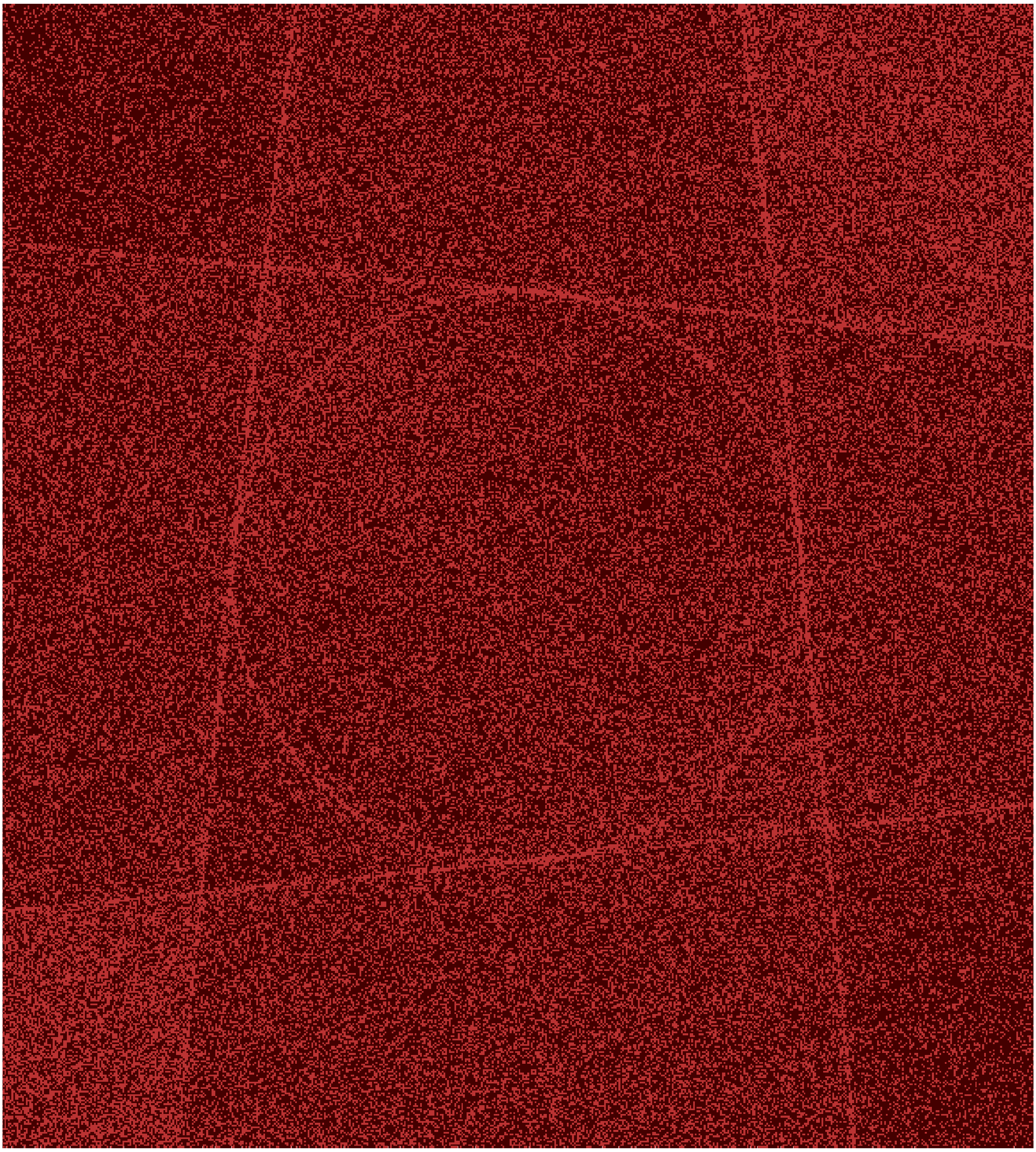
COORDINATE GEOMETRY



Lines • Circles • Curves • Tangents • Normals

COORDINATE GEOMETRY
The Straight Line

Lesson 1

A-Level Pure Mathematics, Year 1
Additional Mathematics
GCSE
Coordinate Geometry

1.1 Gradient of Straight Lines

Most straight lines can be written in the form, $y = mx + c$

where m is the gradient of the line
and c is the y axis intercept

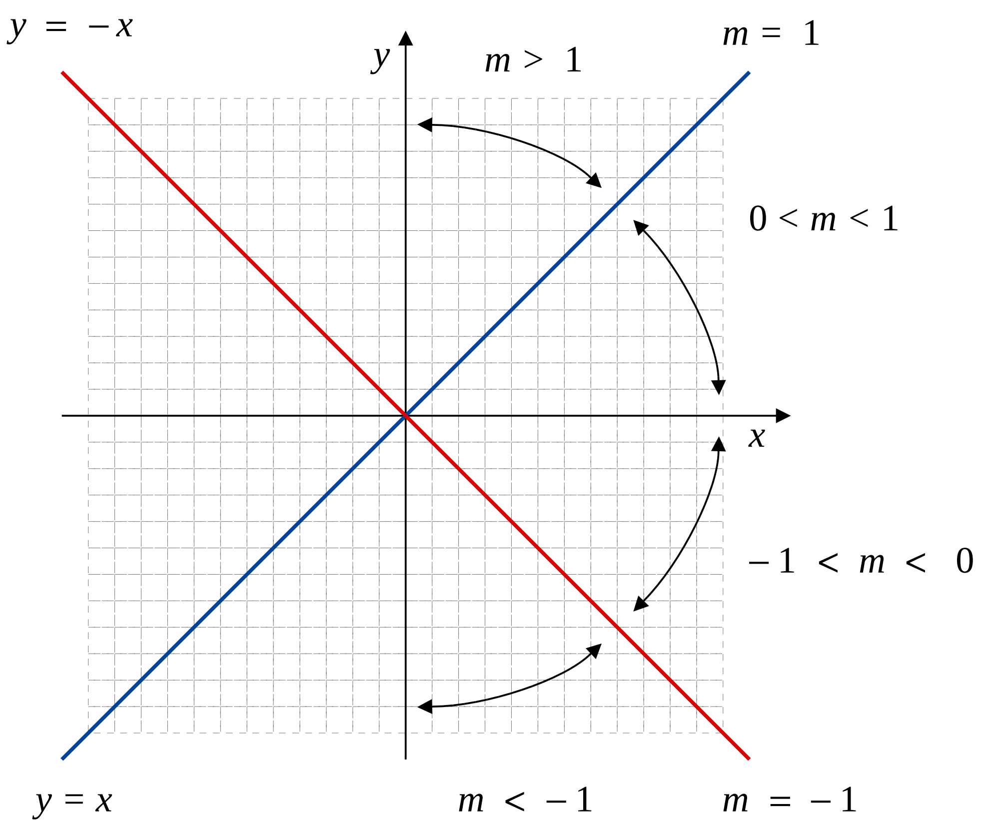
The exception is vertical lines, such as, for example $x = 3$.

The gradient between two points $A(x_a, y_a)$ and $B(x_b, y_b)$ is given by,

$$m = \frac{y_b - y_a}{x_b - x_a}$$

This is often written, $m = \dfrac{\Delta Y}{\Delta X}$, which some remember as, $m = \dfrac{rise}{run}$

By eye, graphs are always read from left to right.
A line with height that is increasing, left to right, has a positive gradient.
By making $\Delta X = 1$ the gradient becomes what you go up by, for every 1 moved across.



1.2 "Together" Exercise

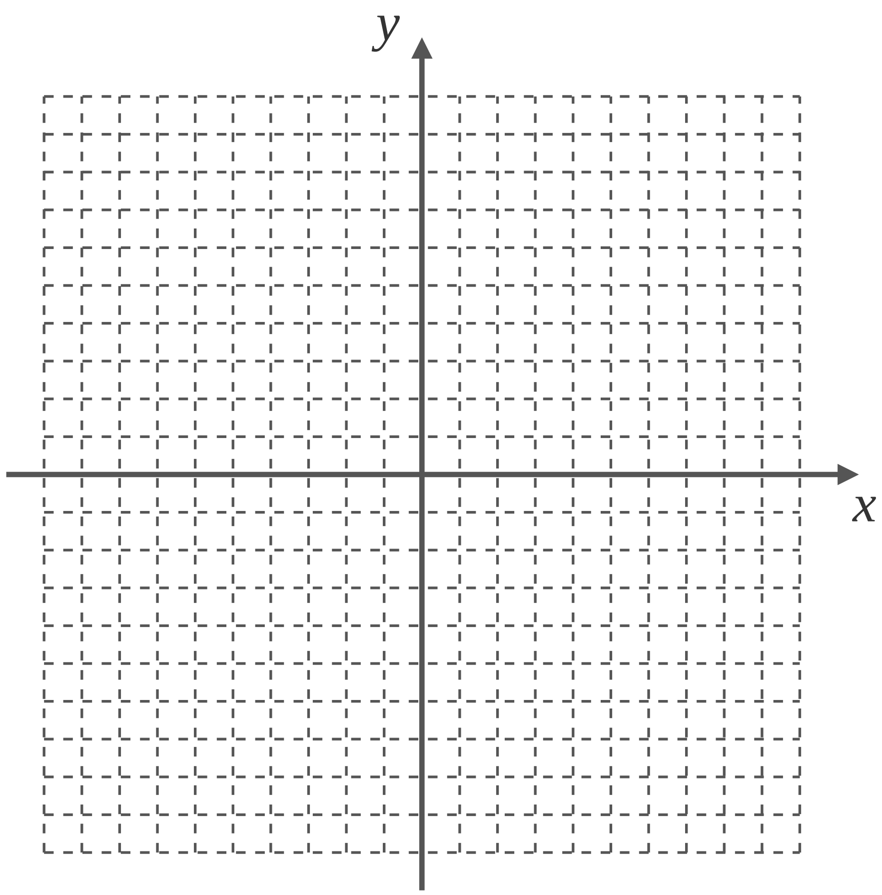
Question 1

(i) On the graph below plot the lines with equations;

$y = 3x + 1$ $\quad\quad$ $y = \frac{1}{2}x - 4$ $\quad\quad$ $y = -2x + 11$

Clearly show which equation goes with which line

(ii) Shade in the triangle formed and mark on the triangle's right angle.



[5 marks]

Teaching Video : http://www.NumberWonder.co.uk/v9033/1a.mp4

Question 2

Without plotting a graph, find the equation of the line with gradient 2 through (5, 1)
Write your answer in the form $y = mx + c$

[3 marks]

Question 3

Without plotting a graph, find the equation of the line between the points A (2, 11) and B (5, 20).

[3 marks]

Teaching Video : http://www.NumberWonder.co.uk/v9033/1b.mp4

1.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

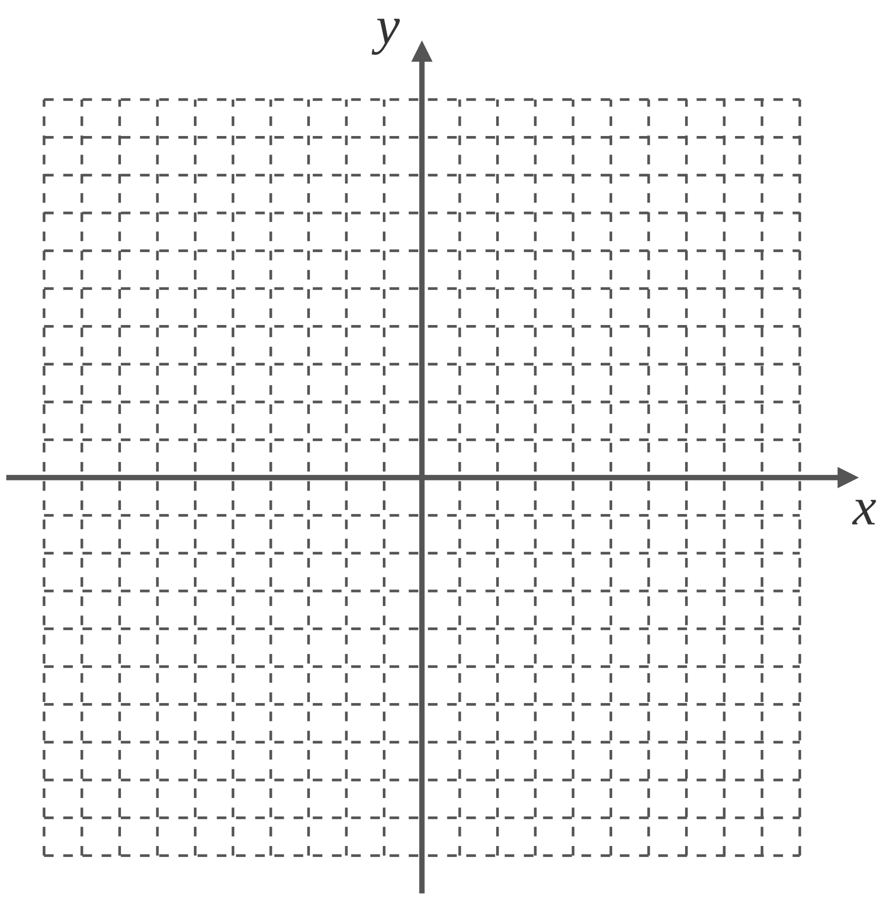
Question 1

(i) On the graph below plot the lines with equations;

$$y = 3x - 2 \qquad y = \frac{1}{2}x + 3 \qquad y = -2x - 7$$

Clearly show which equation goes with which line.

(ii) Shade in the triangle formed and mark on the triangle's right angle.



[5 marks]

Question 2

Without drawing a graph, find the equation of the line with gradient 3 through (2, 13)
Write your answer in the form $y = mx + c$

[3 marks]

Question 3
Without drawing a graph find the equation of the line between the points $A(2, 5)$ and $B(5, 17)$

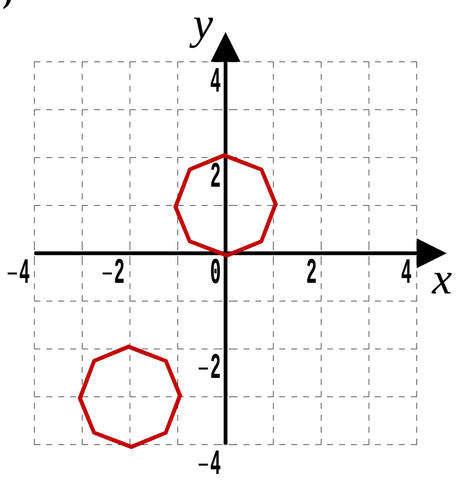
[3 marks]

Question 4
On each of the following graphs,

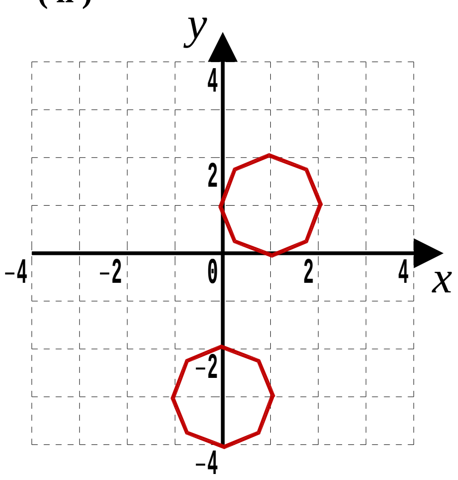
(a) Carefully draw a line that passes exactly through the two points at the centre of the octagons

(b) Write down the equation of the line, where possible, in the form $y = mx + c$

(i)

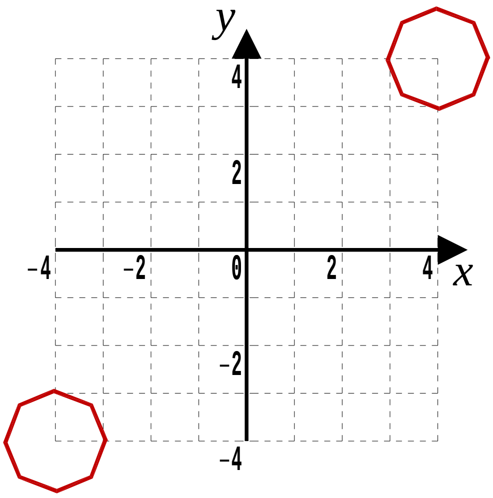


(ii)

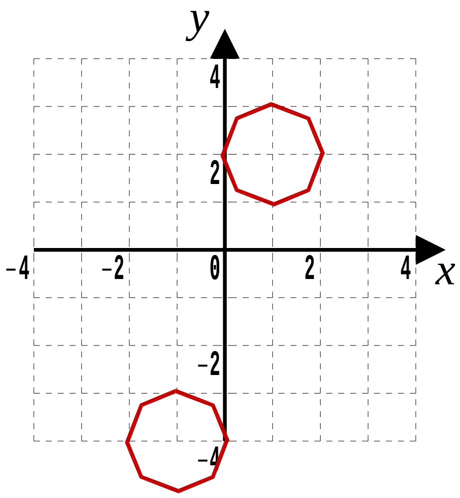


[2, 2 marks]

(iii)

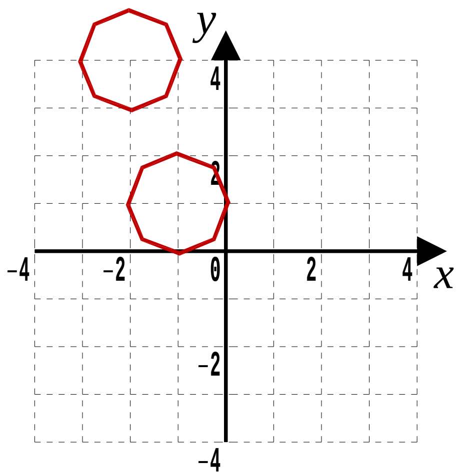


(iv)

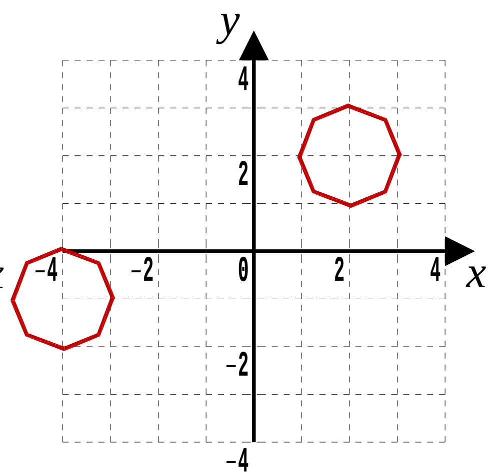


[2, 2 marks]

(v)

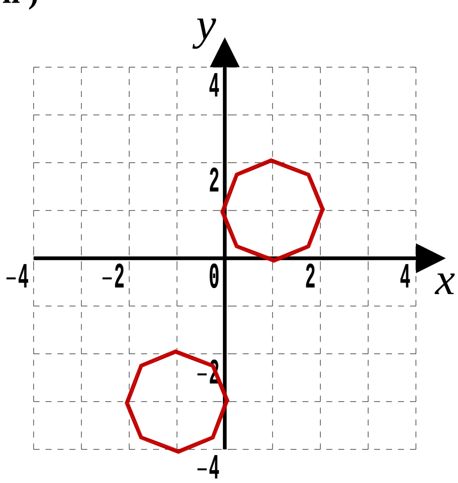


(vi)

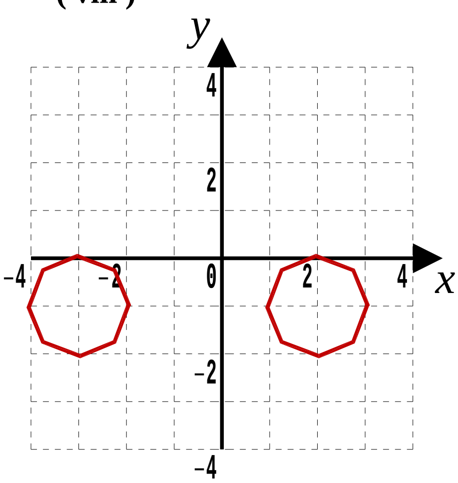


[2, 2 marks]

(vii)

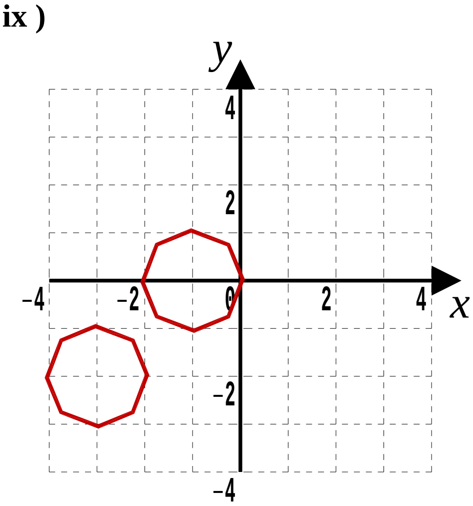


(viii)

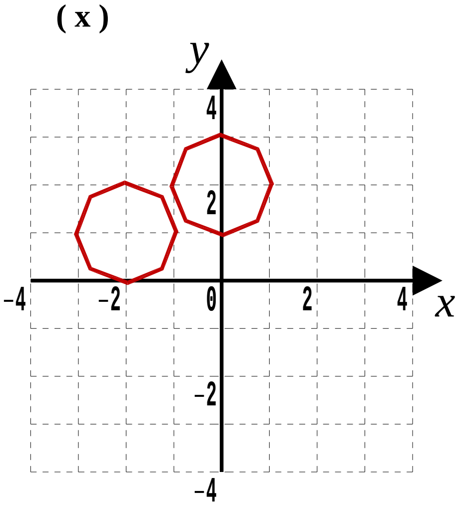


[2, 2 marks]

(ix)



(x)



[2, 2 marks]

Question 5

A question on the "throw a box around" method to find the area of a triangle.

On the graph below the *x*-axis runs from –10 to +10 and the *y*-axis does the same. Three straight lines are plotted.

(i) Next to each line, at a suitable place, write that line's equations.

[3 marks]

(ii) Calculate the area of triangle *A*

[1 mark]

(iii) Calculate the area of triangle *B*

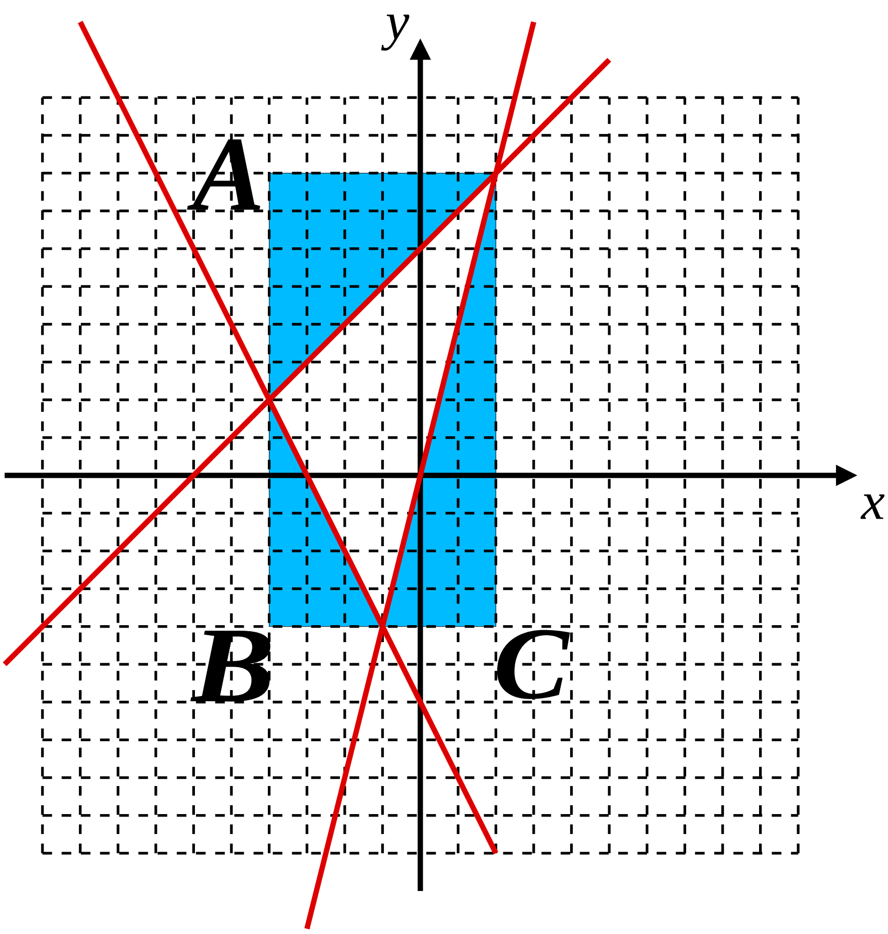
[1 mark]

(iv) Calculate the area of triangle *C*

[1 mark]

Hence, or otherwise, determine the area of the triangle enclosed by the three lines

[3 marks]



Question 6
Find the distance between the points A (2, 11) and B (5, 20).
Give your answer in the form $p\sqrt{10}$ where p is an integer to be found.

HINT : The Theorem of Pythagoras

[3 marks]

Question 7
Without drawing a graph, find the equation of the line with gradient 0.5 that passes through the point (12, 2), writing your answer in the form $y = mx + c$

[3 marks]

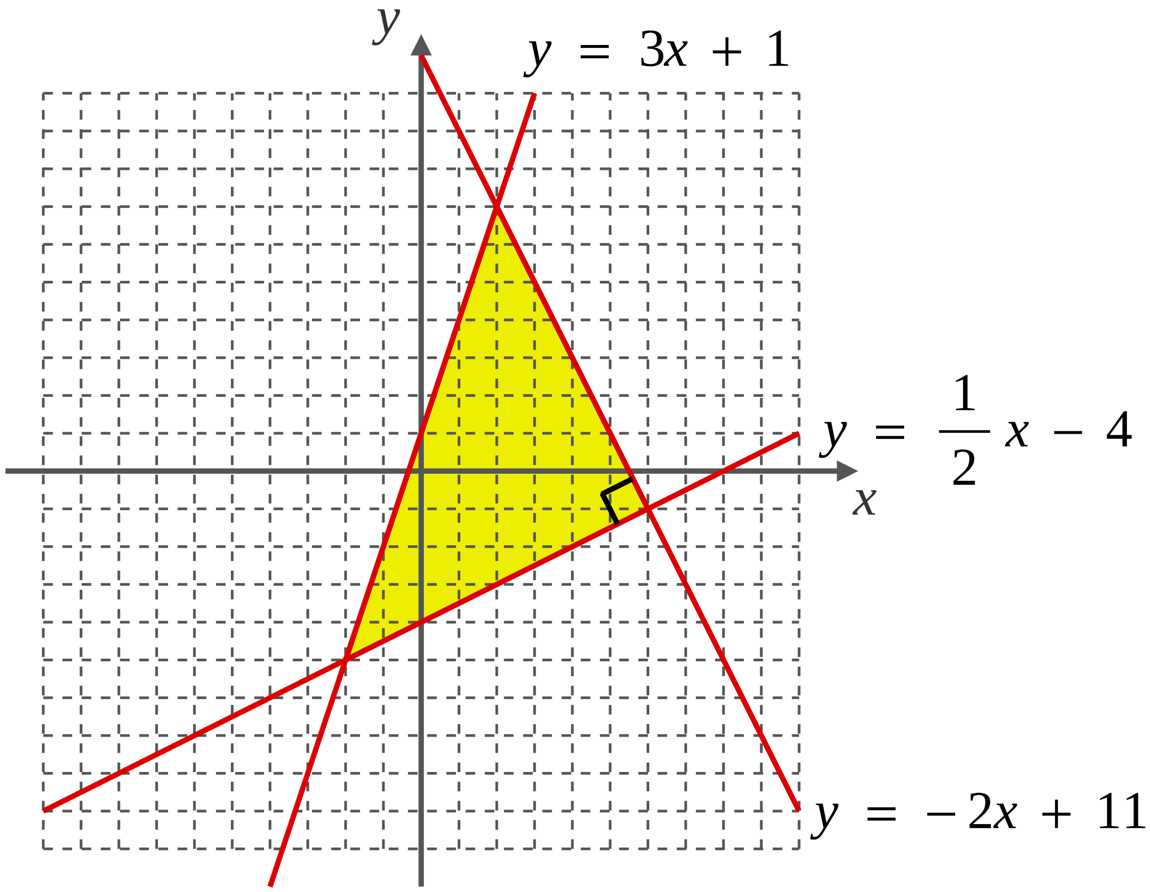
Question 8
Without drawing a graph, find the equation of the line between the points A (3, 5) and B (7, – 11). Show your working.

