0$, $T = 75°C$

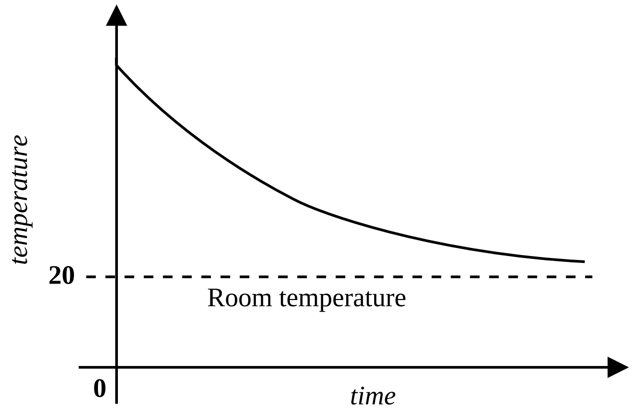
[1 mark]

(c)

$$\begin{aligned} 50 &= 55\, e^{-0.125t} + 20 \\ 30 &= 55\, e^{-0.125t} \\ \frac{30}{55} &= e^{-0.125t} \\ ln\left(\frac{30}{55}\right) &= -0.125t \\ t &= \frac{ln\left(\frac{30}{55}\right)}{-0.125} \\ &= 4.849086 \\ &= 4 \text{ minutes } \; 51 \text{ seconds} \end{aligned}$$

[3 marks]

(d)



The tea cannot become colder than the surrounding temperature.

[2 marks]

Answer 4

(a)

$$\begin{aligned} x &= D e^{-0.2t} \\ &= 15 e^{-0.2 \times 4} \\ &= 6.740 \text{ mg} \end{aligned}$$

[2 marks]

(b)

$$\begin{aligned} amount &= 15 e^{-0.2 \times 7} + 15 e^{-0.2 \times 2} \\ &= 13.754 \text{ mg} \end{aligned}$$

[2 marks]

Answer 5

$$\begin{aligned} y &= 20 x^n \\ 590 &= 20 \times 15^n \\ 29.5 &= 15^n \\ n &= \frac{ln\, 29.5}{ln\, 15} \\ &= 1.25 \end{aligned}$$

[5 marks]

Answer 6

(a)

$$log_{10} P = 2 + 0.01\, t$$

[3 marks]

(b)

$$P = a\, b^t$$

$$a = 10^2$$

$$= 100$$

$\therefore$ The initial population is 100

[3 marks]

(c)

$$b = 10^{0.01}$$

$$= 1.023$$

[2 marks]

(d) Thus,

$$P = 100 \times 1.023^t$$

$$= 100 \times 1.023^{30}$$

$$= 198$$

(Accept answers between 195 and 200)

The more accurate answer is 199.526...

[1 mark]

Answer 7

(a) For 1400, $t = 0$

$$y = A\, e^{kt}$$

$$375 = A\, e^0$$

$$A = 375 \quad \text{as expected}$$

[1 mark]

(b)

$$y = 375\, e^{kt}$$

$$945 = 375\, e^{4k}$$

$$e^{4k} = 2.52$$

$$4k = ln\, 2.52$$

$$4k = 0.9243$$

$$k = 0.231$$

[4 marks]

(c)

$$y = 375\,e^{0.231t}$$
$$= 375\,e^{0.231 \times 2}$$
$$= 595$$

It's an overestimate of the actual population which was 540 million
The percentage error on the actual measured population is,

$$\frac{594 - 540}{540} \times 100 = 10\,\%$$

This can be justified as being reasonable or not depending on your expectations of the mathematics and how accurate you think the figures in the table were in the first place.

[2 marks]

Answer 8

$$2^{2x} + 2^3 = 2^{x+3}$$
$$(2^x)^2 + 8 = 8 \times 2^x$$
$$(2^x)^2 - 8 \times (2^x) + 8 = 0$$
$$2^x = \frac{8 \pm \sqrt{8^2 - 4 \times 1 \times 8}}{2 \times 1}$$
$$2^x = 4 \pm 2\sqrt{2}$$
$$x = \frac{ln\,(4 \pm 2\sqrt{2})}{ln\,2}$$
$$= 2.772 \text{ or } 0.228$$

[4 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

Lesson 11

Revision

11.1 Summary

The First Rule

$$\log_c (ab) = \log_c a + \log_c b$$

The Second Rule

$$\log_c \left(\frac{a}{b}\right) = \log_c a - \log_c b$$

The Third Rule

$$\log_c a^n = n \log_c a$$

The "Jump Out of Logs" Manoeuvre

$$\log_c a = b$$
$$\Leftrightarrow \quad c^b = a$$

Two Special Results

$$\log_c c = 1 \qquad\qquad \log_c 1 = 0$$

11.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 40

Question 1

Solve the equation

$$7^x = 5$$

Give your answer correct to 3 decimal places.

