9.5 Answers

9.5.1 Solution to 9.2 Example #1 (1st Teaching Video)

From the equation of the radius, $y = \frac{3}{2}x - \frac{1}{2}$ the gradient of the radius m_r is $\frac{3}{2}$

$\therefore$ The tangent will have gradient m_t that is the sign changed reciprocal, $-\frac{2}{3}$

The tangent is a straight line through the point (6, 5),

$$y = mx + c$$

$$5 = -\frac{2}{3} \times 6 + c \Rightarrow c = 9$$

$\therefore$ The equation of the tangent is, $y = -\frac{2}{3}x + 9$

[2 marks]

9.5.2 Solution to 9.3 Example #2 (2nd Teaching Video)

(i) $Q(4, 5)$ Draw radius of 5 vertically between Q and (4, 0) to see this.

(ii) $(x - 4)^2 + (y - 5)^2 = 5^2$

(iii) Apply the Theorem of Pythagoras to ΔQTP which has a right angle at T
First get length PQ

$$|PQ| = \sqrt{(17 - 5)^2 + (8 - 4)^2}$$

$$= \sqrt{12^2 + 4^2}$$

$$= \sqrt{160}$$

$$|PT|^2 = |PQ|^2 - |QT|^2$$

$$= 160 - 5^2$$

$$= 135$$

$\therefore$ $|PT| = 11.6$ (requested to 3 significant figures)

[6 marks]

9.5.3 Solutions to 9.4 Exercise

Answer 1

(i) $(1, -2)$

[1 mark]

(ii) $\text{LHS} = (x - 1)^2 + (y + 2)^2$

$= (3 - 1)^2 + (1 + 2)^2$ if the circle passes through $(3, 1)$

$= 4 + 9$

$= 13$

$= \text{RHS}$

$\therefore$ $(3, 1)$ is on the circle ☐

[2 marks]

(iii) To find the gradient of the radius between centre $(1, -2)$ and $(3, 1)$

$$m_r = \frac{1 - (-2)}{3 - 1}$$

$$= \frac{3}{2}$$

[2 marks]

(iv) The straight line tangent will have a gradient of $-\dfrac{2}{3}$ through $(3, 1)$

$$y = mx + c$$

$$1 = -\frac{2}{3} \times 3 + c \Rightarrow c = 3$$

Equation of tangent is $y = -\dfrac{2}{3}x + 3$

[3 marks]

Answer 2

Centre is $(-5, 1)$ Point on circumference $(3, 2)$

$$m_r = \frac{2 - 1}{3 - (-5)} = \frac{1}{8}$$

Gradient of tangent, $m_t = -8$ (sign changed reciprocal)

$$y = mx + c$$

$$2 = -8 \times 3 + c \Rightarrow c = 26$$

Equation of tangent is $y = -8x + 26$

[5 marks]

Answer 3

$$x^2 + y^2 - 6x + 4y = 7$$

$$\left[x^2 - 6x \right] + \left[y^2 + 4y \right] = 7$$

$$\left[(x - 3)^2 - 9 \right] + \left[(y + 2)^2 - 4 \right] = 7$$

$$(x - 3)^2 + (y + 2)^2 = 20$$

$\therefore$ centre is at $(3, -2)$

Point on the circumference is $(1, 2)$

$$m_r = \frac{2 - (-2)}{1 - 3} = \frac{4}{(-2)} = -2$$

Gradient of tangent, $m_t = \dfrac{1}{2}$ (sign changed reciprocal)

$$y = mx + c$$

$$2 = \frac{1}{2} \times 1 + c \Rightarrow c = \frac{3}{2}$$

Equation of tangent is $y = \dfrac{1}{2}x + \dfrac{3}{2}$

[6 marks]

Answer 4

(i) $Q(8, 3)$ Draw radius of 8 horizontally between Q and $(0, 3)$ to see this.