[4 marks]

1.4 Answers

1.4.1 Solutions to 1.2 "Together" Exercise

Answer 1



[5 marks]

Answer 2

$$y = mx + c$$

$$1 = 2 \times 5 + c \Rightarrow c = -9 \Rightarrow \text{The line is thus, } y = 2x - 9$$

[3 marks]

Answer 3

$$m = \frac{\Delta Y}{\Delta X} = \frac{20 - 11}{5 - 2} = \frac{9}{3} = 3$$

$$y = mx + c$$

$$11 = 3 \times 2 + c$$

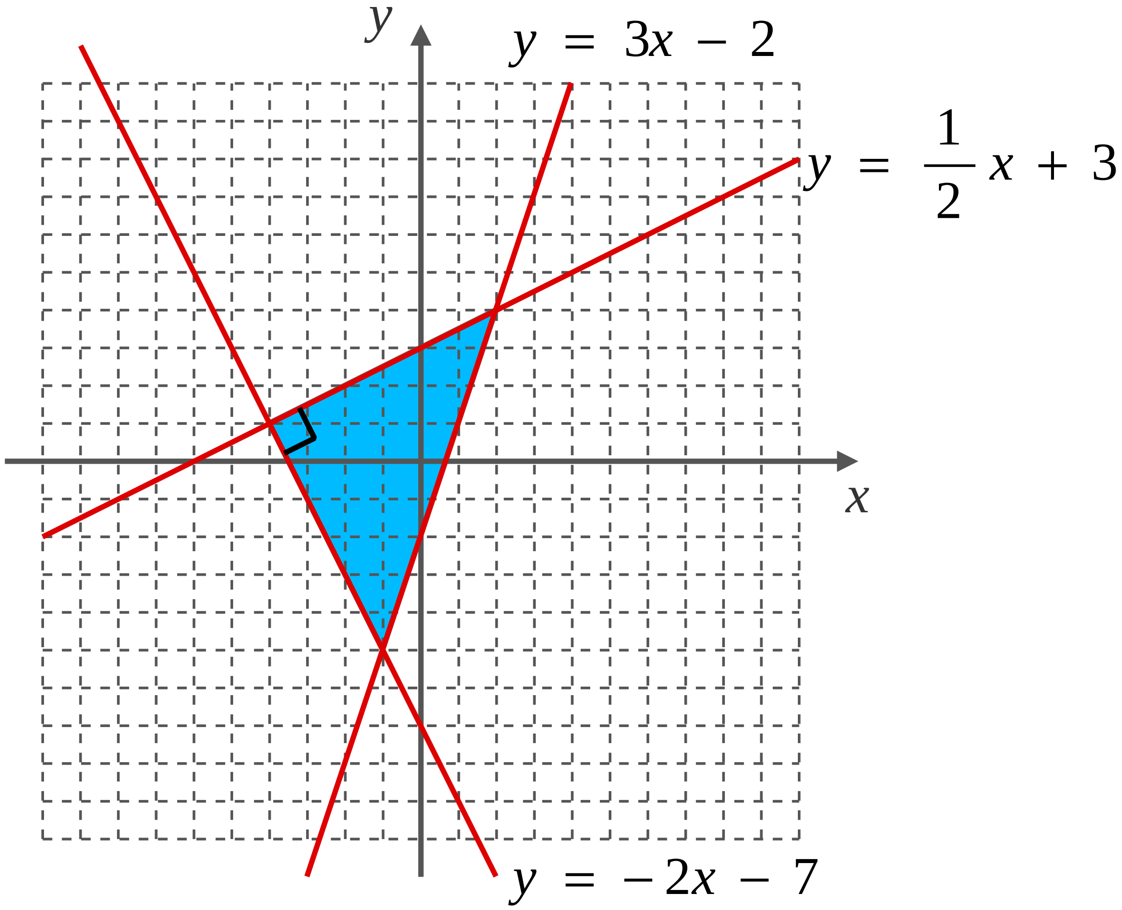
$$c = 5$$

The line is thus, $y = 3x + 5$

[3 marks]

1.4.2 Solutions to 1.3 Exercise

Answer 1



[5 marks]

Answer 2

$$y = mx + c$$

$$13 = 3 \times 2 + c$$

$$c = 7$$

The line is thus, $y = 3x + 7$

[3 marks]

Answer 3

$$m = \frac{\Delta Y}{\Delta X} = \frac{17 - 5}{5 - 2} = \frac{12}{3} = 4$$

$$y = mx + c$$

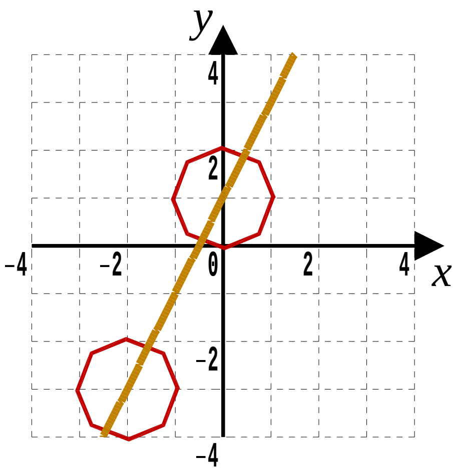
$$5 = 4 \times 2 + c$$

$$c = -3$$

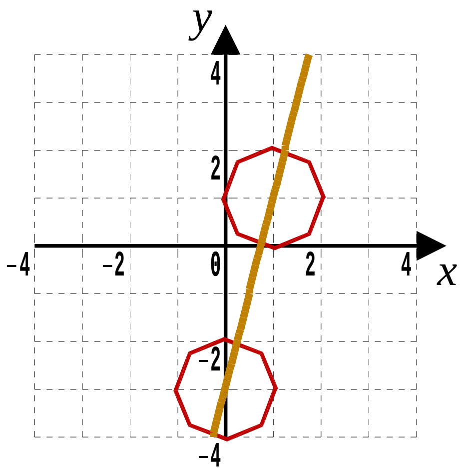
The line is thus, $y = 4x - 3$

[3 marks]

Answer 4

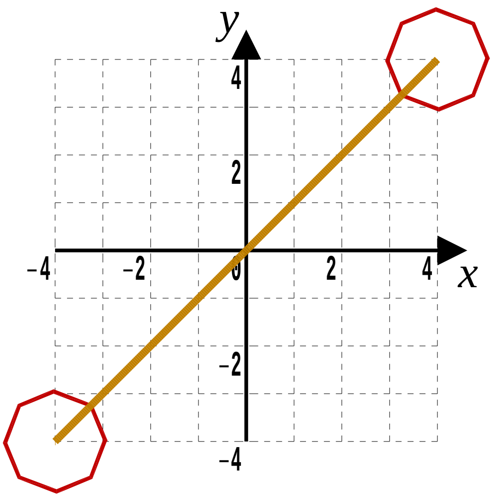


(i) $y = 2x + 1$

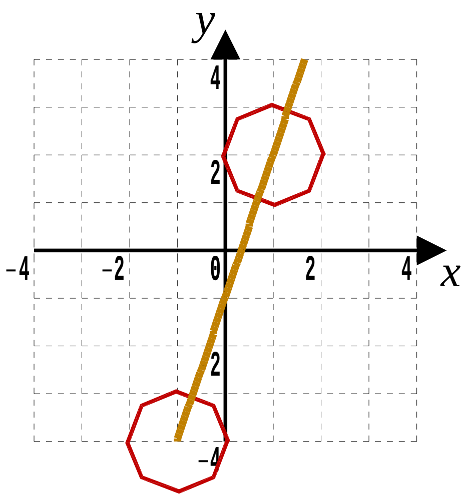


(ii) $y = 4x - 3$

[2, 2 marks]

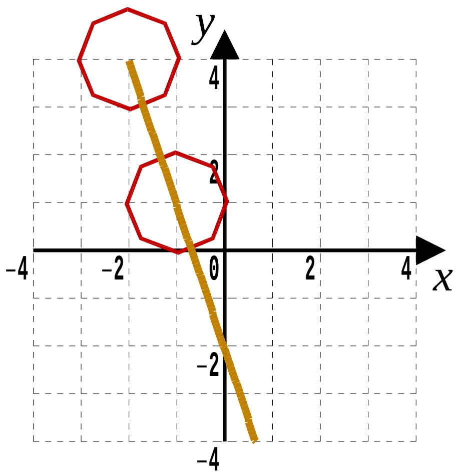


(iii) $y = x$

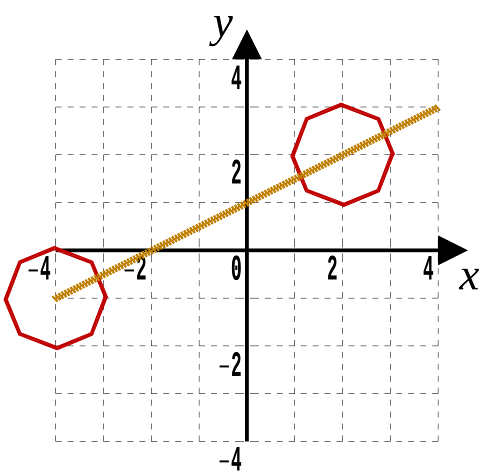


(iv) $y = 3x - 1$

[2, 2 marks]

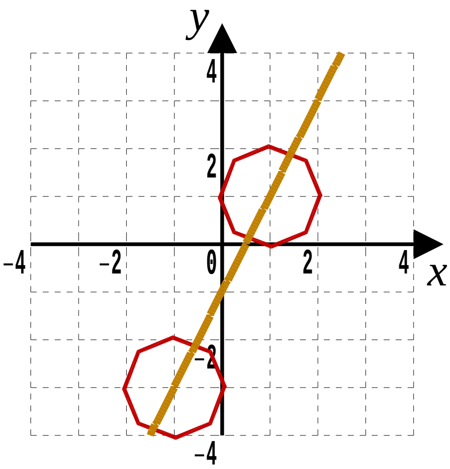


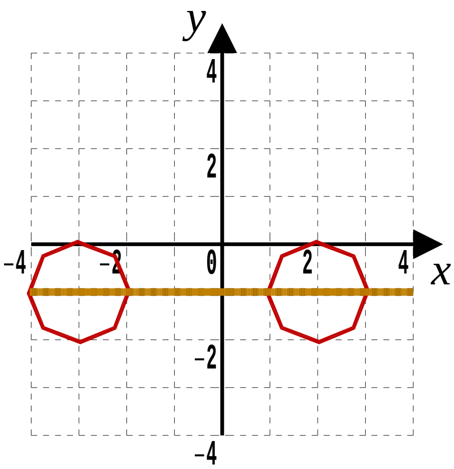
(v) $y = -3x - 2$



(vi) $y = \frac{1}{2}x + 1$

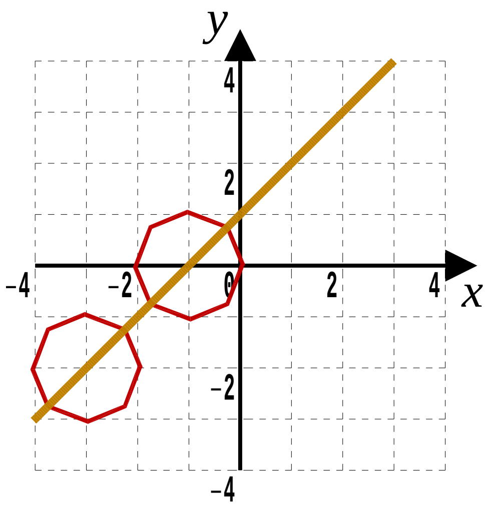
[2, 2 marks]

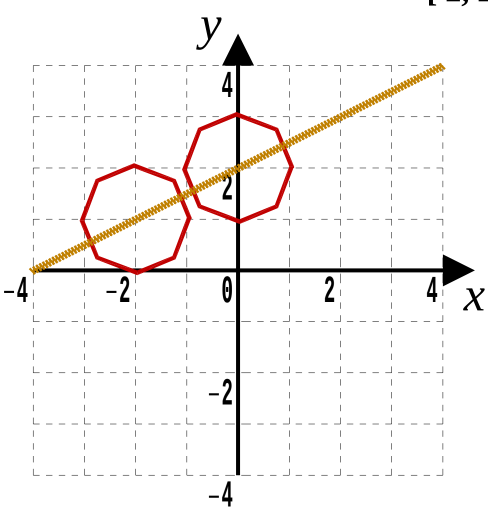




(vii) $y = 2x - 1$ **(viii)** $y = -1$

[2, 2 marks]





(ix) $y = x + 1$ **(x)** $y = \frac{1}{2}x + 2$

[2, 2 marks]

Answer 5

(i) $y = 4x$ $y = x + 6$ $y = -2x - 6$

(ii) Area ΔA =18

(iii) Area ΔB = 9

(iv) Area ΔC = 18

$\therefore$ The area of the enclosed triangle is $6 \times 12 - 45 = 27$

Answer 6

$3\sqrt{10}$

Answer 7

$y = \frac{1}{2}x - 4$

Answer 8

$y = -4x + 17$

2.1 Line Segments

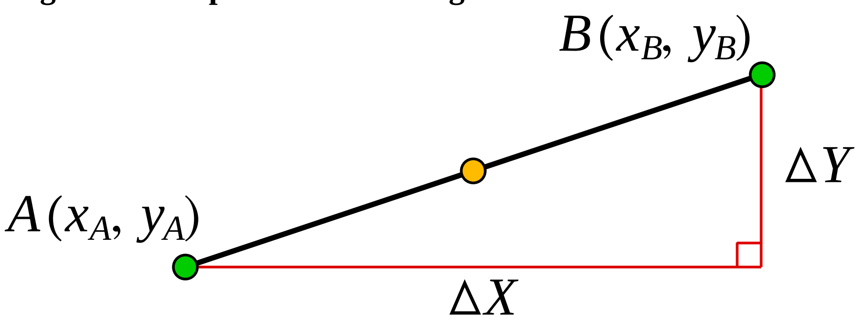
A line segment is simply a piece of straight line between two endpoints.

The length of a line segment is found by using the theorem of Pythagoras.

The coordinates of the midpoint of a line segments are found by taking the average (the mean) of the *x* parts of the two endpoints, to give the *x* part of the midpoint, and, separately, taking the average (the mean) of the *y* parts of the two endpoints to give the *y* part of the midpoint.

These comments are formalised in the following theorem;

The Length and Midpoint of a Line Segment



A line segment with endpoints $A(x_A, y_A)$ and $B(x_B, y_B)$ has a length given by;

$$|AB| = \sqrt{(\Delta X)^2 + (\Delta Y)^2}$$

$$\text{where } \Delta X = x_B - x_A \text{ and } \Delta Y = y_B - y_A$$

Furthermore,

$$Midpoint\ AB = \left(\frac{x_B + x_A}{2}, \frac{y_B + y_A}{2}\right)$$

The ΔX and ΔY are the same as that used to calculate gradient, $m = \dfrac{\Delta Y}{\Delta X}$

In three dimensions with endpoints $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ the theorem is only marginally more complicated;

$$|AB| = \sqrt{(\Delta X)^2 + (\Delta Y)^2 + (\Delta Z)^2}$$

$$\text{where } \Delta X = x_B - x_A,\ \Delta Y = y_B - y_A \text{ and } \Delta Z = z_B - z_A$$

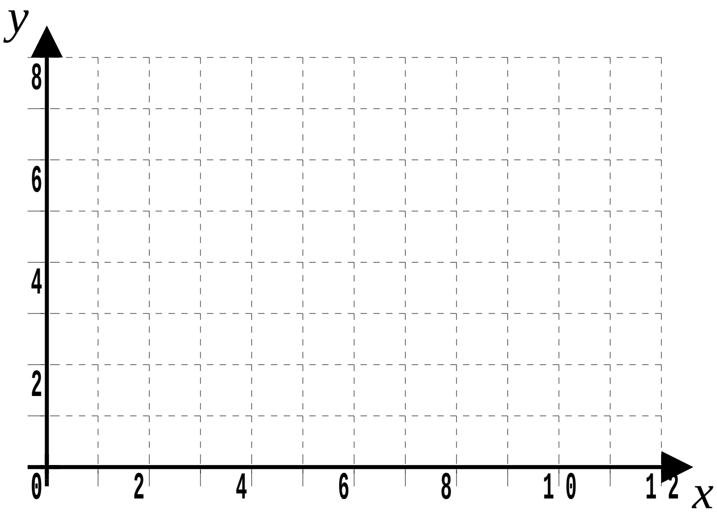
$$Midpoint\ AB = \left(\frac{x_B + x_A}{2}, \frac{y_B + y_A}{2}, \frac{z_B + z_A}{2}\right)$$

2.2 Example

The graph of this function is a line segment.

$$f(x) = \frac{1}{2}x + 1, \quad x \in \mathbb{R}, \; 2 \leqslant x \leqslant 10$$

Sketch the line segment on the grid below, and determine its length and midpoint using the theorem, “The Length and Midpoint of a Line Segment”



Teaching Video : http://www.NumberWonder.co.uk/v9033/2.mp4

[7 marks]

2.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

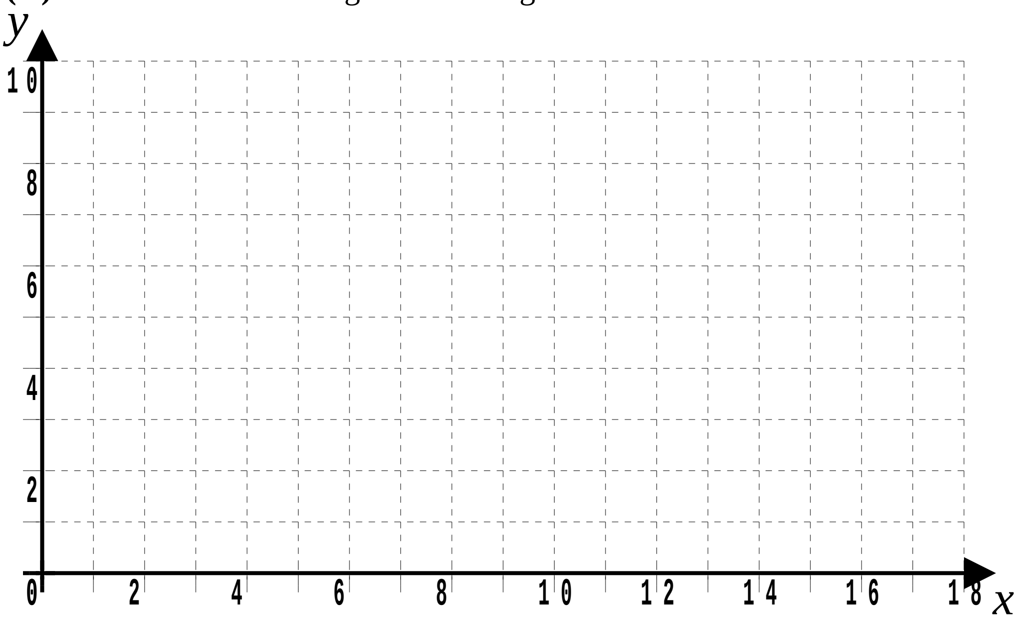
Marks Available : 55

Question 1

The graph of this function is a line segment.

$$g(x) = -\frac{1}{3}x + 8, \quad x \in \mathbb{R}, \ 3 \leqslant x \leqslant 15$$

(i) Sketch the line segment on the grid below



[3 marks]

(ii) Use the theorem "The Length and Midpoint of a Line Segment" to calculate the exact length of the line segment.

[2 marks]

(iii) Use the theorem "The Length and Midpoint of a Line Segment" to determine midpoint of the line segment.

[2 marks]

Question 2

Without drawing a graph, and showing your working, determine the exact distance between the two points $A(-15, -2)$ and $B(9, 5)$

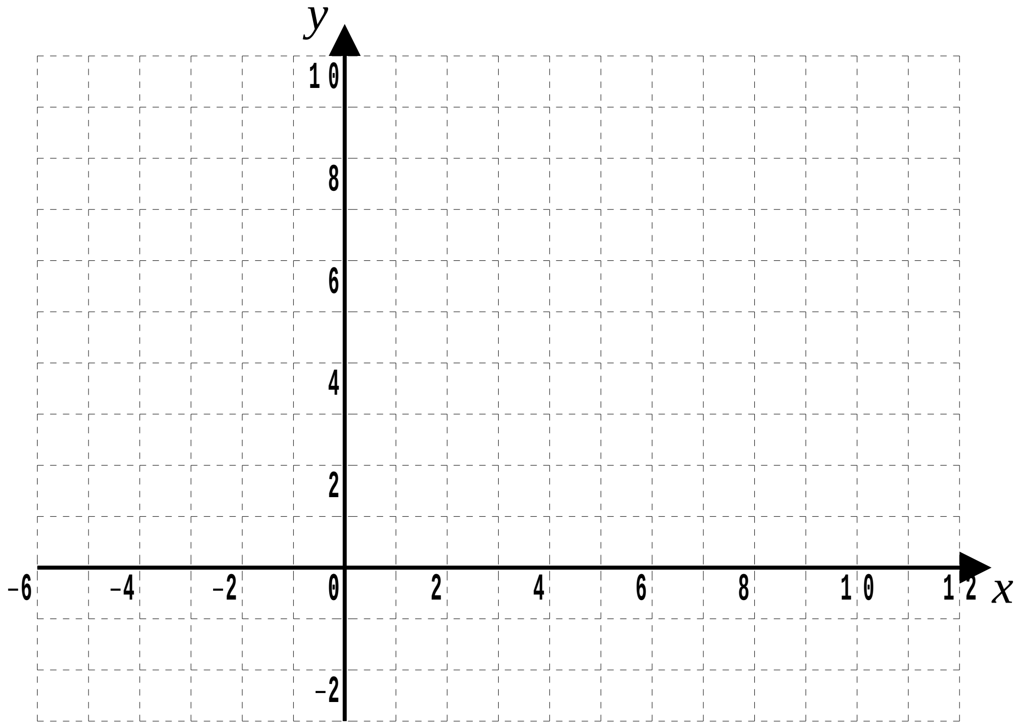
[3 marks]

Question 3

The graph of this function is a line segment.

$$h(x) = \frac{2}{3}x + 1, \quad x \in \mathbb{R}, \quad -3 \leqslant x \leqslant 9$$

(i) Sketch the line segment on the grid below



[3 marks]

(ii) Use the theorem "The Length and Midpoint of a Line Segment" to calculate the exact length of the line segment.

[2 marks]

(iii) Use the theorem "The Length and Midpoint of a Line Segment" to determine midpoint of the line segment.

[2 marks]

Question 4

A straight line has equation $4y - 12x + 3 = 0$

Write this equation in the form $y = mx + c$

[2 marks]

Question 5

IGCSE Examination Question from January 2020, Paper 1H, Q1 (Edexcel)

The point A has coordinates $(5, -4)$

The point B has coordinates $(13, 1)$

(a) Work out the coordinates of the midpoint of AB

[2 marks]

Line L has equation $y = 2 - 3x$

(b) Write down the gradient of line L

[1 mark]

(c) Does the point with coordinates $(100, -302)$ lie on L ?

You must give a reason for your answer

[1 mark]

Question 6

IGCSE Examination Question from January 2018, Paper 3H, Q10 (Edexcel)

The straight line L is parallel to the line with equation $2y + 8x = 5$

L passes through the point with coordinates $(2, 3)$

Find an equation for L

[3 marks]

Question 7

Additional Mathematics Examination Question from June 2017, Q4 (OCR, FSMQ)

The coordinates of A and B are $(1,\ 5)$ and $(-3,\ 7)$ respectively

(i) Calculate the exact length of AB

[2 marks]

(ii) Find the coordinates of the midpoint of AB

[1 mark]

Question 8

Additional Mathematics Examination Question from June 2009, Q3 (OCR, FSMQ)

A is the point $(1,\ 5)$ and C is the point $(3,\ p)$

(i) Find the equation of the line through A which is parallel to $2x + 5y = 7$

[2 marks]

(ii) This line also passes through the point C.
Find the value of p

[2 marks]

Question 9

IGCSE Examination Question from June 2017, Paper 4H, Q13 (Edexcel)

Here are the equations of four straight lines,

Line **A** $y = 2x + 3$

Line **B** $2y = 6 - 3x$

Line **C** $4x - 2y = 3$

Line **D** $y = 3 - 2x$

Two of these lines are parallel.

(a) Which two lines ?

[2 marks]

Line **L** has a gradient of $-\dfrac{5}{2}$ and passes through the point with coordinates $(1,\ 3)$

(b) Find an equation of **L**

Give your answer in the form $ax + by = c$ where a, b and c are integers

[3 marks]

Question 10

A-Level Examination Question from January 2008, C1, Q4.

The point $A(-6,\ 4)$ and the point $B(8,\ -3)$ lie on the line L.

(a) Find an equation for L in the form $ax + by + c = 0$,
where a, b and c are integers.

[4 marks]

(b) Find the distance AB, giving your answer in the form
$k\sqrt{5}$, where k is an integer.

[3 marks]

Question 11

A-Level Examination Question from May 2010, C1, Q8

(a) Find an equation of the line joining A (7, 4) and B (2, 0),
giving your answer in the form $ax + by + c = 0$,
where a, b and c are integers

[3 marks]

(b) Find the length of AB, leaving your answer in surd form

[2 marks]

The point C has coordinates (2, t), where $t > 0$, and $AC = AB$

(c) Find the value of t

[1 mark]

(d) Find the area of triangle ABC

[2 marks]

2.4 Answers

A-Level Pure Mathematics, Year 1
Additional Mathematics
GCSE
Coordinate Geometry

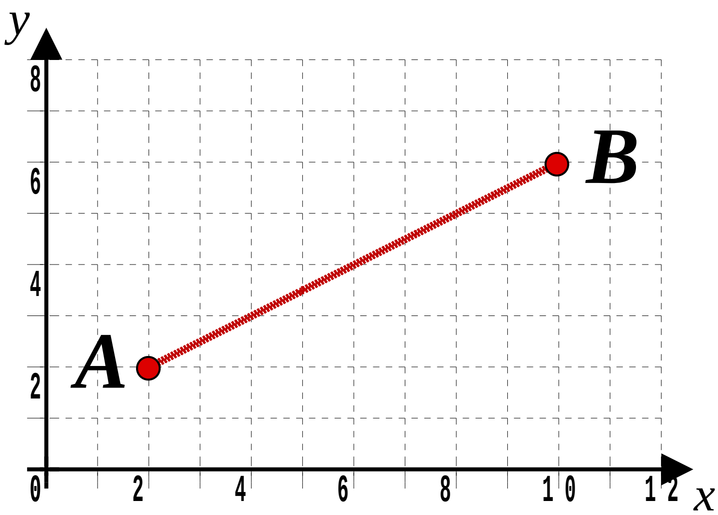
2.4.1 Solution to 2.2 Example (Teaching Video)

$$f(x) = \frac{1}{2}x + 1, \quad x \in \mathbb{R},\ 2 \leqslant x \leqslant 10$$

Endpoints are,

$$f(2) = \frac{1}{2} \times 2 + 1 = 2 \Rightarrow A(2,\ 2)$$

$$f(10) = \frac{1}{2} \times 10 + 1 = 6 \Rightarrow B(10,\ 6)$$



[3 marks]

$$\begin{aligned} |AB| &= \sqrt{(\Delta X)^2 + (\Delta Y)^2} \\ &= \sqrt{(10 - 2)^2 + (6 - 2)^2} \\ &= \sqrt{8^2 + 4^2} \\ &= \sqrt{80} \\ &= 4\sqrt{5} \end{aligned}$$

is the length of the line segment

[2 marks]

$$\begin{aligned} \text{Midpoint } AB &= \left(\frac{x_B + x_A}{2},\ \frac{y_B + y_A}{2} \right) \\ &= \left(\frac{10 + 2}{2},\ \frac{6 + 2}{2} \right) \\ &= (6,\ 4) \end{aligned}$$

[2 marks]

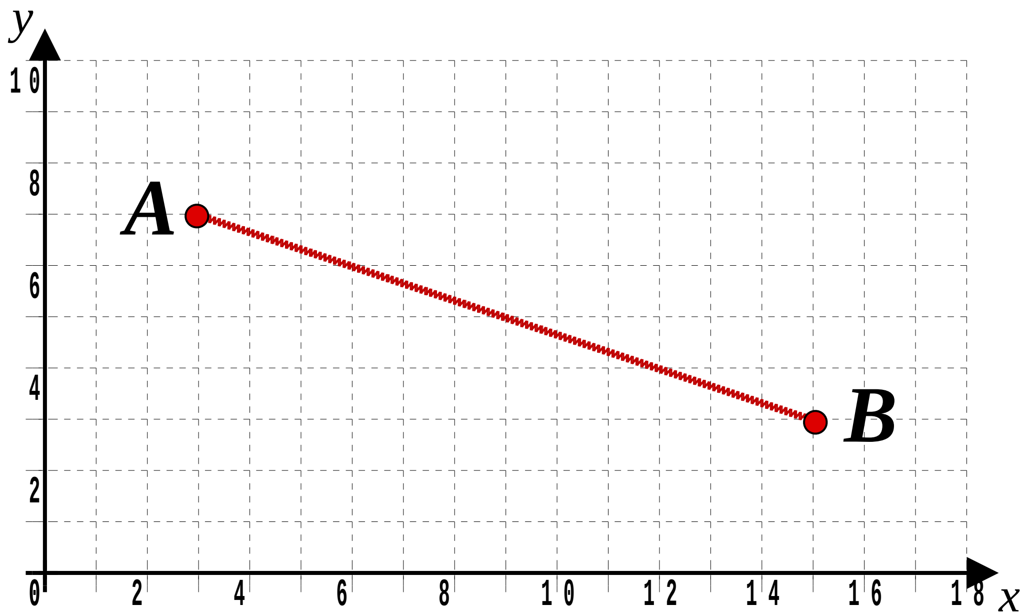
2.4.2 Solutions to 2.3 Exercise

Answer 1

(i) $$g(x) = -\frac{1}{3}x + 8, \quad x \in \mathbb{R}, \ 3 \leqslant x \leqslant 15$$

$$\text{Endpoint } A : g(3) = -\frac{1}{3} \times 3 + 8 = 7 \Rightarrow A(3, 7)$$

$$\text{Endpoint } B : g(15) = -\frac{1}{3} \times 15 + 8 = 3 \Rightarrow B(15, 3)$$



[3 marks]

(ii)
$$|AB| = \sqrt{(\Delta X)^2 + (\Delta Y)^2}$$
$$= \sqrt{(15 - 3)^2 + (3 - 7)^2}$$
$$= \sqrt{12^2 + 4^2} = \sqrt{160} = 4\sqrt{10}$$

[2 marks]

(iii)
$$\textit{Midpoint } AB = \left(\frac{x_B + x_A}{2}, \frac{y_B + y_A}{2} \right)$$
$$= \left(\frac{15 + 3}{2}, \frac{3 + 7}{2} \right)$$
$$= (9, 5)$$

[2 marks]

Answer 2

$$|AB| = \sqrt{(\Delta X)^2 + (\Delta Y)^2}$$
$$= \sqrt{(9 - (-15))^2 + (5 - (-2))^2}$$
$$= \sqrt{24^2 + 7^2} = 25$$

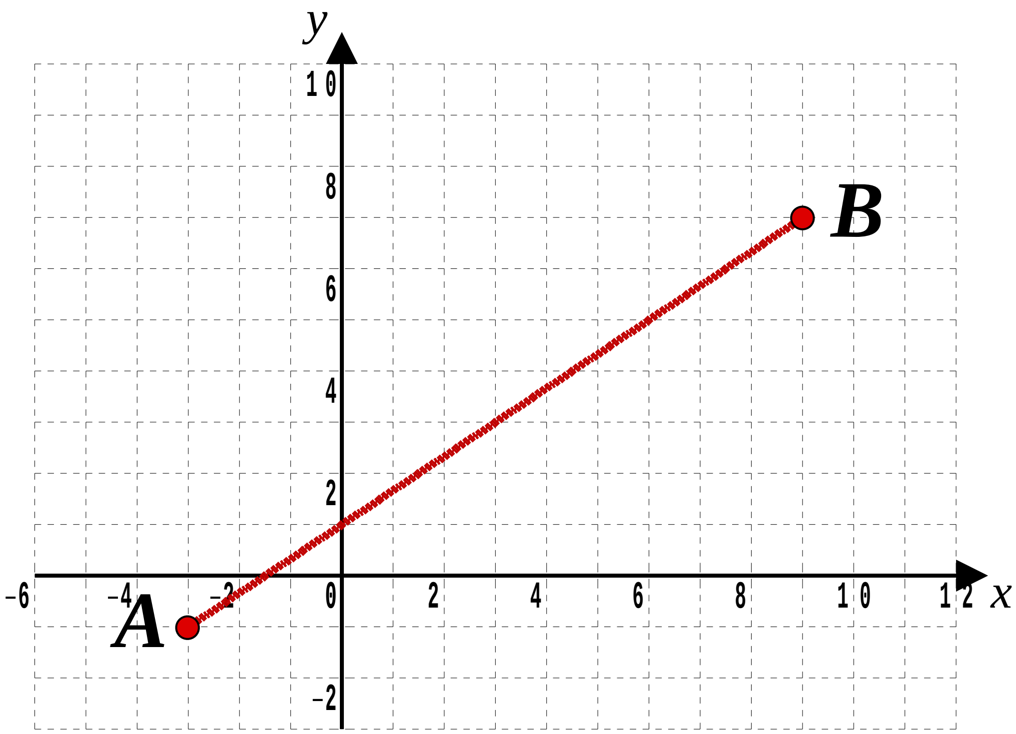
[3 marks]

Answer 3

(i) $$h(x) = \frac{2}{3}x + 1, \quad x \in \mathbb{R}, \quad -3 \leqslant x \leqslant 9$$

Endpoint A : $h(-3) = \frac{2}{3} \times (-3) + 1 = 7 \Rightarrow A(-3, \ -1)$

Endpoint B : $h(9) = \frac{2}{3} \times 9 + 1 = 3 \Rightarrow B(9, \ 7)$



[3 marks]

(ii)

$$|AB| = \sqrt{(\Delta X)^2 + (\Delta Y)^2}$$
$$= \sqrt{(9 - (-3))^2 + (7 - (-1))^2}$$
$$= \sqrt{12^2 + 8^2} = \sqrt{208} = 4\sqrt{13}$$

[2 marks]

(iii)

$$Midpoint\ AB = \left(\frac{x_B + x_A}{2}, \ \frac{y_B + y_A}{2} \right)$$
$$= \left(\frac{9 + (-3)}{2}, \ \frac{7 + (-1)}{2} \right)$$
$$= (3, \ 3)$$

[2 marks]

Answer 4

$$4y - 12x + 3 = 0 \Rightarrow y = 3x - \frac{3}{4}$$

[2 marks]

Answer 5

(a) $$\textit{Midpoint } AB = \left(\frac{x_B + x_A}{2}, \; \frac{y_B + y_A}{2} \right)$$

$$= \left(\frac{13 + 5}{2}, \; \frac{1 + (-4)}{2} \right)$$

$$= \left(9, -\frac{3}{2} \right)$$

[2 marks]

(b) Gradient is $-\ 3$

[1 mark]

(c) Put $x = 100$ into $y = 2 - 3x \Rightarrow y = -298$

$\therefore$ As the answer was not $-\ 302$, the point $(\ 100, -302)$ is not on L

[1 mark]

Answer 6

$$2y + 8x = 5 \Rightarrow y = -4x + \frac{5}{2} \text{ which has gradient of } -\ 4$$

$$y = mx + c$$

$$3 = -4 \times 2 + c$$

$$c = 11$$

$\therefore$ L has equation $y = -4x + 11$

[3 marks]

Answer 7

(i) $$|AB| = \sqrt{(\Delta X)^2 + (\Delta Y)^2}$$

$$= \sqrt{(\ (-3) - 1\ )^2 + (\ 7 - 5\ )^2}$$

$$= \sqrt{4^2 + 2^2}$$

$$= \sqrt{20}$$

$$= 2\sqrt{5} \quad \text{is the length of the line segment}$$

[2 marks]

(ii) $$\textit{Midpoint } AB = \left(\frac{x_B + x_A}{2}, \; \frac{y_B + y_A}{2} \right)$$

$$= \left(\frac{(-3) + 1}{2}, \; \frac{7 + 5}{2} \right)$$

$$= (-1, \; 6)$$

[1 mark]

Answer 8

(i) $2x + 5y = 7 \Rightarrow y = -\frac{2}{5}x + \frac{7}{5}$ which has gradient of $-\frac{2}{5}$

$$y = mx + c$$

$$5 = -\frac{2}{5} \times 1 + c$$

$$c = \frac{27}{5}$$

$\therefore$ The line through A has equation $y = -\frac{2}{5}x + \frac{27}{5}$

[2 marks]

(ii)

$$y = -\frac{2}{5}x + \frac{27}{5}$$

$$p = -\frac{2}{5} \times 3 + \frac{27}{5}$$

$$= \frac{21}{5}$$

[2 marks]

Answer 9

(a) Putting all of the lines into the $y = mx + c$ format;

Line **A** $y = 2x + 3$

Line **B** $2y = 6 - 3x \Rightarrow y = -\frac{3}{2}x + 3$

Line **C** $4x - 2y = 3 \Rightarrow y = 2x - \frac{3}{2}$

Line **D** $y = 3 - 2x$

$\therefore$ Line **A** and Line **C** are parallel, both with a gradient of 2

[2 marks]

(b)

$$y = mx + c$$

$$3 = -\frac{5}{2} \times 1 + c$$

$$c = \frac{11}{2}$$

The line is thus, $y = -\frac{5}{2}x + \frac{11}{2}$

$$2y = -5x + 11$$

$5x + 2y = 11$ is the requested format

[3 marks]

Answer 10

(a)

$$m = \frac{\Delta Y}{\Delta X} \qquad\qquad y = mx + c$$

$$= \frac{-3 - 4}{8 - (-6)} \qquad\qquad 4 = -\frac{1}{2} \times (-6) + c$$

$$= -\frac{7}{14} \qquad\qquad c = 1$$

$= -\frac{1}{2}$ The line is thus, $y = -\frac{1}{2}x + 1$

That is $x + 2y - 2 = 0$

[4 marks]

(b)

$$|AB| = \sqrt{(\Delta X)^2 + (\Delta Y)^2}$$

$$= \sqrt{(8 - (-6))^2 + (-3 - 4)^2}$$

$$= \sqrt{14^2 + 7^2}$$

$$= \sqrt{245}$$

$= 7\sqrt{5}$ is the length of the line segment

[3 marks]

Answer 11

(a)

$$m = \frac{\Delta Y}{\Delta X} \qquad\qquad y = mx + c$$

$$= \frac{0 - 4}{2 - 7} \qquad\qquad 0 = \frac{4}{5} \times 2 + c$$

$$= \frac{4}{5} \qquad\qquad c = -\frac{8}{5}$$

The line is thus, $y = \frac{4}{5}x - \frac{8}{5}$

That is, $4x - 5y - 8 = 0$

[3 marks]

(b)

$$|AB| = \sqrt{(\Delta X)^2 + (\Delta Y)^2}$$

$$= \sqrt{(2 - 7)^2 + (0 - 4)^2}$$

$$= \sqrt{5^2 + 4^2}$$

$$= \sqrt{41}$$

[3 marks]

(c) $t = 8$

[1 mark]

(d)

$$Area\,\Delta = \frac{1}{2}bh$$

$$= \frac{1}{2} \times 8 \times (7 - 2)$$

$$= 20$$

[2 marks]

Lesson 3

A-Level Pure Mathematics, Year 1
Additional Mathematics
GCSE
Coordinate Geometry

3.1 Short Questions Homework

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

Find the length of the line segment between $A(2,\ 9)$ and $B(3,\ 12)$

[2 marks]

Question 2

Determine the gradient of the line through the points $A(4,\ 3)$ and $B(8,\ 11)$

[2 marks]

Question 3

A line has a gradient of $\frac{5}{7}$ and cuts the y-axis at the point $\left(0,\ \frac{9}{7}\right)$

Write the equation of the line in the form $y = mx + c$, where m and c are constants.