[2 marks]

Question 2

By first using a Law of Indices, or otherwise, solve the equation,

$$5^x \times 5^{2x+1} = 10$$

Give your answer to three decimal places.

[3 marks]

Question 3

Solve the following equation

$$log_5 (2x + 3) - log_5 x = log_5 8$$

[3 marks]

Question 4

Solve the following equation

$$2\, log_7 (x - 6) - log_7 x = log_7 3$$

[5 marks]

Question 5
Solve the equation

$$2^{2x+1} - 2^{x} = 15$$

giving your solutions correct to three decimal places.

[5 marks]

Question 6

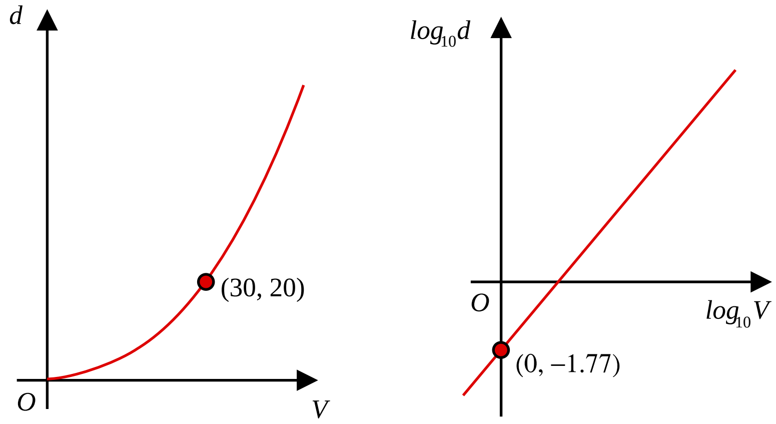
A-Level Examination Question from June 2019, Paper 2, Q9 (Edexcel)

A research engineer is testing the effectiveness of the braking system of a car when it is driven in wet conditions.

The engineer measures and records the braking distance, d metres, when the brakes are applied from a speed of V km h^{-1}

Graphs of d against V and $log_{10} d$ against $log_{10} V$ were plotted.

The results are shown below together with a data point from each graph.



(a) Explain how the rightmost graph would lead the engineer to believe that the braking distance should be modelled by the formula,

$d = k V^n$ where k and n are constants with $k \approx 0.017$

[3 marks]

Using the information given in the leftmost graph, with $k = 0.017$

(b) find a complete equation for the model giving the value of n to 3 significant figures.

[3 marks]

Sean is driving this car at 60 km h^{-1} in wet conditions when he notices a large puddle in the road 100 m ahead.
It takes him 0.8 seconds to react before applying the brakes.

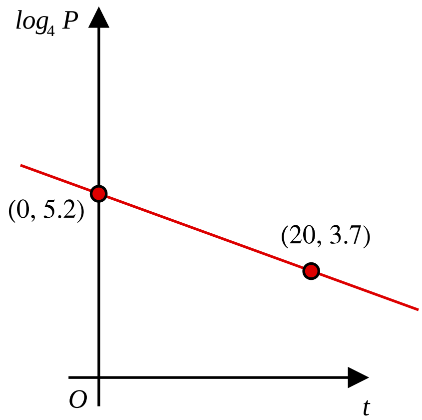
(c) Use your formula to find out if Sean will be able to stop before reaching the puddle.

[3 marks]

Question 7

The number of people, P, left in a concert venue t minutes after the band stops playing can be modelled by an equation of the form $P = a\,b^t$

The diagram shows the graph of $log_4 P$ against t



(a) Write down the equation of the line shown on the graph.

[2 marks]

(b) Find the number of people in the venue when $t = 0$, correct to 3 significant figures.

[1 mark]

(c) Find the values of a and b in the model, correct to 3 significant figures.

[2 marks]

(d) Find the number of people half an hour after the band stopped playing.

[1 mark]

Question 8

A-Level Examination Question from May 2016, Paper C2, Q8 (Edexcel)

(i) Given that

$$log_3(3b + 1) - log_3(a - 2) = -1, \qquad a > 2$$

express b in terms of a

[3 marks]

(ii) Solve the equation

$$2^{2x+5} - 7\left(2^{x}\right) = 0$$

giving your answer to 2 decimal places.