[2 marks]

(ii) $(x - 8)^2 + (y - 3)^2 = 8^2$

[2 marks]

(iii) Apply the Theorem of Pythagoras to ΔQTP which has a right angle at T
First get length PQ

$$|PQ| = \sqrt{(19 - 8)^2 + (13 - 3)^2} = \sqrt{11^2 + 10^2} = \sqrt{221}$$

$$|PT|^2 = |PQ|^2 - |QT|^2 = 221 - 8^2 = 157$$

$\therefore |PT| = 12.5$ (requested to 3 significant figures)

[3 marks]

Answer 5

(i)

$$\text{LHS} = (x - 2)^2 + (y + 1)^2$$

$$= (8 - 2)^2 + (5 + 1)^2 \qquad \text{if the circle passes through } (8,\ 5)$$

$$= 6^2 + 6^2$$

$$= 36 + 36$$

$$= 72$$

$$\neq \text{RHS of } 16 \qquad \therefore (8,\ 5) \text{ is not on the circle} \qquad \square$$

[2 marks]

(ii) This is similar to Question 4 but without a diagram being provided.

First, obtain the distance, D, between the circle's centre $(2, -1)$ and $(8,\ 5)$

$$D = \sqrt{(8 - 2)^2 + (5 - (-1))^2}$$

$$= \sqrt{6^2 + 6^2}$$

$$= \sqrt{72}$$

$$= 6\sqrt{2}$$

Second, apply Pythagoras' Theorem to the right angled triangle with vertices at $(2, -1)$, $(8,\ 5)$ and where the radius meets the tangent to get the length requested, X

$$X = \sqrt{\left(\sqrt{72}\right)^2 - 4^2}$$

$$= \sqrt{72 - 16}$$

$$= 2\sqrt{14} \qquad \text{(about 7.48)}$$

[5 marks]

Answer 6

(a) **(i)**

$$x^2 + y^2 - 20x - 24y + 195 = 0$$

$$\left[x^2 - 20x \right] + \left[y^2 - 24y \right] + 195 = 0$$

$$\left[(x - 10)^2 - 100 \right] + \left[(y - 12)^2 - 144 \right] + 195 = 0$$

$$(x - 10)^2 + (y - 12)^2 = 49$$

$\therefore$ Centre is $M (10, \ 12)$

[3 marks]

(ii)

Radius is 7

[2 marks]

(b)

$$| MN | = \sqrt{(25 - 10)^2 + (32 - 12)^2}$$

$$= \sqrt{15^2 + 20^2}$$

$$= 25$$

[2 marks]

(c) Use the Theorem of Pythagoras to get the unknown side in ΔMNP

$$| NP |^2 = | MN |^2 - | MP |^2$$

$$= 25^2 - 7^2$$

$$= 24$$

[2 marks]

Answer 7

(a) **(i)**

$$x^2 + y^2 - 8x - 10y + 16 = 0$$

$$\left[x^2 - 8x \right] + \left[y^2 - 10y \right] + 16 = 0$$

$$\left[(x - 4)^2 - 16 \right] + \left[(y - 5)^2 - 25 \right] + 16 = 0$$

$$(x - 4)^2 + (y - 5)^2 = 25$$

$\therefore$ Centre is $T\,(4,\ 5)$

[2 marks]

(ii)

Radius is 5

[1 mark]

(b)

$$|MT| = \sqrt{(20 - 4)^2 + (12 - 5)^2}$$

$$= \sqrt{16^2 + 7^2}$$

$$= \sqrt{305}$$

[2 marks]

(c) Use the Theorem of Pythagoras to get the unknown side in $\triangle MPT$

$$|PM|^2 = |MT|^2 - |PT|^2$$

$$= 305 - 25$$

$$= 280$$

$$|PM| = \sqrt{280}$$

$$= 2\sqrt{70}$$

$$Area\ \triangle MPT = \frac{1}{2}\,b\,h$$

$$= \frac{1}{2} \times 2\sqrt{70} \times 5$$

$$= 5\sqrt{70} \qquad \text{(about 41.8)}$$

[3 marks]

Answer 8

(i) $L_1 \;:\; 3x - y = 1 \Rightarrow y = 3x - 1 \Rightarrow L_1$ has gradient 3

$\therefore L_2$ will have sign changed reciprocal gradient $-\dfrac{1}{3}$ through $P(8,\ 3)$

$$y = mx + c$$

$$3 = -\frac{1}{3} \times 8 + c \Rightarrow c = \frac{17}{3}$$

$$L_2 \text{ has equation } y = -\frac{1}{3}x + \frac{17}{3}$$

[3 marks]