[2 marks]

Question 4

A line has equation $3y + 4x + 6 = 0$

Find **(i)** the gradient

and **(ii)** the y-intercept

of the line

[2 marks]

Question 5

Find the equation of the line with gradient $\frac{1}{2}$ which passes through (3, 2)

[3 marks]

Question 6

Find the equation of the line that passes through $(\ 2,\ -1\ )$ and $(\ 3,\ \ 7\ )$

[3 marks]

Question 7

Find the coordinates of the point where the line $y \ = \ -2x \ - \ 7$ cuts

(i) the *y*-axis

[1 mark]

(ii) the *x*-axis

[1 mark]

Question 8

For each of the following lines, decide if the point $(\ 4,\ 8\ )$ is on the line, or not.

(i) $y = 2x$

[1 mark]

(ii) $y = x + 4$

[1 mark]

(iii) $y = 5x - 12$

[1 mark]

(iv) $y = -2x \ - \ 8$

[1 mark]

Question 9
Find the equation of the line with gradient 3 that passes through the point $(1, -0.5)$

[3 marks]

Question 10
Find the equation of the straight line through $(-1, 4)$ and $(2, 3)$

[3 marks]

Question 11
The lines with equations $5x + 6y = 45$ and $3y - x - 5 = 0$ meet at the point A.
Find an equation of the line through A whose gradient is 2.

[4 marks]

3.2 Answers

A-Level Pure Mathematics, Year 1
Additional Mathematics
GCSE
Coordinate Geometry

Answer 1

$$
\begin{aligned}
|AB| &= \sqrt{(3-2)^2 + (12-9)^2} \\
&= \sqrt{1^2 + 3^2} \\
&= \sqrt{10}
\end{aligned}
$$

[2 marks]

Answer 2

$$m_{AB} = \frac{11-3}{8-4} = \frac{8}{4} = 2$$

[2 marks]

Answer 3

$$y = \frac{5}{7}x + \frac{9}{7}$$

[2 marks]

Answer 4

$$3y + 4x + 6 = 0 \Rightarrow y = -\frac{4}{3}x - 2$$

(i) gradient is $-\dfrac{4}{3}$

(ii) y-axis intercept is -2 That is, $(0, -2)$

[2 marks]

Answer 5

$$
\begin{aligned}
y &= mx + c \\
2 &= \frac{1}{2} \times 3 + c \\
c &= \frac{1}{2}
\end{aligned}
$$

$\therefore$ The equation of the line is $y = \dfrac{1}{2}x + \dfrac{1}{2}$

[3 marks]

Answer 6

$$m = \frac{7-(-1)}{3-2} = 8$$

$$y = mx + c$$

$$7 = 8 \times 3 + c \Rightarrow c = -17$$

$\therefore$ The equation of the line is $y = 8x - 17$

[3 marks]

Answer 7

(i) $(0, -7)$ Direct from the equation

(ii) Cuts x-axis when y is zero, $0 = -2x - 7 \Rightarrow x = -\dfrac{7}{2} \quad \therefore \left(-\dfrac{7}{2},\ 0\right)$

[2 marks]

Answer 8

(i) Yes, on the line as $8 = 2 \times 4$

(ii) Yes, on the line as $8 = 4 + 4$

(iii) Yes, on the line as $8 = 5 \times 4 - 12$

(iv) No, not on the line as $8 \neq -2 \times 4 - 8$

[4 marks]

Answer 9

$$y = mx + c$$

$$-0.5 = 3 \times 1 + c \Rightarrow c = -3.5$$

The line has equation $y = 3x - 3.5$

[3 marks]

Answer 10

$$m = \frac{3 - 4}{2 - (-1)} = -\frac{1}{3}$$

$$y = mx + c \Rightarrow 3 = -\frac{1}{3} \times 2 + c \Rightarrow c = \frac{11}{3}$$

$\therefore$ The equation of the line is $y = -\dfrac{1}{3}x + \dfrac{11}{3}$

[3 marks]

Answer 11

$$\begin{aligned} 5x + 6y &= 45 \\ -x + 3y &= 5 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 5x + 6y &= 45 \\ -\ \ 2x + 6y &= 10 \\ \hline 7x &= 35 \end{aligned}$$

$$x = 5 \Rightarrow y = \frac{10}{3}$$

$$y = mx + c \Rightarrow \frac{10}{3} = 2 \times 5 + c \Rightarrow c = -\frac{20}{3}$$

$$y = 2x - \frac{20}{3}$$

[4 marks]

4.1 Lines at Right Angles

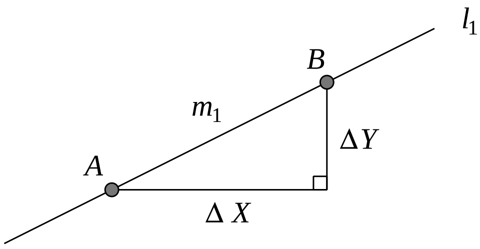
The Perpendicular Lines Theorem (Version 1)

If the gradient of the line l_1 is m_1 and the gradient of the line l_2 is m_2 then the lines l_1 and l_2 are perpendicular if and only if

$$m_1 \times m_2 = -1$$

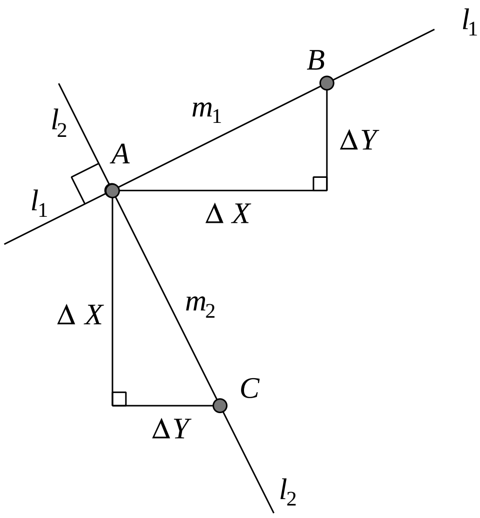
Proof

Consider two points, A and B on the line l_1 which has gradient m_1



Clearly, $\quad m_1 = \dfrac{\Delta Y}{\Delta X}$

Now, consider a rotation of – 90° about the point A which gives a line l_2 with gradient m_2 which is perpendicular to l_1.



Clearly, $\quad m_2 = -\dfrac{\Delta X}{\Delta Y}$

Observe that, $\quad m_1 \times m_2 = \dfrac{\Delta Y}{\Delta X} \times -\dfrac{\Delta X}{\Delta Y}$

$$= -1$$

□

In many questions, the gradient of a first line will be known and the gradient of a second, perpendicular to the first, sought.
In consequense the following version of the theorem is often of more use;

The Perpendicular Lines Theorem (Version 2)

Given a line l_1 with gradient m_1 then the gradient m_2 of any perpendicular line l_2 is the ***sign changed reciprocal*** of m_1.

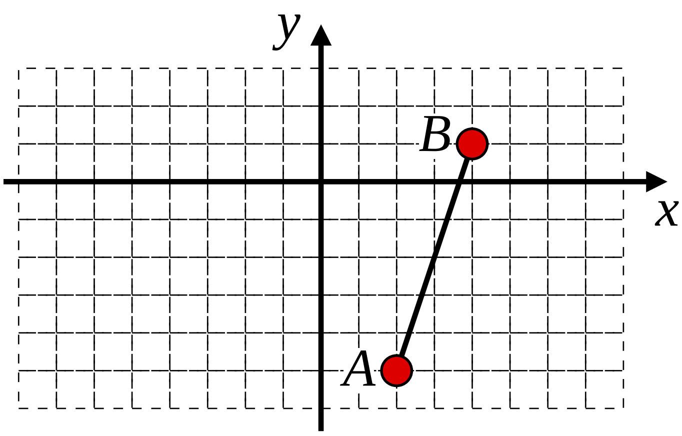
$$\text{That is, } m_2 = -\frac{1}{m_1}$$

4.2 Example

Find the equation of the perpendicular bisector of the line segment AB where A is $(2, -5)$ and B is $(4, 1)$

(i) Give your answer in the form $y = mx + c$

(ii) Illustrate your answer with a sketch graph.



Teaching Video : http://www.NumberWonder.co.uk/v9033/4.mp4

[4 marks]

4.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

A line, L, has equation

$$y = \frac{2}{3}x + \frac{1}{3}$$

(i) What is the gradient of L ?

[1 mark]

(ii) What would be the gradient of a line, perpendicular to L ?

[1 mark]

Question 2

A line has equation $2x + 5y - 4 = 0$

(i) Write this line's equation in the form $y = mx + c$

[1 marks]

(ii) Hence state the gradient of the line $2x + 5y - 4 = 0$

[1 mark]

(iii) What is the gradient of a line, perpendicular to $2x + 5y - 4 = 0$?

[1 mark]

Question 3

A line has equation $5x - 3y - 2 = 0$

(i) What is the gradient of this line ?

[2 marks]

(ii) What is the gradient of a line perpendicular to $5x - 3y - 2 = 0$?

[1 mark]

Question 4

Additional Mathematics Examination Question from June 2015, Q1, (OCR, FSMQ)

Find the equation of the line which is perpendicular to the line $2x + 3y = 5$ and which passes through the point $(3, 4)$

[3 marks]

Question 5

A-Level Examination question from May 2011, C1, Q3 (Edexcel)

The points P and Q have coordinates $(-1, 6)$ and $(9, 0)$ respectively.
The line l is perpendicular to PQ and passes through the mid-point of PQ.
Find an equation for l, giving your answer in the form $ax + by + c = 0$,
where a, b and c are integers.

[5 marks]

Question 6

A-Level Examination Question from January 2010, C1, Q3 (Edexcel)

The line l_1 has equation $3x + 5y - 2 = 0$

(a) Find the gradient of l_1

[2 marks]

The line l_2 is perpendicular to l_1 and passes through the point (3, 1)

(b) Find the equation of l_2 in the form $y = mx + c$, where m and c are constants

[3 marks]

Question 7

A-Level Examination Question from January 2006, C1, Q3 (Edexcel)

The line L has equation $y = 5 - 2x$

(a) Show that the point $P(3, -1)$ lies on L

[1 mark]

(b) Find an equation of the line perpendicular to L, which passes through P. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[4 marks]

Question 8

Additional Mathematics Examination Question from June 2014, Q8 (OCR)

Four points have coordinates $A(-5, -1)$, $B(0, 4)$, $C(7, 3)$ and $D(2, -2)$

(i) Using gradients of lines, prove that *ABCD* is a parallelogram

[2 marks]

(ii) Using lengths of lines, prove that *ABCD* is a rhombus

[2 marks]

(iii) Prove that *ABCD* is not a square

[2 marks]

Question 9

A-Level Examination Question from January 2011, C1, Q9 (Edexcel)

The line L_1 has equation $2y - 3x - k = 0$, where k is a constant.

Given that the point $A(1, 4)$ lies on L_1 find,

(a) the value of k,

[1 mark]

(b) the gradient of L_1

[2 marks]

The line L_2 passes through A and is perpendicular to L_1

(c) Find an equation of L_2 giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[4 marks]

The line L_2 crosses the x-axis at the point B

(d) Find the coordinates of B

[2 marks]

(e) Find the exact length of AB

[2 marks]

Question 10

A-Level Examination Question from May 2007, C1, Q11 (Edexcel)

The line l_1 has equation $y = 3x + 2$,

and the line l_2 has equation $3x + 2y - 8 = 0$

(a) Find the gradient of the line l_2

[2 marks]

The point of intersection of l_1 and l_2 is P

(b) Find the coordinates of P

[3 marks]

The lines l_1 and l_2 cross the line $y = 1$ at the points A and B respectively.

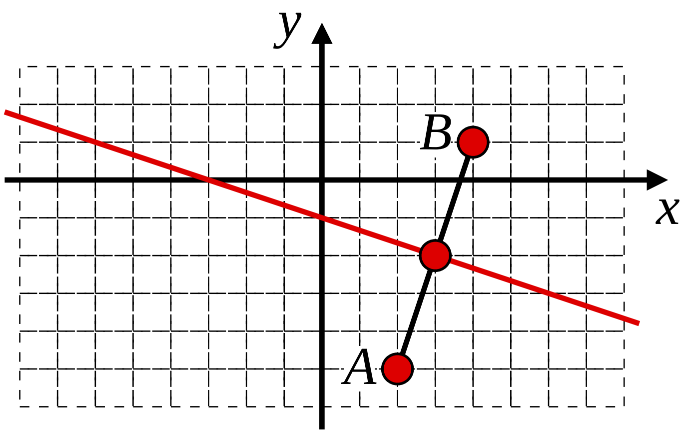
(c) Find the area of triangle ABP

[4 marks]

4.4 Answers

A-Level Pure Mathematics, Year 1
Additional Mathematics
GCSE
Coordinate Geometry

4.4.1 Solutions to 4.2 Example (Teaching Video)



$$\text{Midpoint of } AB = \left(\frac{x_B + x_A}{2}, \frac{y_B + y_A}{2} \right)$$

$$= \left(\frac{4 + 2}{2}, \frac{1 + (-5)}{2} \right)$$

$$= (3, -2)$$

$$\text{Gradient of } AB = \frac{\Delta Y}{\Delta X} = \frac{1 - (-5)}{4 - 2} = \frac{6}{2} = 3$$

The line perpendicular to AB will have gradient $-\dfrac{1}{3}$

("Sign changed reciprocal")

Equation of perpendicular bisector:

$$y = mx + c$$

$$-2 = -\frac{1}{3} \times 3 + c$$

$$\therefore c = -1$$

Thus, $\quad y = -\dfrac{1}{3}x - 1$

[4 marks]

4.4.2 Solutions to 4.3 Exercise

Answer 1

(i) $\frac{2}{3}$ **(ii)** $-\frac{3}{2}$

[1, 1 mark]

Answer 2

(i) $y = -\frac{2}{5}x + \frac{4}{5}$ **(ii)** $-\frac{2}{5}$ **(iii)** $\frac{5}{2}$

[1, 1, 1 mark]

Answer 3

$$y = \frac{5}{3}x - \frac{2}{3}$$

(i) $\frac{5}{3}$ **(ii)** $-\frac{3}{5}$

[2, 1 mark]

Answer 4

$2x + 3y = 5 \Rightarrow y = -\frac{2}{3}x + \frac{5}{3}$ which has gradient $-\frac{2}{3}$

Require a line with the sign changed reciprocal gradient through $(3, 4)$

$$y = mx + c$$

$$4 = \frac{3}{2} \times 3 + c \Rightarrow c = -\frac{1}{2}$$

$$\therefore y = \frac{3}{2}x - \frac{1}{2}$$

[3 marks]

Answer 5

$$Midpoint\ PQ = \left(\frac{9 + (-1)}{2}, \frac{0 + 6}{2} \right) = (4, 3)$$

$$Gradient\ PQ = \frac{0 - 6}{9 - (-1)} = -\frac{6}{10} = -\frac{3}{5}$$

Require a line with the sign changed reciprocal gradient through $(4, 3)$

$$y = mx + c$$

$$3 = \frac{5}{3} \times 4 + c \Rightarrow c = -\frac{11}{3}$$

$$\therefore y = \frac{5}{3}x - \frac{11}{3} \Rightarrow 5x - 3y - 11 = 0$$

[5 marks]

Answer 6

(a) $3x + 5y - 2 = 0 \Rightarrow y = -\frac{3}{5}x + \frac{2}{5}$

$\therefore$ The line l_1 has a gradient of $-\frac{3}{5}$

[2 marks]

(b) Require l_2 to have sign changed reciprocal gradient through $(3,\ 1)$

$$y = mx + c$$

$$1 = \frac{5}{3} \times 3 + c \Rightarrow c = -4$$

$$\therefore y = \frac{5}{3}x - 4$$

[3 marks]

Answer 7

(a) Put $x = 3$ into $y = 5 - 2x \Rightarrow y = -1$

$\therefore$ As the answer was -1, the point $(3, -1)$ is on L

[1 mark]

(b) The sign changed reciprocal gradient through $(3, -1)$ leads to,

$$y = mx + c$$

$$-1 = \frac{1}{2} \times 3 + c \Rightarrow c = -\frac{5}{2}$$

$$\therefore y = \frac{1}{2}x - \frac{5}{2} \Rightarrow x - 2y - 5 = 0$$

[4 marks]

Answer 8

(i) $m_{AB} = \dfrac{4 - (-1)}{0 - (-5)} = \dfrac{5}{5} = 1$ $\qquad m_{CD} = \dfrac{(-2) - 3}{2 - 7} = \dfrac{-5}{-5} = 1$

$\therefore$ As $m_{AB} = m_{CD}$, AB is parallel to CD

$m_{BC} = \dfrac{3 - 4}{7 - 0} = \dfrac{-1}{7}$ $\qquad m_{DA} = \dfrac{-2 - (-1)}{2 - (-5)} = \dfrac{-1}{7}$

$\therefore$ As $m_{BC} = m_{DA}$, BC is parallel to DA

Taken together, these two results show that $ABCD$ is a parallelogram ☐

[2 marks]

(ii)

$$|AB| = \sqrt{(0 - (-5))^2 + (4 - (-1))^2} = \sqrt{50}$$

$$|BC| = \sqrt{(7 - 0)^2 + (3 - 4)^2} = \sqrt{50}$$

$$|CD| = \sqrt{(2 - 7)^2 + (-2 - 3)^2} = \sqrt{50}$$

$$|DA| = \sqrt{(2 - (-5))^2 + ((-1) - (-2))^2} = \sqrt{50}$$

$\therefore$ As $|AB| = |BC| = |CD| = |DA|$, $ABCD$ is a rhombus ☐

(All sides have the same length, $\sqrt{50}$)

[2 marks]

(iii) As $m_{AB} \times m_{BC} \neq -1$ the lines AB and BC are not perpendicular which they would be, if $ABCD$ where a square.

$\therefore$ $ABCD$ is not a square ☐

[2 marks]

Answer 9

(a) $$2y - 3x - k = 0 \text{ with } A(1,\ 4)$$

$$\Rightarrow 2 \times 4 - 3 \times 1 - k = 0$$

$$k = 5$$

[1 mark]

(b) $$2y - 3x - 5 = 0 \Rightarrow y = \frac{3}{2}x + \frac{5}{2}$$

$$\therefore \text{ The gradient of } L_1 \text{ is } \frac{3}{2}$$

[2 marks]

(c) The sign changed reciprocal gradient through $A(1,\ 4)$ leads to,

$$y = mx + c$$

$$4 = -\frac{2}{3} \times 1 + c \Rightarrow c = \frac{14}{3}$$

$$\therefore y = -\frac{2}{3}x + \frac{14}{3} \Rightarrow 2x + 3y - 14 = 0$$

[4 marks]

(d) Where a line crosses the x-axis, $y = 0$

$$\therefore 2x + 3 \times 0 - 14 = 0$$

$$2x = 14$$

$$x = 7 \qquad \Rightarrow B(7,\ 0)$$

[2 marks]

(e) $A(1,\ 4),\ B(7,\ 0)$

$$|AB| = \sqrt{(7-1)^2 + (0-4)^2}$$

$$= \sqrt{6^2 + 4^2}$$

$$= \sqrt{52}$$

$$= 2\sqrt{13}$$

[2 marks]

Answer 10

(a) $3x + 2y - 8 = 0 \Rightarrow y = -\frac{3}{2}x + 4$

$\therefore$ the line l_2 has a gradient of $-\frac{3}{2}$

[2 marks]

(b)

$$3x + 2 = -\frac{3}{2}x + 4$$

multiple through by 2

$$6x + 4 = -3x + 8$$

$$9x = 4$$

$$x = \frac{4}{9}$$

$$y = 3 \times \frac{4}{9} + 2$$

$$= \frac{10}{3}$$

$\therefore$ P is the point $\left(\frac{4}{9}, \frac{10}{3}\right)$

[3 marks]

(c) To find the point A, $1 = 3x + 2 \Rightarrow x = -\frac{1}{3} \Rightarrow A\left(-\frac{1}{3}, 1\right)$

To find the point B, $1 = -\frac{3}{2}x + 4$

multiply through by 2

$$2 = -3x + 8$$

$$x = 2 \Rightarrow B(2, 1)$$

$$Area\ \Delta = \frac{1}{2}bh$$

$$= \frac{1}{2} \times \left(2 - \left(-\frac{1}{3}\right)\right)\left(\frac{10}{3} - 1\right)$$

$$= \frac{1}{2} \times \frac{7}{3} \times \frac{7}{3}$$

$$= \frac{49}{18} \qquad \text{(about 2.72)}$$

[4 marks]

Lesson 5

A-Level Pure Mathematics, Year 1
Additional Mathematics
GCSE
Coordinate Geometry

5.1 TEST

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 74

Question 1

Determine the gradient, m, of the line between the points A (7, 2) and B (15, 6) by using the formula;

$$m = \frac{\Delta y}{\Delta x}$$

[2 marks]

Question 2

$$(x_A , y_A) \text{ and } (x_B , y_B) \text{ have midpoint } \left(\frac{x_B + x_A}{2} , \frac{y_B + y_A}{2} \right)$$

Find the midpoint of the line segment with end points A (4, – 9) and B (6, 7)

[2 marks]

Question 3

Find the exact distance, D, between the points A (3, – 1) and B (15, 4) by using the formula;

$$D = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

[2 marks]

Question 4

A line, L, has the following equation,

$$y = -\frac{4}{5}x + \frac{2}{5}$$

(i) Write down the gradient of a line that is parallel with L

[1 mark]

(ii) write down the gradient of a line that is perpendicular with L

[1 mark]

Question 5

What is the equation of the line with gradient 5, through the point (11, 21) ?
Give your answer in the form $y = mx + c$, with m and c being constants.

[3 marks]

Question 6

What is the gradient of the line with equation $5x - 4y - 12 = 0$?

[2 marks]

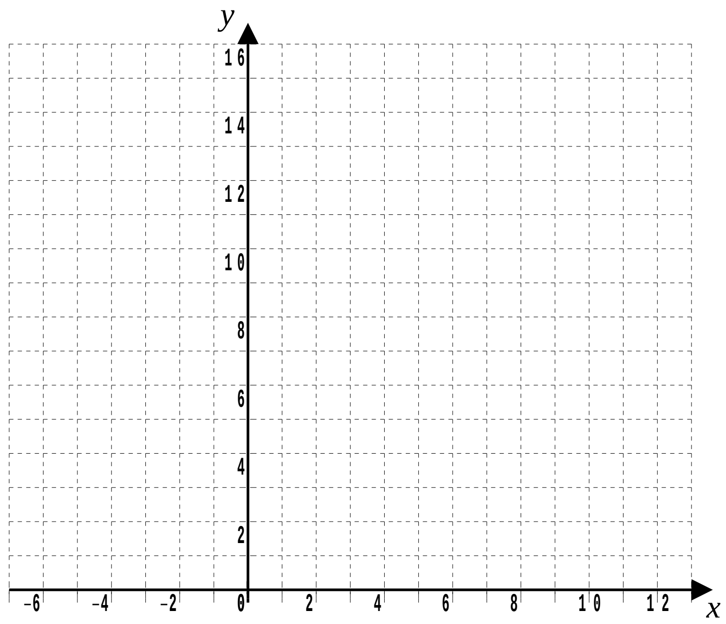
Question 7

The graph of this function is a line segment.

$$f(x) = \frac{1}{3}x + 4, \quad x \in \mathbb{R}, \quad -6 \leqslant x \leqslant 12$$

(i) Plot the line segment on the grid.

[3 marks]



(ii) Find the equation of the perpendicular bisector of the line segment

[5 marks]

(iii) Add a plot of the perpendicular bisector onto the grid

[2 marks]

Question 8

IGCSE Examination Question from May 2019, Paper 1H, Q23 (Edexcel)

$ABCD$ is a kite with $AB = AD$ and $CB = CD$

B is the point with coordinates $(10, 19)$

D is the point with coordinates $(2, 7)$

Find an equation of the line AC

Give your answer in the form $py + qx = r$ where p, q and r are integers

[5 marks]

Question 9

The line *l* passes through the point (3, 7) and is perpendicular to the line with equation;

$$y = \frac{1}{2}x - 2$$

(i) Find an equation for *l*

[3 marks]

(ii) Find the coordinates of the point where the lines intersect

[3 marks]

Question 10

The straight line through the points $A\,(-3,\ 3)$ and $B\,(2, 18)$ meets the coordinate axes at the points P and Q.

Find the area of a square having PQ as one of its sides.

[5 marks]

Question 11

A line whose gradient is $\frac{1}{2}$ passes through the point of intersection of the lines;

$$y = 3x + 8 \qquad \text{and} \qquad y = -2x + 13$$

Find the line's equation in the form $ax + by + c = 0$ where a, b and c are integers.

[6 marks]

Question 12

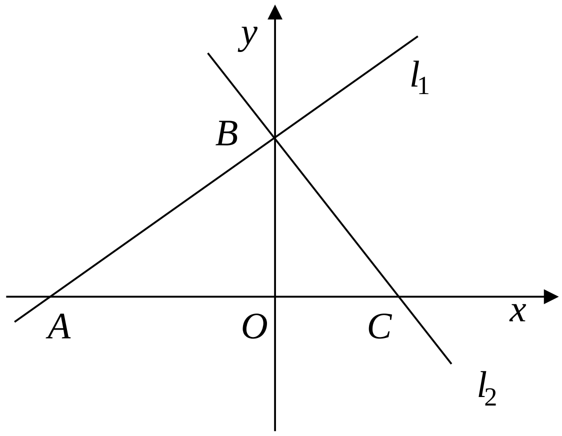
The line with equation $y = 3x + 1$ cuts the curve with equation $y = x^2 + 4x - 5$ at the points A and B.

Calculate the exact length of AB.

[6 marks]

Question 13

A-Level Examination Question from January 2012, C1, Q6 (Edexcel)



The line l_1 has equation $2x - 3y + 12 = 0$

(a) Find the gradient of l_1

[1 mark]

The line l_1 crosses the x-axis at the point A and the y-axis at the point B.

The line l_2 is perpendicular to l_1 and passes through B.

(b) Find an equation of l_2

[3 marks]

The line l_2 crosses the x-axis at the point C.

(c) Find the area of triangle ABC.

[4 marks]

Question 14

A-Level Examination Question from May 2006, C1, Q11 (Edexcel)

The line l_1 passes through the points $P\ (-1, 2)$ and $Q\ (11, 8)$.

(a) Find an equation for l_1 in the form $y = mx + c$, where m and c are constants.

[3 marks]

The line l_2 passes through the point $R\ (10, 0)$ and is perpendicular to l_1.
The lines l_1 and l_2 intersect at the point S.

(b) Calculate the coordinates of S.

[4 marks]

(c) Show that the length of RS is $3\sqrt{5}$

[2 marks]

(d) Hence, or otherwise, find the exact area of triangle PQR.