[4 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted “**Independent School of the Year 2020**”
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

11.3 Answers

Answer 1

$$7^x = 5$$

Take the natural log of both sides

$$\ln 7^x = \ln 5$$

$$x \ln 7 = \ln 5$$

$$x = \frac{\ln 5}{\ln 7}$$

$$= 0.827$$

[2 marks]

Answer 2

$$5^x \times 5^{2x+1} = 10$$

$$5^{3x+1} = 10$$

Take the natural log of both sides

$$\ln 5^{3x+1} = \ln 10$$

$$(3x + 1) \ln 5 = \ln 10$$

$$3x + 1 = \frac{\ln 10}{\ln 5}$$

$$3x = 1.4307 - 1$$

$$x = \frac{0.4307}{3}$$

$$= 0.144$$

[3 marks]

Answer 3

$$\log_5 (2x + 3) - \log_5 x = \log_5 8$$

$$\log_5 \left(\frac{2x + 3}{x} \right) = \log_5 8$$

$$\frac{2x + 3}{x} = 8$$

$$2x + 3 = 8x$$

$$6x = 3$$

$$x = \frac{1}{2}$$

[3 marks]

Answer 4

$$2\,log_7(\,x - 6\,) - log_7 x = log_7 3$$
$$log_7\,(\,x - 6\,)^2 - log_7 x = log_7 3$$
$$log_7\left(\frac{(\,x - 6\,)^2}{x}\right) = log_7 3$$
$$\frac{(\,x - 6\,)^2}{x} = 3$$
$$x^2 - 12x + 36 = 3x$$
$$x^2 - 15x + 36 = 0$$
$$(\,x - 3\,)(\,x - 12\,) = 0$$

Either $x = 3$ or $x = 12$

Reject $x = 3$ as it gives rise to $log_7(\,3 - 6\,)$ which is not a valid entity.

$\therefore$ The only solution is $x = 12$

[5 marks]

Answer 5

$$2^{2x+1} - 2^x = 15$$
$$2^{2x} \times 2^1 - 2^x - 15 = 0$$
$$2\left(\,2^x\,\right)^2 - \left(\,2^x\,\right) - 15 = 0$$

Let $w = 2^x$,

$$2\,w^2 - w - 15 = 0$$
$$(\,2w + 5\,)(\,w - 3\,) = 0$$

Either $w = -\frac{5}{2}$ which has no solutions

or $w = 3$

$$2^x = 3$$
$$ln\,2^x = ln\,3$$
$$x\,ln\,2 = ln\,3$$
$$x = \frac{ln\,3}{ln\,2}$$
$$= 1.585$$

[5 marks]

Answer 6

(a) The rightmost graph is a straight line and so has an equation of the form,

$$y = mx + c$$

The y-axis intercept is given as, $c = -1.77$ and so,

$$y = mx - 1.77$$

$$log_{10} d = m\, log_{10} V - 1.77$$

$$log_{10} d = log_{10} V^m - 1.77$$

$$log_{10} V^m - log_{10} d = 1.77$$

$$log_{10}\left(\frac{V^m}{d} \right) = 1.77$$

$$10^{1.77} = \frac{V^m}{d}$$

$$d = \frac{V^m}{10^{1.77}}$$

$$\therefore \ d = 0.017\, V^m$$

which is of the form $d = k\, V^n$ with $k \approx 0.017$ □

[3 marks]

(b) From the leftmost graph it is known that $d = 20$ when $V = 30$

$$\therefore \ 20 = 0.017 \times 30^n$$

$$\left(\frac{20}{0.017} \right) = 30^n$$

Take the natural log of both sides

$$ln\left(\frac{20}{0.017} \right) = ln\, 30^n$$

$$n\, ln\, 30 = ln\,(1176.471)$$

$$n = \frac{ln\, 1176.471}{ln\, 30}$$

$$= 2.08$$

$$\therefore \ d = 0.017 \times V^{2.08}$$

[3 marks]

(c)

$$d = 0.017 \times 60^{2.08}$$

$= 84.92$ metres is the stopping distance.

$$Reaction\ Distance = speed \times time = \frac{60 \times 1000}{60 \times 60} \times 0.8 = 13.33 \text{ metres}$$

As $84.92 + 13.33 = 98.25 < 100$ Sean is able to stop in time.

[3 marks]

Answer 7

(a)

$$m = \frac{\Delta y}{\Delta x} = \frac{3.7 - 5.2}{20 - 0} = -\frac{3}{40}$$

$$y = mx + c \Rightarrow y = -\frac{3}{40}x + 5.2$$

$$log_4 P = -\frac{3}{40}t + 5.2 \qquad \left(-\frac{3}{40} = -0.075\right)$$

[2 marks]

(b) When $t = 0$,

$$log_4 P = 5.2$$

$$4^{5.2} = P$$

$$P = 1350 \quad \text{(3 significant figures asked for)}$$

[1 mark]

(c)

$$log_4 P = -\frac{3}{40}t + 5.2$$

$$P = 4^{-0.075t + 5.2}$$

$$P = 4^{5.2} \times 4^{-0.075t}$$

$$P = 1350 \times \left(4^{-0.075}\right)^t$$

$$P = 1350 \times 0.901^t$$

$$\therefore\ a = 1350 \text{ and } b = 0.901$$

[2 marks]