(ii) L_1 and L_2 intersect when $3x - 1 = -\dfrac{1}{3}x + \dfrac{17}{3}$

multiply through by 3

$$9x - 3 = -x + 17$$

$$10x = 20$$

$$x = 2$$

$\therefore$ Point of intersection is $Q(2,\ 5)$

[3 marks]

(iii)
$$|PQ| = \sqrt{(2 - 8)^2 + (5 - 3)^2}$$

$$= \sqrt{6^2 + 2^2}$$

$$= \sqrt{40} = 2\sqrt{10}$$

[2 marks]

(iv) Draw a diagram with $P(8,\ 3)$, $Q(2,\ 5)$, L_1 of $y = 3x - 1$,

L_2 of $y = -\dfrac{1}{3}x + \dfrac{17}{3}$ and perpendicular to L_1 and $|PQ| = \sqrt{40}$

It's then obvious that the circle described will have equation,

$$(x - 8)^2 + (y - 3)^2 = 40$$

[1 mark]

(v) The vector to go from $Q(2,\ 5) \rightarrow P(8,\ 3)$ is $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$

So this vector applied to P gets to $(14,\ 1)$

$$y = mx + c$$

$$1 = 3 \times 14 + c \Rightarrow c = -41$$

Other tangent is $y = 3x - 41$

[3 marks]

Answer 9

(a) The radius can be read off the diagram as it's the horizontal line between $F(8,\ 4)$ and $A(3,\ 4)$. That is, 5 cm

$$(x - 8)^2 + (y - 4)^2 = 25$$

[3 marks]

(b) **(i)** $m_{FB} = \dfrac{8 - 4}{11 - 8} = \dfrac{4}{3}$

$\therefore\ m_{BC} = -\dfrac{3}{4}$ (sign changed reciprocal)

$y = mx + c$

$8 = -\dfrac{3}{4} \times 11 + c \Rightarrow c = \dfrac{65}{4}$

$$y = -\frac{3}{4}x + \frac{65}{4}$$

[4 marks]

(ii) C is where the tangent of (b) (i) cuts the x-axis

$$0 = -\frac{3}{4}x + \frac{65}{4}$$

$$0 = -3x + 65$$

$$3x = 65$$

$$x = \frac{65}{3} \qquad \therefore C\left(\frac{65}{3},\ 0\right)$$

$$|BC| = \sqrt{\left(\frac{65}{3} - 11\right)^2 + 8^2}$$

$$= \sqrt{\left(\frac{32}{3}\right)^2 + 64}$$

$$= \frac{40}{3}$$

$$\text{Total Length} = 4 + 11.07 + \frac{40}{3}$$

$$= 28.4 \text{ cm}$$

[5 marks]

Answer 10

(i)

$$(x - 0)^2 + (y - 3)^2 = 3^2$$
$$x^2 + y^2 - 6y + 9 = 9$$
$$x^2 + y^2 - 6y = 0$$
$$\therefore\ k = 6$$

[2 marks]

(ii) There is a clever way of doing this from a diagram but the algebraic approach is as follows,

The circle and the line intersect when,

$$x^2 + y^2 - 6y = 0 \text{ meets } y = mx - 2$$
$$x^2 + (mx - 2)^2 - 6 (mx - 2) = 0$$
$$x^2 + m^2 x^2 - 4mx + 4 - 6mx + 12 = 0$$
$$\left(m^2 + 1 \right) x^2 - 10mx + 16 = 0$$

For $a x^2 + bx + c = 0$ to have only one solution, $b^2 - 4ac = 0$

$$\text{from } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore (-10m)^2 - 4\left(m^2 + 1 \right) \times 16 = 0$$
$$100 m^2 - 64 m^2 - 64 = 0$$
$$36 m^2 = 64$$
$$m^2 = \frac{64}{36}$$
$$= \frac{16}{9}$$
$$m = \pm \frac{4}{3}$$

[6 marks]

(iii) In $\triangle PCA$, PC is of length 5 (along the y-axis) and CA is a radius of the circle of length 3
So the Theorem of Pythagoras gives the length PA to be 4
By symmetry PB has the same length as PA

[4 marks]