[4 marks]

5.2 Answers

A-Level Pure Mathematics, Year 1
Additional Mathematics
GCSE
Coordinate Geometry

Answer 1

$$m_{AB} = \frac{6 - 2}{15 - 7} = \frac{4}{8} = \frac{1}{2}$$

[2 marks]

Answer 2

$$Midpoint\ AB = \left(\frac{6 + 4}{2}, \ \frac{7 + (-9)}{2} \right) = (5, -1)$$

[2 marks]

Answer 3

$$D = \sqrt{(15 - 3)^2 + (4 - (-1))^2} = \sqrt{12^2 + 5^2} = 13$$

[2 marks]

Answer 4

(i) $m_{parallel} = -\frac{4}{5}$

(ii) $m_{perpendicular} = \frac{5}{4}$

[1, 1 mark]

Answer 5

$$y = mx + c$$

$$21 = 5 \times 11 + c \Rightarrow c = -34$$

$\therefore$ The line will have equation $y = 5x - 34$

[3 marks]

Answer 6

$$5x - 4y - 12 = 0$$

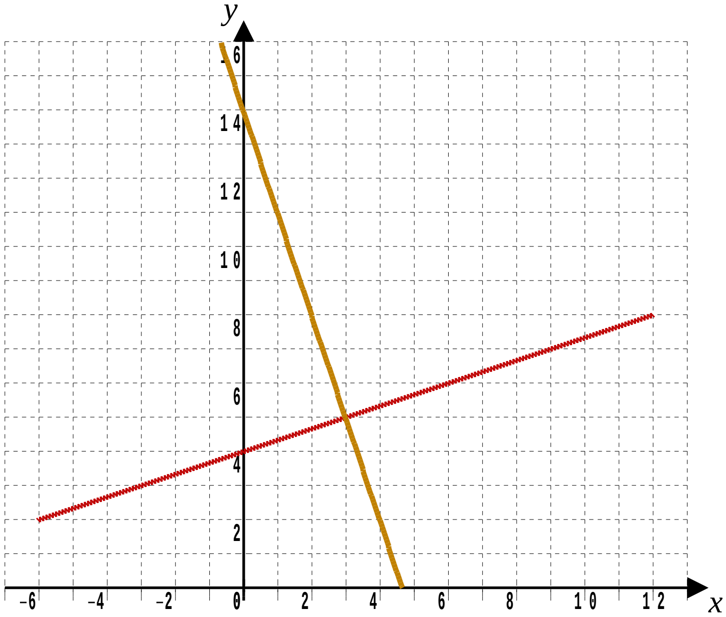
$$4y = 5x - 12$$

$$y = \frac{5}{4}x - 3$$

The gradient is $\frac{5}{4}$

[2 marks]

Answer 7
(i), (iii)



[3, 2 marks]

(ii)

$$f(-6) = \frac{1}{3} \times (-6) + 4 = 2 \Rightarrow (-6,\ 2) \text{ is an endpoint}$$

$$f(12) = \frac{1}{3} \times 12 + 4 = 8 \Rightarrow (12,\ 8) \text{ is an endpoint}$$

$$Midpoint = \left(\frac{12 + (-6)}{2},\ \frac{8 + 2}{2} \right) = (3,\ 5)$$

$$Gradient = \frac{1}{3} \text{ from the equation given in the question}$$

Require a line with the sign changed reciprocal gradient through $(3,\ 5)$

$$y = mx + c$$

$$5 = -3 \times 3 + c \Rightarrow c = 14$$

$$\therefore\ y = -3x + 14$$

[5 marks]

Answer 8

$$Midpoint\ BD = \left(\frac{2 + 10}{2},\ \frac{7 + 19}{2} \right) = (6,\ 13)$$

$$Gradient\ BD = \frac{7 - 19}{2 - 10} = \frac{-12}{-8} = \frac{3}{2}$$

Require a line with the sign changed reciprocal gradient through $(6,\ 13)$

$$y = mx + c$$

$$13 = -\frac{2}{3} \times 6 + c \Rightarrow c = 17$$

$$\therefore\ y = -\frac{2}{3}x + 17$$

multiply through by 3

$$3y = -2x + 51$$

$$3y + 2x = 51$$

[5 marks]

Answer 9

(i) The sign changed reciprocal gradient is – 2

$$y = mx + c$$

$$7 = -2 \times 3 + c \Rightarrow c = 13$$

$\therefore$ The line *l* has equation $y = -2x + 13$

[3 marks]

(ii)

$$-2x + 13 = \frac{1}{2}x - 2$$

multiply through by 2

$$-4x + 26 = x - 4$$

$$5x = 30$$

$$x = 6 \Rightarrow y = 1$$

The two lines intersect at the point $(6,\ 1)$

[3 marks]

Answer 10

$$m_{AB} = \frac{18 - 3}{2 - (-3)} = \frac{15}{5} = 3$$

$$y = mx + c$$

$$18 = 3 \times 2 + c \Rightarrow c = 12$$

$\therefore$ The line through A and B is $y = 3x + 12$

This line cuts the x-axis when $y = 0 \Rightarrow (-4,\ 0)$

and cuts the y-axis at $(0, 12)$

Square has sides of length $\sqrt{12^2 + 4^4} = \sqrt{160} \ \left(= 4\sqrt{10}\right)$

$\therefore$ Area of the square is 160 units^2

[5 marks]

Answer 11

$$3x + 8 = -2x + 13$$

$$5x = 5$$

$$x = 1 \Rightarrow y = 11$$

So a line through $(1,\ 11)$ with gradient $\frac{1}{2}$ is required

$$y = mx + c$$

$$11 = \frac{1}{2} \times 1 + c \Rightarrow c = \frac{21}{2}$$

$$y = \frac{1}{2}x + \frac{21}{2}$$

$$\therefore\ x - 2y + 21 = 0$$

[6 marks]

Answer 12

$$x^2 + 4x - 5 = 3x + 1$$

$$x^2 + x - 6 = 0$$

$$(x + 3)(x - 2) = 0$$

Either $x = -3$ or $x = 2$

Points of intersection are thus $(-3, -8)$, $(2, 7)$

$$D = \sqrt{(7 - (-8))^2 + (2 - (-3))^2}$$

$$= \sqrt{15^2 + 5^2}$$

$$= \sqrt{250}$$

$$= 5\sqrt{10} \qquad \text{(about 15.8)}$$

[6 marks]

Answer 13

(a) $2x - 3y + 12 = 0 \Rightarrow y = \dfrac{2}{3}x + 4 \quad \therefore\ m = \dfrac{2}{3}$

[1 mark]

(b) $A(-6, 0) \qquad B(0, 4)$

l_2 has a sign changed reciprocal gradient of $-\dfrac{3}{2}$ through $B(0, 4)$

$$\therefore\ y = -\frac{3}{2}x + 4$$

[3 marks]

(c) $C\left(\dfrac{8}{3}, 0\right)$

$$\text{Area } \Delta ABC = \frac{1}{2} \times \left(6 + \frac{8}{3}\right) \times 4$$

$$= \frac{1}{2} \times \frac{26}{3} \times 4$$

$$= \frac{52}{3} = 17\frac{1}{3}$$

[4 marks]

Answer 14

(a) $m = \dfrac{8 - 2}{11 - (-1)} = \dfrac{6}{12} = \dfrac{1}{2}$

$$y = mx + c$$

$$8 = \frac{1}{2} \times 11 + c \Rightarrow c = \frac{5}{2}$$

$$\therefore\ y = \frac{1}{2}x + \frac{5}{2}$$

[3 marks]

(b) l_2 has a sign changed reciprocal gradient of -2 through $R(10,\ 0)$

$$y = mx + c$$

$$0 = -2 \times 10 + c \Rightarrow c = 20$$

$\therefore\ y = -2x + 20$ is the equation of l_2

Intersection is when, $\dfrac{1}{2}x + \dfrac{5}{2} = -2x + 20$

multiply through by 2

$$x + 5 = -4x + 40$$

$$5x = 35$$

$$x = 7 \Rightarrow y = 6$$

$$\therefore\ S(7,\ 6)$$

[4 marks]

(c) $R(10,\ 0),\ S(7,\ 6)$

$$|RS| = \sqrt{(7 - 10)^2 + (6 - 0)^2}$$

$$= \sqrt{3^2 + 6^2}$$

$$= \sqrt{45}$$

$$= 3\sqrt{5} \qquad \square$$

[2 marks]

(d) $P(-1,\ 2),\ Q(11,\ 8)$

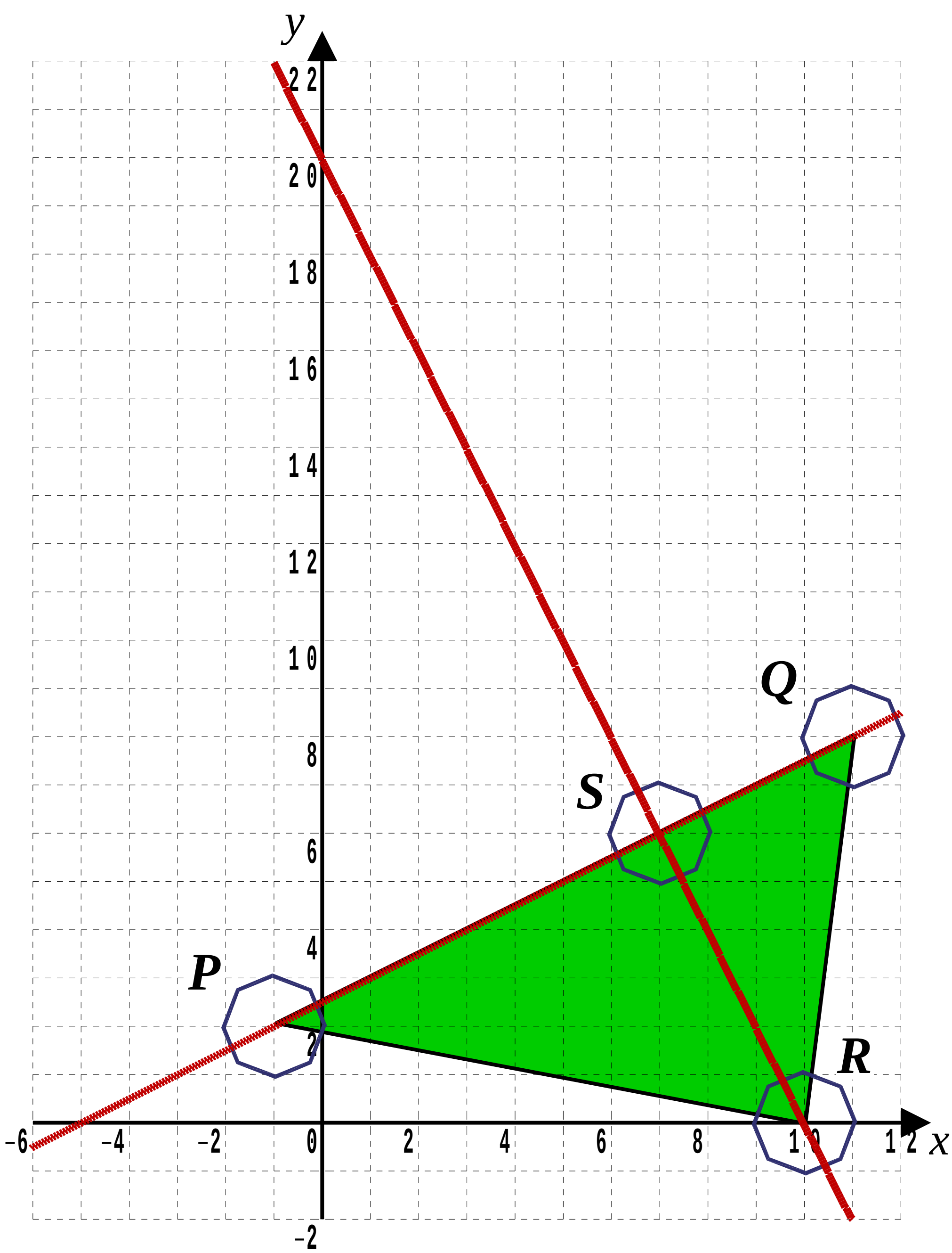
$$|PQ| = \sqrt{(8 - 2)^2 + (11 - (-1))^2}$$

$$= \sqrt{6^2 + 12^2}$$

$$= \sqrt{180}$$

$$= 6\sqrt{5} \qquad \square$$

A diagram is needed to properly see what is going on...



(d) $P(-1,\ 2),\ Q(11,\ 8),\ R(10,\ 0),\ S(7,\ 6)$

$$\text{Area } \Delta PQR = \frac{1}{2} \times |PQ| \times |RS|$$

$$= \frac{1}{2} \times 6\sqrt{5} \times 3\sqrt{5}$$

$$= 45$$

[4 marks]

6.1 The Equation of a Circle

In the work of straight lines, a geometric object was analysed using algebra. Extensive use was made of the equation $y = mx + c$ and an understanding of how the various parts of that equation related to the particular straight line under consideration; that the value of m gave information about gradient, and that the value of c gave information about where the line intercepted the y-axis.

A similar approach is taken with circles.

The Equation of a Circle

$$(x - a)^2 + (y - b)^2 = r^2$$

represents a circle with centre (a, b) and radius r

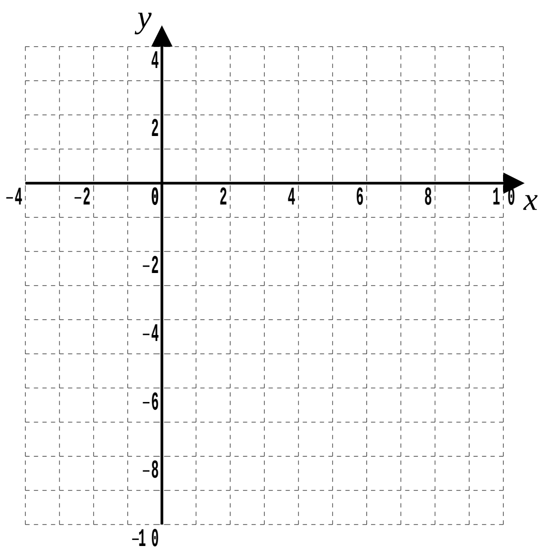
6.2 Example

For the circle with equation $(x - 4)^2 + (y + 3)^2 = 25$

(i) Write down the coordinates of the circle's centre and state its radius.

(ii) Prove that the circle passes through the origin.

(iii) Sketch the circle, using the NEWS method.



[2, 2, 2 marks]

Teaching Video : http://www.NumberWonder.co.uk/v9033/6.mp4

The video walks through the above example

6.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 60

Question 1

$$(x - 16)^2 + (y - 9)^2 = 100$$

State the radius and the centre of this circle

[2 marks]

Question 2

$$(x + 25)^2 + (y + 121)^2 = 17^2$$

State the radius and the centre of this circle

[2 marks]

Question 3

$$(x - 12)^2 + (y + 5)^2 = 400$$

(i) State the radius and the centre of this circle, *C*

[2 marks]

(ii) What is the distance between the centre of the circle, *C*, and the origin, *O* ?

[2 marks]

Question 4

Write down the equation of a circle, centre (2, 8) and radius 7

[2 marks]

Question 5

Write down the equation of a circle, centre $(-2.1, \ 4.8)$ and radius 3.6

[2 marks]

Question 6

Write down the equation of a circle, centre $(-1, -5)$ and radius $\sqrt{13}$

[2 marks]

Question 7

Additional Mathematics Examination Question from June 2006, Q4 (OCR)

(i) Find the distance between the points (2, 3) and (7, 9)

[2 marks]

(ii) Hence find the equation of the circle with centre (2, 3) and passing through the point (7, 9)

[2 marks]

Question 8

Additional Mathematics Examination Question from June 2009, Q4, (OCR)

AB is a diameter of a circle, where *A* is (1, 1) and *B* is (5, 3)

Find,

(i) the exact length of *AB*

[2 marks]

(ii) the coordinates of the midpoint of *AB*

[1 mark]

(iii) the equation of the circle

[3 marks]

Question 9

Additional Mathematics Examination Question from June 2011, Q1 (OCR)

Determine whether the point (5, 2) lies inside or outside the circle whose equation is $x^2 + y^2 = 30$

You must show your working.

[3 marks]

Question 10

Consider the circle with equation $(x - 10)^2 + (y - 8)^2 = 100$

(i) Write down the coordinates of the circle's centre

[1 mark]

(ii) What is the radius of the circle ?

[1 mark]

(iii) Prove that the point (18, 14) is on the circle

[2 marks]

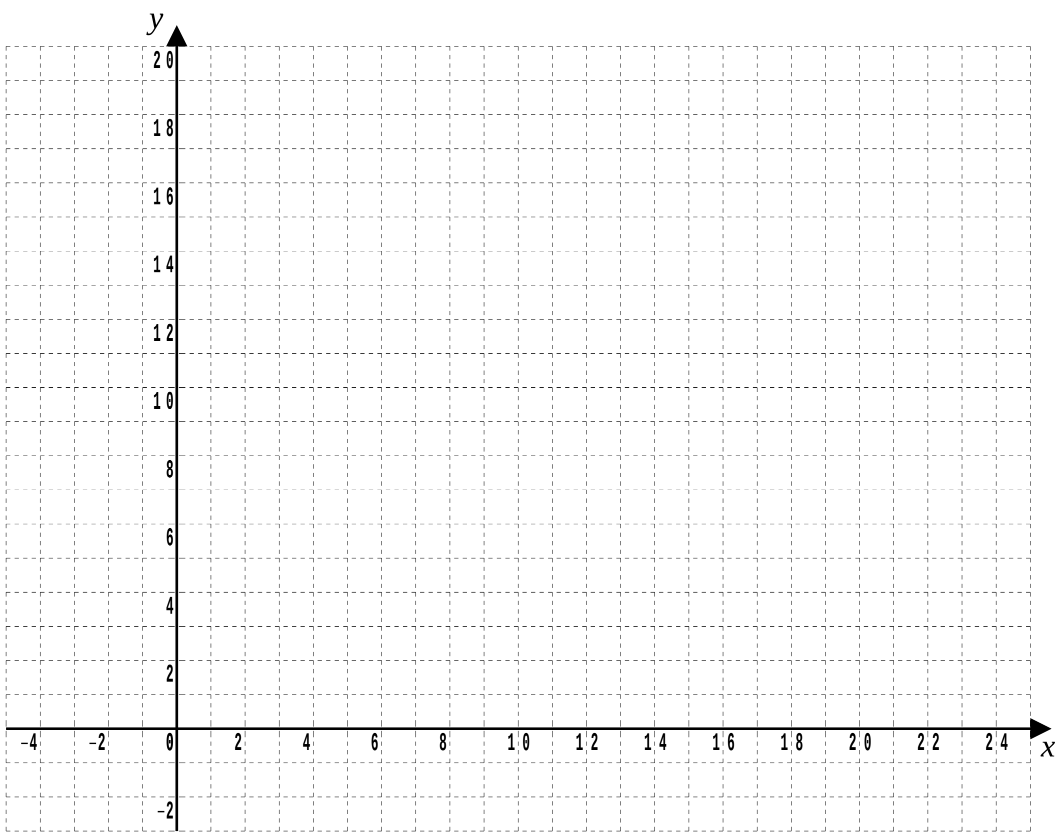
(iv) Find the coordinates of the two points where the circle crosses the *x*-axis

[2 marks]

(v) Sketch the circle by first plotting points

- with the help of the NEWS method
- using the answers to parts (iii) and (iv)

and making use of the circle's horizontal and vertical lines of mirror symmetry



[3 marks]

Question 11

Additional Mathematics Examination Question from May 2012, Q10 (OCR)

$A(1,\ 10)$, $B(8,\ 9)$ and $C(7,\ 2)$ are three points

(i) Find the coordinates of the midpoint, M, of AC

[1 mark]

(ii) Find the equation of the circle with AC as diameter

[4 marks]

(iii) Show that B lies on this circle

[1 mark]

(iv) Prove that AM and BM are perpendicular

[3 marks]

(v) BD is a diameter of this circle. Find the coordinates of D

[3 marks]

Question 12

Additional Mathematics Examination Question from June 2013, Q11 (OCR)

A circle has equation $(x - 2)^2 + y^2 = 100$

(a) Write down the radius and the coordinates of the centre, C, of this circle

[2 marks]

The line $y = 2x + 6$ cuts the circle at two points, A and B

(b) Find

(i) the coordinates of A and B

[5 marks]

(ii) the midpoint, M, of AB

[1 mark]

(iii) the length AB

[2 marks]

(c) Hence find the distance of the centre of the circle from the line AB

[2 marks]

6.4 Answers

A-Level Pure Mathematics, Year 1
Additional Mathematics
Coordinate Geometry

6.4.1 Solution to 6.2 Example (Teaching Video)

Answer 1

(i) The centre is $(4, -3)$ and the radius is 5

[2 marks]

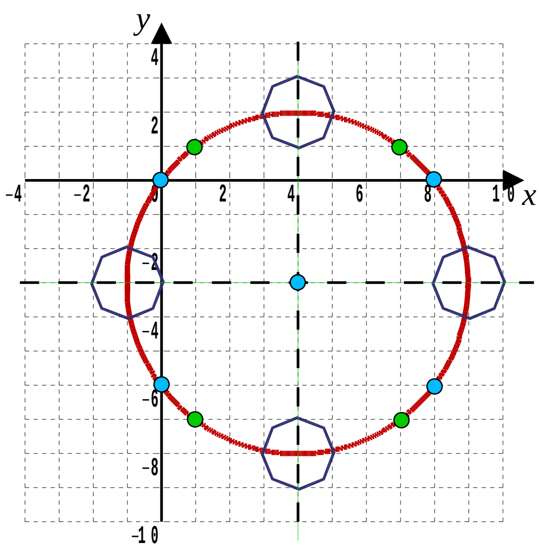
(ii)

$$\begin{aligned} \text{LHS} &= (x-4)^2 + (y+3)^2 \\ &= (0-4)^2 + (0+3)^2 \qquad \text{if the circle passes through } (0,\ 0) \\ &= (-4)^2 + 3^2 \\ &= 16 + 9 \\ &= 25 \\ &= \text{RHS} \end{aligned}$$

$\therefore$ The circle passes through $(0,\ 0)$ $\square$

[2 marks]

(iii)



[2 marks]

6.4.2 Solutions to 6.3 Exercise

Answer 1

$$(x - 16)^2 + (y - 9)^2 = 100$$

The centre is $(16,\ 9)$ and the radius is 10

[2 marks]

Answer 2

$$(x + 25)^2 + (y + 121)^2 = 17^2$$

The centre is $(-25, -121)$ and the radius is 17

[2 marks]

Answer 3

(i)

$$(x - 12)^2 + (y + 5)^2 = 400$$

The centre is $(12, -5)$ and the radius is 20

[2 marks]

(ii)

$$|OC| = \sqrt{(12 - 0)^2 + (-5 - 0)^2}$$
$$= \sqrt{12^2 + 5^2}$$
$$= 13$$

Notice that the origin is inside the circle

[2 marks]

Answer 4

Centre $(2,\ 8)$, Radius 7

$$(x - 2)^2 + (y - 8)^2 = 7^2$$

[2 marks]

Answer 5

Centre $(-2.1,\ 4.8)$, Radius 3.6

$$(x + 2.1)^2 + (y - 4.8)^2 = 3.6^2$$

[2 marks]

Answer 6

Centre $(-1, -5)$, Radius $\sqrt{13}$

$$(x + 1)^2 + (y + 5)^2 = 13$$

[2 marks]

Answer 7

(i) Radius $= \sqrt{(9 - 3)^2 + (7 - 2)^2}$

$= \sqrt{6^2 + 5^2}$

$= \sqrt{61}$ (about 7.81)

[2 marks]

(ii) Centre (2, 3), Radius $\sqrt{61}$

$(x - 2)^2 + (y - 3)^2 = 61$

[2 marks]

Answer 8

(i) $|AB| = \sqrt{(5 - 1)^2 + (3 - 1)^2}$

$= \sqrt{4^2 + 2^2}$

$= \sqrt{20}$

$= 2\sqrt{5}$

[2 marks]

(ii) $Midpoint\ AB = \left(\dfrac{5 + 1}{2}, \dfrac{3 + 1}{2}\right)$

$= (3, 2)$

[1 mark]

(iii) Centre (3, 2) Radius $\sqrt{5}$

$(x - 3)^2 + (y - 2)^2 = 5$

[3 marks]

Answer 9

For the point (5, 2), $x^2 + y^2 = 5^2 + 2^2$

$= 25 + 4$

$= 29$

As $29 < 30$ the point (5, 2) is inside the circle

[3 marks]

Answer 10

(i) (10, 8) **(ii)** 10

[1, 1 marks]

(iii) LHS $= (x - 10)^2 + (y - 8)^2$

$= (18 - 10)^2 + (14 - 8)^2$ if the circle passes through (18, 14)

$= 8^2 + 6^2$

$= 100$

$=$ RHS $\quad \therefore$ The circle passes through (18, 14) ☐

[2 marks]

(iv) Circle crosses the x-axis when $y = 0$ in which case,

$$(x - 10)^2 + (0 - 8)^2 = 100$$

$$x^2 - 20x + 100 + 64 = 100$$

$$x^2 - 20x + 64 = 0$$

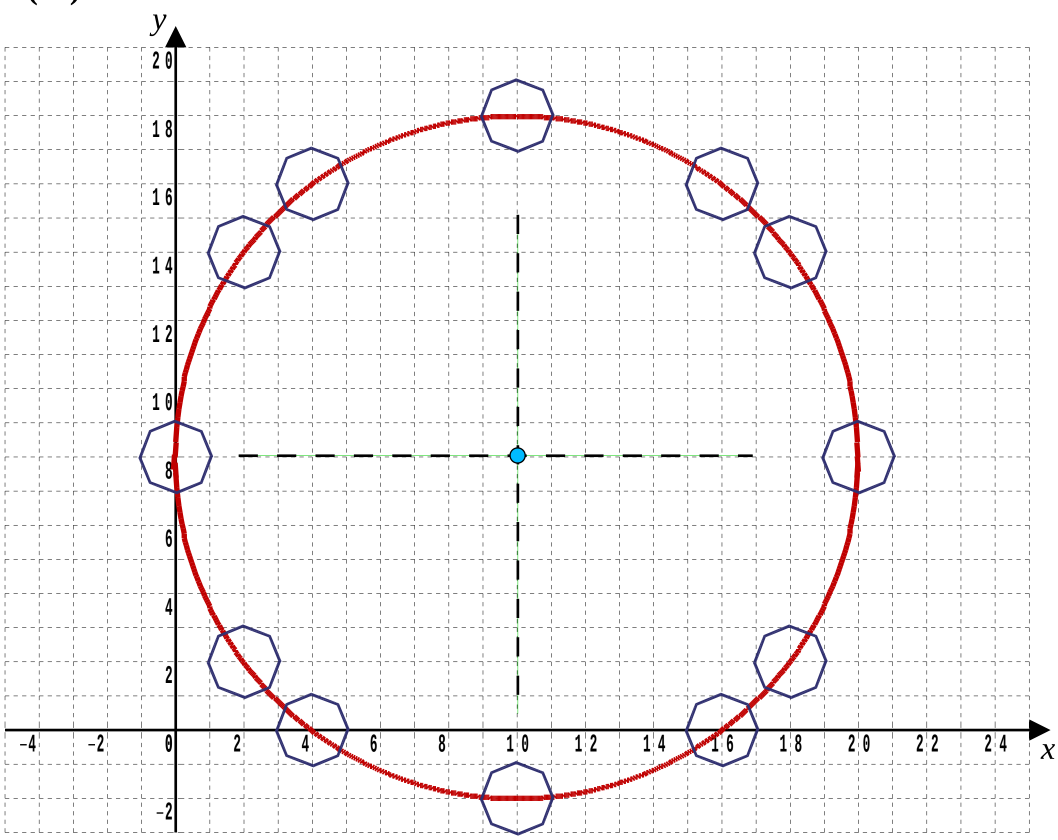
$$(x - 4)(x - 16) = 0$$

Either $x - 4 = 0$ or $x - 16 = 0$

$x = 4$ or $x = 16$, that is, (4, 0) and (16, 0)

[2 marks]

(v)



[3 marks]

Answer 11

$A(1,\ 10)$, $B(8,\ 9)$ and $C(7,\ 2)$

(i) $\quad Midpoint\ AC,\ M = \left(\dfrac{7+1}{2}, \dfrac{2+10}{2} \right) = (4,\ 6)$

[1 mark]

(ii)

$$|AC| = \sqrt{(7-1)^2 + (2-10)^2}$$
$$= \sqrt{6^2 + 8^2}$$
$$= 10$$
$$\therefore\ Radius = 5$$

And the equation of the circle will be $(x-4)^2 + (y-6)^2 = 5^2$

[4 marks]

(iii)

$$\text{LHS} = (x-4)^2 + (y-6)^2$$
$= (8-4)^2 + (9-6)^2$ if the circle passes through $(8,\ 9)$
$$= 4^2 + 3^2$$
$$= 16 + 9$$
$$= 25$$
$= \text{RHS}$ $\quad \therefore$ The circle passes through $B(8,\ 9)$ ☐

[1 mark]

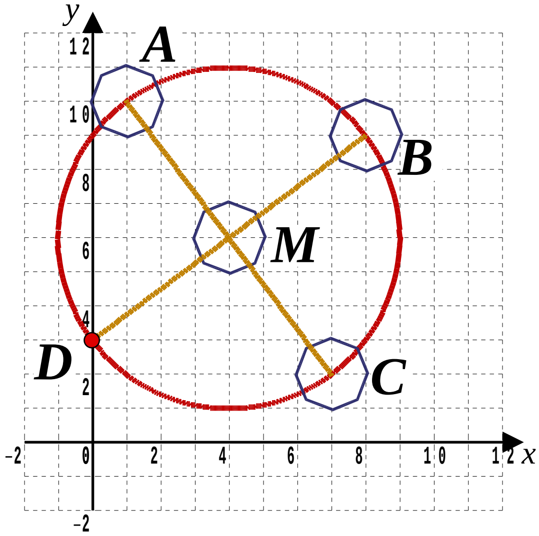
(iv)

$$m_{AM} = \frac{\Delta Y}{\Delta X} = \frac{6-10}{4-1} = -\frac{4}{3}$$
$$m_{BM} = \frac{\Delta Y}{\Delta X} = \frac{6-9}{4-8} = \frac{3}{4}$$

As $m_{AM} \times m_{BM} = -1$ AM is perpendicular to BM ☐

[3 marks]

(v) A diagram is an excellent idea in answering this question



Here, I've used a vector method,

$$\overrightarrow{BM} = m - b$$

$$= \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 8 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$$

$$\overrightarrow{OD} = \overrightarrow{OM} + \overrightarrow{MD} \qquad \text{but } \overrightarrow{BM} = \overrightarrow{MD}$$

$$= \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$\therefore D(0,\ 3)$$

[3 marks]

Answer 12

(a) Radius 10, Centre (2, 0)

[2 marks]

(b) **(i)**

$$(x - 2)^2 + y^2 = 100 \text{ and } y = 2x + 6$$

$$(x - 2)^2 + (2x + 6)^2 = 100$$

$$x^2 - 4x + 4 + 4x^2 + 24x + 36 = 100$$

$$5x^2 + 20x - 60 = 0$$

$$x^2 + 4x - 12 = 0$$

$$(x + 6)(x - 2) = 0$$

Either $x = -6$ or $x = 2$

Points of intersection are, $(-6,\ -6)$, $(2,\ 10)$

[5 marks]

(ii) $\quad Midpoint,\ M \ = \left(\dfrac{2 \ + \ (-6)}{2},\ \ \dfrac{10 \ + \ (-6)}{2} \right) \ = \ (-2,\ \ 2\,)$

[1 mark]

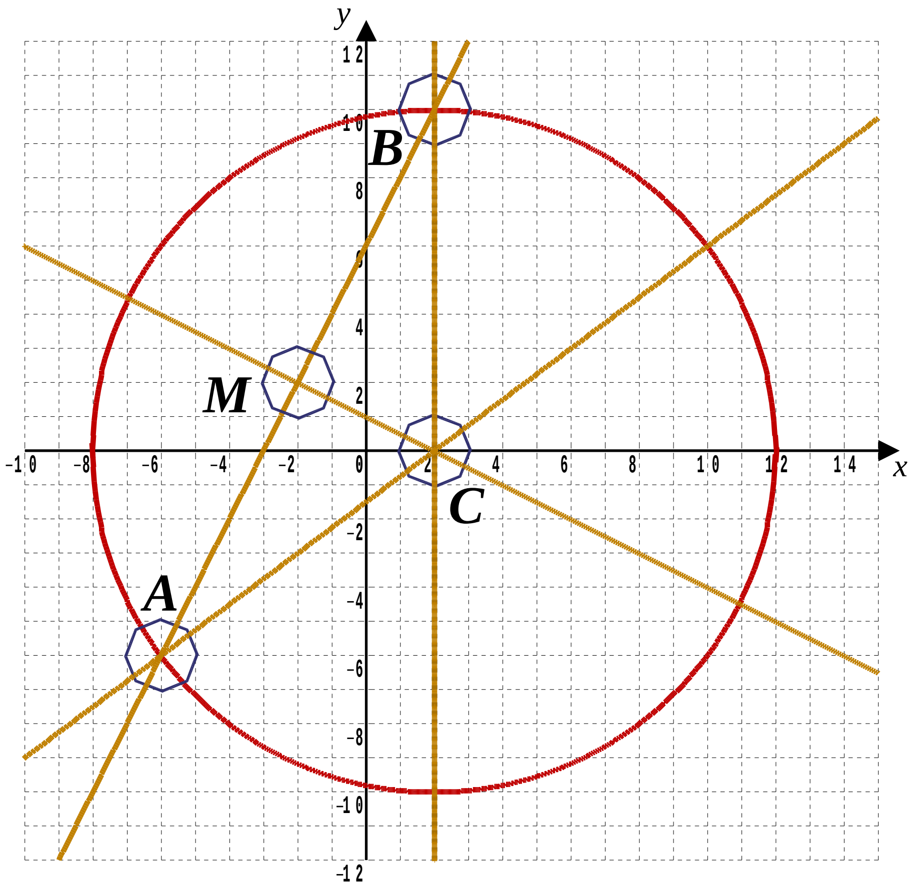
(iii) $\quad \left| AB \right| \ = \ \sqrt{(\,2 \ - \ (-6))^2 \ + \ (10 \ - \ (-6))^2}$

$$= \ \sqrt{8^2 \ + \ 16^2}$$

$$= \ 8\sqrt{5}$$

[2 marks]

(c) A diagram is helpful...



$$\left| MC \right| \ = \ \sqrt{10^2 \ - \ (4\sqrt{5})^2}$$

$$= \ \sqrt{100 \ - \ 16 \ \times \ 5}$$

$$= \ \sqrt{20}$$

$$= \ 2\sqrt{5}$$

[2 marks]

Lesson 7

A-Level Pure Mathematics, Year 1
Additional Mathematics
Coordinate Geometry

7.1 Circles in Disguise

Consider the equation, $x^2 + y^2 + 12x - 2y = 27$

A friend of mine claims that this is a circle. If they are correct then it must be possible to algebraically manipulate this equation into the form

$$(x - a)^2 + (y - b)^2 = r^2$$

where a, b and r are constants the values of which need to be found.
Then, the circle's centre would be $(a, \ b)$ and its radius r.

7.2 Completing the Square

The technique employed is called "completing the square".

Teaching Video : http://www.NumberWonder.co.uk/v9033/7.mp4

[6 marks]

7.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 70

Question 1

Write each of the following in the "completed square" form,

$$y = (x + a)^2 + b$$

(i) $y = x^2 + 8x + 17$

(ii) $y = x^2 + 10x + 7$

(iii) $y = x^2 - 12x + 3$

(iv) $y = x^2 - 6x - 7$

[8 marks]

Question 2

Consider the circle, $x^2 - 4x + y^2 - 8y = 44$

(i) Rewrite this in the form

$$(x - a)^2 + (y - b)^2 = r^2$$

where *a*, *b* and *r* are constants the values of which are to be found.

[4 marks]

(ii) Hence, or otherwise, state;

(a) The coordinates of the centre of the circle

[1 mark]

(b) The radius of the circle.