(d)

$$P = 1350 \times 0.901^{30}$$

$$= 59 \text{ people}$$

[1 mark]

Answer 8

(i)

$$\log_3(3b+1) - \log_3(a-2) = -1$$

$$\log_3\left(\frac{3b+1}{a-2}\right) = -1$$

$$3^{-1} = \frac{3b+1}{a-2}$$

$$3b+1 = \frac{a-2}{3}$$

$$3b = \frac{a-2}{3} - \frac{3}{3}$$

$$3b = \frac{a-5}{3}$$

$$b = \frac{a-5}{9}$$

[3 marks]

(ii)

$$2^{2x+5} - 7(2^x) = 0$$

$$2^{2x} \times 2^5 - 7(2^x) = 0$$

$$32(2^x)^2 - 7(2^x) = 0$$

$$(2^x)(32 \times 2^x - 7) = 0$$

Either $2^x = 0$ which has no solutions

Or $2^x = \frac{7}{32}$

$$\ln 2^x = \ln\left(\frac{7}{32}\right)$$

$$x \ln 2 = \ln\left(\frac{7}{32}\right)$$

$$x = \frac{\ln\left(\frac{7}{32}\right)}{\ln 2}$$

$$= -2.19$$

[4 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

Lesson 12

A-Level Pure Mathematics : Year 1
Exponentials and Logarithms

12.1 TEST

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 50

Question 1

Solve the equation

$$3^x = 65$$

Give your answer correct to 3 significant figures.

[2 marks]

Question 2

By first using a Law of Indices, or otherwise, solve the equation,

$$7^{3x} \times 7^{2x+3} = 43$$

Give your answer to three decimal places.

[3 marks]

Question 3
Solve the following equation, giving the exact answer.

$$log_2(5x + 3) - log_2 x = log_2 12$$

[3 marks]

Question 4
Solve the following equation

$$2\,log_7(x - 4) - log_7 x = log_7 2$$

[5 marks]

Question 5
Solve the equation

$$5^{2x} - 13\left(5^{x}\right) + 22 = 0$$

giving your answers correct to three decimal places.

[4 marks]

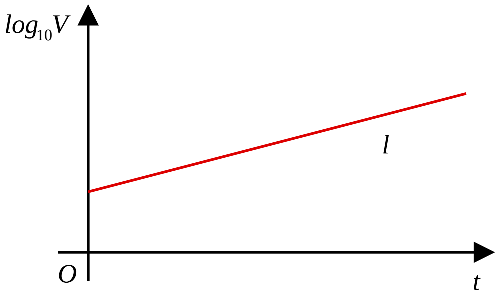
Question 6
Solve the equation

$$log_2 (3x + 1) + log_2 (2x) = 3$$

[4 marks]

Question 7

As-Level Examination Question from May 2018, Paper 1, Q13 (Edexcel)



The value of a rare painting, £V, is modelled by the equation $V = pq^t$, where p and q are constants and t is the number of years since the value of the painting was first recorded on 1st January 1980.

The line l shown on the graph illustrates the linear relationship between t and $log_{10}V$ since 1st January 1980.

The equation of line l is $log_{10}V = 0.05t + 4.8$

(a) Find, to 4 significant figures, the value of p and the value of q.

[4 marks]

(b) With reference to the model interpret

(i) the value of the constant p

[1 mark]

(ii) the value of the constant q

[1 mark]

(c) Find the value of the painting, as predicted by the model, on 1st January 2010.
Give your answer to the nearest thousand pounds.

[2 marks]

Question 8

A-Level Sample Assessment Materials from 2017, Paper 1, Q12 (Edexcel)

In a controlled experiment, the number of microbes, N, present in a culture T days after the start of the experiment were counted.