[1 mark]

Question 3

Expand the brackets and simplify;

(i) $y = (x + 3)^2 + 4$

(ii) $y = (x + 7)^2 - 10$

(iii) $y = (x + 1)^2 + 7$

(iv) $y = \left(x + \frac{1}{2}\right)^2 + 10$

[8 marks]

Question 4

Consider the circle, $x^2 + y^2 + 8x - 14y + 29 = 0$

(i) Rewrite this in the form

$$(x - a)^2 + (y - b)^2 = r^2$$

where *a*, *b* and *r* are constants the values of which are to be found.

[4 marks]

(ii) Hence, or otherwise, state;

(a) The coordinates of the centre of the circle.

[1 mark]

(b) The radius of the circle.

[1 mark]

Question 5

Additional Mathematics Examination Question from June 2007, Q3 (OCR)

A circle has equation $x^2 + y^2 - 4x - 6y + 3 = 0$

Find the coordinates of the centre and the radius of the circle.

[4 marks]

Question 6

Additional Mathematics Examination Question from June 2015, Q9 (OCR)

The equation of a circle is $x^2 + y^2 - 8x + 2y - 19 = 0$

(i) Express the equation of C in the form $(x - a)^2 + (y - b)^2 = r^2$

[4 marks]

(ii) Hence or otherwise, use an algebraic method to decide whether the point (8, 3) lies inside, outside or on the circumference of the circle. Show all your working.

[2 marks]

Question 7

Additional Mathematics Examination Question from June 2010, Q9 (OCR)

The diameter of a circle is PQ, where $P\ (1, 3)$ and $Q\ (15, 1)$

(i) Find the centre of the circle.

[2 marks]

(ii) Show that the radius of the circle is $5\sqrt{2}$

[2 marks]

(iii) Hence find the equation of the circle in the form,

$$x^2 + y^2 + ax + by + c = 0$$

[2 marks]

Question 8

Consider the circle $x^2 + y^2 + 41 = 10(x + y)$

(i) Rewrite this in the form

$$(x - a)^2 + (y - b)^2 = r^2$$

where a, b and r are constants the values of which are to be found.

[4 marks]

(ii) Hence, or otherwise, state;

(a) The coordinates of the centre of the circle.

[1 mark]

(b) The radius of the circle.

[1 mark]

Question 9

Additional Mathematics Examination Question from June 2005, Q12 (OCR)

(i) A circle has equation $x^2 + y^2 - 2x - 4y - 20 = 0$

Find the coordinates of its centre, C, and its radius.

[3 marks]

(ii) Find the coordinates of the points, A and B, where the line $y = x + 2$ cuts the circle

[5 marks]

(iii) Find angle ACB

[4 marks]

Question 10

A-Level Examination Question from May 2011, Paper C2, Q4 (Edexcel)

The circle C has equation

$$x^2 + y^2 + 4x - 2y - 11 = 0$$

Find

(a) the coordinates of the centre of C,

[2 marks]

(b) the radius of C,

[2 marks]

(c) the coordinates of the points where C crosses the y-axis, giving your answers as simplified surds.

[4 marks]

7.4 Answers

7.4.1 Solution to 7.2 Completing the Square (Teaching Video)

$$x^2 + y^2 + 12x - 2y = 27$$
$$[\, x^2 + 12x \,] + [\, y^2 - 2y \,] = 27$$
$$[\, (x + 6)^2 - 36 \,] + [\, (y - 1)^2 - 1 \,] = 27$$
$$(x + 6)^2 + (y - 1)^2 = 27 + 36 + 1$$
$$(x + 6)^2 + (y - 1)^2 = 64$$

This is the equation of a circle, centre $(-6,\ 1)$ and radius 8

7.4.2 Solution to 7.3 Exercise

Answer 1

(i)
$$\begin{aligned} y &= x^2 + 8x + 17 \\ &= [\, x^2 + 8x \,] + 17 \\ &= [\, (x + 4)^2 - 16 \,] + 17 \\ &= (x + 4)^2 + 1 \end{aligned}$$

(ii)
$$\begin{aligned} y &= x^2 + 10x + 7 \\ &= [\, x^2 + 10x \,] + 7 \\ &= [\, (x + 5)^2 - 25 \,] + 7 \\ &= (x + 5)^2 - 18 \end{aligned}$$

(iii)
$$\begin{aligned} y &= x^2 - 12x + 3 \\ &= [\, x^2 - 12x \,] + 3 \\ &= [\, (x - 6)^2 - 36 \,] + 3 \\ &= (x - 6)^2 - 33 \end{aligned}$$

(iv)
$$\begin{aligned} y &= x^2 - 6x - 7 \\ &= [\, x^2 - 6x \,] - 7 \\ &= [\, (x - 3)^2 - 9 \,] - 7 \\ &= (x - 3)^2 - 16 \end{aligned}$$

[8 marks]

Answer 2

(i)
$$x^2 - 4x + y^2 - 8y = 44$$
$$[\, x^2 - 4x \,] + [\, y^2 - 8y \,] = 44$$
$$[\, (x - 2)^2 - 4 \,] + [\, (y - 4)^2 - 16 \,] = 44$$
$$(x - 2)^2 + (y - 4)^2 = 64$$

[4 marks]

(ii) Centre is $(2,\ 4)$

[1 mark]

(iii) Radius is 8

[1 mark]

Answer 3

(i) $y = (x + 3)^2 + 4$

$= x^2 + 6x + 9 + 4$

$= x^2 + 6x + 13$

(ii) $y = (x + 7)^2 - 10$

$= x^2 + 14x + 49 - 10$

$= x^2 + 14x + 39$

(iii) $y = (x + 1)^2 + 7$

$= x^2 + 2x + 1 + 7$

$= x^2 + 2x + 8$

(iv) $y = \left(x + \frac{1}{2}\right)^2 + 10$

$= x^2 + x + \frac{1}{4} + 10$

$= x^2 + x + 10\frac{1}{4}$

[8 marks]

Answer 4

(i)

$$x^2 + y^2 + 8x - 14y + 29 = 0$$

$$\left[x^2 + 8x \right] + \left[y^2 - 14y \right] + 29 = 0$$

$$\left[(x + 4)^2 - 16 \right] + \left[(y - 7)^2 - 49 \right] + 29 = 0$$

$$(x + 4)^2 + (y - 7)^2 = 36$$

[4 marks]

(ii) Centre is $(-4, \ 7)$

[1 mark]

(iii) Radius is 6

[1 mark]

Answer 5

$$x^2 + y^2 - 4x - 6y + 3 = 0$$

$$\left[x^2 - 4x \right] + \left[y^2 - 6y \right] + 3 = 0$$

$$\left[(x - 2)^2 - 4 \right] + \left[(y - 3)^2 - 9 \right] + 3 = 0$$

$$(x - 2)^2 + (y - 3)^2 = 10$$

This is a circle, centre $(2, \ 3)$ and radius $\sqrt{10}$ (about 3.16)

[4 marks]

Answer 6

(i)

$$x^2 + y^2 - 8x + 2y - 19 = 0$$

$$\left[x^2 - 8x \right] + \left[y^2 + 2y \right] - 19 = 0$$

$$\left[(x - 4)^2 - 16 \right] + \left[(y + 1)^2 - 1 \right] - 19 = 0$$

$$(x - 4)^2 + (y + 1)^2 = 36$$

[4 marks]

(ii)

$$\text{LHS} = (x - 4)^2 + (y + 1)^2$$

$$= (8 - 4)^2 + (3 + 1)^2 \quad \text{if the circle passes through } (8,\ 3)$$

$$= 4^2 + 4^2$$

$$= 16 + 16$$

$$= 32$$

$< \text{RHS of } 36 \qquad \therefore (8,\ 3)$ is inside the circle □

[2 marks]

Answer 7

(i) The midpoint of PQ will be the centre, C, of the circle

$$Midpoint\ PQ = \left(\frac{15 + 1}{2},\ \frac{1 + 3}{2} \right) = (8,\ 2)$$

[2 marks]

(ii) The distance PQ will be the diameter of the circle

$$|PQ| = \sqrt{(15 - 1)^2 + (1 - 3)^2}$$

$$= \sqrt{14^2 + 2^2}$$

$$= \sqrt{196 + 4}$$

$$= 10\sqrt{2}$$

$\therefore$ Radius is $5\sqrt{2}$ □

[2 marks]

(iii) The centre at $(8, 2)$ and radius of $5\sqrt{2}$ give,

$$(x - 8)^2 + (y - 2)^2 = 25 \times 2$$

$$x^2 - 16x + 64 + y^2 - 4y + 4 = 50$$

$$x^2 + y^2 - 16x - 4y + 18 = 0$$

That is, $a = -16$, $b = -4$ and $c = 18$

[2 marks]

Answer 8

(i)

$$x^2 + y^2 + 41 = 10(x + y)$$

$$[x^2 - 10x] + [y^2 - 10y] + 41 = 0$$

$$[(x - 5)^2 - 25] + [(y - 5)^2 - 25] + 41 = 0$$

$$(x - 5)^2 + (y - 5)^2 = 9$$

[4 marks]

(ii) Centre is (5, 5)

[1 mark]

(iii) Radius is 3

[1 mark]

Answer 9

(i)

$$x^2 + y^2 - 2x - 4y - 20 = 0$$

$$[x^2 - 2x] + [y^2 - 4y] - 20 = 0$$

$$[(x - 1)^2 - 1] + [(y - 2)^2 - 4] - 20 = 0$$

$$(x - 1)^2 + (y - 2)^2 = 25$$

This represents a circle with centre (1, 2) and radius 5

[3 marks]

(ii)

$$(x - 1)^2 + (y - 2)^2 = 25 \text{ and } y = x + 2$$

$$(x - 1)^2 + ((x + 2) - 2)^2 = 25$$

$$(x - 1)^2 + x^2 = 25$$

$$x^2 - 2x + 1 + x^2 = 25$$

$$2x^2 - 2x - 24 = 0$$

divide through by 2

$$x^2 - x - 12 = 0$$

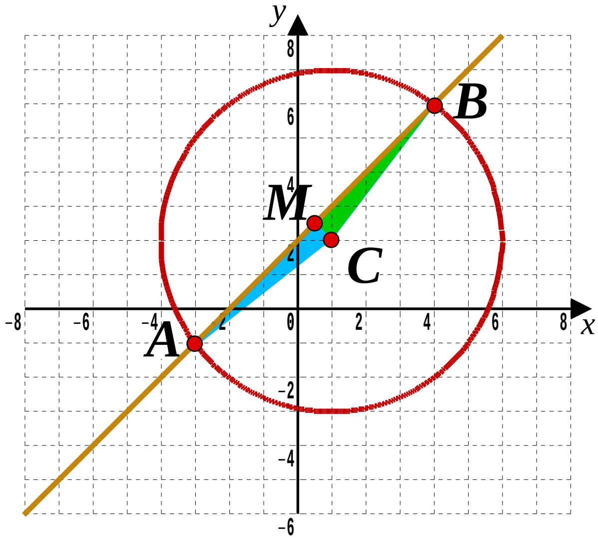
$$(x - 4)(x + 3) = 0$$

Either $x = 4$ or $x = -3$

$\therefore$ Using, $y = x + 2$, the points are $(4, 6)$, $(-3, -1)$

[5 marks]

(iii) A diagram is needed for this part of the question.
Using the part (i) answer, draw the circle with centre (1, 2) and radius 5 (using the NEWS method)
Add the straight line $y = x + 2$ and the part (ii) intersections A at $(-3, -1)$ and B at $(4, 6)$



$$|AB| = \sqrt{(-3-4)^2 + (-1-6)^2} = \sqrt{7^2 + 7^2} = \sqrt{98} = 7\sqrt{2}$$

Let M be the midpoint at AB in which case $\angle CMA$ is a right angle.

$$\angle ACM = arcsin\left(\frac{3.5\sqrt{2}}{5}\right) = 81.87 \Rightarrow \angle ACB = 163.7$$

[4 marks]

Answer 10

$$x^2 + y^2 + 4x - 2y - 11 = 0$$

$$[\,x^2 + 4x\,] + [\,y^2 - 2y\,] - 11 = 0$$

$$[\,(x+2)^2 - 4\,] + [\,(y-1)^2 - 1\,] - 11 = 0$$

$$(x+2)^2 + (y-1)^2 = 16$$

(a) Centre is $(-2,\ 1)$

[2 marks]

(b) Radius is 4

[2 marks]

(c) The circle will cross the y-axis when $x = 0$

$$(0+2)^2 + (y-1)^2 = 16$$

$$4 + (y-1)^2 = 16$$

$$y - 1 = \pm\sqrt{12}$$

$$y = 1 \pm 2\sqrt{3}$$

The points are thus $\left(0,\ 1 - 2\sqrt{3}\right),\ \left(0,\ 1 + 2\sqrt{3}\right)$

[4 marks]

8.1 Diameter

In general, two points will fix the location of any straight line and three points the location of any circle, provided the three points are not collinear.

What does collinear mean ?

☞

[1 mark]

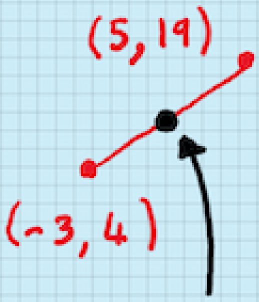
However, two points can fix the location of a circle, if those points are the end points of its diameter.

8.2 Example

The line joining the points $(-3, 4)$ and $(5, 19)$ is the diameter of the circle C.
Find an equation for C.

[6 marks]

8.3 Solution to 8.2 Example



GET THE LENGTH OF THE DIAMETER

$$D = \sqrt{(5-(-3))^2 + (19-4)^2}$$

$$= \sqrt{8^2 + 15^2} = 17$$

$\therefore$ RADIUS IS $\underline{8.5}$

GET THE MIDPOINT WHICH WILL BE CENTRE

$$M = \left(\frac{5+(-3)}{2}, \frac{19+4}{2}\right) = \underline{\underline{(1, 11.5)}}$$

$\therefore$ EQUATION OF CIRCLE IS

$$\underline{\underline{(x-1)^2 + (y-11.5)^2 = 8.5^2}}$$

[6 marks]

8.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 61

Question 1

A-Level Examination Question from January 2005, Paper C2, Q2 (Edexcel)

The points A and B have coordinates (5, – 1) and (13, 11) respectively

(a) Find the coordinates of the mid-point of AB

[2 marks]

Given that AB is a diameter of the circle C

(b) find an equation for C

[4 marks]

Question 2

A-Level Examination Question from January 2007, Paper C2, Q3 (Edexcel)

The line joining the points (– 1, 4) and (3, 6) is a diameter of the circle C

Find an equation for C

[6 marks]

Question 3

A-Level Examination Question from January 2012, Paper C2, Q2 (Edexcel)

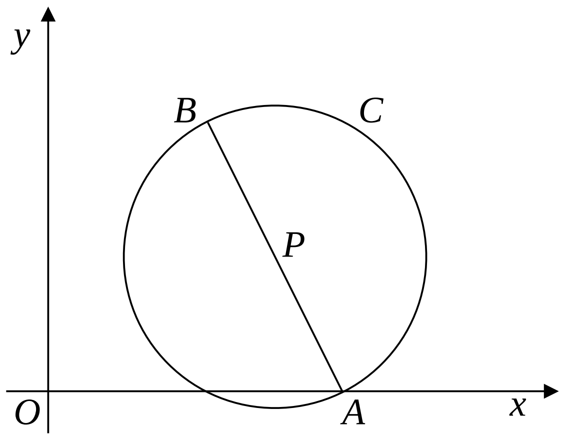
A circle C has centre $(-1, 7)$ and passes through the point $(0, 0)$

Find an equation for C

[4 marks]

Question 4

A-Level Examination Question from January 2006, Paper C2, Q3 (Edexcel)



The end points of a diameter of the circle C are $A(4, 0)$ and $B(3, 5)$

Find

(a) the exact length of AB

[2 marks]

(b) the coordinates of the midpoint P of AB

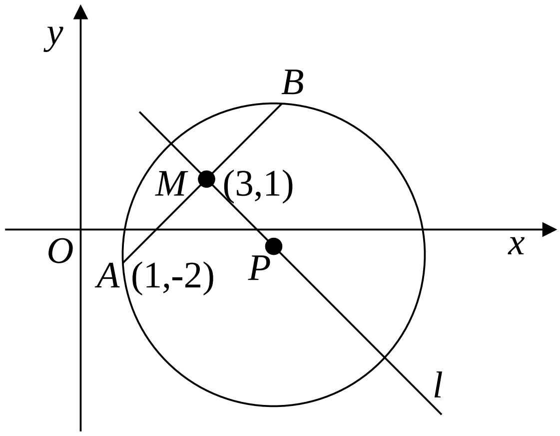
[2 marks]

(c) an equation for the circle C

[3 marks]

Question 5

A-Level Examination Question from May 2007, Paper C2, Q7 (Edexcel)



The points A and B lie on a circle with centre P, as shown.
The point A has coordinates $(1, -2)$
The mid-point M of AB has coordinates $(3, 1)$
The line l passes through the points M and P

(a) Find an equation for l

[4 marks]

Given that the x-coordinate of P is 6,

(b) use your answer to part (a) to show that the y-coordinate of P is -1,

[1 mark]

(c) find an equation for the circle.

[4 marks]

Question 6

A-Level Examination Question from May 2018, Paper C12, Q13 (Edexcel)

The point A (9, $-$ 13) lies on a circle C with centre the origin and radius r

(a) Find the exact value of r

[2 marks]

(b) Find an equation of the circle C

[1 mark]

A straight line through point A has equation $2y + 3x = k$, where k is a constant

(c) Find the value of k

[1 mark]

This straight line cuts the circle again at the point B

(d) Find the exact coordinates of point B

[6 marks]

Question 7

A-Level Examination Question from June 2009, Paper C2, Q6 (Edexcel)

The circle C has equation,

$$x^2 + y^2 - 6x + 4y = 12$$

(a) Find the centre and the radius of C

[5 marks]

The point $P\,(-1, 1)$ and the point $Q\,(7, -5)$ both lie on C

(b) Show that PQ is a diameter of C

[2 marks]

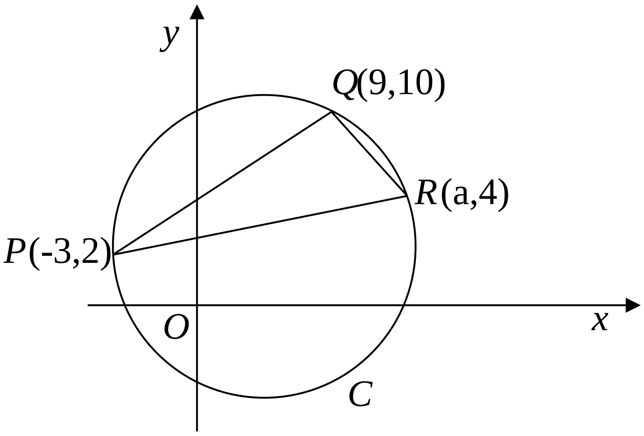
The point R lies on the positive y-axis and the angle $PRQ = 90°$

(c) Find the coordinates of R

[4 marks]

Question 8

A-Level Examination Question from January 2009, Paper C2, Q5 (Edexcel)



The points $P\ (-3, 2)$, $Q\ (9, 10)$ and $R\ (a, 4)$ lie on the circle C, as shown.
Given that PR is a diameter of C,

(a) show that $a = 13$,

[3 marks]

(b) find an equation for C.

[5 marks]

8.5 Answers

8.5.1 Solution to 8.2 Example

The midpoint of the two given points will be the centre of the circle.

$$\left(\frac{5 + (-3)}{2} , \frac{19 + 4}{2} \right) = (1, 11.5)$$

Half the distance between the two given points will be the radius.

$$\begin{aligned} r &= \frac{1}{2} \sqrt{\Delta y^2 + \Delta x^2} \\ &= \frac{1}{2} \sqrt{(19 - 4)^2 + (5 - (-3))^2} \\ &= \frac{1}{2} \sqrt{15^2 + 8^2} \\ &= \frac{17}{2} \\ &= 8.5 \end{aligned}$$

Thus, the equation of the circle is;

$$(x - 1)^2 + (y - 11.5)^2 = 8.5^2$$

8.5.2 Solutions to 8.4 Exercise

Answer 1

(a) $\quad Midpoint\ AB = \left(\frac{13 + 5}{2}, \frac{11 + (-1)}{2} \right) = (9, 5)$

[2 marks]

(b) $\quad |AB| = \sqrt{(13 - 5)^2 + (11 - (-1))^2} = \sqrt{8^2 + 12^2} = 4\sqrt{13}$

Centre (9, 5) Radius $2\sqrt{13}$

The Equation of the circle is, $(x - 9)^2 + (y - 5)^2 = 52$

[4 marks]

Answer 2

$$Midpoint = \left(\frac{3 + (-1)}{2}, \frac{6 + 4}{2} \right) = (1, 5)$$

$$\begin{aligned} Length\ of\ diameter &= \sqrt{(3 - (-1))^2 + (6 - 4)^2} \\ &= \sqrt{4^2 + 2^2} = 2\sqrt{5} \end{aligned}$$

The Equation of the circle is, $(x - 1)^2 + (y - 5)^2 = 5$

[6 marks]

Answer 3

$$Radius = \sqrt{(0 - (-1))^2 + (0 - 7)^2}$$
$$= \sqrt{1^2 + 7^2} = \sqrt{50} = 5\sqrt{2}$$

The Equation of the circle is, $(x + 1)^2 + (y - 7)^2 = 50$

[4 marks]

Answer 4

(a) $|AB| = \sqrt{(3 - 4)^2 + (5 - 0)^2} = \sqrt{1^2 + 5^2} = \sqrt{26}$

[2 marks]

(b) $Midpoint\ AB = \left(\frac{3 + 4}{2}, \frac{5 + 0}{2}\right) = (3.5, 2.5)$

[2 marks]

(c) The Equation of the circle is,

$$(x - 3.5)^2 + (y - 2.5)^2 = \left(\frac{\sqrt{26}}{2}\right)^2$$
$$(x - 3.5)^2 + (y - 2.5)^2 = \frac{26}{4}$$
$$(x - 3.5)^2 + (y - 2.5)^2 = 6.5$$

[3 marks]

Answer 5

(a) $m_{AB} = \frac{1 - (-2)}{3 - 1} = \frac{3}{2}$ $\therefore$ l will have gradient $-\frac{2}{3}$

$$y = mx + c$$
$$1 = -\frac{2}{3} \times 3 + c \Rightarrow c = 3$$

l has equation $y = -\frac{2}{3}x + 3$

[4 marks]

(b) As P lies on l, $y = -\frac{2}{3} \times 6 + 3$

$$= -4 + 3$$
$$= -1 \text{ as expected} \quad \square$$

[1 mark]

(c) Centre $P(6, -1)$, Radius is distance between P and A

$$\text{Radius} = \sqrt{(6 - 1)^2 + (-1 - (-2))^2}$$
$$= \sqrt{5^2 + 1^2} = \sqrt{26}$$

The circle has equation $(x - 6)^2 + (y + 1)^2 = 26$

[4 marks]

Answer 6

(a) $r = \sqrt{9^2 + (-13)^2} = \sqrt{250} = 5\sqrt{10}$

[2 marks]

(b) $x^2 + y^2 = 250$

[1 mark]

(c) $k = 2 \times (-13) + 3 \times 9 = 1$

[1 mark]

(d) From $2y + 3x = 1 \Rightarrow y = -\dfrac{3}{2}x + \dfrac{1}{2}$

$$x^2 + y^2 = 250$$

$$x^2 + \left(-\frac{3}{2}x + \frac{1}{2}\right)^2 = 250$$

multiply through by 4

$$4x^2 + (-3x + 1)^2 = 1000$$

$$4x^2 + 9x^2 - 6x + 1 = 1000$$

$$13x^2 - 6x - 999 = 0$$

It's known that $x = 9$ must be a solution

$$(13x + 111)(x - 9) = 0$$

$\therefore$ B has x coordinate $-\dfrac{111}{13}$

$$y_B = -\frac{3}{2} \times -\frac{111}{13} + \frac{1}{2} = -\frac{173}{13}$$

[6 marks]

Answer 7

(a)

$$x^2 + y^2 - 6x + 4y = 12$$

$$\left[x^2 - 6x \right] + \left[y^2 + 4y \right] = 12$$

$$\left[(x - 3)^2 - 9 \right] + \left[(y + 2)^2 - 4 \right] = 12$$

$$(x - 3)^2 + (y + 2)^2 = 25$$

The circle has centre $(3, -2)$ and radius 5

[5 marks]

(b)

$$|PQ| = \sqrt{(7 - (-1))^2 + (-5 - 1)^2}$$

$$= \sqrt{8^2 + 6^2} = 10$$

As this is twice the radius of 5, PQ is a diameter of C

[2 marks]

(c)

R must lie on the circle, by the circle theorem that any angle which stands on a diameter is 90° (argued backwards)

As R is on the y-axis, $x = 0$

$$(0 - 3)^2 + (y + 2)^2 = 25$$

$$(y + 2)^2 = 25 - 9$$

$$y + 2 = \pm 4$$

$$y = -6 \text{ or } 2$$

Question states that R is on the positive y-axis, so $R\,(0, 2)$

[4 marks]

Answer 8

(a) By the circle theorem that any angle which stands on a diameter is 90°

$$m_{PQ} \times m_{RQ} = -1$$

$$\frac{10 - 2}{9 - (-3)} \times \frac{4 - 10}{a - 9} = -1$$

$$\frac{8}{12} \times \frac{-6}{a - 9} = -1$$

$$\frac{-4}{a - 9} = -1$$

$$a - 9 = 4$$

$$a = 13 \qquad \square$$

[3 marks]

(b) $\left| PR \right| = \sqrt{(13 - (-3))^2 + (4 - 2)^2}$

$= \sqrt{16^2 + 2^2} = \sqrt{260} = 2\sqrt{65} \quad \therefore \text{Radius} = \sqrt{65}$

$$Midpoint\ AB = \left(\frac{13 + (-3)}{2}, \ \frac{4 + 2}{2} \right) = (5, \ 3)$$

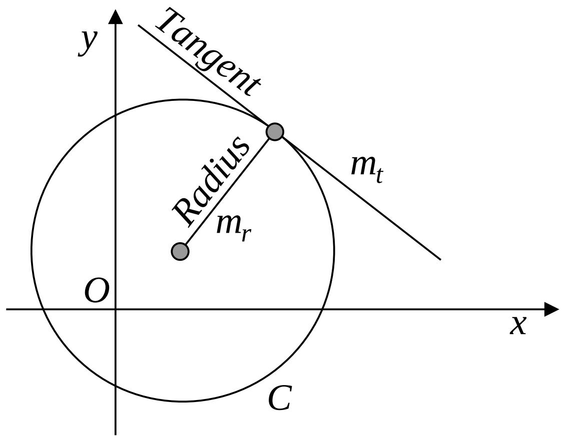
The Equation of the circle is, $(x - 5)^2 + (y - 3)^2 = 65$

[5 marks]

9.1 Tangents

A tangent to a circle is a straight line that touches the circle at a single point. If a radius of the circle is drawn to the point touched by the tangent, then that radius makes a right angle with the tangent.

In other words, the radius and the tangent are mutually perpendicular.



If the tangent has gradient m_t and the perpendicular radius has gradient m_r then ***each is the sign changed reciprocal of the other***.

That is, $m_t \times m_r = -1$

Keeping this relationship between m_t and m_r is mind is often the key to answering a question about a circle that involves a tangent.

9.2 Example #1

In the above diagram, suppose that the equation of the radius is,

$$y = \frac{3}{2}x - \frac{1}{2}$$

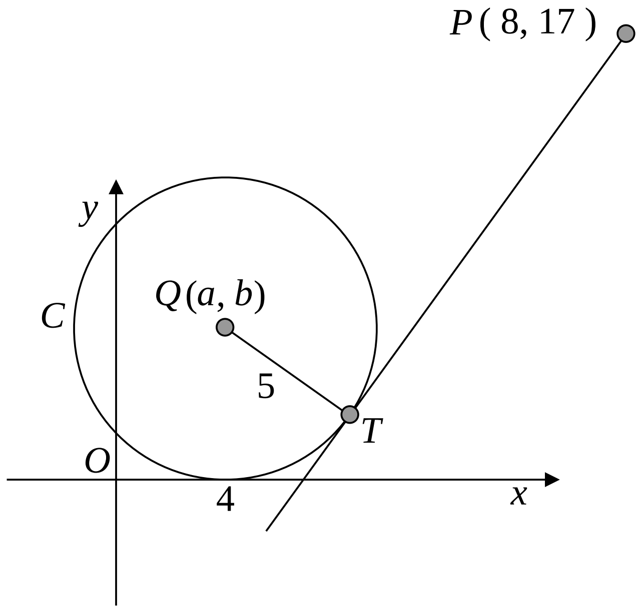
and that the point where radius and tangent meet is (6, 5).

What is the equation of the tangent ?

Teaching Video : http://www.NumberWonder.co.uk/v9033/9a.mp4

[2 marks]

9.3 Example #2



A circle C, with centre Q (a, b) and radius 5, touches the x-axis at (4, 0).

(i) Write down the value of a and the value of b.

(ii) Find a Cartesian equation of C.

A tangent to the circle, drawn from the point P(8, 17), touches the circle at T.

(iii) Find, to 3 significant figures, the length of PT.

Teaching Video : http://www.NumberWonder.co.uk/v9033/9b.mp4

[7 marks]

9.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 86

Question 1

A circle has equation,

$$(x - 1)^2 + (y + 2)^2 = 13$$

(i) State the coordinates of the centre of the circle.

[1 mark]

(ii) Show that the point (3, 1) is on this circle.

[2 marks]

(iii) What is the gradient of the radius of the circle to the point (3, 1) ?

[2 marks]

A tangent to the circle touches the point (3, 1)

(iv) Find the equation of this tangent in the form $y = mx + c$

[3 marks]

Question 2

A circle has equation,

$$(x + 5)^2 + (y - 1)^2 = 65$$

The point (3, 2) is on this circle

Find the equation of the tangent to the circle at the point (3, 2)

[5 marks]

Question 3

A circle has equation,

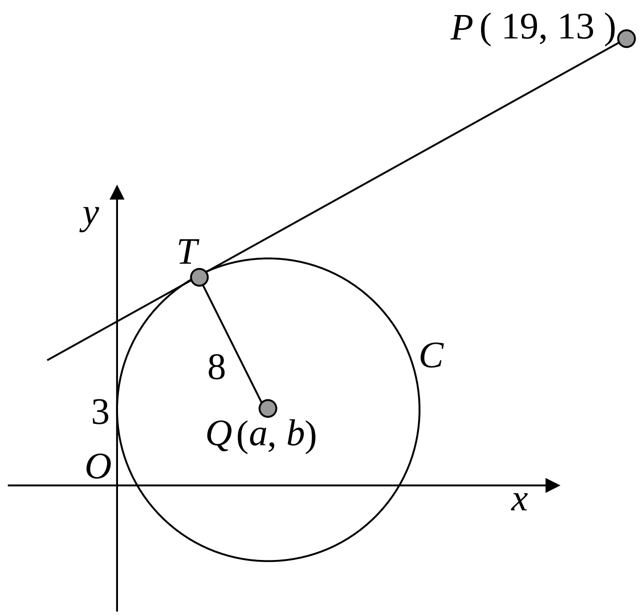
$$x^2 + y^2 - 6x + 4y = 7$$

The point (1, 2) is on this circle.

Find the equation of the tangent to the circle at the point (1, 2)

[6 marks]

Question 4



A circle C, with centre Q (a, b) and radius 8, touches the y-axis at (0, 3)

(i) Write down the value of a and the value of b

[2 marks]

(ii) Find a Cartesian equation of C

[2 marks]

A tangent to the circle, drawn from the point P(19, 13), touches the circle at T

(iii) Find, to 3 significant figures, the length of PT

[3 marks]

Question 5

A circle has equation,

$$(x - 2)^2 + (y + 1)^2 = 16$$

(i) Show that the point (8, 5) is NOT on the circle.

[2 marks]

(ii) Find the length of a tangent from the point (8, 5) to the circle.

[5 marks]

Question 6

A-Level Examination Question from January 2013, Paper C2, Q5 (Edexcel)

The circle *C* has equation

$$x^2 + y^2 - 20x - 24y + 195 = 0$$

The centre of *C* is at the point *M*

(a) Find

(i) the coordinates of the point *M*

(ii) the radius of the circle *C*

[5 marks]

N is the point with coordinates (25, 32)

(b) Find the length of the line *MN*

[2 marks]

The tangent to *C* at a point *P* on the circle passes through point *N*

(c) Find the length of the line *NP*

[2 marks]

Question 7

A-Level Examination Question from January 2018, Paper C12, Q11 (Edexcel)

The circle C has equation

$$x^2 + y^2 - 8x - 10y + 16 = 0$$

The centre of C is at the point T

(a) Find,

(i) the coordinates of the point T

[2 marks]

(ii) the radius of the circle C

[1 mark]

The point M has coordinates (20, 12)

(b) Find the exact length of the line MT

[2 marks]

Point P lies on the circle C such that the tangent at P passes through the point M

(c) Find the exact area of triangle MTP, giving your answer as a simplified surd.