N and T are expected to satisfy a relationship of the form,

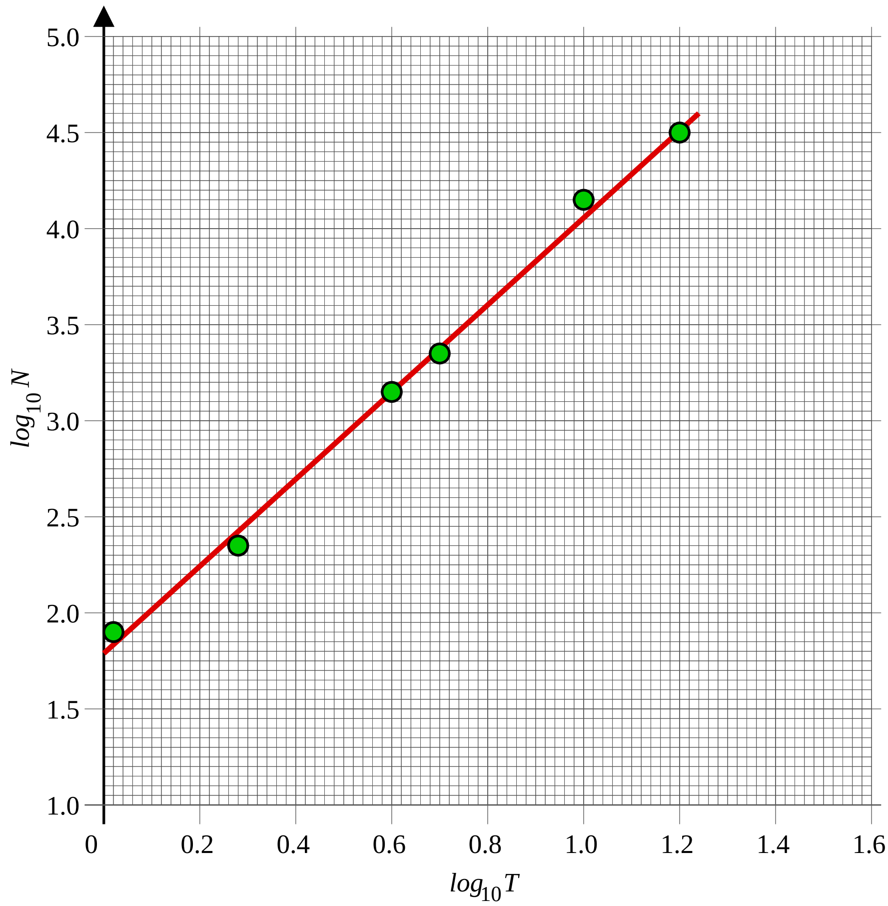
$$N = aT^b, \qquad \text{where } a \text{ and } b \text{ are constants}$$

(a) Show that this relationship can be expressed in the form

$$log_{10} N = m\, log_{10} T + c$$

giving m and c in terms of the constants a and/or b.

[2 marks]



The graph shows the line of best fit for values of $log_{10} N$ against values of $log_{10} T$

(b) Use the information provided to estimate the number of microbes present in the culture 3 days after the start of the experiment.

[4 marks]

(c) Explain why the information provided could not reliably be used to estimate the day when the number of microbes in the culture first exceeds 1000000.

[2 marks]

(d) With reference to the model, interpret the value of the constant a

[1 mark]

Question 9

The density of a pesticide in a section of field, P mg/m^2, can be modelled by the equation,

$$P = 160\, e^{-0.063\,t}$$

where t is the time in days since the pesticide was first applied.

(a) What was the initial density of the pesticide immediately after it had been applied to the field ?

[1 mark]

(b) By solving an appropriate mathematical equation, find the number of days that must pass before the density of the pesticide has fallen to 80 mg/m^2

[3 marks]

(c) The farmer wishes to harvest the crop of carrots that are being grown in another field in twenty days time but has a problem with Carrot Rust Flies and needs to spray the field.
The field has not been sprayed in the past.
When he harvests the crop, health and safety regulations state that the density of pesticide must be below 36 mg/m^2
What is the maximum density of pesticide that he can immediately apply such that he can still harvest the crop as planned ?

[3 marks]

Question 10

(i) Given that

$$9^x = 3^a$$

express a in terms of x

[1 mark]

(ii) Hence, or otherwise, solve the following equation

$$9^x - 2 \times 3^{x+1} + 8 = 0$$

Give your answers to 3 decimal places.

[4 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

12.2 Answers

Answer 1

$$3^x = 65$$

Take the natural log of both sides

$$ln\,3^x = ln\,65$$

$$x\,ln\,3 = ln\,65$$

$$x = \frac{ln\,65}{ln\,3}$$

$$= 3.80$$

[2 marks]

Answer 2

$$7^{3x} \times 7^{2x+3} = 43$$

$$7^{5x+3} = 43$$

Take the natural log of both sides

$$ln\,7^{5x+3} = ln\,43$$

$$(5x + 3)\,ln\,7 = ln\,43$$

$$5x + 3 = \frac{ln\,43}{ln\,7}$$

$$5x = 1.9337 - 3$$

$$x = -\frac{1.0671}{5}$$

$$= -0.213$$

[3 marks]

Answer 3

$$log_2(5x + 3) - log_2 x = log_2 12$$

$$log_2\left(\frac{5x + 3}{x}\right) = log_2 12$$

$$\frac{5x + 3}{x} = 12$$

$$5x + 3 = 12x$$

$$7x = 3$$

$$x = \frac{3}{7}$$

[3 marks]

Answer 4

$$2\,log_7\,(x - 4) - log_7\,x = log_7\,2$$

$$log_7\,(x - 4)^2 - log_7\,x = log_7\,2$$

$$log_7\left(\frac{(x - 4)^2}{x}\right) = log_7\,2$$

$$\frac{(x - 4)^2}{x} = 2$$

$$x^2 - 8x + 16 = 2x$$

$$x^2 - 10x + 16 = 0$$

$$(x - 2)(x - 8) = 0$$

Either $x = 2$ or $x = 8$

Reject $x = 2$ as it gives rise to $log_7(2 - 6)$ which is not a valid entity.

$\therefore$ The only solution is $x = 8$

[5 marks]

Answer 5

$$5^{2x} - 13\left(5^x\right) + 22 = 0$$

$$\left(5^x\right)^2 - 13\left(5^x\right) + 22 = 0$$

Let $w = 5^x$,

$$w^2 - 13w + 22 = 0$$

$$(w - 2)(w - 11) = 0$$

Either $w = 2$ or $w = 11$

$5^x = 2$	$5^x = 11$
$ln\,5^x = ln\,2$	$ln\,5^x = ln\,11$
$x\,ln\,5 = ln\,2$	$x\,ln\,5 = ln\,11$
$x = \dfrac{ln\,2}{ln\,5}$	$x = \dfrac{ln\,11}{ln\,5}$
$= 0.431$	$x = 1.490$

[4 marks]

Answer 6

$$\log_2(3x+1)+\log_2(2x)=3$$
$$\log_2\big((3x+1)(2x)\big)=3$$
$$6x^2+2x=2^3$$
$$3x^2+x=4$$
$$3x^2+x-4=0$$
$$(3x+4)(x-1)=0$$

Either $x=-\dfrac{4}{3}$ which is rejected as it does not satisfy the original equation.

or $x=1$

[4 marks]

Answer 7

(a)

$$\log_{10}V=0.05t+4.8$$
$$V=10^{0.05t+4.8}$$
$$=10^{0.05t}\times 10^{4.8}$$
$$=63100\times\left(10^{0.05}\right)^t$$
$$=63100\times 1.122^t$$
$$\therefore\ p=63100 \text{ and } q=1.122$$

[4 marks]

(b) **(i)** p is the value of the painting on 1st January 1980.

[1 mark]

(ii) q is the annual rate at which the value of the painting is increasing.
1.122 corresponds to an annual growth rate of 12.2 %

[1 mark]

(c) As $t=0$ corresponded to 1st January 1980,
$t=30$ will correspond to 1st January 2010.

$$\therefore\ V=63100\times 1.122^{30}$$
$$=\pounds\,1\,994\,000$$

[2 marks]

Answer 8

(a) $$N = a\,T^b$$

Take the log to the base 10 of both sides

$$log_{10} N = log_{10}\left(a\,T^b \right)$$

$$log_{10} N = log_{10} a + log_{10} T^b$$

$$log_{10} N = b\, log_{10} T + log_{10} a$$

which is in the form requested with $m = b$ and $c = log_{10} a$

[2 marks]

(b) From the graph, the y-axis intercept is 1.8

$$log_{10} a = 1.8$$

$$a = 10^{1.8}$$

$$= 63.1$$

From the graph, picking two points on the line, far apart,

$$(0,\ 1.8) \text{ and } (1.2,\ 4.5) \Rightarrow m = \frac{4.5 - 1.8}{1.2 - 0} = \frac{9}{4} = 2.25 \ (= b)$$

$$N = a\,T^b$$

$$= 63.1 \times 3^{2.25}$$

$$= 747 \text{ microbes}$$

[4 marks]

(c) From the graph, the largest value looked at was,

$$log_{10} N = 4.5$$

$$N = 10^{4.5}$$

$$= 31600 \text{ microbes}$$

As 1000000 microbes is well outside of the known range of data, no reliable estimate can be made.

[2 marks]

(d) The constant a is the number of microbes, when $T = 1$.
That is, about 63 microbes, one day after the experiment began.