[3 marks]

Question 8

Additional Mathematics Examination Question from June 2016, Q12 (OCR)

The line L_1 has equation $3x - y = 1$ and the point P has coordinates $(8,\ 3)$

(i) Find the equation of the line L_2 which passes through P and is perpendicular to line L_1

[3 marks]

(ii) Find the coordinates of the point Q where L_1 and L_2 intersect

[3 marks]

(iii) Find length PQ

[2 marks]

(iv) Write down the equation of the circle that has centre P and line L_1 as a tangent

[1 mark]

(v) Find the equation of the other line that is a tangent to the circle and is parallel to line L_1

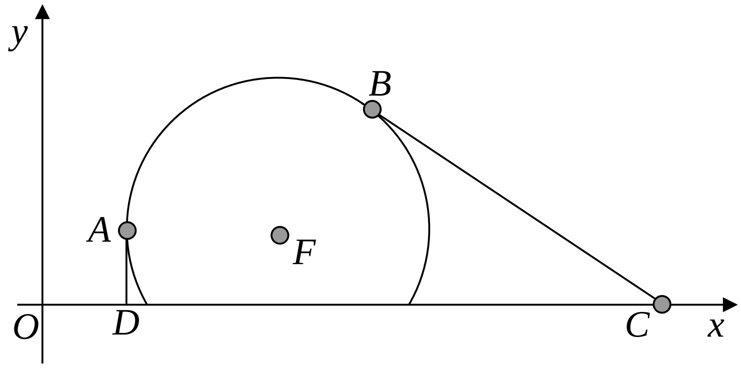
[3 marks]

Question 9

Additional Mathematics Examination Question from June 2004, Q12 (OCR)

The shape shown in the diagram is part of a circle. The centre of the circle is F (8, 4) and AD and BC are tangents at A and B respectively.
A is the point (3, 4) and B is the point (11, 8)

A wire is stretched from D to A, round the circumference to the circle to B and then to C, where D and C are on the x-axis. Units are centimetres.



(a) Find the equation of the circle.

[3 marks]

(b) **(i)** Find the gradient of FB and hence the equation of the tangent BC.

[4 marks]

(ii) Given that the length of the wire from A to B in contact with the circle is 11.07 cm, correct to 2 decimal places, find the total length of the wire.

[5 marks]

Question 10

Additional Mathematics Examination Question from, June 2018, Q11 (OCR)

A circle has centre (0, 3) and radius 3

(i) Show that the equation of the circle is $x^2 + y^2 - ky = 0$ where k is to be determined.

[2 marks]

The line $y = mx - 2$ passes through the point $P\,(0, -2)$ and is a tangent to the circle

(ii) Find the two possible values of m

[6 marks]

The two tangents from P meet the circle at the points A and B respectively.

(iii) Find the lengths PA and PB

[4 marks]

9.5 Answers

A-Level Pure Mathematics, Year 1
Additional Mathematics
Coordinate Geometry

9.5.1 Solution to 9.2 Example #1 (1st Teaching Video)

From the equation of the radius, $y = \frac{3}{2}x - \frac{1}{2}$ the gradient of the radius m_r is $\frac{3}{2}$

$\therefore$ The tangent will have gradient m_t that is the sign changed reciprocal, $-\frac{2}{3}$

The tangent is a straight line through the point (6, 5),

$$y = mx + c$$

$$5 = -\frac{2}{3} \times 6 + c \Rightarrow c = 9$$

$$\therefore \text{ The equation of the tangent is, } y = -\frac{2}{3}x + 9$$

[2 marks]

9.5.2 Solution to 9.3 Example #2 (2nd Teaching Video)

(i) $Q(4, 5)$ Draw radius of 5 vertically between Q and (4, 0) to see this.

(ii) $(x - 4)^2 + (y - 5)^2 = 5^2$

(iii) Apply the Theorem of Pythagoras to ΔQTP which has a right angle at T
First get length PQ

$$|PQ| = \sqrt{(17 - 5)^2 + (8 - 4)^2}$$
$$= \sqrt{12^2 + 4^2}$$
$$= \sqrt{160}$$
$$|PT|^2 = |PQ|^2 - |QT|^2$$
$$= 160 - 5^2$$
$$= 135$$
$$\therefore |PT| = 11.6 \text{ (requested to 3 significant figures)}$$

[6 marks]

9.5.3 Solutions to 9.4 Exercise

Answer 1

(i) $\left(1, -2\right)$

[1 mark]

(ii) LHS $= (x - 1)^2 + (y + 2)^2$

$= (3 - 1)^2 + (1 + 2)^2$ if the circle passes through $(3,\ 1)$

$= 4 + 9$

$= 13$

$=$ RHS

$\therefore\ (3,\ 1)$ is on the circle ☐

[2 marks]

(iii) To find the gradient of the radius between centre $(1, -2)$ and $(3,\ 1)$

$$m_r = \frac{1 - (-2)}{3 - 1}$$

$$= \frac{3}{2}$$

[2 marks]

(iv) The straight line tangent will have a gradient of $-\dfrac{2}{3}$ through $(3,\ 1)$

$$y = mx + c$$

$$1 = -\frac{2}{3} \times 3 + c \Rightarrow c = 3$$

Equation of tangent is $y = -\dfrac{2}{3}x + 3$

[3 marks]

Answer 2

Centre is $(-5,\ 1)$ Point on circumference $(3,\ 2)$

$$m_r = \frac{2 - 1}{3 - (-5)} = \frac{1}{8}$$

Gradient of tangent, $m_t = -8$ (sign changed reciprocal)

$$y = mx + c$$

$$2 = -8 \times 3 + c \Rightarrow c = 26$$

Equation of tangent is $y = -8x + 26$

[5 marks]

Answer 3

$$x^2 + y^2 - 6x + 4y = 7$$

$$\left[x^2 - 6x \right] + \left[y^2 + 4y \right] = 7$$

$$\left[(x - 3)^2 - 9 \right] + \left[(y + 2)^2 - 4 \right] = 7$$

$$(x - 3)^2 + (y + 2)^2 = 20$$

$\therefore$ centre is at $(3, -2)$

Point on the circumference is $(1, 2)$

$$m_r = \frac{2 - (-2)}{1 - 3} = \frac{4}{(-2)} = -2$$

Gradient of tangent, $m_t = \dfrac{1}{2}$ (sign changed reciprocal)

$$y = mx + c$$

$$2 = \frac{1}{2} \times 1 + c \Rightarrow c = \frac{3}{2}$$

Equation of tangent is $y = \dfrac{1}{2}x + \dfrac{3}{2}$

[6 marks]

Answer 4

(i) $Q(8, 3)$ Draw radius of 8 horizontally between Q and $(0, 3)$ to see this.

[2 marks]

(ii) $(x - 8)^2 + (y - 3)^2 = 8^2$

[2 marks]

(iii) Apply the Theorem of Pythagoras to ΔQTP which has a right angle at T
First get length PQ

$$|PQ| = \sqrt{(19 - 8)^2 + (13 - 3)^2} = \sqrt{11^2 + 10^2} = \sqrt{221}$$

$$|PT|^2 = |PQ|^2 - |QT|^2 = 221 - 8^2 = 157$$

$\therefore$ $|PT| = 12.5$ (requested to 3 significant figures)

[3 marks]

Answer 5

(i)

$$\begin{aligned} \text{LHS} &= (x - 2)^2 + (y + 1)^2 \\ &= (8 - 2)^2 + (5 + 1)^2 \qquad \text{if the circle passes through } (8,\ 5) \\ &= 6^2 + 6^2 \\ &= 36 + 36 \\ &= 72 \\ &\neq \text{RHS of } 16 \qquad \therefore (8,\ 5) \text{ is not on the circle} \qquad \square \end{aligned}$$

[2 marks]

(ii) This is similar to Question 4 but without a diagram being provided.

First, obtain the distance, D, between the circle's centre $(2, -1)$ and $(8,\ 5)$

$$\begin{aligned} D &= \sqrt{(8 - 2)^2 + (5 - (-1))^2} \\ &= \sqrt{6^2 + 6^2} \\ &= \sqrt{72} \\ &= 6\sqrt{2} \end{aligned}$$

Second, apply Pythagoras' Theorem to the right angled triangle with vertices at $(2, -1)$, $(8,\ 5)$ and where the radius meets the tangent to get the length requested, X

$$\begin{aligned} X &= \sqrt{\left(\sqrt{72}\,\right)^2 - 4^2} \\ &= \sqrt{72 - 16} \\ &= 2\sqrt{14} \qquad \text{(about 7.48)} \end{aligned}$$

[5 marks]

Answer 6

(a) **(i)**

$$x^2 + y^2 - 20x - 24y + 195 = 0$$

$$\left[x^2 - 20x \right] + \left[y^2 - 24y \right] + 195 = 0$$

$$\left[(x - 10)^2 - 100 \right] + \left[(y - 12)^2 - 144 \right] + 195 = 0$$

$$(x - 10)^2 + (y - 12)^2 = 49$$

$\therefore$ Centre is $M (10, \ 12)$

[3 marks]

(ii)

Radius is 7

[2 marks]

(b)

$$\left| MN \right| = \sqrt{(25 - 10)^2 + (32 - 12)^2}$$

$$= \sqrt{15^2 + 20^2}$$

$$= 25$$

[2 marks]

(c) Use the Theorem of Pythagoras to get the unknown side in ΔMNP

$$| NP |^2 = | MN |^2 - | MP |^2$$

$$= 25^2 - 7^2$$

$$= 24$$

[2 marks]

Answer 7

(a) **(i)**

$$x^2 + y^2 - 8x - 10y + 16 = 0$$

$$\left[x^2 - 8x \right] + \left[y^2 - 10y \right] + 16 = 0$$

$$\left[(x - 4)^2 - 16 \right] + \left[(y - 5)^2 - 25 \right] + 16 = 0$$

$$(x - 4)^2 + (y - 5)^2 = 25$$

$\therefore$ Centre is $T\,(4,\ 5)$

[2 marks]

(ii)

Radius is 5

[1 mark]

(b)

$$|MT| = \sqrt{(20 - 4)^2 + (12 - 5)^2}$$

$$= \sqrt{16^2 + 7^2}$$

$$= \sqrt{305}$$

[2 marks]

(c) Use the Theorem of Pythagoras to get the unknown side in $\triangle MPT$

$$|PM|^2 = |MT|^2 - |PT|^2$$

$$= 305 - 25$$

$$= 280$$

$$|PM| = \sqrt{280}$$

$$= 2\sqrt{70}$$

$$Area\ \triangle MPT = \frac{1}{2}\, b\, h$$

$$= \frac{1}{2} \times 2\sqrt{70} \times 5$$

$$= 5\sqrt{70} \qquad \text{(about 41.8)}$$

[3 marks]

Answer 8

(i) $L_1 \ : \ 3x - y = 1 \Rightarrow y = 3x - 1 \Rightarrow L_1$ has gradient 3

$\therefore L_2$ will have sign changed reciprocal gradient $-\dfrac{1}{3}$ through $P(8, \ 3)$

$$y = mx + c$$

$$3 = -\frac{1}{3} \times 8 + c \Rightarrow c = \frac{17}{3}$$

L_2 has equation $y = -\dfrac{1}{3}x + \dfrac{17}{3}$

[3 marks]

(ii) L_1 and L_2 intersect when $3x - 1 = -\dfrac{1}{3}x + \dfrac{17}{3}$

multiply through by 3

$$9x - 3 = -x + 17$$

$$10x = 20$$

$$x = 2$$

$\therefore$ Point of intersection is $Q(2, \ 5)$

[3 marks]

(iii) $|PQ| = \sqrt{(2-8)^2 + (5-3)^2}$

$= \sqrt{6^2 + 2^2}$

$= \sqrt{40} = 2\sqrt{10}$

[2 marks]

(iv) Draw a diagram with $P(8, \ 3)$, $Q(2, \ 5)$, L_1 of $y = 3x - 1$,

L_2 of $y = -\dfrac{1}{3}x + \dfrac{17}{3}$ and perpendicular to L_1 and $|PQ| = \sqrt{40}$

It's then obvious that the circle described will have equation,

$$(x - 8)^2 + (y - 3)^2 = 40$$

[1 mark]

(v) The vector to go from $Q(2, \ 5) \rightarrow P(8, \ 3)$ is $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$

So this vector applied to P gets to $(14, \ 1)$

$$y = mx + c$$

$$1 = 3 \times 14 + c \Rightarrow c = -41$$

Other tangent is $y = 3x - 41$

[3 marks]

Answer 9

(a) The radius can be read off the diagram as it's the horizontal line between $F(8,\ 4)$ and $A(3,\ 4)$. That is, 5 cm

$$(x - 8)^2 + (y - 4)^2 = 25$$

[3 marks]

(b) **(i)** $m_{FB} = \dfrac{8 - 4}{11 - 8} = \dfrac{4}{3}$

$\therefore\ m_{BC} = -\dfrac{3}{4}$ (sign changed reciprocal)

$y = mx + c$

$8 = -\dfrac{3}{4} \times 11 + c \Rightarrow c = \dfrac{65}{4}$

$$y = -\frac{3}{4}x + \frac{65}{4}$$

[4 marks]

(ii) C is where the tangent of (b) (i) cuts the x-axis

$$0 = -\frac{3}{4}x + \frac{65}{4}$$

$$0 = -3x + 65$$

$$3x = 65$$

$$x = \frac{65}{3} \qquad \therefore C\left(\frac{65}{3},\ 0\right)$$

$$|BC| = \sqrt{\left(\frac{65}{3} - 11\right)^2 + 8^2}$$

$$= \sqrt{\left(\frac{32}{3}\right)^2 + 64}$$

$$= \frac{40}{3}$$

$$\text{Total Length} = 4 + 11.07 + \frac{40}{3}$$

$$= 28.4 \text{ cm}$$

[5 marks]

Answer 10

(i)

$$(x - 0)^2 + (y - 3)^2 = 3^2$$
$$x^2 + y^2 - 6y + 9 = 9$$
$$x^2 + y^2 - 6y = 0$$
$$\therefore\ k = 6$$

[2 marks]

(ii) There is a clever way of doing this from a diagram but the algebraic approach is as follows,

The circle and the line intersect when,

$$x^2 + y^2 - 6y = 0 \text{ meets } y = mx - 2$$
$$x^2 + (mx - 2)^2 - 6(mx - 2) = 0$$
$$x^2 + m^2x^2 - 4mx + 4 - 6mx + 12 = 0$$
$$\left(m^2 + 1\right)x^2 - 10mx + 16 = 0$$

For $ax^2 + bx + c = 0$ to have only one solution, $b^2 - 4ac = 0$

$$\text{from } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore (-10m)^2 - 4\left(m^2 + 1\right) \times 16 = 0$$
$$100m^2 - 64m^2 - 64 = 0$$
$$36m^2 = 64$$
$$m^2 = \frac{64}{36}$$
$$= \frac{16}{9}$$
$$m = \pm\frac{4}{3}$$

[6 marks]

(iii) In $\triangle PCA$, PC is of length 5 (along the y-axis) and CA is a radius of the circle of length 3
So the Theorem of Pythagoras gives the length PA to be 4
By symmetry PB has the same length as PA

[4 marks]

10.1 Tangent from Curve

Previously looked at was "How to find the equation of the tangent to a circle". The mathematics needed to find the tangents to more general curves is straight forward, provided an ability to *differentiate* is already in place.

10.2 Differentiation

This is the remarkable ability to take the equation of many curves and, without any working, simply write down the gradient equation of that curve.

The Power Rule

If $y = x^n$ then $\dfrac{dy}{dx} = n\,x^{n-1}$ for any constant, n

---

Example #1

Write down the derivative of the following curve then check your answer with that given at the foot of the next page.

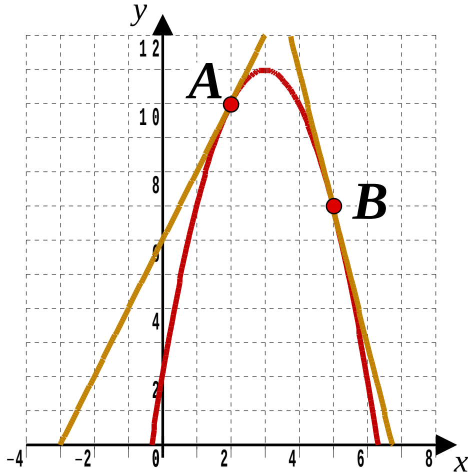
$$y = 7x^4 + 0.5x + 17$$

$$\Rightarrow \frac{dy}{dx} =$$

[2 marks]

10.3 The Key Idea

The gradient of the tangent to a point on a curve is the same as the gradient of the the curve at that point.



The red parabola, P, has equation $y = 2 + 6x - x^2$ and so $\dfrac{dy}{dx} = 6 - 2x$

At $A(2,\ 10)$, P has gradient 2 and the tangent is $y = 2x + 6$

At $B(5,\ 7)$, P has gradient -4 and the tangent is $y = -4x + 27$

10.4 Making Use of The Key Idea

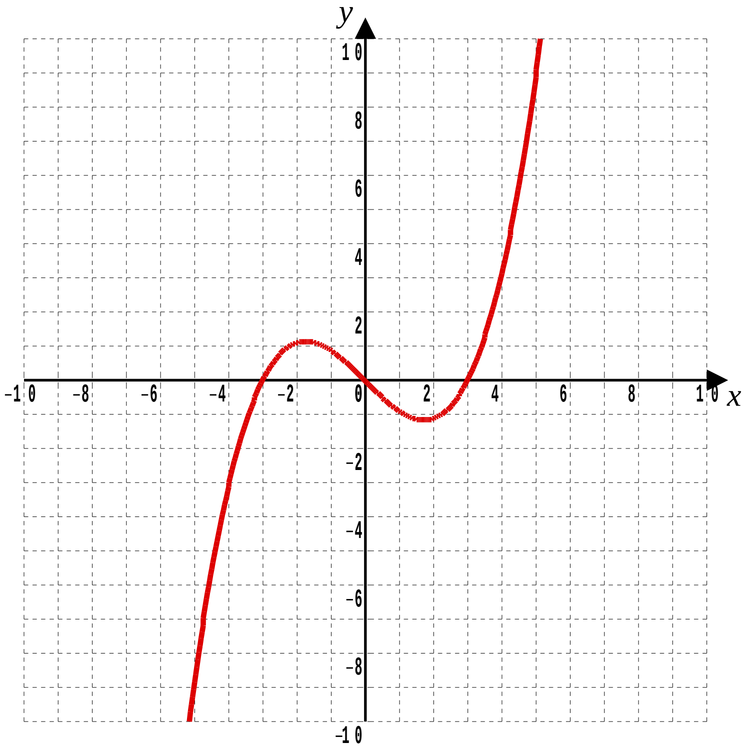
The key idea is often used to get the equation of the tangent to a curve at a given point starting from only the equation of the curve.

Example #2

The equation of a curve is $y = \dfrac{x^3}{9} - x$

(i) Find the equation of the tangent to this curve at the point (3, 0)

(ii) To the graph below add the part (i) tangent



Teaching Video : http://www.NumberWonder.co.uk/v9033/10.mp4

[6 marks]

10.5 Example #1 Answer : $y = 7x^4 + 0.5x + 17 \Rightarrow \dfrac{dy}{dx} = 28x^3 + 0.5$

10.6 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1
Differentiate

$$y = 4x^8 - 3x^5$$

[2 marks]

Question 2
Write down the derivative of

$$y = 15x^{0.4}$$

[2 marks]

Question 3
Differentiate

$$y = 7 + \frac{1}{2}x^6$$

[2 marks]

Question 4
Determine $\frac{dy}{dx}$ of the following expression,

$$y = 0.2x^9 + 0.1x$$

[2 marks]

Question 5
Write down the gradient equation for the following curve

$$y = x^{-5}$$

[2 marks]

Question 6
By first expanding the brackets, differentiate the following

$$y = (3x + 8)(2x + 1)$$

[3 marks]

Question 7

Find the derivative of the following

$$y = \frac{x^3}{9} + 4$$

[2 marks]

Question 8

(i) Differentiate;

$$y = \frac{12}{x^2}$$

[2 marks]

Question 9

Find the numerical value of the gradient at the point (1, 5) on the curve

$$y = x^3 + 4$$

[3 marks]

Question 10

Find the numerical value of the gradient at the point (2, 3.2) on the curve,

$$y = 0.1\,x^5$$

[2 marks]

Question 11

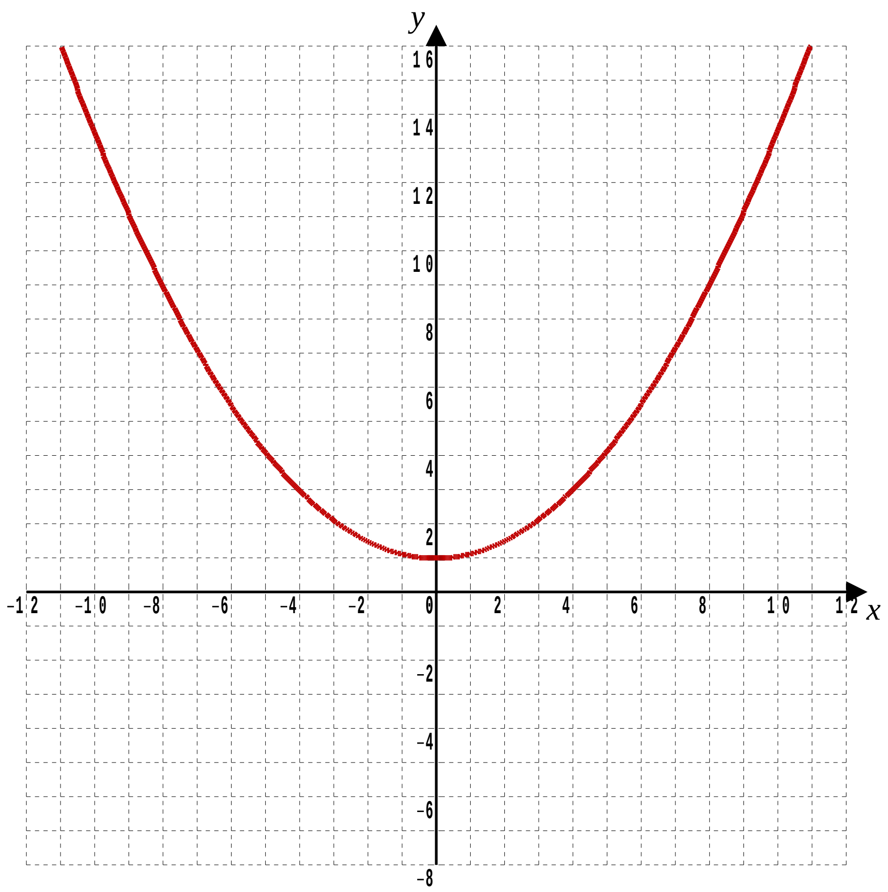
The equation of a curve is

$$y = \frac{x^2}{8} + 1$$

(i) Find the equation of the tangent to this curve at the point (8, 9)

[4 marks]

(ii) To the graph of the curve add your part (i) tangent



[2 marks]

Question 12

Additional Mathematics Examination Question from June 2011, Q2 (OCR)

The equation of a curve is

$$y = x^3 - x^2 - 2x - 3$$

Find the equation of the tangent to this curve at the point (3, 9)

[5 marks]

Question 13

Additional Mathematics Examination Question from June 2007, Q6 (OCR)

Find the equation of the tangent to the curve

$$y = x^3 - 3x + 4$$

at the point (2, 6)

[4 marks]

Question 14

$$f(x) = \frac{x^3 - x + 8}{16x}, \qquad x > 0$$

(i) Show that $f(x) = Ax^2 + B + Cx^{-1}$, where A, B and C are constants to be determined.

[4 marks]

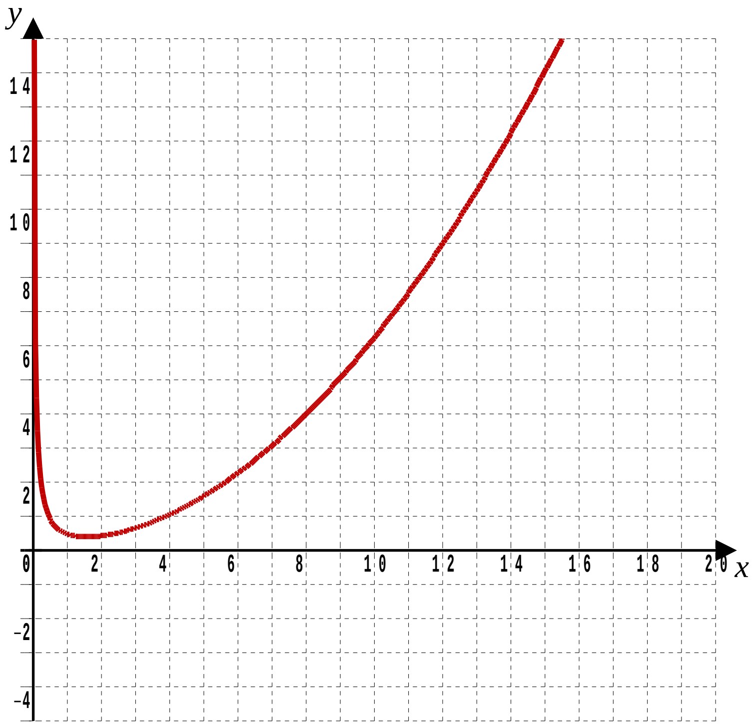
(ii) Hence find $f'(x)$

[2 marks]

(iii) Find an equation of the tangent to the curve $y = f(x)$ when $x = 8$

[4 marks]

(iv) The graph of $f(x)$ is plotted below.
Add the tangent calculated in part (iii) to this graph.



[3 marks]

10.7 Answers

A-Level Pure Mathematics, Year 1
Additional Mathematics
Coordinate Geometry

10.7.1 Solution to 10.4 Making Use of The Key Idea (Teaching Video)

$$y = \frac{x^3}{9} - x + 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{9} - 1$$

$$= \frac{x^2}{3} - 1$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=3} = \frac{3^2}{3} - 1$$

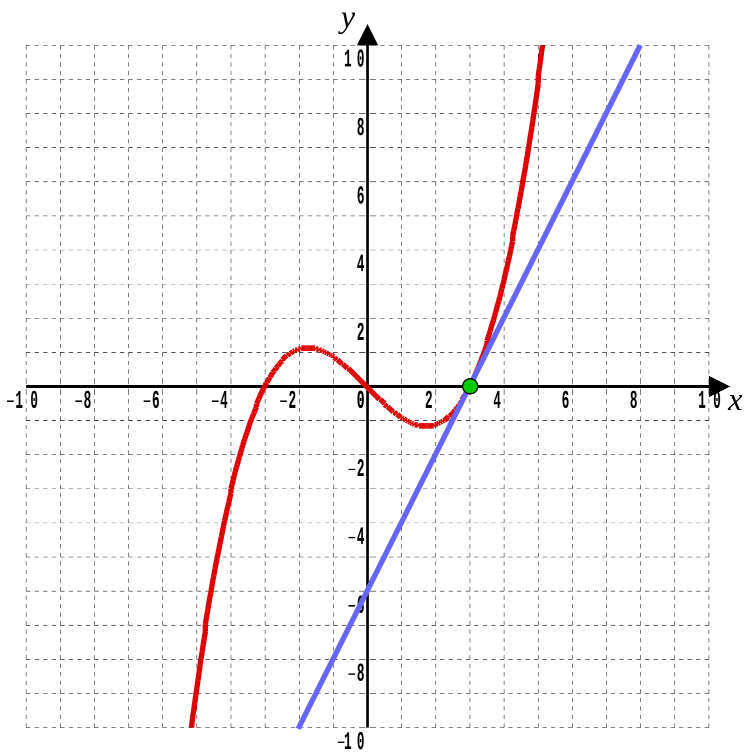
$$= 2$$

$\therefore$ The tangent is the straight line with gradient 2 through $(3,\ 0)$

$$y = mx + c$$

$$0 = 2 \times 3 + c \Rightarrow c = -6$$

The equation of the tangent is, $y = 2x - 6$



[6 marks]

10.7.2 Solutions to 10.6 Exercise

Answer 1

$$y = 4x^8 - 3x^5 \qquad \Rightarrow \qquad \frac{dy}{dx} = 32x^7 - 15x^4$$

[2 marks]

Answer 2

$$y = 16x^{0.4} \qquad \Rightarrow \qquad \frac{dy}{dx} = 6x^{-0.6}$$

[2 marks]

Answer 3

$$y = 7 + \frac{1}{2}x^6 \qquad \Rightarrow \qquad \frac{dy}{dx} = 3x^5$$

[2 marks]

Answer 4

$$y = 0.2x^9 + 0.1x \qquad \Rightarrow \qquad \frac{dy}{dx} = 1.8x^8 + 0.1$$

[2 marks]

Answer 5

$$y = x^{-5} \qquad \Rightarrow \frac{dy}{dx} = -5x^{-6}$$

[2 marks]

Answer 6

$$\begin{aligned} y &= (3x + 8)(2x + 1) \\ &= 6x^2 + 19x + 8 \qquad \Rightarrow \qquad \frac{dy}{dx} = 12x + 19 \end{aligned}$$

[3 marks]

Answer 7

$$y = \frac{x^3}{9} + 4 \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{x^2}{3}$$

[2 marks]

Answer 8

$$\begin{aligned} y &= \frac{12}{x^2} \\ &= 12x^{-2} \qquad \Rightarrow \qquad \frac{dy}{dx} = -24x^{-3} \\ &\qquad\qquad\qquad\qquad\qquad = -\frac{24}{x^3} \end{aligned}$$

[2 marks]

Answer 9

$$y = x^3 + 4 \Rightarrow \frac{dy}{dx} = 3x^2 \text{ and so when } x = 1, \frac{dy}{dx} = 3$$

[3 marks]

Answer 10

$$y = 0.1x^5 \Rightarrow \frac{dy}{dx} = 0.5x^4 \text{ and so when } x = 2, \frac{dy}{dx} = 8$$

[2 marks]

Answer 11

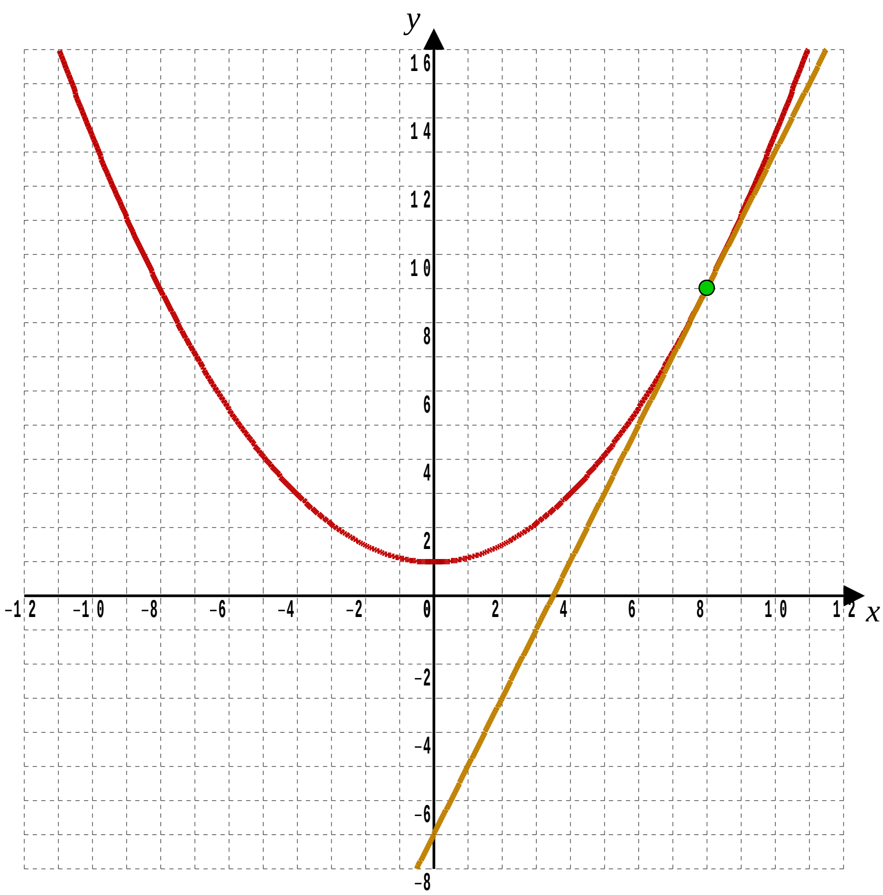
$$y = \frac{x^2}{8} + 1 \Rightarrow \frac{dy}{dx} = \frac{x}{4} \text{ and so when } x = 8, \frac{dy}{dx} = 2$$

$$y = mx + c$$

$$9 = 2 \times 8 + c \Rightarrow c = -7$$

$$\therefore \text{ The equation of the tangent is } y = 2x - 7$$

[4 marks]



[2 marks]