[1 mark]

Answer 9

(a) Let $t = 0$ in the equation $P = 160\,e^{-0.063\,t}$ to get $P_0 = 160$ mg/m^2

[1 mark]

(b) Let $P = 80$,

$$80 = 160\,e^{-0.063\,t}$$

Divide both sides by 160

$$0.5 = e^{-0.063\,t}$$

Take the natural log of both sides

$$ln\,0.5 = -0.063\,t$$

$$t = \frac{ln\,0.5}{-0.063}$$

$$= 11 \text{ days}$$

[3 marks]

(c)

$$36 = a\,e^{-0.063 \times 20}$$

$$a = \frac{36}{e^{-1.26}}$$

$$= 127 \text{ mg/m}^2$$

[3 marks]

Answer 10

(i) $a = 2x$

[1 mark]

(ii)

$$9^x - 2 \times 3^{x+1} + 8 = 0$$

$$\left(3^x \right)^2 - 6\left(3^x \right) + 8 = 0$$

Let $w = 3^x$

$$w^2 - 6w + 8 = 0$$

$$(w - 2)(w - 4) = 0$$

Either $w = 2$ or $w = 4$

$3^x = 2$ $\qquad$ $3^x = 4$

$x = \dfrac{ln\,2}{ln\,3}$ $\qquad$ $x = \dfrac{ln\,4}{ln\,3}$

$= 0.631$ $\qquad$ $= 1.262$

[4 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

Later Date Revision

13.1 The Rules Of Logs

The First Rule

$$log_c\,(ab) \;=\; log_c\,a \;+\; log_c\,b$$

The Second Rule

$$log_c\left(\frac{a}{b}\right) \;=\; log_c\,a \;-\; log_c\,b$$

The Third Rule

$$log_c\,a^n \;=\; n\,log_c\,a$$

The "Jump Out of *logs*" Manoeuvre

$$log_c\,a \;=\; b$$

$$\Leftrightarrow \quad c^b \;=\; a$$

Two Special Results

$$log_c\,c \;=\; 1 \qquad\qquad log_c\,1 \;=\; 0$$

13.2 STARTER

- Toby wishes to solve the equation

$$3^x = -1$$

By sketching a graph, explain why this equation has no solutions.

- Karen wishes to solve the equation

$$x = log(-2.4)$$

By sketching a graph, explain why this equation has no solutions.

13.3 The Worked Examples

Solve for x,

(i) $9^x - 2 \times 3^{1+x} - 7 = 0$

(ii) $2 \log_2 x - 3 = \log_2 (3x - 10)$

(iii) $3^{2x-1} = 2^{x+4}$

(iv) $4^{2x-1} = 2^{x+4}$

Answer (i)

$$9^x - 2 \times 3^{1+x} - 7 = 0$$
$$(3^2)^x - 2 \times 3 \times (3^x) - 7 = 0$$
$$(3^x)^2 - 6(3^x) - 7 = 0$$
$$z^2 - 6z - 7 = 0$$
$$(z - 7)(z + 1) = 0$$

Either $z - 7 = 0$ or $z + 1 = 0$

$z = 7$ $z = -1$

$3^x = 7$ $3^x = -1$

$\log 3^x = \log 7$ *no solutions*

$x \log 3 = \log 7$

$$x = \frac{\log 7}{\log 3}$$

$$x = 1.771$$

Answer (ii)

$$2\,log_2\,x - 3 = log_2\,(3x - 10)$$
$$2\,log_2\,x - log_2\,(3x - 10) = 3$$
$$log_2\,x^2 - log_2\,(3x - 10) = 3$$
$$log_2\left(\frac{x^2}{3x - 10}\right) = 3$$
$$2^3 = \frac{x^2}{3x - 10}$$
$$24x - 80 = x^2$$
$$x^2 - 24x + 80 = 0$$
$$(x - 20)(x - 4) = 0$$

Either $x = 20$ or $x = 4$

Answer (iii)

$$3^{2x-1} = 2^{x+4}$$
$$log\,3^{2x-1} = log\,2^{x+4}$$
$$(2x - 1)\,log\,3 = (x + 4)\,log\,2$$
$$2x\,log\,3 - log\,3 = x\,log\,2 + 4\,log\,2$$
$$2x\,log\,3 - x\,log\,2 = 4\,log\,2 + log\,3$$
$$x(2\,log\,3 - log\,2) = log\,2^4 + log\,3$$
$$x\left(log\,3^2 - log\,2\right) = log\,(16 \times 3)$$
$$x\,log\left(\frac{9}{2}\right) = log\,(48)$$
$$x = \frac{log\,48}{log\left(\frac{9}{2}\right)}$$

Answer (iv)

$$4^{2x-1} = 2^{x+4}$$
$$\left(2^2\right)^{2x-1} = 2^{x+4}$$
$$2^{4x-2} = 2^{x+4}$$
$$4x - 2 = x + 4$$
$$3x = 6$$
$$x = 2$$

13.4 Exercise

Question 1
Solve the equation

$$3^x(3^x + 4) = 5$$

[4 marks]

Question 2
Solve the equation

$$3^x = 65$$

Give your answer correct to 3 significant figures.

[3 marks]

Question 3

By first using a Law of Indices, or otherwise, solve the equation,

$$7^{3x} \times 7^{2x+3} = 43$$

Give your answer to three decimal places.