Answer 12

$$y = x^3 - x^2 - 2x - 3 \Rightarrow \frac{dy}{dx} = 3x^2 - 2x - 2$$

$$\text{when } x = 3, \ \frac{dy}{dx} = 27 - 6 - 2$$

$$= 19$$

$$y = mx + c$$

$$9 = 19 \times 3 + c \Rightarrow c = -48$$

Equation of the tangent is $y = 19x - 48$

[5 marks]

Answer 13

$$y = x^3 - 3x + 4 \Rightarrow \frac{dy}{dx} = 3x^2 - 3$$

$$\text{when } x = 2, \ \frac{dy}{dx} = 9$$

$$y = mx + c$$

$$6 = 9 \times 2 + c \Rightarrow c = -12$$

Equation of the tangent is $y = 9x - 12$

[4 marks]

Answer 14

(i)

$$f(x) = \frac{x^3 - x + 8}{16x}, \qquad x > 0$$

$$= \frac{x^3}{16x} - \frac{x}{16x} + \frac{8}{16x}$$

$$= \frac{x^2}{16} - \frac{1}{16} + \frac{x^{-1}}{2}$$

$$\therefore A = \frac{1}{16}, \ B = -\frac{1}{16} \text{ and } C = \frac{1}{2}$$

[4 marks]

(ii)

$$f'(x) = \frac{2x}{16} - \frac{x^{-2}}{2}$$

$$= \frac{x}{8} - \frac{1}{2x^2}$$

[2 marks]

(iii) When $x = 8$, $f(8) = \dfrac{8^3 - 8 + 8}{16 \times 8} = \dfrac{8 \times 8 \times 8}{2 \times 8 \times 8} = 4$

$\therefore$ (8, 4) is the point at which the tangent to the curve is wanted

When $x = 8$, $f'(8) = \dfrac{8}{8} - \dfrac{1}{2 \times 8 \times 8}$

$$= 1 - \frac{1}{128} = \frac{127}{128}$$

$$y = mx + c$$

$$4 = \frac{127}{128} \times 8 + c$$

$$4 = \frac{127}{16} + c$$

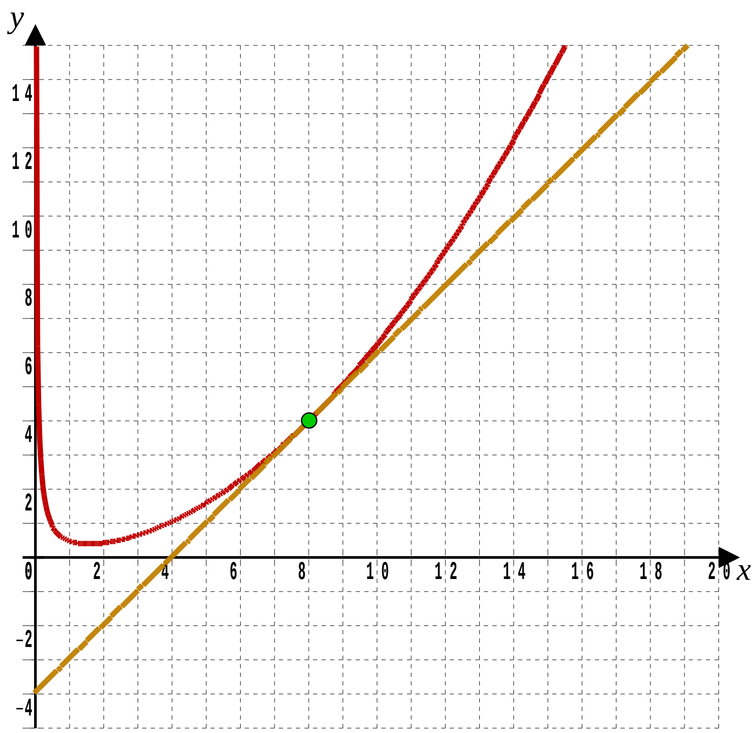
$$c = 4 - \frac{127}{16} = -\frac{63}{16}$$

Equation of tangent is $y = \dfrac{127}{128}x - \dfrac{63}{16}$

(This is very close to being $y = x - 4$)

[4 marks]

(iv)



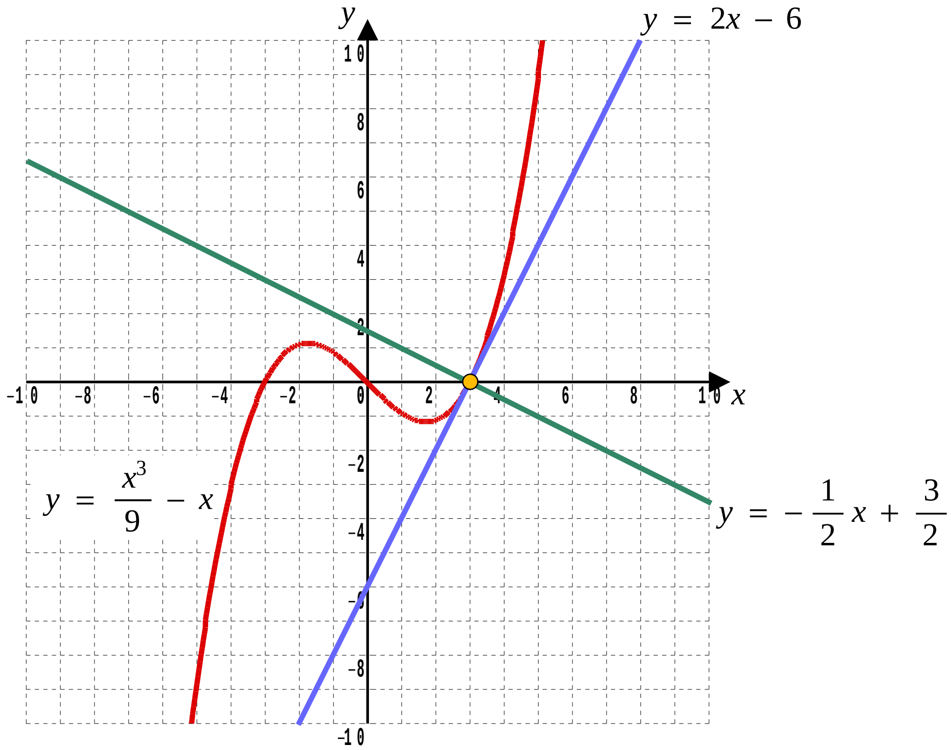
[3 marks]

Lesson 11

A-Level Pure Mathematics, Year 1
Additional Mathematics
Coordinate Geometry

11.1 Normal from Curve

Previously, the curve with equation $y = \dfrac{x^3}{9} - x$ was studied and the tangent to it at the point $(3, 0)$ found to be $y = 2x - 6$



There is a second line of interest, called the “normal” that is a right angles to the tangent at any specified point. At the point $(3, 0)$ the normal to the curve $y = \dfrac{x^3}{9} - x$ turns out to be $y = -\dfrac{1}{2}x + \dfrac{3}{2}$.

Notice that the gradient of the tangent, m_t, and the gradient of the normal, m_n have the property of any pair of mutually perpendicular lines; $m_t \times m_n = -1$

In other words, each is the sign changed reciprocal of the other.

11.2 Why the Normal is of Interest

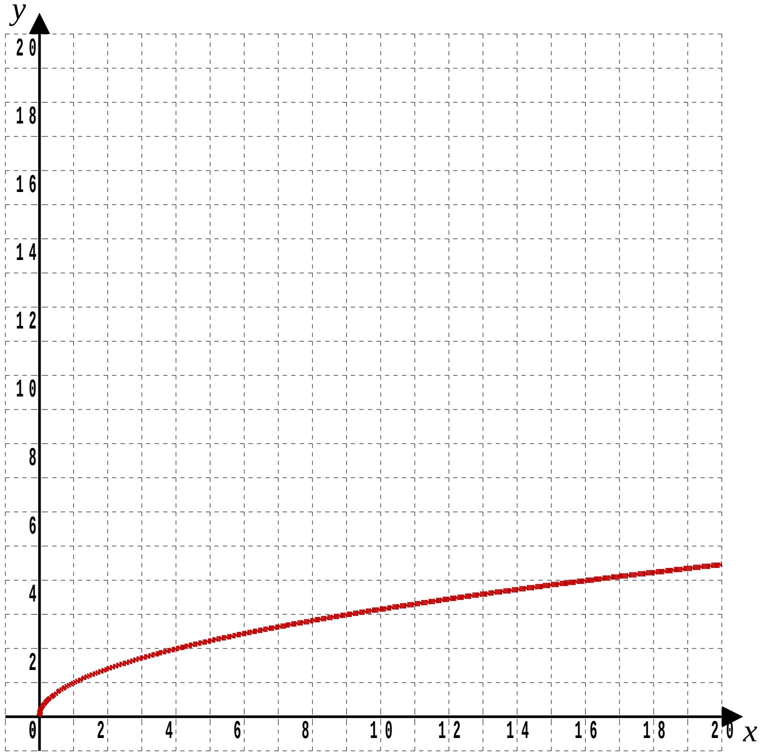
Imagine the graph to be a road map and the curve a road on that map. A car moves along the road with constant speed. The tangent represents the direction a car moving along the road has at any moment. The normal represents the direction along which the force felt by a person in the car acts as it moves around each bend. Like the tangent the normal gives only a direction. It does not give the magnitude of the force; that depends on how sharply the road is bending and, indeed, on a straight piece of road the force along the normal has magnitude zero. The force along the normal is often referred to as centripetal force.

11.3 Example

The equation of a curve is $y = \sqrt{x}$

(i) Find the equation of the normal to this curve at the point where $x = 4$

(ii) To the graph below add the part (i) normal.



Teaching Video : http://www.NumberWonder.co.uk/v9033/11.mp4

[5 marks]

11.4 Exercise

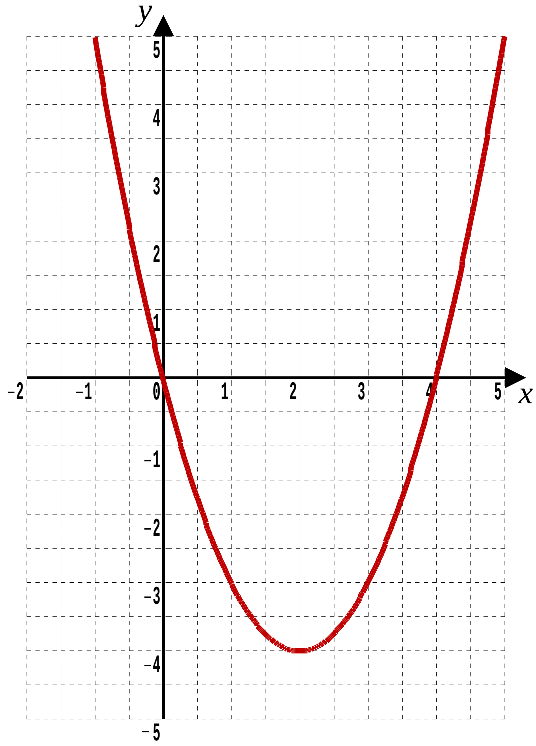
Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 52

Question 1

The equation of a curve is $y = x^2 - 4x$

(i) Find the equation of the normal to this curve at the point where $x = 4$

(ii) To the graph below add the part (i) normal



[5 marks]

Question 2

Additional Mathematics Examination Question from June 2009, Q2 (OCR)

Find the equation of the normal to the curve

$$y = x^3 + 5x - 7$$

at the point $(1, -1)$

[5 marks]

Question 3

Additional Mathematics Examination Question from June 2019, Paper 1, Q3 (OCR)

Find the equation of the normal to the curve

$$y = x^3 - 2x^2 + 2x + 4$$

at the point $(2, 8)$

[6 marks]

Question 4

Additional Mathematics Examination Question from June 2018, Q7 (OCR)

(i) Find the coordinates of the points where the line $y = 7x - 9$ cuts the curve $y = x^2 + 2x - 5$

[4 marks]

(ii) Determine whether the line is a normal to the curve at either of the points of intersection

[3 marks]

Question 5

Additional Mathematics Examination Question from June 2014, Q10 (OCR)

(i) Find the coordinates of the point *P* on the curve $y = 2x^2 + x - 5$ where the gradient of the curve is 5

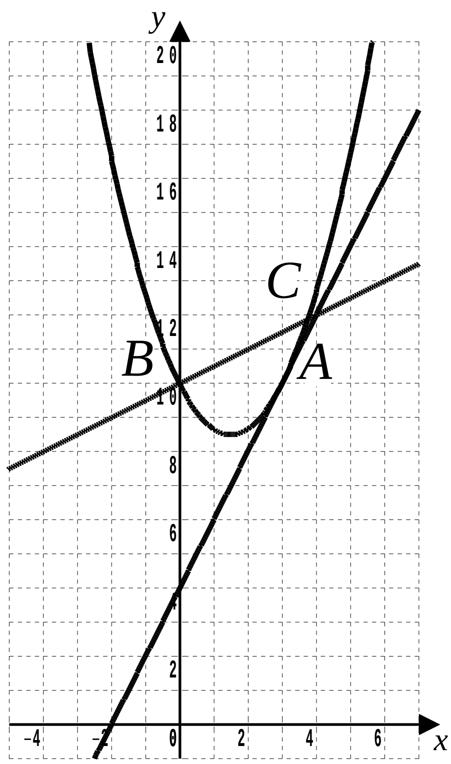
[3 marks]

(ii) Find the equation of the normal to the curve at the point *P*

[3 marks]

Question 6

Additional Mathematics Examination Question from June 2005, Q10 (OCR)



The curve shown has equation;

$$y = \frac{2}{3}x^2 - 2x + 10$$

(i) Find the equation of the tangent to the curve at A (3, 10)

[4 marks]

(ii) Show that the equation of the normal to the curve at B (0, 10) is

$$2y - x = 20$$

[3 marks]

(iii) Find the coordinates of the point C where these two lines intersect

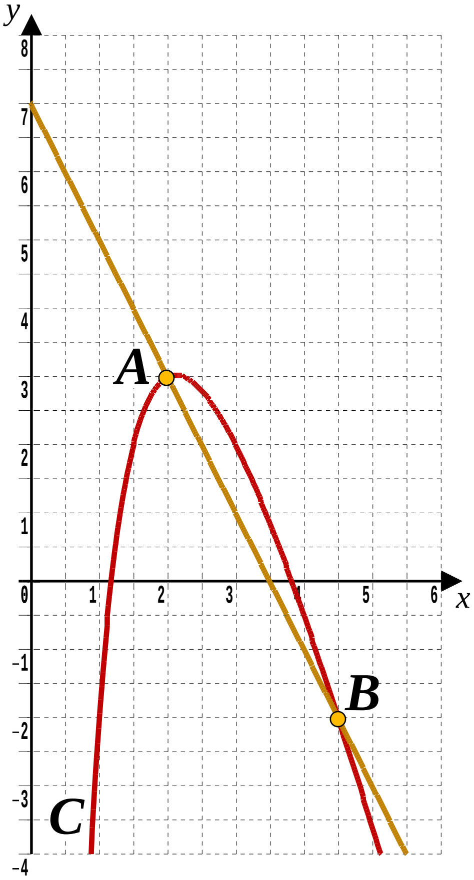
[3 marks]

(iv) Calculate the length BC

[2 marks]

Question 7

A-Level Examination Question from May 2014, IAL, Paper C1(R), Q11 (Edexcel)



The sketch is of part of the curve C with equation $y = 20 - 4x - \dfrac{18}{x}$, $x > 0$

Point A lies on C and has an x coordinate equal to 2

(a) Show that the equation of the normal to C at A is $y = -2x + 7$

[6 marks]

The normal to C at A meets C again at the point B

(b) Use algebra to find the coordinates of B

[5 marks]

11.5 Answers

A-Level Pure Mathematics, Year 1
Additional Mathematics
Coordinate Geometry

11.5.1 Solution to 11.3 Example (Teaching Video)

(i)

$$y = \sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \times \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

When $x = 4$, $y = 2$ and $\left.\frac{dy}{dx}\right|_{x=4} = \frac{1}{2\sqrt{4}}$

$$= \frac{1}{4}$$

If after the tangent, this would be the tangent's gradient

For the normal, the sign changed reciprocal gradient is -4

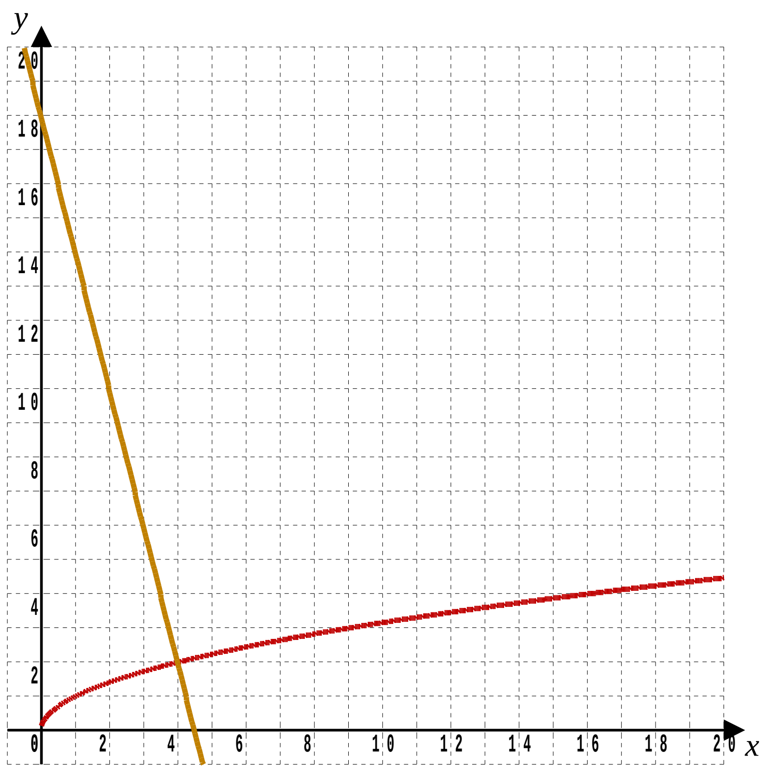
So the normal sought is a straight line through $(4, 2)$ with gradient -4

$$y = mx + c$$

$$2 = -4 \times 4 + c \Rightarrow c = 18$$

The normal has equation $y = -4x + 18$

(ii)



[5 marks]

11.5.2 Solutions to 11.4 Exercise

Answer 1

(i)

$$y = x^2 - 4x$$

$$\frac{dy}{dx} = 2x - 4$$

$$\text{When } x = 4, \; y = 0 \text{ and } \frac{dy}{dx} = 4$$

If after the tangent, this would be the tangent's gradient

For the normal, the sign changed reciprocal gradient is $-\frac{1}{4}$

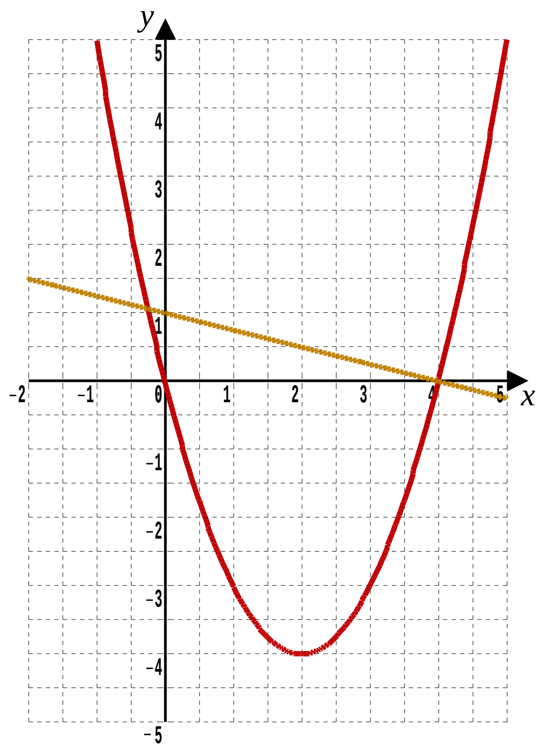
So the normal sought is a straight line through $(4, \; 0)$ with gradient $-\frac{1}{4}$

$$y = mx + c$$

$$0 = -\frac{1}{4} \times 4 + c \Rightarrow c = 1$$

The normal has equation $y = -\frac{1}{4}x + 1$

(ii)



[5 marks]

Answer 2

$$y = x^3 + 5x - 7$$

$$\frac{dy}{dx} = 3x^2 + 5$$

$$\text{When } x = 1, \quad \frac{dy}{dx} = 8$$

$\therefore$ Normal has gradient $-\dfrac{1}{8}$ and passes through the point $(1, -1)$

$$y = mx + c$$

$$-1 = -\frac{1}{8} \times 1 + c \Rightarrow -\frac{7}{8}$$

$$\text{Equation of the normal is } y = -\frac{1}{8}x - \frac{7}{8}$$

$$\text{Alternatively}: x + 8y + 7 = 0$$

[5 marks]

Answer 3

$$y = x^3 - 2x^2 + 2x + 4$$

$$\frac{dy}{dx} = 3x^2 - 4x + 2$$

$$\text{When } x = 2, \quad \frac{dy}{dx} = 12 - 8 + 2$$

$$= 6$$

$\therefore$ Normal has gradient $-\dfrac{1}{6}$ and passes through the point $(2, 8)$

$$y = mx + c$$

$$8 = -\frac{1}{6} \times 2 + c \Rightarrow \frac{25}{3}$$

$$\text{Equation of the normal is } y = -\frac{1}{6}x + \frac{25}{3}$$

$$\text{Alternatively}: x + 6y - 50 = 0$$

[6 marks]

Answer 4

(i)

$$x^2 + 2x - 5 = 7x - 9$$

$$x^2 - 5x + 4 = 0$$

$$(x - 1)(x - 4) = 0$$

$$\text{Either } x = 1 \text{ or } x = 4$$

$$\therefore \text{ Points of intersection are } (1, -2), \ (4, 19)$$

[4 marks]

(ii) The line has a gradient of 7 which, if it is a normal will correspond to a gradient on the curve being $-\dfrac{1}{7}$

$$y = x^2 + 2x - 5$$

$$\frac{dy}{dx} = 2x + 2$$

At $(1, -2)$, $\dfrac{dy}{dx} = 4$ so not the line is not normal to the curve here

At $(4, 19)$, $\dfrac{dy}{dx} = 10$ so the line is not normal to the curve here

[3 marks]

Answer 5

(i)

$$y = 2x^2 + x - 5$$

$$\frac{dy}{dx} = 4x + 1$$

The gradient is 5 when $4x + 1 = 5$

$$4x = 4$$

$$x = 1 \qquad \Rightarrow P(1, -2)$$

[3 marks]

(ii) The normal will have gradient $-\dfrac{1}{5}$ and pass through $P(1, -2)$

$$y = mx + c$$

$$-2 = -\frac{1}{5} \times 1 + c \Rightarrow c = -\frac{9}{5}$$

Equation of the normal is $y = -\dfrac{1}{5}x - \dfrac{9}{5}$

Alternatively : $x + 5y + 9 = 0$

[3 marks]

Answer 6

(i)

$$y = \frac{2}{3}x^2 - 2x + 10$$

$$\frac{dy}{dx} = \frac{4}{3}x - 2$$

$$\text{When } x = 3, \ \frac{dy}{dx} = 2$$

$\therefore$ Tangent has gradient 2 through $A(3, \ 10)$

$$y = mx + c$$

$$10 = 2 \times 3 + c \Rightarrow c = 4$$

Equation of tangent is $y = 2x + 4$

[4 marks]

(ii)

$$\text{When } x = 0, \ \frac{dy}{dx} = -2$$

$\therefore$ Normal has gradient $\frac{1}{2}$ through $B(0, \ 10)$

$$y = mx + c$$

$$10 = -2 \times 0 + c \Rightarrow c = 10$$

Equation of normal is $y = \frac{1}{2}x + 10$

That is, $2y - 1 = 20$ $\square$

[3 marks]

(iii)

$$2x + 4 = \frac{1}{2}x + 10$$

multiply through by 2

$$4x + 8 = x + 20$$

$$3x = 12$$

$$x = 4 \qquad \Rightarrow C(4, \ 12)$$

[3 marks]

(iv)

$$|BC| = \sqrt{(4 - 0)^2 + (12 - 10)^2}$$

$$= \sqrt{4^2 + 2^2}$$

$$= \sqrt{20}$$

$$= 2\sqrt{5}$$

[2 marks]

Answer 7
(a)

$$y = 20 - 4x - \frac{18}{x}$$

$$\text{When } x = 2, \quad y = 20 - 4 \times 2 - \frac{18}{2}$$

$$= 3 \qquad \therefore A(2,\ 3)$$

$$y = 20 - 4x - 18x^{-1}$$

$$\frac{dy}{dx} = -4 + 18x^{-2}$$

$$= -4 + \frac{18}{x^2}$$

$$\text{When } x = 2, \quad \frac{dy}{dx} = \frac{1}{2}$$

$\therefore$ Normal has gradient -2 through $A(2,\ 3)$

$$y = mx + c$$

$$3 = -2 \times 2 + c \Rightarrow c = 7$$

Equation of normal is $y = -2x + 7$ ☐

[6 marks]

(b)

$$-2x + 7 = 20 - 4x - \frac{18}{x}$$

$$2x - 13 + \frac{18}{x} = 0$$

multiply through by x

$$2x^2 - 13x + 18 = 0$$

$$(x - 2)(2x - 9) = 0$$

$$\text{Either } x = 2 \text{ or } x = \frac{9}{2}$$

$$A(2,\ 3),\ B\left(\frac{9}{2}, -2\right)$$

[5 marks]

Lesson 12

A-Level Pure Mathematics, Year 1
Additional Mathematics
Coordinate Geometry

12.1 Revision

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 65

Question 1

The equation of a circle is;

$$(x + 7)^2 + (y - 3)^2 = 81$$

State the coordinates of the circle's centre, and its radius

[2 marks]

Question 2

Write down the equation of the circle with centre (5, 12) and which passes through the origin.

[2 marks]

Question 3

By completing the square, or otherwise, determine the centre and radius of the following circle;

$$x^2 + y^2 + 10x - 6y + 18 = 0$$

[5 marks]

Question 4

Differentiate

(i)

$$y = 7x^{11} - 5x^3$$

[2 marks]

(ii)

$$y = 24x^{0.25}$$

[2 marks]

(iii)

$$y = \frac{4}{5}x^{-5}$$

[2 marks]

Question 5

The equation of the curve, plotted on the next page, is

$$y = \frac{x^3}{24} - x$$

(i) Write down the gradient equation, $\frac{dy}{dx}$, for the curve

[2 marks]

(ii) Use your part (i) answer to find the value of the gradient on the curve when $x = 6$

[1 mark]

(iii) Use your part (ii) answer to determine the equation of the tangent to the curve at the point (6, 3)

[2 marks]

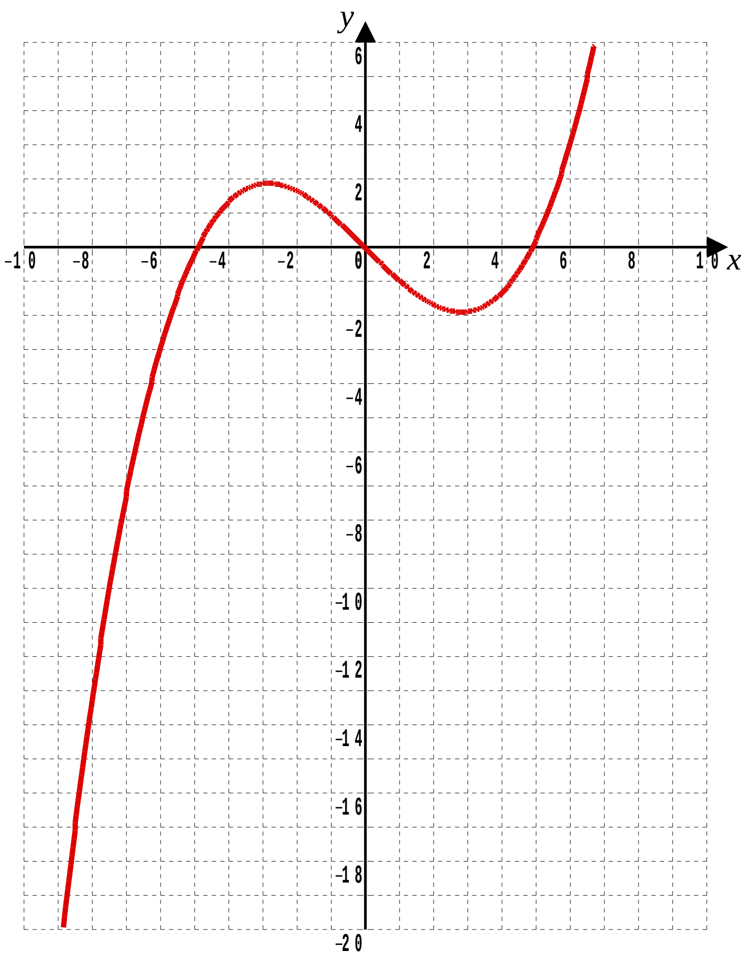
(iv) Use your part (ii) answer to determine the gradient of the normal to the curve at the point (6, 3)

[1 mark]

(v) Find the equation of the normal to the curve at the point (6, 3)

[2 marks]

(vi) On the graph, add your part (iii) tangent, and your part (v) normal, clearly indicating which is which and making sure they both pass through the point (6, 3)



[2 marks]

Question 6

$$(x + 1)^2 + y^2 = 9^2$$

Is the point (5, 7) inside, outside, or on the circumference of this circle ?
Justify your answer.

[3 marks]

Question 7

Find the point(s) of intersection, if any, of the circle

$$(x - 3)^2 + (y + 2)^2 = 13$$

and the line

$$y = 3x - 2$$

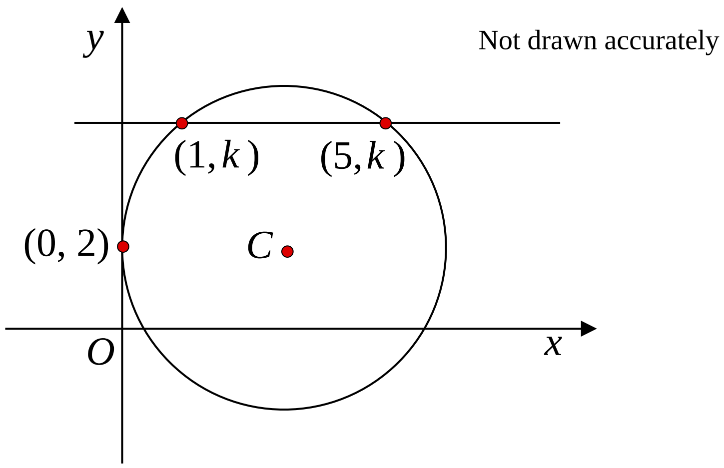
[6 marks]

Question 8

Further Mathematics Specimen Examination Question 2020, Paper 1, Q11 (AQA)

A circle, centre C, touches the y-axis at the point (0, 2)

The line $y = k$ intersects the circle at the points (1, k) and (5, k)



Work out the equation of the circle

[3 marks]

Question 9

Find the coordinates of the point on the curve $y = (4x - 5)^2$ such that the gradient of the normal to the curve is $\frac{1}{8}$

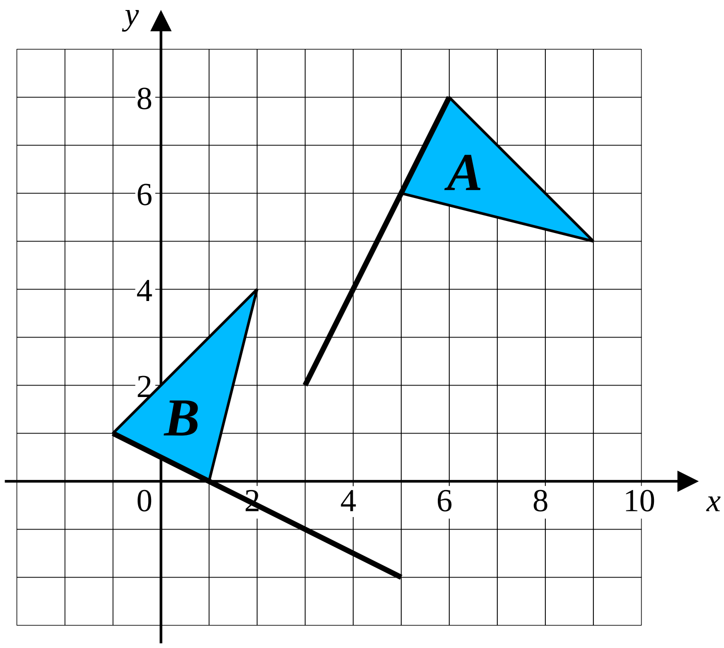
[4 marks]

Question 10

The diagram shows a flag A

The image of flag A when it is rotated by 90° is flag B

This question walks you through the steps involved in mathematically finding the centre of the rotation.



(i) **Step 1** :

Pick two matching points, say the foot of each flag pole, (3, 2) and (5, – 2)

Step 2 :

Find the midpoint of these two points.

[1 mark]

(ii) **Step 3** :

Find the gradient of the straight line between these two points.

[1 mark]

(iii) **Step 4** :

Use the answers to Steps 2 and 3 to determine the equation of the perpendicular bisector of the two points.

[2 marks]

(iv) **Step 5** :

Pick two other matching points, say the tip of the flag, (9, 5) and (2, 4)

Step 6 :

Find the equation of the perpendicular bisector between these two points.

[4 marks]

(v) **Step 7** :

Find the point of intersection of the two perpendicular bisectors.

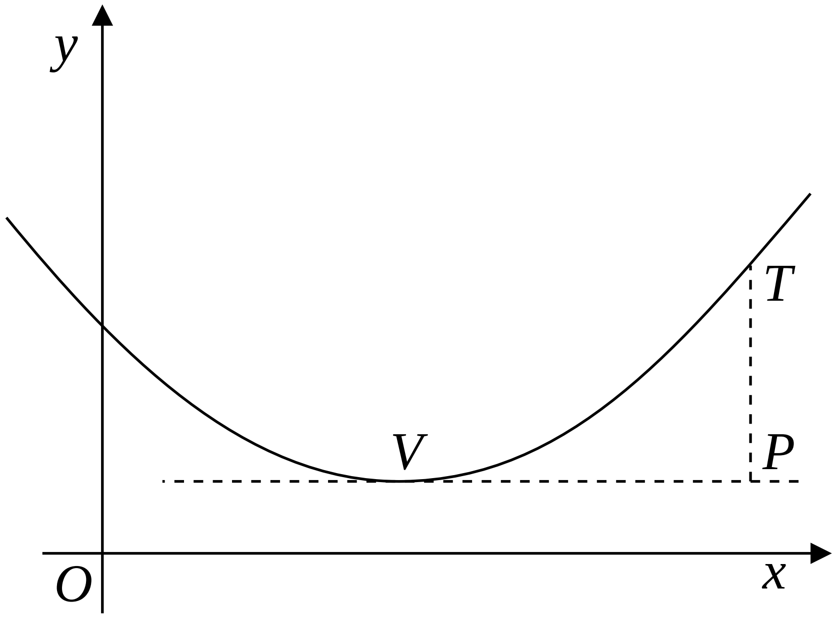
This is the centre of the rotation.

[4 marks]

Check your mathematics is correct by using tracing paper, and verifying that when rotated by 90° about the point you claimed was the answer, flag *A* does indeed map onto flag *B*

Question 11

Additional Mathematics Examination Question from June 2006, Q14 (OCR)



The diagram shows the quadratic curve

$$y = x^2 - 4x + 5$$

V (2, 1) is the minimum point of the curve.

T (5, 10) is a point on the curve.

The line VP is the tangent to the curve at V and TP is perpendicular to this line.

(i) Write down the coordinates of P

[1 mark]

(ii) Find the coordinates of M, the midpoint of VP

[2 marks]

(iii) Find the equation of the tangent of the curve at T

[4 marks]

(iv) Show that the tangent to the curve at T passes through the point M

[2 marks]

(v) Use the result in part (iv) to suggest a way of drawing a tangent to a point on a quadratic curve without involving calculus.

[3 marks]

12.2 Answers

A-Level Pure Mathematics, Year 1
Additional Mathematics
Coordinate Geometry

Answer 1

Centre $(-7,\ 3)$ Radius 9

[2 marks]

Answer 2

Working out the distance between the centre and the origin, which will be the radius...

$$\begin{aligned} R &= \sqrt{(5-0)^2 + (12-0)^2} \\ &= \sqrt{5^2 + 12^2} \\ &= 13 \end{aligned}$$

$$\therefore\ (x-5)^2 + (y-12)^2 = 13^2$$

[2 marks]

Answer 3

$$x^2 + y^2 + 10x - 6y + 18 = 0$$

$$\left[x^2 + 10x\right] + \left[y^2 - 6y\right] + 18 = 0$$

$$\left[(x+5)^2 - 25\right] + \left[(y-3)^2 - 9\right] + 18 = 0$$

$$(x+5)^2 + (y-3)^2 = 16$$

$\therefore$ Centre $(-5,\ 3)$ Radius 4

[5 marks]

Answer 4

(i) $y = 7x^{11} - 5x^3 \quad \Rightarrow \quad \dfrac{dy}{dx} = 77x^{10} - 15x^2$

(ii) $y = 24x^{0.25} \quad \Rightarrow \quad \dfrac{dy}{dx} = 6x^{-0.75}$

(iii) $y = \dfrac{4}{5}x^{-5} \quad \Rightarrow \quad \dfrac{dy}{dx} = -4x^{-6}$

[2, 2, 2 marks]

Answer 5

(i)

$$y = \frac{x^3}{24} - x$$

$$\frac{dy}{dx} = \frac{x^2}{8} - 1$$

[2 marks]

(ii) When $x = 6$, $\dfrac{dy}{dx} = \dfrac{7}{2}$

[1 mark]

(iii)

$$y = mx + c$$

$$3 = \frac{7}{2} \times 6 + c \Rightarrow c = -18$$

$$\text{Equation of the tangent is } y = \frac{7}{2}x - 18$$

$$\text{Alternatively } 7x - 2y - 36 = 0$$

[2 marks]

(iv) Normal's "sign changed reciprocal" gradient will be $-\dfrac{2}{7}$

[1 mark]

(v)

$$y = mx + c$$

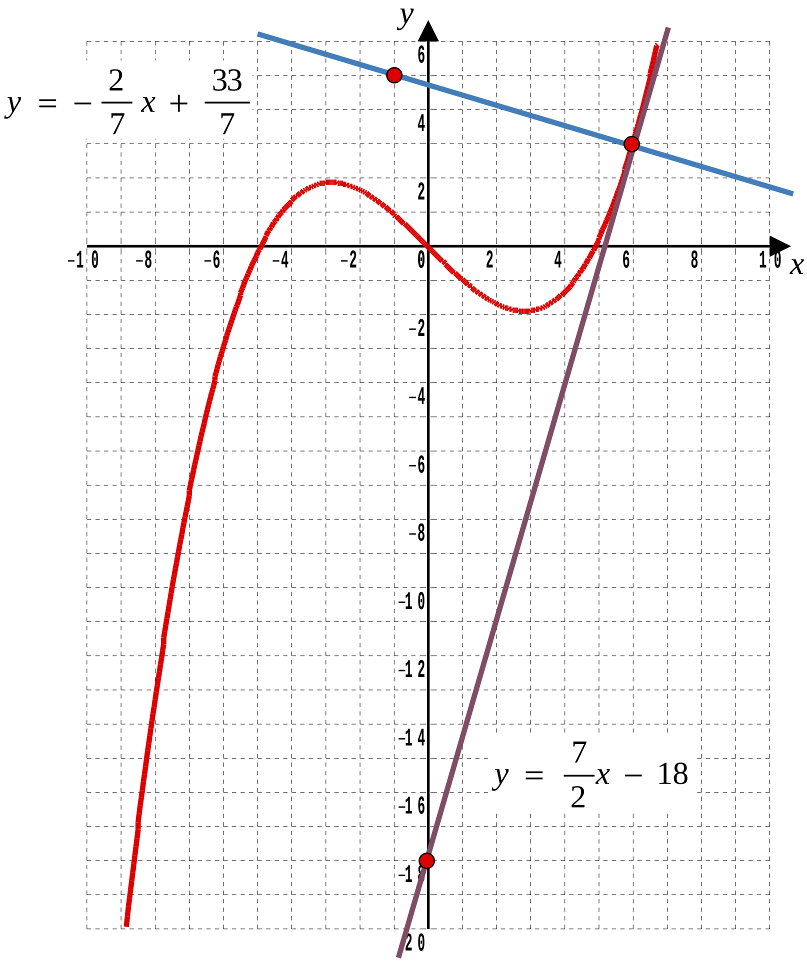
$$3 = -\frac{2}{7} \times 6 + c \Rightarrow c = \frac{33}{7}$$

$$\text{Equation of the normal is } y = -\frac{2}{7}x + \frac{33}{7}$$

$$\text{Alternatively } 7y + 2x - 33 = 0$$

[2 marks]

(vi)



[2 marks]

Answer 6

$$\begin{aligned} \text{LHS} &= (x+1)^2 + y^2 \\ &= (5+1)^2 + 7^2 \qquad \text{if the circle passes through } (5,\ 7) \\ &= 6^2 + 7^2 \\ &= 36 + 49 \\ &= 85 \\ &> \text{RHS} \end{aligned}$$

$\therefore\ (5,\ 7)$ is outside the circle $\square$

[3 marks]

Answer 7

$$(x-3)^2 + (y+2)^2 = 13 \text{ and } y = 3x - 2$$

$$(x-3)^2 + (3x - 2 + 2)^2 = 13$$

$$x^2 - 6x + 9 + 9x^2 = 13$$

$$10x^2 - 6x - 4 = 0$$

divide through by 2

$$5x^2 - 3x - 2 = 0$$

$$(5x+2)(x-1) = 0$$

Either $x = -\dfrac{2}{5}$ or $x = 1$

Points of intersection are $\left(-\dfrac{2}{5}, -\dfrac{16}{5}\right), (1,\ 1)$

[6 marks]

Answer 8

The centre of the circle is $(3,\ 2)$ and the radius is 3

$$\therefore\ (x-3)^2 + (y-2)^2 = 9$$

[3 marks]

Answer 9

The tangent to the curve at the same point must have a gradient of – 8

$$\begin{aligned} y &= (4x-5)^2 \\ &= 16x^2 - 40x + 25 \end{aligned}$$

$$\frac{dy}{dx} = 32x - 40$$

$$\therefore\ 32x - 40 = -8$$

$x = 1 \Rightarrow$ The point is $(1,\ 1)$

[4 marks]

Answer 10

(i)

$$Midpoint \ = \left(\frac{5 \ + \ 3}{2}, \ \frac{-2 \ + \ 2}{2} \right) \ = \ (\, 4, \ 0 \,)$$

[1 mark]

(ii)

$$m \ = \ \frac{\Delta Y}{\Delta X} \ = \ \frac{-2 \ - \ 2}{5 \ - \ 3} \ = \ -\frac{4}{2} \ = \ -2$$

[1 mark]

(iii)

Perpendicular bisector will have gradient $\frac{1}{2}$

$$y \ = \ mx \ + \ c$$

$$0 \ = \ \frac{1}{2} \times 4 + c \ \Rightarrow \ c \ = \ -2$$

Perpendicular bisector has equation $y \ = \ \frac{1}{2} \, x \ - \ 2$

[2 marks]

(iv)

$$Midpoint \ = \left(\frac{2 \ + \ 9}{2}, \ \frac{4 \ + \ 5}{2} \right) \ = \left(\frac{11}{2}, \ \frac{9}{2} \right)$$

$$m \ = \ \frac{\Delta Y}{\Delta X} \ = \ \frac{4 \ - \ 5}{2 \ - \ 9} \ = \ \frac{1}{7}$$

Perpendicular bisector will have gradient – 7

$$y \ = \ mx \ + \ c$$

$$\frac{9}{2} \ = \ -7 \times \frac{11}{2} \ + \ c \ \Rightarrow \ c \ = \ 43$$

Perpendicular bisector has equation $y \ = \ -7x \ + \ 43$

[4 marks]

(v)

$$\frac{1}{2}x \ - \ 2 \ = \ -7x \ + \ 43$$

multiply through by 2

$$x \ - \ 4 \ = \ -14x \ + \ 86$$

$$15x \ = \ 90$$

$$x \ = \ 6$$

Centre of the rotation is $(\, 6, \ 1 \,)$

[4 marks]

Answer 11

(i) $P(5,\ 1)$

[1 mark]

(ii) $V(2,\ 1),\ P(5,\ 1)$

$$\text{Midpoint } VP = \left(\frac{5+2}{2},\ \frac{1+1}{2}\right) = M\left(\frac{7}{2},\ 1\right)$$

[2 marks]

(iii) $T(5,\ 10)$

$$y = x^2 - 4x + 5$$

$$\frac{dy}{dx} = 2x - 4$$

$$\text{When } x = 5,\ \frac{dy}{dx} = 2 \times 5 - 4$$

$$= 6$$

$$y = mx + c$$

$$10 = 6 \times 5 + c \Rightarrow c = -20$$

Equation of the tangent at T is $y = 6x - 20$

[4 marks]

(iv)

$$y = 6x - 20 \qquad M\left(\frac{7}{2},\ 1\right)$$

Does $y = 1$ when $x = \frac{7}{2}$?

$$\text{RHS} = 6x - 20$$

$$= 6 \times \frac{7}{2} - 20$$

$$= 1$$

$$= \text{LHS}$$

(v)

- Draw a line, L, parallel to the x-axis through V
- Drop a perpendicular line from T to the line, L, calling the intersection P
- Calculate the midpoint, M, of VP
- Draw a line through T and M which will be the tangent at T

[3 marks]

Out Of Interest

I have a proof that this will always work.

Lesson 13

A-Level Pure Mathematics, Year 1
Additional Mathematics
Coordinate Geometry

13.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 80

Question 1

The equation of a circle is

$$(x + 11)^2 + (y - 13)^2 = 49$$

State the coordinates of the circle's centre, and its radius.

[2 marks]

Question 2

Write down the equation of the circle with centre $(15, -8)$ which passes through the origin.

[2 marks]

Question 3

By completing the square, or otherwise, determine the centre and radius of the following circle;

$$x^2 + y^2 - 12x + 10y - 20 = 0$$

[5 marks]

Question 4

Differentiate

(i)

$$y = 9x^5 - 7x^3 + 4$$

[2 marks]

(ii)

$$y = 100x^{0.35}$$

[2 marks]

(iii)

$$y = \frac{2}{3}x^{-9}$$

[2 marks]

Question 5

The equation of the curve, plotted on the next page, is

$$y = \frac{x^3}{16} - 3$$

(i) Write down the gradient equation, $\frac{dy}{dx}$, for the curve.

[2 marks]

(ii) Use your part (i) answer to find the value of the gradient on the curve when $x = 4$

[1 mark]

(iii) Use your part (ii) answer to determine the equation of the tangent to the curve at the point (4, 1)

[2 marks]

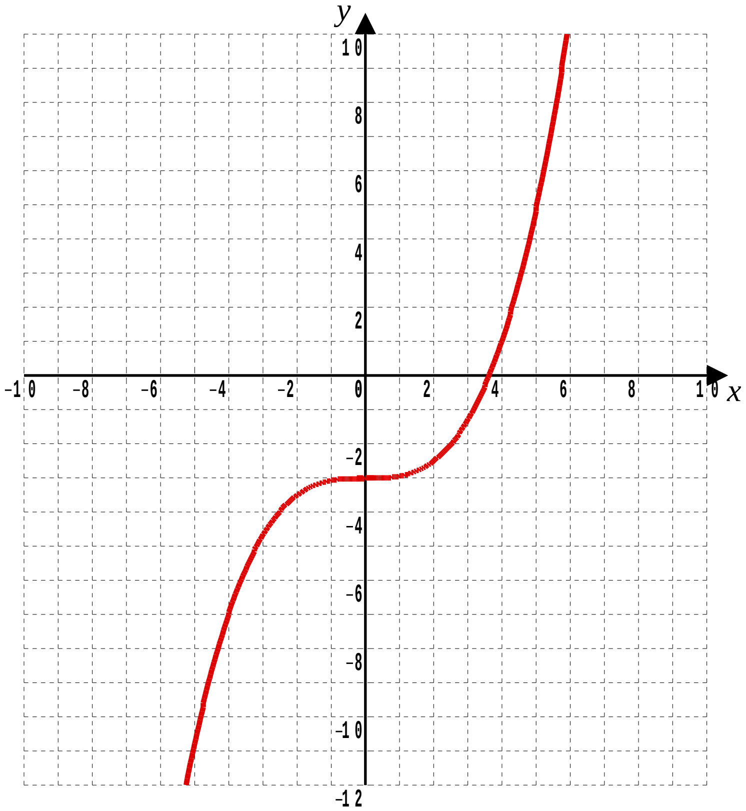
(iv) Use your part (ii) answer to determine the gradient of the normal to the curve at the point (4, 1)

[1 mark]

(v) Find the equation of the normal to the curve at the point (4, 1)

[2 marks]

(vi) On the graph, add your part (iii) tangent, and your part (v) normal, clearly indicating which is which and making sure they both passes through the point (4, 1)



[2 marks]

Question 6

$$x^2 + (y - 2)^2 = 8^2$$

Is the point (7, 6) inside, outside, or on the circumference of this circle ?
Justify your answer.

[3 marks]

Question 7

Find the point(s) of intersection, if any, of the circle

$$(x + 5)^2 + (y - 3)^2 = 32$$

and the line

$$y = 4x + 3$$

[6 marks]

Question 8
AB is a diameter of a circle, where A is $(-2,\ 4)$ and B is $(6, 8)$
Find

(i) the exact length of AB

[2 marks]

(ii) the coordinates of the midpoint of AB

[2 marks]

(iii) the equation of the circle

[3 marks]

(iv) the gradient of the line through A and B

[2 marks]

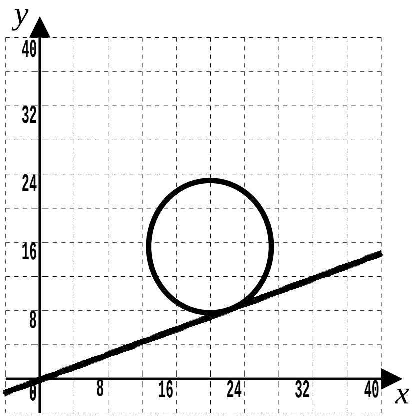
(v) the equation of the tangent to the circle at B

[3 marks]

(vi) the points of intersection of the circle with the y-axis

[3 marks]

Question 9
A circle has centre (20, 15) and radius 7
Find the length of the tangential line from the origin to the circle.



[6 marks]

Question 10

A-Level Examination Question from June 2010, Paper C2, Q10 (Edexcel)

The circle C has centre A (2, 1) and passes through the point B (10, 7)

(a) Find an equation for C

[4 marks]

The line l_1 is the tangent to C at the point B

(b) Find an equation for l_1

[4 marks]

The line l_2 is parallel to l_1 and passes through the mid-point of AB

Given that l_2 intersects C at the points P and Q

(c) Find the length of PQ, giving your answer in its simplest surd form

[3 marks]

Question 11

A-Level Examination Question from January 2016, IAL, Paper C12, Q12 (Edexcel)

$$f(x) = \frac{(4 + 3\sqrt{x})^2}{x}, \qquad x > 0$$

(a) Show that $f(x) = Ax^{-1} + Bx^{k} + C$, where A, B, C and k are constants to be determined.

[4 marks]

(b) Hence find $f'(x)$

[2 marks]

(c) Find an equation of the tangent to the curve $y = f(x)$ at the point where $x = 4$

[4 marks]

Question 12

A-Level Examination Question from January 2019, IAL Paper C12, Q3 (Edexcel)

A curve has equation

$$y = \sqrt{2}\,x^2 - 6\sqrt{x} + 4\sqrt{2}, \qquad x > 0$$

Find the gradient of the curve at the point $P\left(2,\ 2\sqrt{2}\right)$

Write your answer in the form $a\sqrt{2}$, where a is a constant

[4 marks]

13.2 Answers

A-Level Pure Mathematics, Year 1
Additional Mathematics
Coordinate Geometry

Answer 1

Centre $(-11,\ 13)$ Radius 7

[2 marks]

Answer 2

Working out the distance between the centre and the origin, which will be the radius...

$$\begin{aligned} R &= \sqrt{(15 - 0)^2 + (8 - 0)^2} \\ &= \sqrt{15^2 + 8^2} \\ &= 17 \end{aligned}$$

$$\therefore\ (x - 15)^2 + (y + 8)^2 = 17^2$$

[2 marks]

Answer 3

$$\begin{aligned} x^2 + y^2 - 12x + 10y - 20 &= 0 \\ [\,x^2 - 12x\,] + [\,y^2 + 10y\,] - 20 &= 0 \\ [\,(x - 6)^2 - 36\,] + [\,(y + 5)^2 - 25\,] - 20 &= 0 \\ (x - 6)^2 + (y + 5)^2 &= 81 \end{aligned}$$

$$\therefore\ \text{Centre}\ (6, -5)\ \ \text{Radius } 9$$

[5 marks]

Answer 4

(i) $y = 9x^5 - 7x^3 + 4 \quad \Rightarrow \quad \dfrac{dy}{dx} = 45x^4 - 21x^2$

(ii) $y = 100x^{0.35} \quad \Rightarrow \quad \dfrac{dy}{dx} = 35x^{-0.65}$

(iii) $y = \dfrac{2}{3}x^{-9} \quad \Rightarrow \quad \dfrac{dy}{dx} = -6x^{-10}$

[2, 2, 2 marks]

Answer 5

(i)

$$y = \frac{x^3}{16} - 3$$

$$\frac{dy}{dx} = \frac{3x^2}{16}$$

[2 marks]

(ii) When $x = 4$, $\dfrac{dy}{dx} = 3$

[1 mark]

(iii)

$$y = mx + c$$

$$1 = 3 \times 4 + c \Rightarrow c = -11$$

Equation of the tangent is $y = 3x - 11$

[2 marks]

(iv) Normal's "sign changed reciprocal" gradient will be $-\dfrac{1}{3}$

[1 mark]

(v)

$$y = mx + c$$

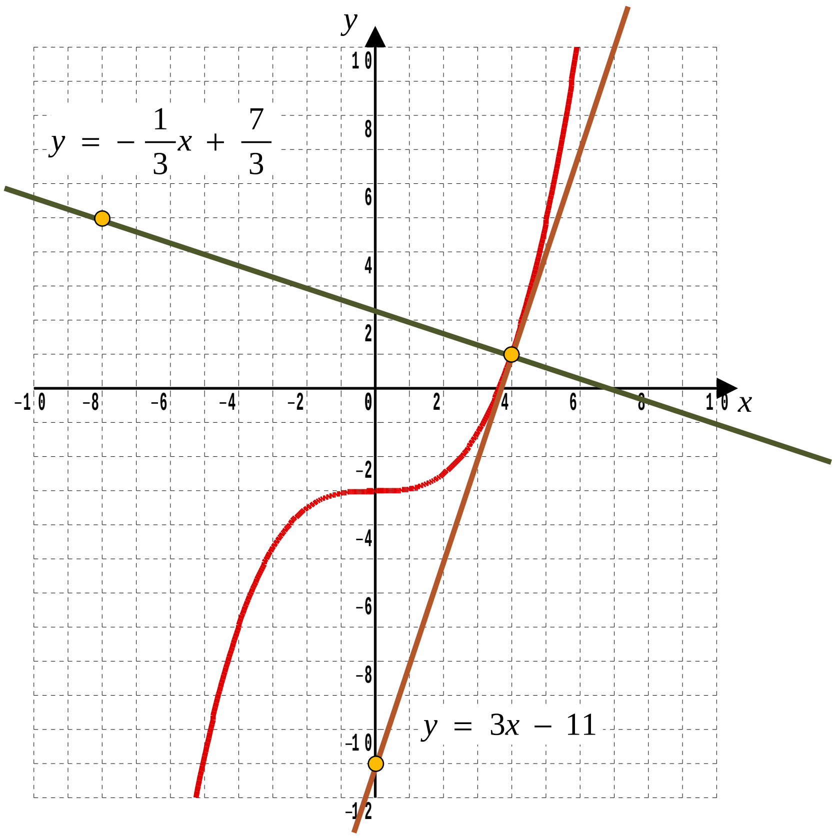
$$1 = -\frac{1}{3} \times 4 + c \Rightarrow c = \frac{7}{3}$$

Equation of the normal is $y = -\dfrac{1}{3}x + \dfrac{7}{3}$

Alternatively $3y + x - 7 = 0$

[2 marks]

(vi)



[2 marks]

Answer 6

$$\begin{aligned} \text{LHS} &= x^2 + (y-2)^2 \\ &= 7^2 + (6-2)^2 \qquad \text{if the circle passes through } (7,\ 6) \\ &= 7^2 + 4^2 \\ &= 49 + 16 \\ &= 65 \\ &> \text{RHS} \end{aligned}$$

$\therefore\ (7,\ 6)$ is outside the circle ☐

[3 marks]

Answer 7

$$(x+5)^2 + (y-3)^2 = 32 \quad \text{and} \quad y = 4x + 3$$

$$(x+5)^2 + (4x+3-3)^2 = 32$$

$$x^2 + 10x + 25 + 16x^2 = 32$$

$$17x^2 + 10x - 7 = 0$$

$$(x+1)(17x-7) = 0$$

$$\text{Either } x = -1 \text{ or } x = \frac{7}{17}$$

Points of intersection are $(-1, -1)$, $\left(\frac{7}{17},\ \frac{79}{17}\right)$

[6 marks]

Answer 8

(i) $$|AB| = \sqrt{(6 - (-2))^2 + (8 - 4)^2}$$
$$= \sqrt{8^2 + 4^2}$$
$$= \sqrt{80}$$
$$= 4\sqrt{5}$$

(This is the diameter)

[2 marks]

(ii) $$Midpoint\ AB = \left(\frac{6 + (-2)}{2}, \frac{8 + 4}{2} \right) = (2, 6)$$

[2 marks]

(iii) $$(x - 2)^2 + (y - 6)^2 = 20$$

[3 marks]

(iv) $$m_{AB} = \frac{\Delta Y}{\Delta X} = \frac{8 - 4}{6 - (-2)} = \frac{4}{8} = \frac{1}{2}$$

[2 marks]

(v) For the tangent, the gradient is -2 through $B(6, 8)$

$$y = mx + c$$
$$8 = -2 \times 6 + c \Rightarrow c = 20$$

The tangent's equation is $y = -2x + 20$

[3 marks]

(vi) The circle will cut the *y*-axis when $x = 0$

$$(x - 2)^2 + (y - 6)^2 = 20$$
$$(0 - 2)^2 + (y - 6)^2 = 20$$
$$(y - 6)^2 = 16$$
$$y - 6 = \pm 4$$

Either $y = 2$ or $y = 10$

Points of intersection are $(0, 2)$, $(0, 10)$

[3 marks]

Answer 9

The distance, D, between the circle's centre and the origin is,

$$D = \sqrt{(20 - 0)^2 + (15 - 0)^2}$$
$$= \sqrt{20^2 + 15^2}$$
$$= \sqrt{625}$$
$$= 25$$

D is the hypotenuse of a right angled triangle with the circle's radius being one of the shorter sides, and the length sought, X, being the other.

$$X = \sqrt{25^2 - 7^2}$$
$$= \sqrt{625 - 49}$$
$$= \sqrt{576}$$
$$= 24$$

[6 marks]

Answer 10

(a) Centre $A(2, 1)$, Radius is distance between A and B

$$\text{Radius} = \sqrt{(10 - 2)^2 + (7 - 1)^2}$$
$$= \sqrt{8^2 + 6^2}$$
$$= 10$$

The circle has equation $(x - 2)^2 + (y - 1)^2 = 100$

[4 marks]

(b) $m_{AB} = \dfrac{\Delta Y}{\Delta X} = \dfrac{7 - 1}{10 - 2} = \dfrac{6}{8} = \dfrac{3}{4}$

The line l_1 will have gradient $-\dfrac{4}{3}$ through $B(10, 7)$

$$y = mx + c$$

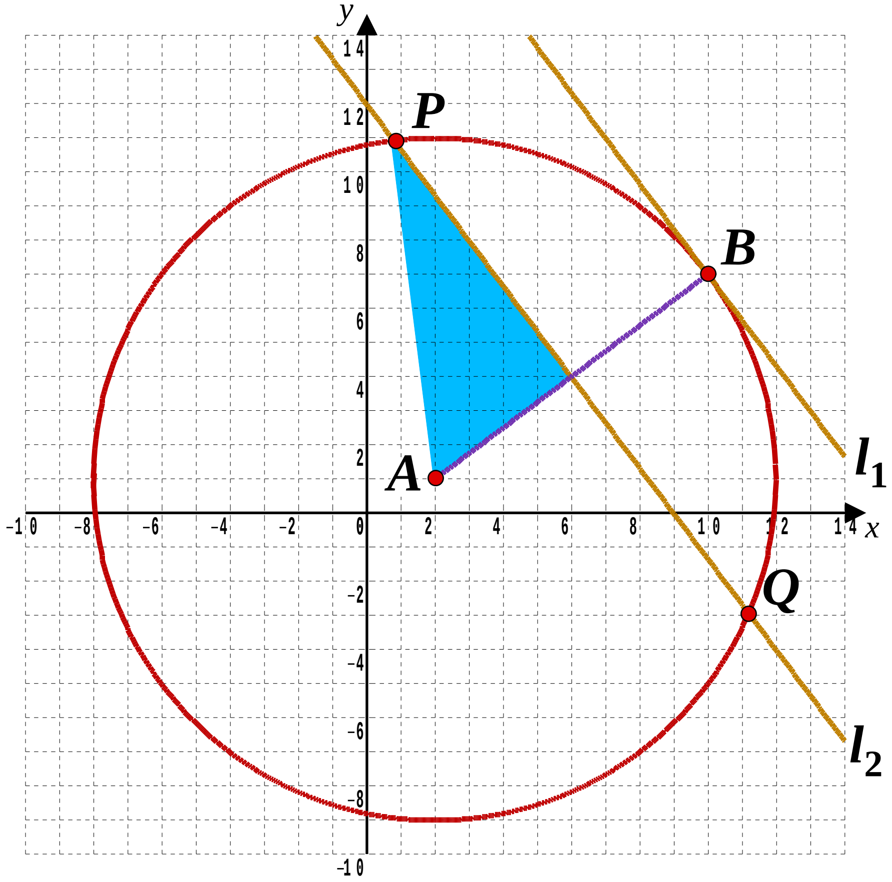
$$7 = -\frac{4}{3} \times 10 + c \Rightarrow c = \frac{61}{3}$$

l_1 has equation $y = -\dfrac{4}{3}x + \dfrac{61}{3}$

Alternatively, $4x + 3y - 61 = 0$

[4 marks]

(c) A rough sketch is helpful in seeing how to answer this part of the question !



The blue triangle has a hypotenuse that is the radius of the circle, 10 and its other side is half the radius running from *A* to *B*

$$
\begin{aligned}
\frac{1}{2}\,|PQ| &= \sqrt{10^2 - 5^2} \\
&= \sqrt{100 - 25} \\
&= \sqrt{75} \\
&= 5\sqrt{3} \\
|PQ| &= 10\sqrt{3}
\end{aligned}
$$

[3 marks]

Answer 11

(a)

$$f(x) = \frac{(4 + 3\sqrt{x})^2}{x}$$

$$= \frac{16 + 24\sqrt{x} + 9x}{x}$$

$$= \frac{16}{x} + \frac{24\sqrt{x}}{x} + \frac{9x}{x}$$

$$= 16x^{-1} + \frac{24\sqrt{x}}{\sqrt{x} \times \sqrt{x}} + 9$$

$$= 16x^{-1} + \frac{24}{\sqrt{x}} + 9$$

$$= 16x^{-1} + \frac{24}{x^{\frac{1}{2}}} + 9$$

$$= 16x^{-1} + 24x^{-\frac{1}{2}} + 9$$

That is, $A = 16$, $B = 24$, $C = 9$ and $k = -\frac{1}{2}$

[4 marks]

(b)

$$f'(x) = -16x^{-2} - 12x^{-\frac{3}{2}}$$

$$= -\frac{16}{x^2} - \frac{12}{x^{\frac{3}{2}}}$$

[2 marks]

(c) When $x = 4$,

$$f(4) = \frac{(4 + 3\sqrt{4})^2}{4} = \frac{100}{4} = 25$$

$$f'(4) = -\frac{16}{4^2} - \frac{12}{4^{\frac{3}{2}}} = -1 - \frac{12}{8} = -\frac{5}{2}$$

$$y = mx + c$$

$$25 = -\frac{5}{2} \times 4 + c \Rightarrow c = 35$$

$$y = -\frac{5}{2} + 35$$

Alternatively, $5x + 2y - 70 = 0$

[4 marks]

Answer 12

$$y = \sqrt{2}\,x^2 - 6\sqrt{x} + 4\sqrt{2}$$

$$= \sqrt{2}\,x^2 - 6x^{\frac{1}{2}} + 4\sqrt{2}$$

$$\frac{dy}{dx} = \sqrt{2} \times 2x - 6 \times \frac{1}{2}x^{-\frac{1}{2}}$$

When $x = 2$, $\dfrac{dy}{dx} = \sqrt{2} \times 2 \times 2 - 3 \times \dfrac{1}{\sqrt{2}}$

$$= 4\sqrt{2} - \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= 4\sqrt{2} - \frac{3\sqrt{2}}{2}$$

$$= \frac{8\sqrt{2} - 3\sqrt{2}}{2}$$

$$= \frac{5\sqrt{2}}{2}$$

$$= \frac{5}{2}\sqrt{2}$$

That is, $a = \dfrac{5}{2}$

[4 marks]