[4 marks]

Question 4

Solve the following equation, giving the exact answer.

$$log_2 (5x + 3) - log_2 x = log_2 12$$

[3 marks]

Question 5
Solve the following equation

$$2\,log_7\,(x - 4) - log_7\,x = log_7\,2$$

[5 marks]

Question 6
Solve the equation,

$$5^{2x} - 13(5^x) + 22 = 0$$

[5 marks]

Question 7

Solve the equation

$$log_2(3x + 1) + log_2(2x) = 3$$

[6 marks]

Question 8

(i) Given that

$$9^x = 3^a$$

express a in terms of x

(ii) Hence, or otherwise, solve the following equation

$$9^x - 2 \times 3^{x+1} + 8 = 0$$

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk

13.5 Answers

13.5.1 Answer to Exercise 13.4

Answer 1

$$3^x(3^x + 4) = 5$$

$$(3^x)^2 + 4(3^x) - 5 = 0$$

Let $z = 3^x$ in which case,

$$z^2 + 4z - 5 = 0$$

$$(z - 1)(z + 5) = 0$$

Either $z = 1$ or $z = -5$

That is, either $3^x = 1$ giving $x = 0$

or $3^x = -5$ which has no solutions.

[4 marks]

Answer 2

$$3^x = 65$$

Take the natural log of both sides

$$\ln 3^x = \ln 65$$

$$x \ln 3 = \ln 65$$

$$x = \frac{\ln 65}{\ln 3}$$

$$= 3.80 \text{ (Must be to 3 sf)}$$

[3 marks]

Answer 3

$$7^{3x} \times 7^{2x+3} = 43$$

$$7^{5x+3} = 43$$

Take the natural log of both sides

$$\ln 7^{5x+3} = \ln 43$$

$$5x + 3 = \frac{\ln 43}{\ln 7}$$

$$5x + 3 = 1.93287$$

$$5x = -1.06713$$

$$x = -0.213$$

[4 marks]

Answer 4

$$log_2(5x+3) - log_2 x = log_2 12$$

$$log_2(5x+3) = log_2 12 + log_2 x$$

$$log_2\left(\frac{5x+3}{x}\right) = log_2 12$$

$$5x+3 = 12x$$

$$7x = 3$$

$$x = \frac{3}{7}$$

[3 marks]

Answer 5

$$2\,log_7(x-4) - log_7 x = log_7 2$$

$$log_7(x-4)^2 = log_7 2 + log_7 x$$

$$log_7(x-4)^2 = log_7 2x$$

$$(x-4)^2 = 2x$$

$$x^2 - 8x + 16 = 2x$$

$$x^2 - 10x + 16 = 0$$

$$(x-2)(x-8) = 0$$

Either $x = 2$ or $x = 8$

However $x = 2$ is an extraneous solution !

The only solution is $x = 8$

[5 marks]

Answer 6

$$5^{2x} - 13(5^x) + 22 = 0$$

Let $z = 5^x$ in which case,

$$z^2 - 13z + 22 = 0$$

$$(z-2)(z-11) = 0$$

Either $z = 2$ or $z = 11$

$$5^x = 2 \qquad 5^x = 11$$

$$x = \frac{ln\,2}{ln\,5} \qquad x = \frac{ln\,11}{ln\,5}$$

$x = 0.431$ or $x = 1.490$

[5 marks]

Answer 7

$$log_2 (3x + 1) + log_2 (2x) = 3$$

$$log_2 \left((3x + 1)(2x) \right) = 3$$

$$log_2 \left(6 x^2 + 2x \right) = 3$$

$$2^3 = 6 x^2 + 2x \qquad \text{Jump out of logs rule}$$

$$6 x^2 + 2x - 8 = 0$$

Divide through by 2

$$3 x^2 + x - 4 = 0$$

$$(3x + 4)(x - 1) = 0$$

$$\text{Either } x = -\frac{4}{3} \text{ or } x = 1$$

However $x = -\frac{4}{3}$ is an extraneous solution !

The only solution is $x = 1$

[6 marks]

Answer 8

(i)

$$9^x = 3^a$$

$$\left(3^2 \right)^x = 3^a$$

$$a = 2x$$

[1 mark]

(ii)

$$9^x - 2 \times 3^{x+1} + 8 = 0$$

$$\left(3^x \right)^2 - 2 \left(3^x \right) 3 + 8 = 0$$

Let $z = 3^x$ in which case,

$$z^2 - 6z + 8 = 0$$

$$(z - 4)(z - 2) = 0$$

$$\text{Either } z = 4 \text{ or } z = 2$$

$$3^x = 4 \qquad 3^x = 2$$

$$x = \frac{ln\, 4}{ln\, 3} \qquad x = \frac{ln\, 2}{ln\, 3}$$

$$x = 1.262 \text{ or } x = 0.631$$

[5 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2023 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk