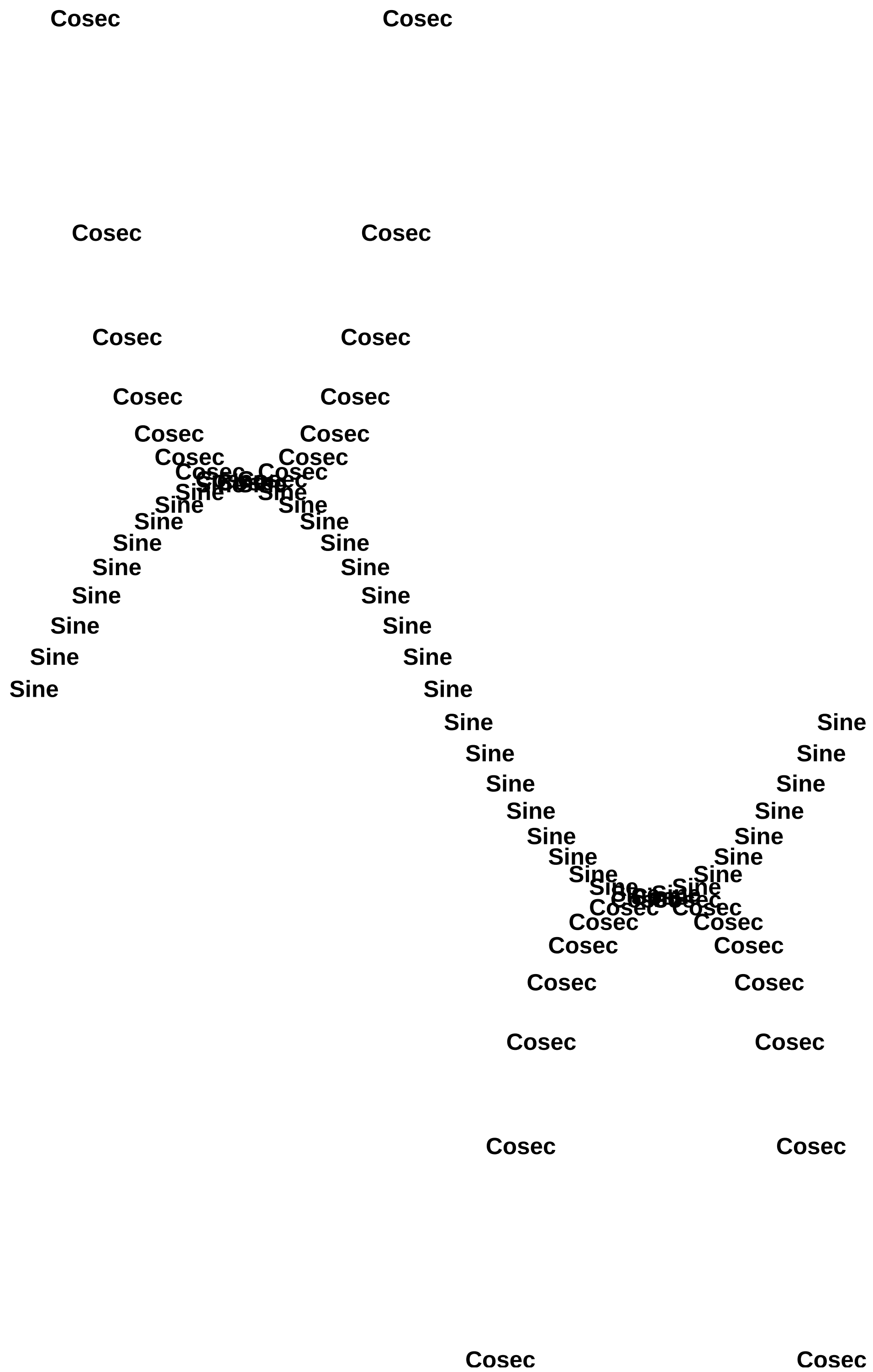
A-Level Pure Mathematics
Year 2

TRIGONOMETRY V

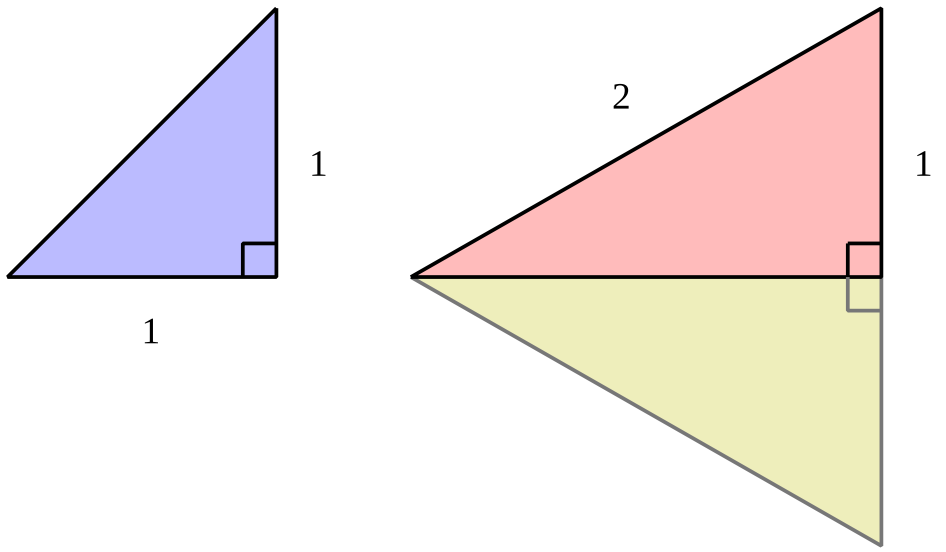
Trigonometric Identities



Lesson 1

1.1 Trigonometric "Exact Values"

The following two simple right angled triangles are of great utility, for they can be used to determine exact values for the *sin*, *cos* and *tan* of certain angles.



The left triangle has a base of length exactly 1 unit and a height of exactly 1 unit. On this triangle in the appropriate place write the exact length of the hypotenuse and the exact size of the two non right angled angles.

The right triangle is equilateral with sides of length exactly 2 units, but has been cut in half. The focus is upon the upper half which has a hypotenuse of exact length 2 units and height of exactly 1 unit. On this triangle in the appropriate place write the exact length of the base and the exact size of the two non right angled angles.

Complete the following "Exact Values" table without using a calculator,

	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					

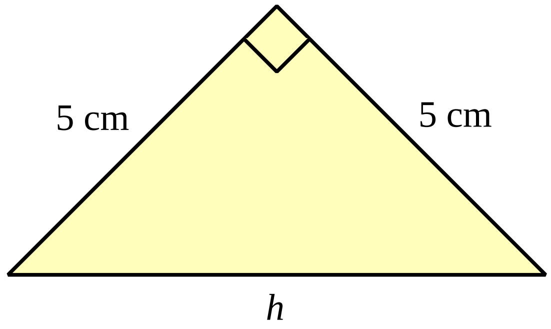
[5 marks]

1.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

A right-angled isosceles triangle has two sides of length 5 cm.

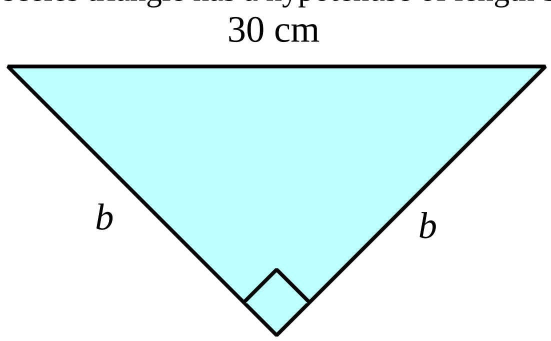


Use the theorem of Pythagoras to find the exact length of its hypotenuse, *h*, in as simplified a form as possible.

[2 marks]

Question 2

A right angled isosceles triangle has a hypotenuse of length 30 cm.

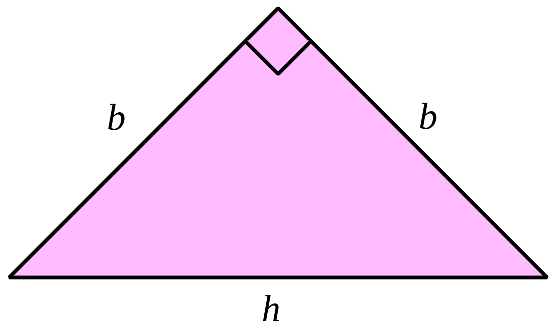


Use the theorem of Pythagoras to find the exact length of one of the other sides, *b*, in as simplified a form as possible.

[3 marks]

Question 3

A right-angled isosceles triangle has two sides of length *b* cm.



(i) Use the theorem of Pythagoras to find the exact length of its hypotenuse, *h*, in terms of *b*, in as simplified a form as possible.

[2 marks]

(ii) Rearrange your part (i) answer to make *b* the subject of the formula and present your answer with a rational denominator.

[2 marks]

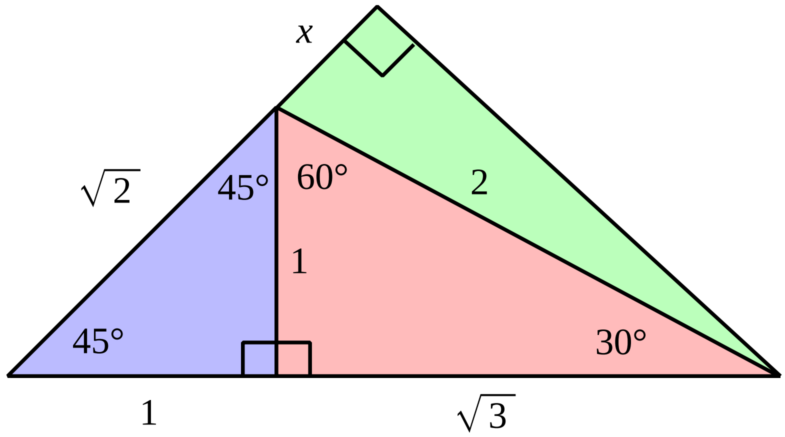
(iii) Given that a right-angled isosceles triangle has two sides of length $1 + \sqrt{5}$, write down the exact length of the hypotenuse by using part (i)'s answer.

[1 mark]

(iv) Given that a right-angled isosceles triangle has a hypotenuse of $1 + \sqrt{3}$, write down the exact length of each of the other sides using part (ii)'s answer. Any denominator in your final answer should be rational.

[2 marks]

Question 4



In the diagram the previous two triangles have been placed back to back.
An extra green triangle has been added such that the entire shape is a right-angled isosceles triangle with two angles of 45°

(i) On the diagram write in the values of the two non right-angled angles of the green triangle.

[1 mark]

(ii) By considering the entire shape, which has a hypotenuse of exact length $1 + \sqrt{3}$, write down an expression for the length of either of the other sides.

[1 mark]

(iii) Hence, or otherwise, determine the exact length of the side of the green triangle marked x

[3 marks]

(iv) Without using a calculator, find an exact value for $\sin 15°$

[2 marks]

(v) Without using a calculator, complete the following "Exact Values" table. Show necessary working, especially for *tan* θ.

	$15°$	$75°$
$\sin\theta$		
$\cos\theta$		
$\tan\theta$		

[5 marks]

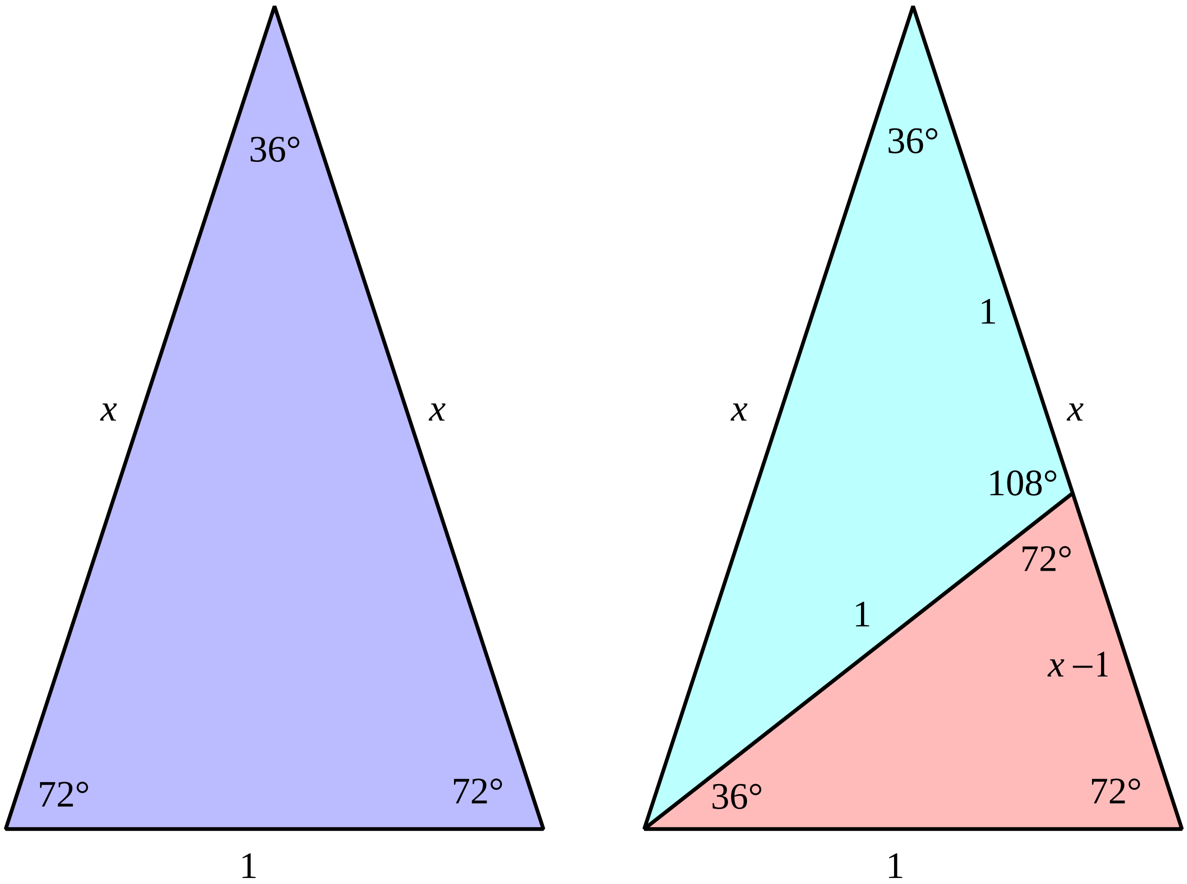
(vi) The cosine rule (reversed) states that in any triangle *ABC*,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Apply this to a suitable triangle to obtain an exact expression for $\cos 105°$

[3 marks]

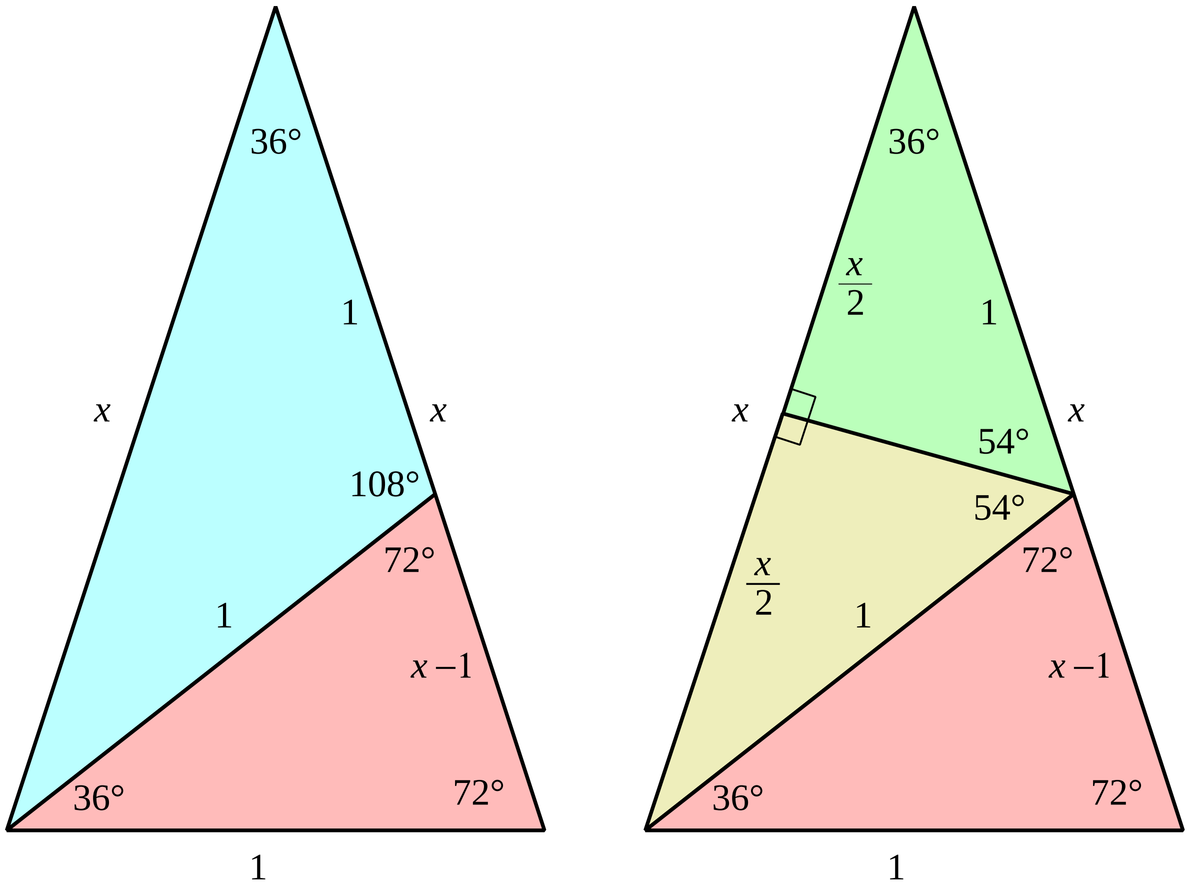
Question 5



To the left is drawn an isosceles purple triangle with base angles of 72°.
The base is 1 unit in length and the congruent sides are of length x.
The angle bisector of the bottom-left base angle is drawn, cutting the original triangle into two triangles, one pink, one blue, as shown right.
These remarkable property of both being isosceles, and so the pink triangle has two sides that must be of length 1 which in turn forces the blue triangle to also have two sides of length 1.
This then means that the base of the pink triangle is $x - 1$

(i) The original blue triangle and the new pink triangle are similar because they both have angles of 72°, 72° and 36°.
Use this fact to find the exact numerical value of x

[5 marks]



The bisector of the 108° angle in the blue triangle is now drawn.
This results in two congruent right angled triangles, one green and one yellow.

(ii) Use the diagram above right, and your knowledge of right angled trigonometry, to find an exact value for *cos* 36°

[3 marks]

(iii) Show that the exact value of $sin\ 36$ is $\dfrac{\sqrt{10 - 2\sqrt{5}}}{4}$

[3 marks]

Question 6

Tom is wondering about how to expand the brackets of,

$$\sin\ (A + B)$$

Lucy suggests that it might always be,

$$\sin A + \sin B$$

for all values of A and B.

(i) Show that Lucy's suggestion is not correct by finding suitable "counterexample" values for A and B taken from an "Exact Values" table.

[2 marks]

Prof Triggy Brain tells Tom that the correct expansion is;

$$\sin\ (A + B) = \sin A \cos B + \cos A \sin B$$

(ii) Provide an illustration that this could be correct by picking suitable values for A and B from an "Exact Values" table.

[2 marks]

1.3 Answers

1.3.1 Solutions 1.1 Trigonometric "Exact Values"

	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Not Defined

1.3.2 Solutions to 1.2 Exercise

Answer 1

$h = 5\sqrt{2}$ cm

[2 marks]

Answer 2

$$30^2 = b^2 + b^2$$
$$900 = 2b^2$$
$$b^2 = 450$$
$$b = 15\sqrt{2}$$

[3 marks]

Answer 3

(i)
$$h^2 = b^2 + b^2$$
$$= 2b^2$$
$$h = \sqrt{2}\,b$$

[2 marks]

(ii)
$$b = \frac{h}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{2}\,h}{2}$$

[2 marks]

(iii) $h = \sqrt{2}\left(1 + \sqrt{5}\right)$

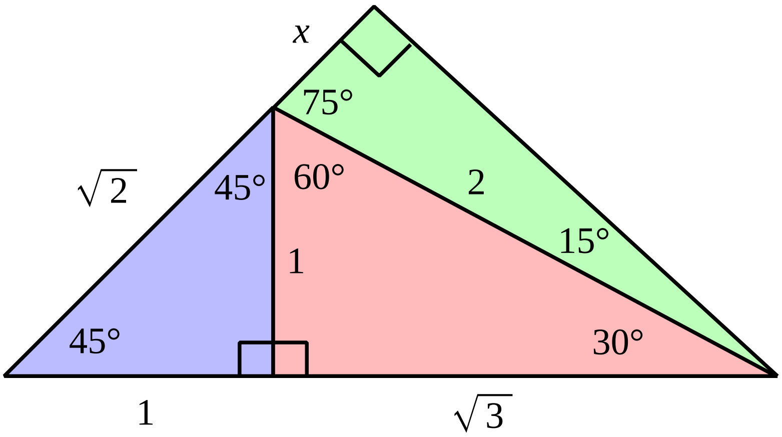
[1 mark]

(iv) $b = \dfrac{\sqrt{2}\left(1 + \sqrt{3}\right)}{2}$

[2 mark]

Answer 4

(i)



[1 mark]

(ii) $\dfrac{\sqrt{2}\,(1+\sqrt{3})}{2} = \dfrac{\sqrt{6}+\sqrt{2}}{2}$

[1 mark]

(iii) It can now be deduced that,

$$\sqrt{2} + x = \frac{\sqrt{2}\,(1+\sqrt{3})}{2}$$

$$x = \frac{\sqrt{2}\,(1+\sqrt{3})}{2} - \sqrt{2}$$

$$= \frac{\sqrt{2}+\sqrt{6}-2\sqrt{2}}{2}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{2}$$

[3 marks]

(iv) $sin\, 15 = \dfrac{x}{2}$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

[2 marks]

(v)

$$\cos 15^\circ = \sin 75^\circ = \left(\frac{\sqrt{6} + \sqrt{2}}{2}\right) \div 2$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

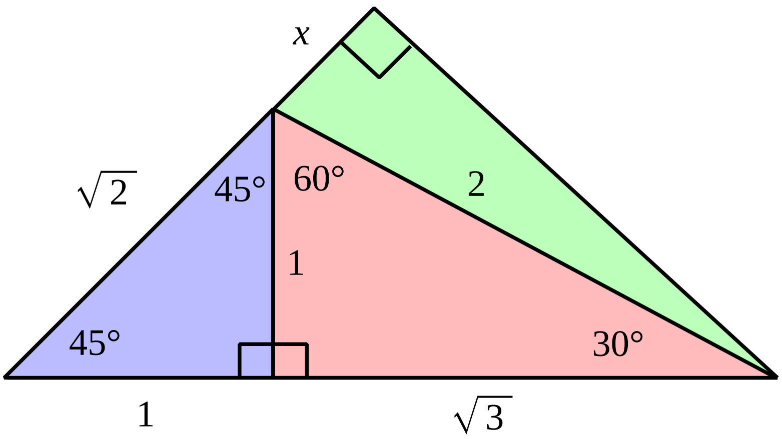
$$\tan 15 = \left(\frac{\sqrt{6} - \sqrt{2}}{2}\right) \div \left(\frac{\sqrt{6} + \sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{6} - \sqrt{2}}{2} \times \frac{2}{\sqrt{6} + \sqrt{2}}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}} \times \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} - \sqrt{2}}$$
$$= \frac{6 - \sqrt{12} - \sqrt{12} + 2}{6 - 2}$$
$$= \frac{8 - 4\sqrt{3}}{4}$$
$$= 2 - \sqrt{3}$$

$$\tan 75 = \frac{1}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$
$$= \frac{2 + \sqrt{3}}{4 - 3}$$
$$= 2 + \sqrt{3}$$

	15°	75°
$\sin\theta$	$\frac{\sqrt{6} - \sqrt{2}}{4}$	$\frac{\sqrt{6} + \sqrt{2}}{4}$
$\cos\theta$	$\frac{\sqrt{6} + \sqrt{2}}{4}$	$\frac{\sqrt{6} - \sqrt{2}}{4}$
$\tan\theta$	$2 - \sqrt{3}$	$2 + \sqrt{3}$

[5 marks]

(vi) The 105° is the (45 + 60)° angle in the diagram



$$\cos 105^\circ = \frac{2^2 + (\sqrt{2})^2 - (1 + \sqrt{3})^2}{2 \times 2 \times \sqrt{2}}$$

$$= \frac{4 + 2 - (1 + 2\sqrt{3} + 3)}{4\sqrt{2}}$$

$$= \frac{4 + 2 - 1 - 2\sqrt{3} - 3}{4\sqrt{2}}$$

$$= \frac{2 - 2\sqrt{3}}{4\sqrt{2}}$$

$$= \frac{1 - \sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

[3 marks]

Answer 5

(i) Looking at the Length Scale Factor moving from the small (pink) triangle to the big purple triangle,

$$\frac{1}{x} = \frac{x - 1}{1}$$

Multiply through by x

$$1 = x^2 - x$$

$$x^2 - x - 1 = 0$$

$$\left[x^2 - x \right] - 1 = 0$$

$$\left[\left(x - \frac{1}{2} \right)^2 - \frac{1}{4} \right] - 1 = 0$$

$$\left(x - \frac{1}{2} \right)^2 = \frac{5}{4}$$

$$x - \frac{1}{2} = \pm \frac{\sqrt{5}}{2}$$

$$x = \frac{\sqrt{5} + 1}{2}$$

This is ϕ, the golden ratio

[5 marks]

(ii) $cos\,36 = \dfrac{x}{2} = \dfrac{\sqrt{5} + 1}{4}$

[1 mark]

(iii) Using the Theorem of Pythagoras, the fact that $\dfrac{x}{2} = \dfrac{\sqrt{5} + 1}{4}$

and the trigonometric identity that $cos^2\,\theta + sin^2\,\theta = 1$,

$$sin\,36° = \sqrt{1 - cos^2\,36}$$

$$= \sqrt{1 - \left(\frac{\sqrt{5} + 1}{4} \right)^2}$$

$$= \sqrt{\frac{16 - 5 - 2\sqrt{5} - 1}{16}}$$

$$= \sqrt{\frac{10 - 2\sqrt{5}}{16}}$$

$$= \frac{\sqrt{10 - 2\sqrt{5}}}{4} \qquad \square$$

[3 marks]

Answer 6

(i) Many counter examples are suitable.

For example let $A = 30°$ and $B = 60°$ in which case,

$$sin\,30° = \frac{1}{2} \quad \text{and} \quad sin\,60° = \frac{\sqrt{3}}{2} \quad \text{and} \quad sin\,90° = 1$$

$$\text{but} \quad \frac{1}{2} + \frac{\sqrt{3}}{2} \neq 1$$

$$\text{so} \quad sin\,30° + sin\,60° \neq sin\,90°$$

$$\therefore \; sin\,A + sin\,B \neq sin\,(A + B) \text{ in general}$$

[2 marks]

(ii) With $A = 30°$ and $B = 60°$ the claim is that,

$$sin\,(30 + 60) = sin\,30\,cos\,60 + cos\,30\,sin\,60$$

LHS $= sin\,(30 + 60)$

$= sin\,90$

$= 1$

RHS $= sin\,30\,cos\,60 + cos\,30\,sin\,60$

$= \frac{1}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$

$= \frac{1}{4} + \frac{3}{4}$

$= 1$

As LHS = RHS it is true that,

$$sin\,(A + B) = sin\,A\,cos\,B + cos\,A\,sin\,B$$

when $A = 30°$ and $B = 60°$

[2 marks]

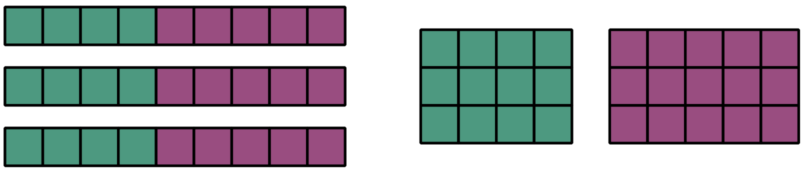
2.1 Law Breaker

Many brackets in mathematics can be expanded by using the Distributive Law. It states that multiplying a number by a group of numbers added together is the same as doing each multiplication separately.

The Distributive Law

$$a \times (b + c) = a \times b + a \times c$$

For example,



$$3 \times (4 + 5) \quad = \quad 3 \times 4 \quad + \quad 3 \times 5$$

To show that the distributive law does not apply to trigonometric functions, let $A = 30°$ and $B = 60°$ such that $A + B = 90°$ in which case,

$$\sin(A + B) \neq \sin A + \sin B$$

because

$$\sin 30 = \frac{1}{2}, \quad \sin 60° = \frac{\sqrt{3}}{2}, \quad \sin 90° = 1 \text{ and } \frac{1}{2} + \frac{\sqrt{3}}{2} \neq 1$$

The correct way that the brackets do expand is as follows,

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

A visual proof of this is given over the page.

This formula for expanding the brackets of $\sin(A + B)$ is one of six formulae that are collectively known as The Addition Formulae,

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The Addition Formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

2.2 Visual Proof

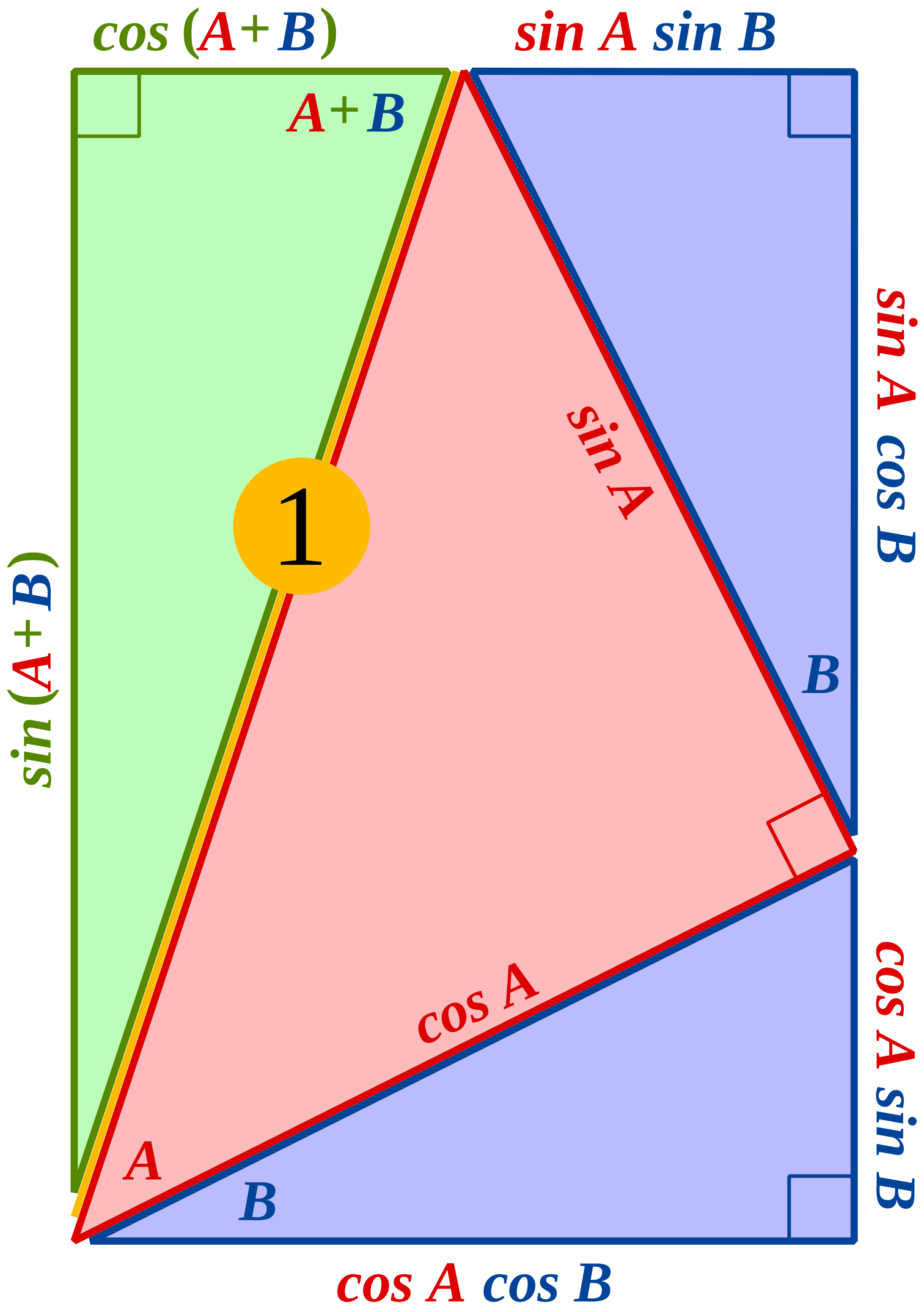
The diagram below is a "proof without words" that,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

and

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

It's a great diagram to pin up on a wall and think about.



2.3 Proving Your Identity

A trigonometric identity is a relationship between trigonometric functions that is essentially true for all values of angle. There may be a necessary technical restriction, on the domain to avoid, for example, a division by zero.

All six of the Addition Formulae are trigonometric identities and so too are,

$$\cos^2\theta + \sin^2\theta = 1, \qquad \theta \in \mathbb{R}$$

and

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \theta \in \mathbb{R},\ \ \theta \neq 90^\circ(1 + 2n) \text{ for } n \in \mathbb{Z}$$

Faced with an unfamiliar trigonometric identity, the immediate task is often to prove that it is true.

2.4 Example

Prove that $\cos(90^\circ - \theta) = \sin\theta$

Teaching Video : http://www.NumberWonder.co.uk/v9040/2.mp4

Watch the Teaching Video which shows the correct technical manner in which to write out the proof.

☞

[3 marks]

2.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 38

Question 1
Prove the following identity.
Be sure to start "LHS =" and conclude with "= RHS" along with the special symbol used to indicate the conclusion of the proof.

$$\sin(90° + \theta) = \cos\theta$$

[3 marks]

Question 2
Prove the following identity.
Be sure to start "LHS =" and conclude with "= RHS" along with the special symbol used to indicate the conclusion of the proof.

$$\sin(180° - \theta) = \sin\theta$$

[3 marks]

Question 3

Prove that

$$sin(A+B)+sin(A-B)=2\,sin\,A\,cos\,B$$

[3 marks]

Question 4

Prove this identity;

$$cos(A-B)-cos(A+B)=2\,sin\,A\,sin\,B$$

[3 marks]

Question 5

Prove,

$$\frac{\sin(A + B)}{\cos A \cos B} = \tan A + \tan B, \qquad A,\ B \neq 90^\circ(1 + 2n),\ n \in \mathbb{Z}$$

[3 marks]

Question 6

By letting $B = A$ in the addition formula $\sin(A + B) = \sin A \cos B + \cos A \sin B$ prove that;

$$\sin 2A = 2 \sin A \cos A$$

[3 marks]

Question 7

Using a similar technique to that of question 6, prove that;

$$\cos 2A = \cos^2 A - \sin^2 A$$

[3 marks]

Question 8

Prove that $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$ $\quad A \neq 45°(1 + 2n),\ n \in \mathbb{Z}$

[3 marks]

Question 9
The three formulae, proved in questions 6, 7 and 8 are called “The Double Angle Formulae”. They are important and useful, and need to be memorised for recall at any moment.

Write out the three double angle formulae.
You may claim a bonus mark if you can do it from memory !

[3 marks]
[1 Bonus]

Question 10
Explain carefully, giving an example of each, the difference between a trigonometric identity and a trigonometric equation.

[4 marks]

Question 11

Solve the following trigonometric equation for x between 0° and 360°

$$2 \sin x = \cos (x + 60°)$$

[6 marks]

2.6 Homework

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 35

Question 1

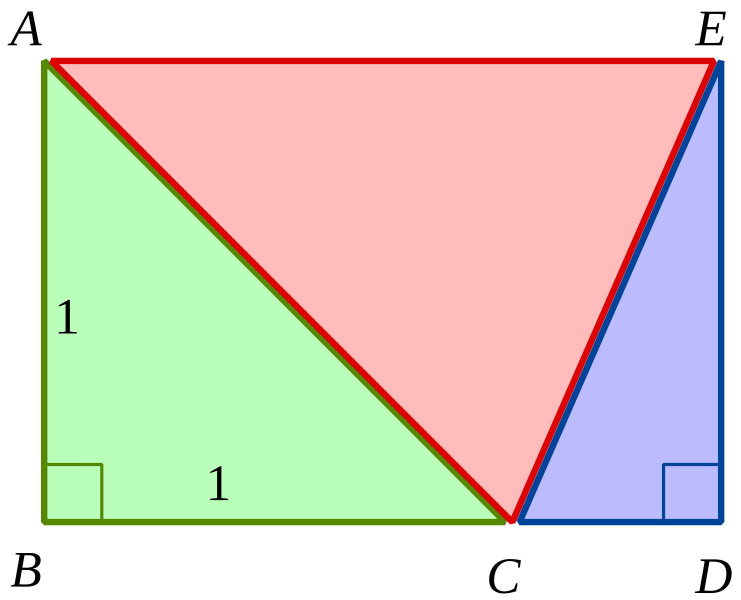
Solve the following trigonometric equation for x between 0° and 360°

$$\sin\ (x - 30°) = \frac{1}{2}\cos x$$

[6 marks]

Question 2

(i) Study the following diagram.
The aim is to work out the exact size of angle *CED*, and then deduce an exact value for *tan CED* that will involve a square root.



The green triangle, $\triangle ABC$ is isosceles and right angled with sides *AB* and *BC* both of length 1. The red triangle, $\triangle ACE$, is also isosceles with sides *AC* and *AE* having the same length. The blue triangle, $\triangle EDC$ is right angled.
All three triangles form a rectangle, *ABDE*, as shown.

Determine the exact value of angle *CED*, and the exact value of *tan CED*.

[5 marks]

(ii) Solve the following trigonometric equation for x between 0° and 360°

$$\cos\,(x - 45°) = \cos x$$

[6 marks]

Question 3

Solve the following trigonometric equation for x between 0° and 360°

$$3\,sin\,(x + 10°) = 4\,cos\,(x - 10°)$$

[6 marks]

Question 4

Prove that, $$\frac{\tan(C + D) - \tan C}{1 + \tan(C + D)\tan C} = \tan D$$

HINT: Let (C + D) = A and C = B, then use the *tan* addition formula backwards.

[5 marks]

Question 5
Given that

$$sin\ (\ x\ -\ \alpha\ )\ =\ cos\ (\ x\ +\ \alpha\ )$$

prove that

$$tan\, x\ =\ 1$$

[7 marks]

2.7 Answers

2.7.1 Solution to 2.4 Example (Teaching Video)

The Addition Formulae

$$\sin\ (A \pm B) = \sin A\ \cos B \pm \cos A \sin B$$

$$\cos\ (A \pm B) = \cos A\ \cos B \mp \sin A\ \sin B$$

$$\tan\ (A \pm B) = \frac{\tan A\ \pm\ \tan B}{1 \mp\ \tan A\ \tan B}$$

$$\begin{aligned} \text{LHS} &= \cos(90 - \theta) \\ &= \cos 90°\ \cos\theta + \sin 90° \sin\theta \\ &= 0 \times \cos\theta + 1 \times \sin\theta \\ &= \sin\theta \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

2.7.2 Solutions to 2.5 Exercise

Answer 1

$$\begin{aligned} \text{LHS} &= \sin\ (90° + \theta) \\ &= \sin 90 \cos\theta + \cos 90 \sin\theta \\ &= 1 \times \cos\theta + 0 \times \sin\theta \\ &= \cos\theta \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

Answer 2

$$\begin{aligned} \text{LHS} &= \sin\ (180° - \theta) \\ &= \sin 180 \cos\theta - \cos 180 \sin\theta \\ &= 0 \times \cos\theta - (-1) \times \sin\theta \\ &= \sin\theta \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

Answer 3

$$\begin{aligned} \text{LHS} &= \sin(A+B) + \sin(A-B) \\ &= \sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B \\ &= 2 \sin A \cos B \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

Answer 4

$$\begin{aligned} \text{LHS} &= \cos(A-B) - \cos(A+B) \\ &= \cos A \cos B + \sin A \sin B - \left[\cos A \cos B - \sin A \sin B\right] \\ &= \cos A \cos B + \sin A \sin B - \cos A \cos B + \sin A \sin B \\ &= 2 \sin A \sin B \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

Answer 5

$$\begin{aligned} \text{LHS} &= \frac{\sin(A+B)}{\cos A \cos B} \\ &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B} \\ &= \frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B} \\ &= \tan A + \tan B \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

Answer 6

$$\begin{aligned} \text{LHS} &= \sin 2A \\ &= \sin(A+A) \\ &= \sin A \cos A + \cos A \sin A \\ &= 2 \sin A \cos A \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

Answer 7

$$\begin{aligned} \text{LHS} &= \cos 2A \\ &= \cos(A + A) \\ &= \cos A \cos A - \sin A \sin A \\ &= \cos^2 A - \sin^2 A \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

Answer 8

$$\begin{aligned} \text{LHS} &= \tan 2A \\ &= \tan(A + A) \\ &= \frac{\tan A + \tan A}{1 - \tan A \tan A} \\ &= \frac{2 \tan A}{1 - \tan^2 A} \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

Answer 9

The Double Angle Formulae

$$\begin{aligned} \sin 2A &= 2 \sin A \cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ \tan 2A &= \frac{2 \tan A}{1 - \tan^2 A} \end{aligned}$$

[3 marks]
[1 Bonus]

Answer 10
A trigonometric identity is a statement that is true for all values of θ
(Sometimes with a technical exclusion)
A trigonometric equation is a statement that is only true for a certain special values of θ that, typically, are to be found.

Plus an example of each.

[4 marks]

Answer 11

$$2\sin x = \cos(x + 60°)$$

$$2\sin x = \cos x\cos 60 - \sin x\sin 60$$

$$2\sin x = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$$

multiply through by 2

$$4\sin x = \cos x - \sqrt{3}\sin x$$

$$\sin x\left(4 + \sqrt{3}\right) = \cos x$$

$$\frac{\sin x}{\cos x} = \frac{1}{4 + \sqrt{3}} \times \frac{4 - \sqrt{3}}{4 - \sqrt{3}}$$

$$\tan x = \frac{4 - \sqrt{3}}{16 - 3}$$

$$\tan x = \frac{4 - \sqrt{3}}{13}$$

$$\tan x = 0.1745$$

$$x = 9.9°,\ 189.9°$$

[6 marks]

2.7.3 Solutions to 2.6 Homework

Answer 1

$$\begin{aligned}
\sin(x - 30°) &= \frac{1}{2}\cos x \\
\sin x \cos 30 - \cos x \sin 30 &= \frac{1}{2}\cos x \\
\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x &= \frac{1}{2}\cos x \\
\frac{\sqrt{3}}{2}\sin x &= \cos x \\
\frac{\sin x}{\cos x} &= \frac{2}{\sqrt{3}} \\
\tan x &= \frac{2\sqrt{3}}{3} \\
x &= 1.155 \\
&= 49.1°,\ 229.1°
\end{aligned}$$

[6 marks]

Answer 2

(i) $\tan 22.5° = \sqrt{2} - 1$

[5 marks]

(ii)

$$\begin{aligned}
\cos(x - 45°) &= \cos x \\
\cos x \cos 45 + \sin x \sin 45 &= \cos x \\
\frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x &= \cos x
\end{aligned}$$

multiply through by 2

$$\begin{aligned}
\sqrt{2}\cos x + \sqrt{2}\sin x &= 2\cos x \\
\sqrt{2}\sin x &= 2\cos x - \sqrt{2}\cos x \\
\sqrt{2}\sin x &= \cos x\left(2 - \sqrt{2}\right) \\
\frac{\sin x}{\cos x} &= \frac{2 - \sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\
\tan x &= \frac{2\sqrt{2} - 2}{2} \\
&= \sqrt{2} - 1 \\
x &= 22.5°,\ 202.5°
\end{aligned}$$

[6 marks]

Answer 3

$$
\begin{aligned}
3\sin(x+10°) &= 4\cos(x-10°) \\
3\sin x\cos 10 + 3\cos x\sin 10 &= 4\cos x\cos 10 + 4\sin x\sin 10 \\
3\sin x\cos 10 - 4\sin x\sin 10 &= 4\cos x\cos 10 - 3\cos x\sin 10 \\
\sin x\,(3\cos 10 - 4\sin 10) &= \cos x\,(4\cos 10 - 3\sin 10) \\
\tan x &= \frac{4\cos 10 - 3\sin 10}{3\cos 10 - 4\sin 10} \\
&= 1.513 \\
x &= 56.5°,\ 236.5°
\end{aligned}
$$

[6 marks]

Answer 4

$$
\begin{aligned}
\text{LHS} &= \frac{\tan(C+D) - \tan C}{1 + \tan(C+D)\tan C} \\
&= \tan\big((C+D) - C\big) \\
&= \tan D \\
&= \text{RHS}
\end{aligned}
$$

[5 marks]

Answer 5

$$
\begin{aligned}
\sin(x-\alpha) &= \cos(x+\alpha) \\
\sin x\cos\alpha - \cos x\sin\alpha &= \cos x\cos\alpha - \sin x\sin\alpha \\
\sin x\cos\alpha + \sin x\sin\alpha &= \cos x\cos\alpha + \cos x\sin\alpha \\
\sin x\,(\cos\alpha + \sin\alpha) &= \cos x\,(\cos\alpha + \sin\alpha) \\
\frac{\sin x}{\cos x} &= \frac{\cos\alpha + \sin\alpha}{\cos\alpha + \sin\alpha} \\
\tan x &= 1 \qquad \square
\end{aligned}
$$

[7 marks]

Lesson 3

3.1 The Double Angle Formulae

In the A-Level examination, candidates are given The Addition Formulae but not The Double Angle Formulae. They should be memorised !

The Double Angle Formulae

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

3.2 Proof : For *sin* 2*A*

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Let $B = A$

$$\sin(A + A) = \sin A \cos A + \cos A \sin A$$

$$\sin 2A = 2\sin A \cos A \qquad \square$$

3.3 Proof : For *cos* 2*A*

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

Let $B = A$

$$\cos(A + A) = \cos A \cos A - \sin A \sin A$$

$$\cos 2A = \cos^2 A - \sin^2 A \qquad \square$$

3.4 Proof : For *tan* 2*A*

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Let $B = A$

$$\tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} \qquad \square$$

3.5 Example

Prove that, $\dfrac{\cos 2A}{1 + \sin 2A} = \dfrac{1 - \sin 2A}{\cos 2A}$

Teaching Video : http://www.NumberWonder.co.uk/v9040/3.mp4

Watch the Teaching Video, write out the proof.

☞

[5 marks]

3.6 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

By changing the 1 for $\cos^2 A + \sin^2 A$, or otherwise, prove this identity;

$$\frac{\sin 2A}{1 + \cos 2A} = \tan A$$

LHS =

[3 marks]

Question 2

By changing both 1s for $\cos^2 A + \sin^2 A$, or otherwise, prove this identity;

$$\sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}} = \tan A$$

LHS =

[3 marks]

Question 3

By use of a difference of two squares, or otherwise, prove this identity;

$$\frac{\cos 2A}{\cos A + \sin A} = \cos A - \sin A$$

LHS =

[3 marks]

Question 4

By use of a difference of two squares, or otherwise, prove this identity;

$$\cos^4 A - \sin^4 A = \cos 2A$$

LHS =

[3 marks]

Question 5

Prove that;

$$\cos 3A = 4\cos^3 A - 3\cos A$$

If you need help with this question there are several video's available as it's a “classic”. Search the internet using “$\cos 3A$”.

[6 marks]

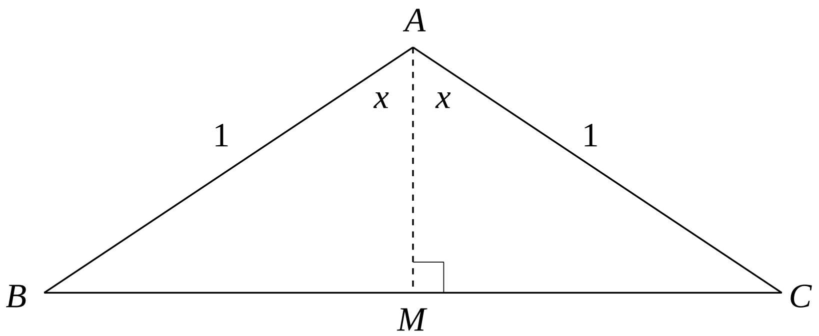
Question 6

(a) By changing the 1 for $cos^2 A + sin^2 A$, prove this identity;

$$1 - 2\,sin^2 x = cos\,2x$$

[2 marks]

(b)



ABC is an isosceles triangle

$AB = AC = 1$

M is the midpoint of *BC*

(i) Use trigonometry to find an expression, in terms of x, for BM

[1 mark]

(ii) Hence write down an expression, in terms of x, for BC

[1 mark]

(iii) Use the cosine rule to find an expression, in terms of $cos\,2x$, for BC^2

[2 marks]

(iv) Hence show that $cos\,2x = 1 - 2\,sin^2 x$

[2 marks]

Question 7

Prove that;

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

[4 marks]

3.7 Answers

3.7.1 Solution to 3.5 Example (Teaching Video)

$$
\begin{aligned}
\text{LHS} &= \frac{\cos 2A}{1 + \sin 2A} \\
&= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A + 2 \sin A \cos A} \\
&= \frac{(\cos A - \sin A)(\cos A + \sin A)}{(\cos A + \sin A)^2} \\
&= \frac{(\cos A - \sin A)(\cos A + \sin A)}{(\cos A + \sin A)\ (\cos A + \sin A)} \\
&= \frac{(\cos A - \sin A)}{(\cos A + \sin A)} \\
&= \frac{(\cos A - \sin A)}{(\cos A + \sin A)} \times \frac{(\cos A - \sin A)}{(\cos A - \sin A)} \\
&= \frac{(\cos A - \sin A)^2}{\cos^2 A - \sin^2 A} \\
&= \frac{\cos^2 A - 2 \sin A \cos A + \sin^2 A}{\cos 2A} \\
&= \frac{1 - \sin 2A}{\cos 2A} \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[6 marks]

As an end-of-lesson filler, here is another way of doing this question,

$$
\begin{aligned}
\text{LHS} &= \frac{\cos 2A}{1 + \sin 2A} \\
&= \frac{\cos 2A}{(1 + \sin 2A)} \times \frac{(1 - \sin 2A)}{(1 - \sin 2A)} \\
&= \frac{\cos 2A\ (1 - \sin 2A)}{1 - \sin^2 2A} \\
&= \frac{\cos 2A\ (1 - \sin 2A)}{\cos^2 2A + \sin^2 2A - \sin^2 2A} \\
&= \frac{\cos 2A\ (1 - \sin 2A)}{\cos^2 2A} \\
&= \frac{1 - \sin 2A}{\cos 2A} \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

3.7.2 Solutions to 3.6 Exercise

Answer 1

$$\begin{aligned}
\text{LHS} &= \frac{\sin 2A}{1 + \cos 2A} \\
&= \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A + \cos^2 A - \sin^2 A} \\
&= \frac{2 \sin A \cos A}{2 \cos^2 A} \\
&= \frac{\sin A}{\cos A} \\
&= \tan A \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 2

$$\begin{aligned}
\text{LHS} &= \sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}} \\
&= \sqrt{\frac{\cos^2 A + \sin^2 A - (\cos^2 A - \sin^2 A)}{\cos^2 A + \sin^2 A + \cos^2 A - \sin^2 A}} \\
&= \sqrt{\frac{2 \sin^2 A}{2 \cos^2 A}} \\
&= \sqrt{\tan^2 A} \\
&= \tan A \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 3

$$\begin{aligned}
\text{LHS} &= \frac{\cos 2A}{\cos A + \sin A} \\
&= \frac{\cos^2 A - \sin^2 A}{\cos A + \sin A} \\
&= \frac{(\cos A + \sin A)\,(\cos A - \sin A)}{\cos A + \sin A} \\
&= \cos A - \sin A \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 4

$$\begin{aligned}\text{LHS} &= \cos^4 A - \sin^4 A \\ &= \left(\cos^2 A\right)^2 - \left(\sin^2 A\right)^2 \\ &= \left(\cos^2 A - \sin^2 A\right)\left(\cos^2 A + \sin^2 A\right) \\ &= \left(\cos^2 A - \sin^2 A\right)(1) \\ &= \cos 2A \\ &= \text{RHS} \qquad \square\end{aligned}$$

[3 marks]

Answer 5

$$\begin{aligned}\text{LHS} &= \cos 3A \\ &= \cos(2A + A) \\ &= \cos 2A \cos A - \sin 2A \sin A \\ &= (\cos^2 A - \sin^2 A)\cos A - 2\sin A\cos A\sin A \\ &= \cos^3 A - \sin^2 A\cos A - 2\sin^2 A\cos A \\ &= \cos^3 A - 3\sin^2 A\cos A \\ &= \cos^3 A - 3\left(1 - \cos^2 A\right)\cos A \\ &= \cos^3 A - 3\cos A + 3\cos^3 A \\ &= 4\cos^3 A - 3\cos A \\ &= \text{RHS} \qquad \square\end{aligned}$$

[6 marks]

Answer 6

(a)

$$\begin{aligned}\text{LHS} &= 1 - 2\sin^2 x \\ &= \cos^2 x + \sin^2 x - 2\sin^2 x \\ &= \cos^2 x - \sin^2 x \\ &= \cos 2x \\ &= \text{RHS} \qquad \square\end{aligned}$$

[2 marks]

(b) **(i)** $\sin x = \dfrac{BM}{1} \Rightarrow BM = \sin x$

[1 mark]

(ii) $BC = 2\,BM \Rightarrow BC = 2\sin x$

[1 mark]

(iii) $BC^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos 2x$

$$BC^2 = 2 - 2\cos 2x$$

$$2\cos 2x = 2 - BC^2$$

$$\cos 2x = 1 - \frac{1}{2} BC^2$$

[2 marks]

(iv)

$$\begin{aligned} \cos 2x &= 1 - \frac{1}{2}(2\sin x)^2 \\ &= 1 - \frac{1}{2} \times 4 \times \sin^2 x \\ &= 1 - 2\sin^2 x \end{aligned}$$

[2 marks]

Answer 7

$$\begin{aligned} \text{LHS} &= \tan 3A \\ &= \tan(2A + A) \\ &= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} \\ &= \frac{\frac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2\tan A}{1 - \tan^2 A} \times \tan A} \times \frac{(1 - \tan^2 x)}{(1 - \tan^2 x)} \\ &= \frac{2\tan A + \tan A\,(1 - \tan^2 x)}{1 - \tan^2 x - 2\tan^2 A} \\ &= \frac{2\tan A + \tan A - \tan^3 A}{1 - \tan^2 x - 2\tan^2 A} \\ &= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \\ &= \text{RHS} \qquad \square \end{aligned}$$

There is a connection with Pascal's Triangle, 1 3 3 1

[4 marks]

Lesson 4

A-Level Pure Mathematics : Year 2
Trigonometric Identities

4.1 Homework

Any solution based entirely on graphical or numerical methods is not acceptable
Marks Available : 40

Question 1

Here is a table of exact trigonometric values that most mathematicians know "off by heart";

Exact values table:

	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Not Defined

Show, using the formula for $\sin(A - B)$ and the table of exact trigonometric values that

$$\sin 15° = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Remember to start, LHS =

[5 marks]

Question 2

(a) Write down the exact value of **(i)** $sin\ 45°$

(ii) $cos\ 45°$

[2 marks]

(b) On one graph sketch the curves of $y = sin\ x$ and $y = cos\ x$ for $0° \leqslant x \leqslant 360°$

[2 marks]

(c) Use the fact that $sin\ 45° = cos\ 45°$ to prove that

$$sin\ (\theta + 45°) = cos\ (\theta - 45°)$$

[5 marks]

Question 3

Prove that, $\dfrac{\cos A}{\sin B} - \dfrac{\sin A}{\cos B} = \dfrac{\cos\,(A + B)}{\sin B \cos B}$

[5 marks]

Question 4

Prove that;

$$cos\ (A + B)\ cos\ (A - B) = cos^2 A - sin^2 B$$

[5 marks]

Question 5

Prove that, $\dfrac{1 \ - \ cos\,2\theta}{sin\,2\theta} \ = \ tan\,\theta$

[5 marks]

Question 6

Prove that;

$$2\,cos^3\theta\,sin\,\theta \;+\; 2\,sin^3\theta\,cos\,\theta \;=\; sin\,2\theta$$

[5 marks]

Question 7

Prove that, $\dfrac{\sin 3\theta}{\sin \theta} - \dfrac{\cos 3\theta}{\cos \theta} = 2$

[6 marks]

4.2 Answers

Answer 1

LHS $= \sin 15°$

$= \sin(45 - 30)$

$= \sin 45 \cos 30 - \cos 45 \sin 30$

$= \dfrac{\sqrt{2}}{2} \times \dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{2}}{2} \times \dfrac{1}{2}$

$= \dfrac{\sqrt{6} - \sqrt{2}}{4}$

$=$ RHS ☐

LHS $= \sin 15$

$= \sin(60 - 45)$

$= \sin 60 \cos 45 - \cos 60 \sin 45$

$= \dfrac{\sqrt{3}}{2} \times \dfrac{\sqrt{2}}{2} - \dfrac{1}{2} \times \dfrac{\sqrt{2}}{2}$

$= \dfrac{\sqrt{6} - \sqrt{2}}{4}$

$=$ RHS ☐

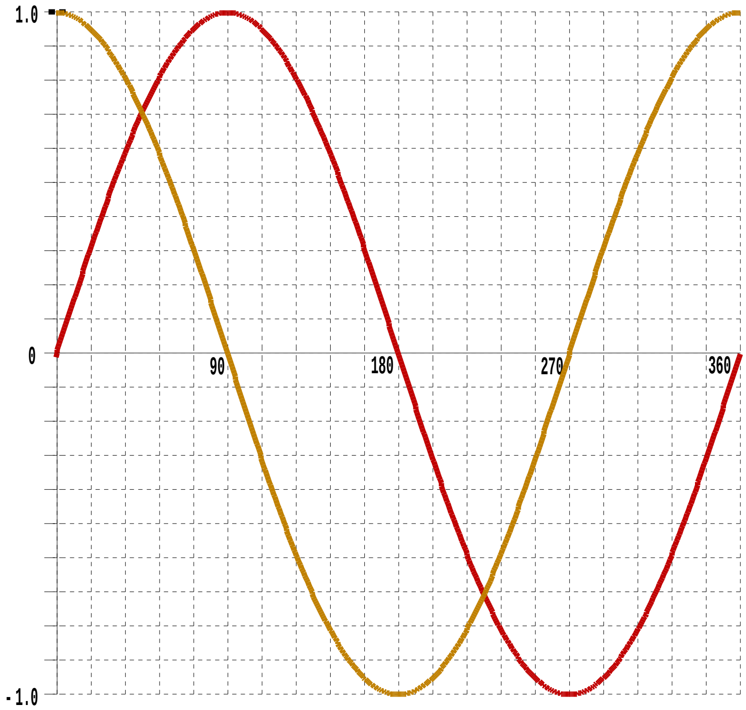
[5 marks]

Answer 2

(a) **(i)** $\sin 45° = \dfrac{\sqrt{2}}{2}$ **(ii)** $\cos 45° = \dfrac{\sqrt{2}}{2}$

[2 marks]

(b)



[2 marks]

(c)

$$
\begin{aligned}
\text{LHS} &= sin\left(\theta + 45° \right) \\
&= sin\, \theta\, cos\, 45 + cos\, \theta\, sin\, 45 \\
&= sin\, \theta\, sin\, 45 + cos\, \theta\, cos\, 45 \\
&= cos\, \theta\, cos\, 45 + sin\, \theta\, sin\, 45 \\
&= cos\left(\theta - 45 \right) \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]

Answer 3

$$
\begin{aligned}
\text{LHS} &= \frac{cos\, A}{sin\, B} - \frac{sin\, A}{cos\, B} \\
&= \frac{cos\, A}{sin\, B} \times \frac{cos\, B}{cos\, B} - \frac{sin\, A}{cos\, B} \times \frac{sin\, B}{sin\, B} \\
&= \frac{cos\, A\, cos\, B - sin\, A\, sin\, B}{sin\, B\, cos\, B} \\
&= \frac{cos\, (A + B)}{sin\, B\, cos\, B} \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]

Answer 4

$$
\begin{aligned}
\text{LHS} &= cos\left(A + B \right) cos\left(A - B \right) \\
&= \left(cos\, A\, cos\, B - sin\, A\, sin\, B \right)\left(cos\, A\, cos\, B + sin\, A\, sin\, B \right)
\end{aligned}
$$

Difference of two squares

$$
\begin{aligned}
&= cos^2 A\, cos^2 B - sin^2 A\, sin^2 B \\
&= cos^2 A \left(1 - sin^2 B \right) - \left(1 - cos^2 A \right) sin^2 B \\
&= cos^2 A - cos^2 A\, sin^2 B - sin^2 B + cos^2 A\, sin^2 B \\
&= cos^2 A - sin^2 B \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]

Answer 5

$$\begin{aligned} \text{LHS} &= \frac{1 - \cos 2\theta}{\sin 2\theta} \\ &= \frac{\cos^2\theta + \sin^2\theta - (\cos^2\theta - \sin^2\theta)}{2\sin\theta\cos\theta} \\ &= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta \\ &= \text{RHS} \qquad \square \end{aligned}$$

[5 marks]

Answer 6

$$\begin{aligned} \text{LHS} &= 2\cos^3\theta\sin\theta + 2\sin^3\theta\cos\theta \\ &= 2\sin\theta\cos\theta\,(\cos^2\theta + \sin^2\theta) \\ &= 2\sin\theta\cos\theta \\ &= \sin 2\theta \\ &= \text{RHS} \qquad \square \end{aligned}$$

[5 marks]

Answer 7

$$\begin{aligned} \text{LHS} &= \frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} \\ &= \frac{\sin 3\theta}{\sin\theta} \times \frac{\cos\theta}{\cos\theta} - \frac{\cos 3\theta}{\cos\theta} \times \frac{\sin\theta}{\sin\theta} \\ &= \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta} \\ &= \frac{\sin(3\theta - \theta)}{\sin\theta\cos\theta} \\ &= \frac{\sin 2\theta}{\sin\theta\cos\theta} \\ &= \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} \\ &= 2 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[6 marks]

5.1 The Reciprocal Trigonometric Functions

The reciprocal trig functions are;

$$sec\ \theta = \frac{1}{cos\ \theta} \qquad csc\ \theta = \frac{1}{sin\ \theta} \qquad cot\ \theta = \frac{1}{tan\ \theta}$$

A further two new identities, derived from an old favourite immediately follow;

$cos^2\ \theta + sin^2\ \theta = 1$	This is the old favourite which yields,
$1 + tan^2\theta = sec^2\ \theta$	upon dividing through the old favourite by $cos^2\ \theta$, and,
$cot^2\ \theta + 1 = csc^2\ \theta$	upon dividing through the old favourite by $sin^2\ \theta$

5.2 Example

Prove that, $\quad csc\ \theta\ (\ cos\ \theta\ cot\ \theta + sin\ \theta\ ) - 1 = cot^2\ \theta$

Teaching Video : http://www.NumberWonder.co.uk/v9040/5.mp4

After watching the Teaching Video, write out the proof,

☞

[3 marks]

5.3 Proofs Strategy

When tackling proof questions involving $sec\ \theta$, $csc\ \theta$ and $cot\ \theta$;

- If no powers are involved, change $sec\ \theta$, $csc\ \theta$ and $cot\ \theta$ into the more familiar trigonometric ratios using

$$sec\ \theta = \frac{1}{cos\ \theta}, \qquad csc\ \theta = \frac{1}{sin\ \theta} \qquad \text{and} \qquad cot\ \theta = \frac{1}{tan\ \theta}$$

- As soon as powers occur, especially a square, try to make use of any of,

$$cos^2\ \theta + sin^2\ \theta = 1$$

$$1 + tan^2\theta = sec^2\ \theta$$

$$cot^2\ \theta + 1 = csc^2\ \theta$$

- It follows from the fact that $tan\ \theta = \dfrac{sin\ \theta}{cos\ \theta}$ that $cot\ \theta = \dfrac{cos\ \theta}{sin\ \theta}$

5.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Prove that, $sin\ \theta\ (\ csc\ \theta - sin\ \theta\ ) + cos\ \theta\ (\ sec\ \theta - cos\ \theta\ ) = 1$

[3 marks]

Question 2

Prove that, $$\frac{\sin\theta\ (\sin\theta - \csc\theta)}{\cos\theta\ (\cos\theta - \sec\theta)} = \cot^2\theta$$

[3 marks]

Question 3

Prove that, $$\frac{(1 - \sin\theta)\ (1 + \sin\theta)}{(1 - \cos\theta)\ (1 + \cos\theta)} = \cot^2\theta$$

[3 marks]

Question 4

Prove that, $sec^2\theta - tan^2\theta + csc^2\theta - cot^2\theta = 2$

[3 marks]

Question 5

Prove that, $sec\,\theta\ (cos\,\theta + sin\,\theta\, tan\,\theta) = sec^2\theta$

[3 marks]

Question 6

Prove that, $$\frac{1}{csc\,\theta\ (\ cos\,\theta\ cot\,\theta\ +\ sin\,\theta\ )}\ =\ sin^2\,\theta$$

[**3 marks]**

Question 7

Prove that, $$\frac{1}{(\ tan^2\,\theta\ +\ 1\ )}\ +\ \frac{1}{(\ cot^2\,\theta\ +\ 1\ )}\ =\ 1$$

[3 marks]

Question 8

Prove that, $(sec^2\theta - 1)(csc^2\theta - 1) = 1$

[3 marks]

Question 9

Prove that, $(sec\,\theta + tan\,\theta)(sec\,\theta - tan\,\theta) = 1$

[3 marks]

Question 10

Prove that, $$\frac{\cos\theta}{\sqrt{1 + \tan^2\theta}} + \frac{\sin\theta}{\sqrt{1 + \cot^2\theta}} = 1$$

[3 marks]

Question 11

Prove this identity, $\cot\theta - \tan\theta = 2\cot 2\theta$

[5 marks]

Question 12

Prove this identity, $\cot\theta + \tan\theta = 2\csc 2\theta$

[5 marks]

5.5 Answers

5.5.1 Solution to 5.2 Example (Teaching Video)

$$\begin{aligned}
\text{LHS} &= csc\,\theta \left(cos\,\theta\, cot\,\theta + sin\,\theta \right) - 1 \\
&= \frac{1}{sin\,\theta} \left(cos\,\theta \times \frac{cos\,\theta}{sin\,\theta} + sin\,\theta \right) - 1 \\
&= \frac{cos^2\,\theta}{sin^2\,\theta} + 1 - 1 \\
&= \left(\frac{cos\,\theta}{sin\,\theta} \right)^2 \\
&= cot^2\,\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

5.5.2 Solutions to 5.4 Exercise

Answer 1

$$\begin{aligned}
\text{LHS} &= sin\,\theta\ (csc\,\theta - sin\,\theta) + cos\,\theta\ (sec\,\theta - cos\,\theta) \\
&= sin\,\theta \left(\frac{1}{sin\,\theta} - sin\,\theta \right) + cos\,\theta \left(\frac{1}{cos\,\theta} - cos\,\theta \right) \\
&= 1 - sin^2\,\theta + 1 - cos^2\,\theta \\
&= 1 + 1 - \left(sin^2\,\theta + cos^2\,\theta \right) \\
&= 1 + 1 - 1 \\
&= 1 \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 2

$$\begin{aligned}
\text{LHS} &= \frac{\sin\theta\ (\sin\theta - \csc\theta)}{\cos\theta\ (\cos\theta - \sec\theta)} \\
&= \frac{\sin\theta\ (\sin\theta - \frac{1}{\sin\theta})}{\cos\theta\ (\cos\theta - \frac{1}{\cos\theta})} \\
&= \frac{\sin^2\theta - 1}{\cos^2\theta - 1} \\
&= \frac{\sin^2\theta - \cos^2\theta - \sin^2\theta}{\cos^2\theta - \cos^2\theta - \sin^2\theta} \\
&= \frac{-\cos^2\theta}{-\sin^2\theta} \\
&= \cot^2\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 3

$$\begin{aligned}
\text{LHS} &= \frac{(1 - \sin\theta)\ (1 + \sin\theta)}{(1 - \cos\theta)\ (1 + \cos\theta)} \\
&= \frac{1 - \sin^2\theta}{1 - \cos^2\theta} \\
&= \frac{\cos^2\theta + \sin^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta - \cos^2\theta} \\
&= \frac{\cos^2\theta}{\sin^2\theta} \\
&= \cot^2\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 4

$$\begin{aligned}
\text{LHS} &= \sec^2\theta - \tan^2\theta + \csc^2\theta - \cot^2\theta \\
&= 1 + \tan^2\theta - \tan^2\theta + \cot^2\theta + 1 - \cot^2\theta \\
&= 1 + 1 \\
&= 2 \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 5

$$\begin{aligned}\text{LHS} &= sec\,\theta\ (\,cos\,\theta + sin\,\theta\,tan\,\theta\,) \\ &= \frac{1}{cos\,\theta}\left(cos\,\theta + sin\,\theta \times \frac{sin\,\theta}{cos\,\theta}\right) \\ &= 1 + \frac{sin^2\,\theta}{cos^2\,\theta} \\ &= 1 + tan^2\,\theta \\ &= sec^2\,\theta \\ &= \text{RHS} \qquad \square\end{aligned}$$

[3 marks]

Answer 6

$$\begin{aligned}\text{LHS} &= \frac{1}{csc\,\theta\ (\,cos\,\theta\,cot\,\theta + sin\,\theta\,)} \\ &= \frac{1}{\frac{1}{sin\,\theta}\left(cos\,\theta \times \frac{cos\,\theta}{sin\,\theta} + sin\,\theta\right)} \\ &= \frac{1}{\frac{cos^2\,\theta}{sin^2\,\theta} + 1} \\ &= \frac{1}{cot^2\,\theta + 1} \\ &= \frac{1}{csc^2\,\theta} \\ &= sin^2\,\theta \\ &= \text{RHS} \qquad \square\end{aligned}$$

[3 marks]

Answer 7

$$\begin{aligned}\text{LHS} &= \frac{1}{(\,tan^2\,\theta + 1\,)} + \frac{1}{(\,cot^2\,\theta + 1\,)} \\ &= \frac{1}{sec^2\,\theta} + \frac{1}{csc^2\,\theta} \\ &= cos^2\,\theta + sin^2\,\theta \\ &= 1 \\ &= \text{RHS} \qquad \square\end{aligned}$$

[3 marks]

Answer 8

$$\begin{aligned}\text{LHS} &= (sec^2\,\theta - 1)\ (csc^2\,\theta - 1)\\ &= (1 + tan^2\,\theta - 1)\ (cot^2\,\theta + 1 - 1)\\ &= tan^2\,\theta\, cot^2\,\theta\\ &= tan^2\,\theta \times \frac{1}{tan^2\,\theta}\\ &= 1\\ &= \text{RHS} \qquad \square\end{aligned}$$

[3 marks]

Answer 9

$$\begin{aligned}\text{LHS} &= (sec\,\theta + tan\,\theta)\ (sec\,\theta - tan\,\theta)\\ &= sec^2\,\theta - tan^2\,\theta\\ &= 1 + tan^2\,\theta - tan^2\,\theta\\ &= 1\\ &= \text{RHS} \qquad \square\end{aligned}$$

[3 marks]

Answer 10

$$\begin{aligned}\text{LHS} &= \frac{cos\,\theta}{\sqrt{1 + tan^2\,\theta}} + \frac{sin\,\theta}{\sqrt{1 + cot^2\,\theta}}\\ &= \frac{cos\,\theta}{\sqrt{sec^2\,\theta}} + \frac{sin\,\theta}{\sqrt{csc^2\,\theta}}\\ &= \frac{cos\,\theta}{sec\,\theta} + \frac{sin\,\theta}{csc\,\theta}\\ &= cos^2\,\theta + sin^2\,\theta\\ &= 1\\ &= \text{RHS} \qquad \square\end{aligned}$$

[3 marks]

Answer 11

$$\begin{aligned}
\text{LHS} &= \cot\theta - \tan\theta \\
&= \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} \\
&= \frac{\cos\theta}{\sin\theta} \times \frac{\cos\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta} \times \frac{\sin\theta}{\sin\theta} \\
&= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} \times \frac{2}{2} \\
&= \frac{2\left(\cos^2\theta - \sin^2\theta\right)}{2\sin\theta\cos\theta} \\
&= \frac{2\cos 2\theta}{\sin 2\theta} \\
&= 2\cot 2\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[5 marks]

Answer 12

$$\begin{aligned}
\text{LHS} &= \cot\theta + \tan\theta \\
&= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \\
&= \frac{\cos\theta}{\sin\theta} \times \frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta} \times \frac{\sin\theta}{\sin\theta} \\
&= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} \times \frac{2}{2} \\
&= \frac{2\left(\cos^2\theta + \sin^2\theta\right)}{2\sin\theta\cos\theta} \\
&= \frac{2}{\sin 2\theta} \\
&= 2\csc 2\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[5 marks]

6.1 Examination Questions

A typical examination question will revolve around the solving of a trigonometric equation that is in essence a quadratic equation. In order to obtain the quadratic it may well be required to use one of the equations,

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

6.2 Example

C3 Examination question from January 2012, Q5.

Solve, for $0 \leqslant \theta \leqslant 180°$, $\quad 2\cot^2 3\theta = 7\,\text{cosec}\,3\theta - 5$

Give your answers in degrees to 1 decimal place.

Teaching Video : http://www.NumberWonder.co.uk/v9040/6a.mp4

Watch the video from “Exam Solutions”
Write out a solution to the question.

☞

[10 marks]

6.3 Exercise
Each question comes with Video Support from “Exam Solutions”
You are not expected to watch these videos.
They are there for when you need a helping hand.

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 55

Question 1
C3 Examination question from January 2011, Q3
Find all the solutions of $2\,cos\,2\theta = 1 - 2\,sin\,\theta$ in the interval $0 \leqslant \theta \leqslant 360°$

[6 marks]

Need help with Question 1 ?
http://www.NumberWonder.co.uk/v9040/6b.mp4

Question 2

C3 Examination question from January 2010, Q8

Solve, $cosec^2 2x - cot\, 2x = 1$ for $0 \leqslant x \leqslant 180°$

[7 marks]

Need help with Question 2 ?

http://www.NumberWonder.co.uk/v9040/6l.mp4

Question 3

C3 Examination question from June 2011, Q6

(a) Prove that, $\dfrac{1}{\sin 2\theta} - \dfrac{\cos 2\theta}{\sin 2\theta} = \tan\theta, \quad \theta \neq 90n°, \ n \in \mathbb{Z}$

[4 marks]

(b) Hence, or otherwise

(i) show that $\tan 15° = 2 - \sqrt{3}$

[3 marks]

(ii) solve, for $0 < x < 360°$, $\quad \operatorname{cosec} 4\theta - \cot 4\theta = 1$

[5 marks]

Need help with Question 3 ?
http://www.NumberWonder.co.uk/v9040/6c.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9040/6d.mp4 (Part 2)
http://www.NumberWonder.co.uk/v9040/6e.mp4 (Part 3)

Part 1 Part 2 Part 3

Question 4

C3 Examination question from June 2006, Q6

(a) Using $\sin^2\theta + \cos^2\theta \equiv 1$, show that $\mathrm{cosec}^2\theta - \cot^2\theta \equiv 1$

[2 marks]

(b) Hence, or otherwise, prove that

$$\mathrm{cosec}^4\theta - \cot^4\theta \equiv \mathrm{cosec}^2\theta + \cot^2\theta$$

[2 marks]

(c) Solve, for $90° \leqslant \theta < 180°$,

$$\mathrm{cosec}^4\theta - \cot^4\theta = 2 - \cot\theta$$

[6 marks]

Need help with Question 4 ?
http://www.NumberWonder.co.uk/v9040/6f.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9040/6g.mp4 (Part 2)

Part 1 Part 2

Question 5

C3 Examination question from June 2008, Q5

(a) Given that $sin^2\,\theta + cos^2\,\theta \equiv 1$, show that $1 + cot^2\,\theta \equiv cosec^2\,\theta$

[2 marks]

(b) Solve, for $0 \leqslant \theta < 180°$, the equation

$$2\,cot^2\,\theta \;-\; 9\,cosec\,\theta \;=\; 3$$

giving your answers to 1 decimal place.

[6 marks]

Need help with Question 5 ?

http://www.NumberWonder.co.uk/v9040/6h.mp4 (Part 1)

http://www.NumberWonder.co.uk/v9040/6i.mp4 (Part 2)

Part 1 Part 2

Question 6

C3 Examination question from June 2010, Q1

(a) Show that

$$\frac{\sin 2\theta}{1 + \cos 2\theta} = \tan \theta$$

[2 marks]

(b) Hence find, for $-180° \leqslant \theta \leqslant 180°$, all the solutions of

$$\frac{2 \sin 2\theta}{1 + \cos 2\theta} = 1$$

Give your answers to 1 decimal place.

[3 marks]

Need help with Question 6 ?

http://www.NumberWonder.co.uk/v9040/6j.mp4 (Part 1)

http://www.NumberWonder.co.uk/v9040/6k.mp4 (Part 2)

Part 1 Part 2

Question 7

C3 Examination question from January 2007, Q1

(a) By writing $\sin 3\theta$ as $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

[5 marks]

(b) Given that $\sin\theta = \frac{\sqrt{3}}{4}$ find the exact value of $\sin 3\theta$

[2 marks]

Need help with Question 7 ?

http://www.NumberWonder.co.uk/v9040/6m.mp4 (Part 1)

http://www.NumberWonder.co.uk/v9040/6n.mp4 (Part 2)

Part 1 Part 2

6.4 Homework

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 32

Question 1

A-Level Paper 1 Examination question from June 2018, Q8

The depth of water, D metres, in a harbour on a particular day is modelled by

$$D = 5 + 2\sin(30t)° \qquad 0 \leqslant t \leqslant 24$$

where t is the number of hours after midnight.

A boat enters the harbour at 6.30 am and it takes 2 hours to load its cargo. The boat requires the depth of water to be at least 3.8 metres before it can leave the harbour.

(a) Find the depth of the water in the harbour when the boat enters the harbour.

[1 mark]

(b) Find, to the nearest minute, the earliest time the boat can leave the harbour.

[4 marks]

Question 2

A-Level Paper 2 Examination question from June 2018, Q12 (a) edited

Prove that

$$1 - \cos 2\theta \equiv \tan\theta \sin 2\theta \qquad \theta \neq \frac{(2n+1)\ 180°}{2},\ n \in \mathbb{Z}$$

[3 marks]

Question 3

A-Level C3 Examination question from June 2017, Q9

(a) Prove that

$$\sin 2x - \tan x \equiv \tan x \cos 2x \qquad x \neq (2n+1)90°, \; n \in \mathbb{Z}$$

[4 marks]

(b) Given that $x \neq 90°$ and $x \neq 270°$ solve for $0 \leqslant x < 360°$

$$\sin 2x - \tan x = 3 \tan x \sin x$$

Give your answers in degrees to one decimal place where appropriate

[5 marks]

Question 4

A-Level Paper 1 Examination question from June 2019, Q6

(a) Solve, for $-180° \leqslant \theta \leqslant 180°$, the equation

$$5 \sin 2\theta = 9 \tan \theta$$

giving your answers, where necessary, to one decimal place.

[6 marks]

(b) Deduce the smallest positive solution to the equation

$$5 \sin(2x - 50°) = 9 \tan(x - 25°)$$

[2 marks]

Question 5

A-Level Paper 2 Examination question from June 2019, Q12

(a) Prove, $\dfrac{\cos 3\theta}{\sin \theta} + \dfrac{\sin 3\theta}{\cos \theta} \equiv 2 \cot 2\theta \qquad \theta \neq (90n)^\circ,\ n \in \mathbb{Z}$

[4 marks]

(b) Hence solve, for $90^\circ < \theta < 180^\circ$, the equation

$$\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = 4$$

giving any solutions to one decimal place

[3 marks]

6.5 Answers

6.5.1 Solution to 6.2 Example (Teaching Video)

$$
\begin{aligned}
2\,cot^2 3\theta &= 7\,csc\,3\theta - 5 \\
2\left(csc^2 3\theta - 1\right) &= 7\,csc\,3\theta - 5 \\
2\,csc^2 3\theta - 2 - 7\,csc\,3\theta + 5 &= 0 \\
2\,csc^2 3\theta - 7\,csc\,3\theta + 3 &= 0 \\
\left(2\,csc\,3\theta - 1\right)\left(csc\,3\theta - 3\right) &= 0
\end{aligned}
$$

Either $2\,csc\,3\theta - 1 = 0$ or $csc\,3\theta - 3 = 0$

$$csc\,3\theta = \frac{1}{2} \qquad\qquad csc\,3\theta = 3$$

$$\frac{1}{sin\,3\theta} = \frac{1}{2} \qquad\qquad \frac{1}{sin\,3\theta} = \frac{3}{1}$$

$$sin\,3\theta = 2 \qquad\qquad sin\,3\theta = \frac{1}{3}$$

No solutions $\qquad 3\theta = 19.5,\ 160.5,\ 379.5,\ 520.5$

$$\theta = 6.5°,\ 53.5°,\ 126.5°,\ 173.5°$$

[10 marks]

6.5.2 Solutions to 6.3 Exercise

Answer 1

$$
\begin{aligned}
2\,cos\,2\theta &= 1 - 2\,sin\,\theta \\
2\left(cos^2\theta - sin^2\theta\right) &= 1 - 2\,sin\,\theta \\
2\left(1 - sin^2\theta - sin^2\theta\right) &= 1 - 2\,sin\,\theta \\
2 - 4\,sin^2\theta &= 1 - 2\,sin\,\theta \\
4\,sin^2\theta - 2\,sin\,\theta - 1 &= 0 \\
sin\,\theta &= \frac{2 \pm \sqrt{(-2)^2 - 4 \times 4 \times (-1)}}{2 \times 4} \\
&= \frac{2 \pm \sqrt{20}}{8} \\
&= \frac{1 \pm \sqrt{5}}{4} \\
\theta &= 54°,\ 126°,\ 198°,\ 342°
\end{aligned}
$$

[6 marks]

Answer 2

$$csc^2\, 2x - cot\, 2x = 1$$

$$cot^2\, 2x + 1 - cot\, 2x = 1$$

$$cot^2\, 2x - cot\, 2x = 0$$

$$cot\, 2x\ (\, cot\, 2x - 1\,) = 0$$

Either $cot\, 2x = 0$ or $cot\, 2x = 1$

$cot\, 2x = 0$ when $tan\, 2x$ is undefined; $2x = 90,\ 270$

$$x = 45,\ 135$$

$cot\, 2x = 1$ when $\dfrac{1}{tan\, 2x} = 1,\ \therefore$ when $tan\, 2x = 1;\ 2x = 45,\ 225$

$$x = 22.5,\ 112.5$$

Solutions are $22.5°,\ 45°,\ 112.5°,\ 135°$

[7 marks]

Answer 3

(a)

$$\text{LHS} = \frac{1}{sin\, 2\theta} - \frac{cos\, 2\theta}{sin\, 2\theta}$$

$$= \frac{(\, cos^2\theta + sin^2\theta\,) - (\, cos^2\theta - sin^2\theta\,)}{2\, sin\,\theta\, cos\,\theta}$$

$$= \frac{2\, sin^2\theta}{2\, sin\,\theta\, cos\,\theta}$$

$$= \frac{sin\,\theta}{cos\,\theta}$$

$$= tan\,\theta$$

$$= \text{RHS} \qquad \square$$

[4 marks]

(b) **(i)**

$$tan\, 15° = \frac{1}{sin\, 30°} - \frac{cos\, 30°}{sin\, 30°}$$

$$= 2 - \sqrt{3}$$

[3 marks]

(ii)

$$cosec\, 4\theta - cot\, 4\theta = 1$$

$$\frac{1}{sin\, 4\theta} - \frac{cos\, 4\theta}{sin\, 4\theta} = 1$$

$$tan\, 2\theta = 1$$

$$2\theta = 45,\ 225,\ 405,\ 585$$

$$\theta = 22.5°,\ 112.5°,\ 202.5°,\ 292.5°$$

[5 marks]

Answer 4

(a) $$\sin^2\theta + \cos^2\theta = 1$$

Divide through by $\sin^2\theta$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\therefore\ \csc^2\theta - \cot^2\theta = 1 \qquad \square$$

[2 marks]

(b) $$\text{LHS} = \csc^4\theta - \cot^4\theta$$

$$= (\csc^2\theta)^2 - (\cot^2\theta)^2$$

$$= (\csc^2\theta + \cot^2\theta)(\csc^2\theta - \cot^2\theta)$$

$$= (\csc^2\theta + \cot^2\theta) \times 1$$

$$= \csc^2\theta + \cot^2\theta$$

$$= \text{RHS} \qquad \square$$

[2 marks]

(c) $$\mathrm{cosec}^4\theta - \cot^4\theta = 2 - \cot\theta$$

$$\csc^2\theta + \cot^2\theta = 2 - \cot\theta$$

$$\cot^2\theta + 1 + \cot^2\theta = 2 - \cot\theta$$

$$2\cot^2\theta + \cot\theta - 1 = 0$$

$$(2\cot\theta - 1)(\cot\theta + 1) = 0$$

Either $\cot\theta = \dfrac{1}{2}$ or $\cot\theta = -1$

$\tan\theta = 2$ $\qquad$ $\tan\theta = -1$

$$\theta = 135°$$

[6 marks]

Answer 5

(a)

$$\sin^2\theta + \cos^2\theta = 1$$

Divide through by $\sin^2\theta$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \csc^2\theta \qquad \square$$

[2 marks]

(b)

$$2\cot^2\theta - 9\csc\theta = 3$$

$$2\left(\csc^2\theta - 1\right) - 9\csc\theta = 3$$

$$2\csc^2\theta - 9\csc\theta - 5 = 0$$

$$\left(2\csc\theta + 1\right)\left(\csc\theta - 5\right) = 0$$

Either $\csc\theta = -\frac{1}{2}$ or $\csc\theta = 5$

$\sin\theta = -2$ $\qquad$ $\sin\theta = \frac{1}{5}$

No solutions $\qquad$ $\theta = 11.5°,\ 168.5°$

[6 marks]

Answer 6

(a)

$$\text{LHS} = \frac{\sin 2\theta}{1 + \cos 2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\cos^2\theta + \sin^2\theta + \cos^2\theta - \sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta$$

$$= \text{RHS} \qquad \square$$

[2 marks]

(b)

$$\frac{2\sin 2\theta}{1 + \cos 2\theta} = 1$$

$$\tan\theta = \frac{1}{2}$$

$$\theta = -153.4°,\ 26.6°$$

[3 marks]

Answer 7

(a)

$$\begin{aligned}
\text{LHS} &= \sin 3\theta \\
&= \sin\,(\,2\theta + \theta\,) \\
&= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \\
&= 2\sin\theta\cos\theta\cos\theta + \left(\cos^2\theta - \sin^2\theta\right)\sin\theta \\
&= 2\sin\theta\cos^2\theta + \cos^2\theta\sin\theta - \sin^3\theta \\
&= 3\sin\theta\cos^2\theta - \sin^3\theta \\
&= 3\sin\theta\left(1 - \sin^2\theta\right) - \sin^3\theta \\
&= 3\sin\theta - 3\sin^3\theta - \sin^3\theta \\
&= 3\sin\theta - 4\sin^3\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[5 marks]

(b)

$$\begin{aligned}
\sin 3\theta &= 3\left(\frac{\sqrt{3}}{4}\right) - 4\left(\frac{\sqrt{3}}{4}\right)^3 \\
&= \frac{3\sqrt{3}}{4} - \frac{4\times 3\sqrt{3}}{4\times 16} \\
&= \frac{12\sqrt{3}}{16} - \frac{3\sqrt{3}}{16} \\
&= \frac{9\sqrt{3}}{16}
\end{aligned}$$

[2 marks]

6.5.3 Solutions to 6.4 Homework

Answer 1

(a) At 6.30 am, $D = 5 + 2\,sin\,(30 \times 6.5)$

$$= 4.5 \text{ metres}$$

[1 mark]

(b) Solve, $5 + 2\,sin\,(30t) = 3.8$

$$2\,sin\,(30t) = -1.2$$

$$sin\,(30t) = -0.6$$

$$30t = 216.9,\ \ 323.1 + 360n$$

$$t = 7.23,\ \ 10.77$$

The boat is not ready to leave at 7.14 am when the tide falls below the depth needed and so has to wait until 10.46 am on the rising tide.

[4 marks]

Answer 2

$$\begin{aligned}
\text{LHS} &= 1 - cos\,2\theta \\
&= \left(cos^2\theta + sin^2\theta \right) - \left(cos^2\theta - sin^2\theta \right) \\
&= 2\,sin^2\theta \\
&= 2\,sin^2\theta \times \frac{cos\,\theta}{cos\,\theta} \\
&= \frac{sin\,\theta}{cos\,\theta} \times 2\,sin\,\theta\,cos\,\theta \\
&= tan\,\theta\,sin\,2\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 3

(a)

$$
\begin{aligned}
\text{LHS} &= \sin 2x - \tan x \\
&= \tan x \left(\sin 2x \times \frac{1}{\tan x} - 1 \right) \\
&= \tan x \left(2 \sin x \cos x \times \frac{\cos x}{\sin x} - 1 \right) \\
&= \tan x \left(2 \cos^2 x - 1 \right) \\
&= \tan x \left(2 \cos^2 x - \left(\cos^2 x + \sin^2 x \right) \right) \\
&= \tan x \left(\cos^2 x - \sin^2 x \right) \\
&= \tan x \cos 2x \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[4 marks]

(b)

$$
\begin{aligned}
\sin 2x - \tan x &= 3 \tan x \sin x \\
\tan x \cos 2x &= 3 \tan x \sin x \\
\tan x \cos 2x - 3 \tan x \sin x &= 0 \\
\tan x \left(\cos 2x - 3 \sin x \right) &= 0
\end{aligned}
$$

Either $\tan x = 0$ or $\cos 2x - 3 \sin x = 0$

No solutions as $x = 0°, \; 180°$ or $\cos^2 x - \sin^2 x - 3 \sin x = 0$

$$
\begin{aligned}
1 - 2 \sin^2 x - 3 \sin x &= 0 \\
2 \sin^2 x + 3 \sin x - 1 &= 0 \\
\sin x &= \frac{-3 \pm \sqrt{3^2 - 4 \times 2 \times (-1)}}{2 \times 2} \\
&= \frac{-3 \pm \sqrt{17}}{4} \\
&= -1.7808, \quad 0.2808
\end{aligned}
$$

$$x = 16.3°, \; 163.7°$$

Solutions are, $0°, \; 16.3°, \; 163.7°, \; 180°$

[5 marks]

Answer 4

(a)

$$5\,sin\,2\theta = 9\,tan\,\theta$$

$$10\,sin\,\theta\,cos\,\theta - 9 \times \frac{sin\,\theta}{cos\,\theta} = 0$$

Multiply through by $cos\,\theta$

$$10\,sin\,\theta\,cos^2\,\theta - 9\,sin\,\theta = 0$$

$$sin\,\theta\,(10\,cos^2\,\theta - 9) = 0$$

Either $sin\,\theta = 0$ or $10\,cos^2\,\theta - 9 = 0$

$$\theta = -180,\ 0,\ 180 \quad \text{or} \quad cos^2\,\theta = \frac{9}{10}$$

$$cos\,\theta = \pm\sqrt{\frac{9}{10}}$$

$$= \pm\frac{3\sqrt{10}}{10}$$

$$= \pm 0.9487$$

$$\theta = \pm 18.4$$

Solutions, $-180°, -18.4°,\ 0°,\ 18.4°,\ 180°$

[6 marks]

(b)

$$5\,sin\,(2x - 50°) = 9\,tan\,(x - 25°)$$

$$\Rightarrow 5\,sin\,2\,(x - 25°) = 9\,tan\,(x - 25°)$$

$\therefore$ Smallest positive solution is, $x - 25 = -18.4$

$$x = 6.6°$$

[2 marks]

Answer 5

(a)

$$\begin{aligned} \text{LHS} &= \frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} \\ &= \frac{\cos 3\theta \cos\theta + \sin 3\theta \sin\theta}{\sin\theta\cos\theta} \times \frac{2}{2} \\ &= \frac{\cos(3\theta - \theta)}{2\sin\theta\cos\theta} \times 2 \\ &= \frac{\cos 2\theta}{\sin 2\theta} \times 2 \\ &= 2\cot 2\theta \\ &= \text{RHS} \qquad \square \end{aligned}$$

[4 marks]

(b)

$$\begin{aligned} \frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} &= 4 \\ 2\cot 2\theta &= 4 \\ \cot 2\theta &= 2 \\ \tan 2\theta &= \frac{1}{2} \\ 2\theta &= 26.6,\ 206.6,\ 386.6,\ \ldots \\ \theta &= 103.3° \end{aligned}$$

[3 marks]

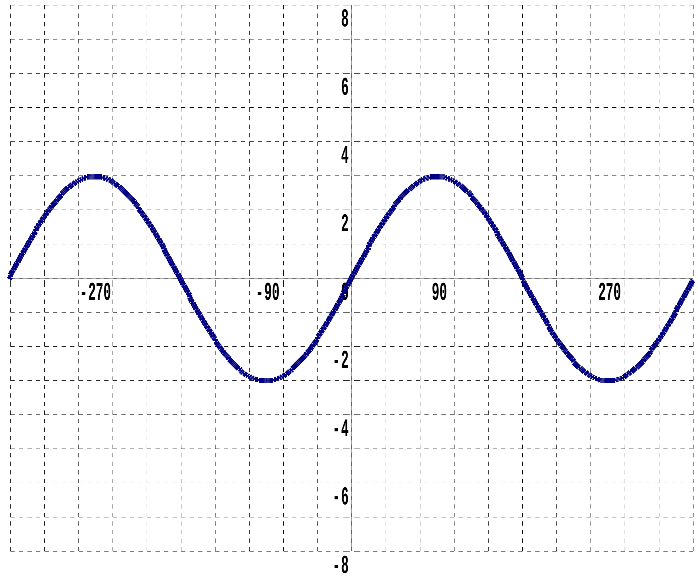
Lesson 7

7.1 Addition of Trigonometric Waveforms

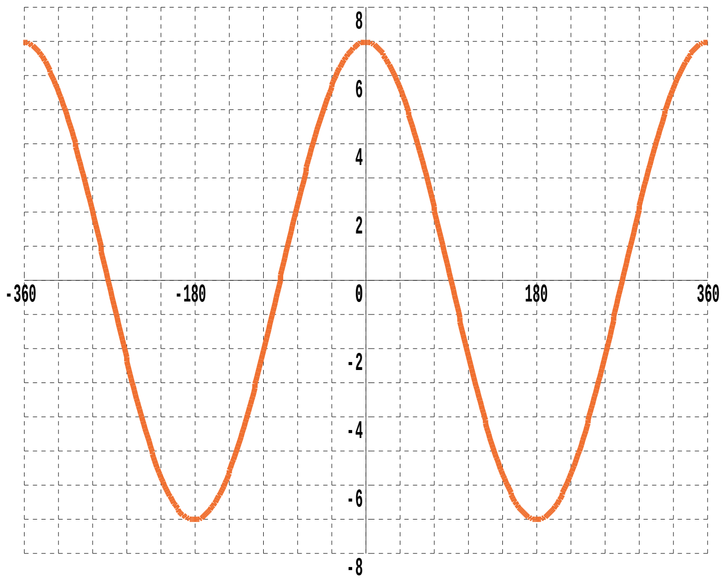
A surprising omission in the trigonometric expressions considered thus far are simple sums of *sine* and *cosine* such as, for example,

$$y = 3\sin\theta + 7\cos\theta$$

What makes this tricky to get a grip of is that there are no squares of trigonometric functions to manipulate. As a fallback strategy, graphs can be considered.
The obvious question to ask is "how complicated is the resulting waveform" ?

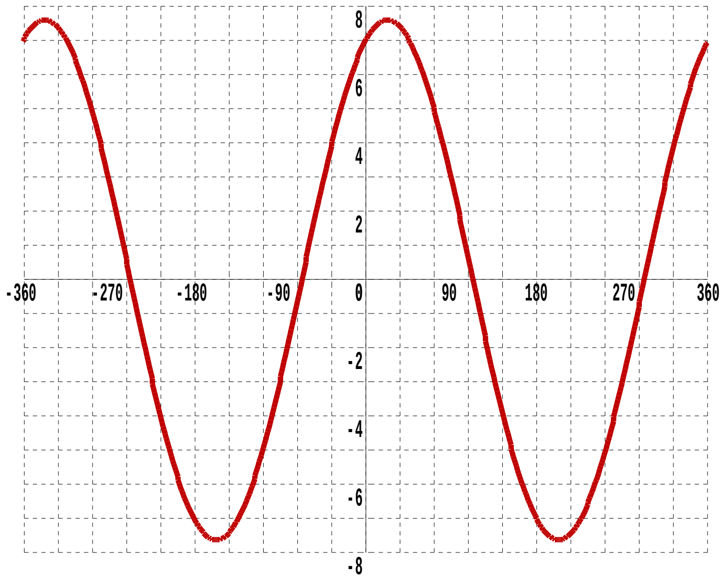


$$y = 3\sin\theta$$



$$y = 7\cos\theta$$

The resulting waveform for $y = 3\sin\theta + 7\cos\theta$ is surprisingly simple !



The waveform for $y = 3\,sin\,\theta + 7\,cos\,\theta$ is not complicated at all !
It's a *sine* wave moved left about 65° and height between 7 and 8.

Knowing that the resulting wave is a *sine* wave is the key to getting an exact answer because the form of the answer has to be,

$$R\,sin\,(\,\theta + \alpha\,)$$

where α is the shift left, and R is the height or *amplitude*.

Question : Express $3\,sin\,\theta + 7\,cos\,\theta$ in the form $R\,sin\,(\,\theta + \alpha\,)$ for $0 < \alpha < 90°$

Answer:

$$R\,sin\,(\,\theta + \alpha\,) = R\,sin\,\theta\,cos\,\alpha + R\,cos\,\theta\,sin\,\alpha$$

which is required to be $3\,sin\,\theta + 7\,cos\,\theta$

$$\therefore R\,cos\,\alpha = 3 \quad \text{and} \quad R\,sin\,\alpha = 7$$

Solving these two equations simultaneously by division,

$$\frac{R\,sin\,\alpha}{R\,cos\,\alpha} = \frac{7}{3}$$

$$\alpha = arctan\left(\frac{7}{3}\right)$$

$$\alpha = 66.8°$$

For reasons to be explained shortly,

$$R = \sqrt{7^2 + 3^2}$$

$$= 7.62$$

Conclusion :

$$3\,sin\,\theta + 7\,cos\,\theta = 7.62\,sin\,(\,\theta + 66.8°\,)$$

7.2 Why applying Pythagoras' theorem gives the value of *R*

$$\begin{aligned}\sqrt{(R\sin\alpha)^2 + (R\cos\alpha)^2} &= \sqrt{R^2\sin^2\alpha + R^2\cos^2\alpha}\\ &= \sqrt{R^2(\sin^2\alpha + \cos^2\alpha)}\\ &= \sqrt{R^2} \qquad \textit{because} \quad \cos^2\alpha + \sin^2\alpha = 1\\ &= R\end{aligned}$$

7.3 Example

Write $8\sin\theta + 15\cos\theta$ in the form $R\sin(\theta + \alpha)$ for $0 < \alpha < 90°$

Teaching Video : http://www.NumberWonder.co.uk/v9040/7.mp4

After watching the Teaching Video, write out a solution in the space below,

☞

[4 marks]

7.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

(i) Write $2\,sin\,\theta + \sqrt{5}\,cos\,\theta$ in the form $R\,sin\,(\,\theta + \alpha\,)$ for $0 < \alpha < 90°$

[4 marks]

(ii) Keeping in mind that the maximum that $sin\,(\,\theta + \alpha\,)$ can be, regardless of α, is 1, what is the maximum value of $2\,sin\,\theta + \sqrt{5}\,cos\,\theta$?

[1 mark]

(iii) Keeping in mind that the minimum that $sin\,(\,\theta + \alpha\,)$ can be, regardless of α, is – 1, what is the minimum value of $2\,sin\,\theta + \sqrt{5}\,cos\,\theta$?

[1 mark]

(iv) Solve the equation $2\,sin\,\theta + \sqrt{5}\,cos\,\theta = 1.5$
Give both solutions that are in the interval $0° < \theta < 360°$

[4 marks]

Question 2

(i) Write $\sqrt{3}\ sin\,\theta\ +\ cos\,\theta$ in the form $R\,sin\,(\ \theta\ +\ \alpha\ )$ for $0\ <\ \alpha\ <\ 90°$

[4 marks]

(ii) What is the minimum value of $\sqrt{3}\ sin\,\theta\ +\ cos\,\theta$?

[1 mark]

(iii) What is the maximum value of $3\,sin\,\theta\ +\ \sqrt{3}\ cos\,\theta$?

[1 mark]

(iv) Solve the equation $\sqrt{3}\ sin\,\theta\ +\ cos\,\theta\ =\ \sqrt{2}$

Give both solutions that are in the interval $0° < \theta < 360°$

[4 marks]

Question 3

(i) Write $sin\,\theta + cos\,\theta$ in the form $R\,sin(\theta + \alpha)$ for $0 < \alpha < 90°$

[3 marks]

(ii) What is the minimum value of $sin\,\theta + cos\,\theta$?

[1 mark]

(iii) What is the exact maximum value of $\dfrac{1}{sin\,\theta + cos\,\theta + 3}$?

[2 mark]

(iv) Solve the equation $sin\,\theta + cos\,\theta = 1$ over the interval $0 \leqslant \theta \leqslant 360°$ giving all solutions as exact values.

[4 marks]

Question 4

Find a formula for α in terms of A and B when $A\ sin\ \theta + B\ cos\ \theta$ is written in the form $R\ sin\,(\,\theta\ +\ \alpha\,)$ for $0\ <\ \alpha\ <\ 90°$

[3 marks]

Question 5

(i) Expand the brackets; $cos\,(\,\theta\ +\ \alpha\,)$

[1 mark]

(ii) Write $9\cos\theta - 12\ sin\ \theta$ in the form $R\ cos\,(\,\theta + \alpha\,)$ for $0 < \alpha < 90°$
Give α accurate to three significant figures.

[3 marks]

(iii) Solve the equation $9\cos\theta - 12\ sin\ \theta\ = 15$ for $0 < \theta < 360°$

[3 marks]

7.5 Answers

7.5.1 Solution to 7.3 Example (Teaching Video)

$$R\,sin\,(\,\theta\;+\;\alpha\,)\;=\;R\,sin\,\theta\;cos\,\alpha\;+\;R\,cos\,\theta\;sin\,\alpha$$

which is required to be $8\;sin\,\theta\;+\;15\;cos\,\theta$

$$\therefore\;R\,cos\,\alpha\;=\;8\quad\text{and}\quad R\,sin\,\alpha\;=\;15$$

Solving these two equations simultaneously by division,

$$\frac{R\,sin\,\alpha}{R\,cos\,\alpha}\;=\;\frac{15}{8}$$

$$\alpha\;=\;arctan\left(\frac{15}{8}\right)$$

$$\alpha\;=\;61.9°$$

$$R\;=\;\sqrt{8^2\;+\;15^2}$$

$$=\;17$$

Conclusion :

$$8\;sin\,\theta\;+\;15\;cos\,\theta\;=\;17\;sin\,(\,\theta\;+\;61.9°\,)$$

[4 marks]

7.5.2 Solutions to 7.4 Exercise

Answer 1
(i)

$$R\,sin\,(\,\theta\,+\,\alpha\,)\;=\;R\,sin\,\theta\,cos\,\alpha\;+\;R\,cos\,\theta\,sin\,\alpha$$

which is required to be $2\,sin\,\theta\;+\;\sqrt{5}\;cos\,\theta$

$$\therefore\;R\,cos\,\alpha\;=\;2\quad\text{and}\quad R\,sin\,\alpha\;=\;\sqrt{5}$$

Solving these two equations simultaneously by division,

$$\frac{R\,sin\,\alpha}{R\,cos\,\alpha}\;=\;\frac{\sqrt{5}}{2}$$

$$\alpha\;=\;arctan\left(\frac{\sqrt{5}}{2}\right)$$

$$\alpha\;=\;48.2°$$

$$R\;=\;\sqrt{2^2\;+\;\left(\sqrt{5}\,\right)^2}$$

$$=\;3$$

Conclusion :

$$2\,sin\,\theta\;+\;\sqrt{5}\;cos\,\theta\;=\;3\,sin\,(\,\theta\;+\;48.2°\,)$$

[4 marks]

(ii) 3

[1 mark]

(iii) $(-\,3)$

[1 mark]

(iv)

$$2\,sin\,\theta\;+\;\sqrt{5}\;cos\,\theta\;=\;1.5$$

$$3\,sin\,(\,\theta\;+\;48.2\,)\;=\;1.5$$

$$sin\,(\,\theta\;+\;48.2\,)\;=\;\frac{1}{2}$$

$$\theta\;+\;48.2\;=\;30,\;\;150,\;\;390,\;\;...$$

$$\theta\;=\;101.8°,\;\;341.8°$$

[4 marks]

Answer 2

(i)

$$R\sin(\theta + \alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

which is required to be $\sqrt{3}\sin\theta + \cos\theta$

$$\therefore R\cos\alpha = \sqrt{3} \quad \text{and} \quad R\sin\alpha = 1$$

Solving these two equations simultaneously by division,

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{\sqrt{3}}$$

$$\alpha = arctan\left(\frac{1}{\sqrt{3}}\right)$$

$$\alpha = 30°$$

$$R = \sqrt{(\sqrt{3})^2 + 1^2}$$

$$= 2$$

Conclusion :

$$\sqrt{3}\sin\theta + \cos\theta = 2\sin(\theta + 30°)$$

[4 marks]

(ii) (-2)

[1 mark]

(iii) $3\sin\theta + \sqrt{3}\cos\theta = \sqrt{3}\left(\sqrt{3}\sin\theta + \cos\theta\right)$

$$= \sqrt{3} \times 2\sin(\theta + 30)$$

which has a maximum value of $2\sqrt{3}$

[1 mark]

(iv)

$$\sqrt{3}\sin\theta + \cos\theta = \sqrt{2}$$

$$2\sin(\theta + 30) = \sqrt{2}$$

$$\sin(\theta + 30) = \frac{\sqrt{2}}{2}$$

$$\theta + 30 = 45,\ 135,\ 405,\ \ldots$$

$$\theta = 15°,\ 105°$$

[4 marks]

Answer 3

(i)

$$R\,sin\,(\,\theta\,+\,\alpha\,)\;=\;R\,sin\,\theta\,cos\,\alpha\,+\,R\,cos\,\theta\,sin\,\alpha$$

which is required to be $sin\,\theta\,+\,cos\,\theta$

$$\therefore\;R\,cos\,\alpha\;=\;1\quad\text{and}\quad R\,sin\,\alpha\;=\;1$$

Solving these two equations simultaneously by division,

$$\frac{R\,sin\,\alpha}{R\,cos\,\alpha}\;=\;\frac{1}{1}$$

$$\alpha\;=\;arctan\left(1\right)$$

$$\alpha\;=\;45°$$

$$R\;=\;\sqrt{1^2\,+\,1^2}$$

$$=\;\sqrt{2}$$

Conclusion :

$$sin\,\theta\,+\,cos\,\theta\;=\;\sqrt{2}\;sin\,(\,\theta\,+\,45°\,)$$

[3 marks]

(ii) $\left(-\sqrt{2}\,\right)$

[1 mark]

(iii) $\dfrac{1}{sin\,\theta\,+\,cos\,\theta\,+\,3}\;=\;\dfrac{1}{\sqrt{2}\;sin\,(\,\theta\,+\,45\,)\;+\,3}$

which has a maximum value of $\dfrac{1}{-\sqrt{2}\;+\,3}\,\times\,\dfrac{3\;+\,\sqrt{2}}{3\,+\,\sqrt{2}}$

$$=\;\frac{3\,+\,\sqrt{2}}{7}$$

[2 mark]

(iv)

$$sin\,\theta\,+\,cos\,\theta\;=\;1$$

$$\sqrt{2}\;sin\,(\,\theta\,+\,45\,)\;=\;1$$

$$sin\,(\,\theta\,+\,45\,)\;=\;\frac{1}{\sqrt{2}}$$

$$\theta\,+\,45\;=\;45,\;\;135,\;\;405,\;\;...$$

$$\theta\;=\;0°,\;\;90°,\;\;360°$$

[4 marks]

Answer 4

$$R\,sin\,(\,\theta\,+\,\alpha\,)\;=\;R\,sin\,\theta\,cos\,\alpha\;+\;R\,cos\,\theta\,sin\,\alpha$$

which is required to be $A\,sin\,\theta\,+\,B\,cos\,\theta$

$$\therefore\;R\,cos\,\alpha\;=\;A\quad\text{and}\quad R\,sin\,\alpha\;=\;B$$

Solving these two equations simultaneously by division,

$$\frac{R\,sin\,\alpha}{R\,cos\,\alpha}\;=\;\frac{B}{A}$$

$$\alpha\;=\;arctan\left(\frac{B}{A}\right)$$

[3 marks]

Answer 5

(i) $cos\,(\,\theta\,+\,\alpha\,)\;=\;cos\,\theta\,cos\,\alpha\;-\;sin\,\theta\,sin\,\alpha$

[1 mark]

(ii)

$$R\,cos\,(\,\theta\,+\,\alpha\,)\;=\;R\,cos\,\theta\,cos\,\alpha\;-\;R\,sin\,\theta\,sin\,\alpha$$

which is required to be $9\,cos\,\theta\,-\,12\,sin\,\theta$

$$\therefore\;R\,cos\,\alpha\;=\;9\quad\text{and}\quad R\,sin\,\alpha\;=\;12$$

Solving these two equations simultaneously by division,

$$\frac{R\,sin\,\alpha}{R\,cos\,\alpha}\;=\;\frac{12}{9}$$

$$\alpha\;=\;arctan\left(\frac{4}{3}\right)$$

$$\alpha\;=\;53.1°$$

$$R\;=\;\sqrt{9^2\,+\,12^2}$$

$$=\;15$$

Conclusion :

$$9\,cos\,\theta\,-\,12\,sin\,\theta\;=\;15\,cos\,(\,\theta\,+\,53.1°\,)$$

[3 marks]

(iii)

$$9\,cos\,\theta\,-\,12\,sin\,\theta\;=\;15$$

$$15\,cos\,(\,\theta\,+\,53.1\,)\;=\;15$$

$$cos\,(\,\theta\,+\,53.1\,)\;=\;1$$

$$\theta\,+\,53.1\;=\;0,\;360,\;...$$

$$\theta\;=\;306.9°$$

[3 marks]

8.1 Waveforms Expressed as $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$

In lesson 7 we looked at rewriting expressions of the form

$$a\sin\theta + b\cos\theta$$

as

$$R\sin(\theta + \alpha)$$

This is one of four possible variations on the same theme.

Here are all four rewrites, given that a and b are positive valued;

- $a\sin\theta \pm b\cos\theta \equiv R\sin(\theta \pm \alpha)$
- $a\cos\theta \pm b\sin\theta \equiv R\cos(\theta \mp \alpha)$

with $R > 0$ and $0 < \alpha < 90°$

where $R\cos\alpha = a$ and $R\sin\alpha = b$ and $R = \sqrt{a^2 + b^2}$

The recommended method of tackling problems involving such rewrites is to, as we did in lesson 7, use the addition formula to expand whichever of $\sin(\theta \pm \alpha)$ or $\cos(\theta \mp \alpha)$ is desired, then equate coefficients of $\sin\theta$ and $\cos\theta$.

8.2 Example

Express $4\cos\theta + 3\sin\theta$ in the form $R\cos(\theta - \alpha)$ for $R > 0$ and $0 < \alpha < 90°$

Teaching Video : http://www.NumberWonder.co.uk/v9040/8.mp4

After watching the video
write out a solution to
the example

☞

[4 marks]

Note : Rearranging the expression as $3\sin\theta + 4\cos\theta$ gives rise to a rewrite of $5\sin(\theta + 53.1°)$. The two answers are equivalent because $\cos\varphi = \sin(\varphi + 90°)$

8.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

(i) Express $15\,cos\,\theta + 36\,sin\,\theta$ in the form $R\,cos\,(\theta - \alpha)$

where $R > 0$ and $0 < \alpha < 90°$

[3 marks]

(ii) Hence solve, for $0 < \theta < 360°$, the equation $15\,cos\,\theta + 36\,sin\,\theta = 13$

[3 marks]

Question 2

C3 Examination Question from January 2006, Q6

$$f(x) = 12\cos x - 4\sin x$$

Given that $f(x) = R\cos(x + \alpha)$, where $R \geqslant 0$ and $0 \leqslant \alpha \leqslant 90°$

(a) find the value of R and the value of α

[4 marks]

(b) Hence solve the equation $12\cos x - 4\sin x = 7$

for $0 \leqslant x \leqslant 360°$, giving your answer to one decimal place.

[5 marks]

(c) **(i)** Write down the minimum value of $12\cos x - 4\sin x$

[1 mark]

(ii) Find, to 2 decimal places, the smallest positive value of x for which this minimum value occurs.

[2 marks]

Question 3

C3 Examination Question from January 2009, Q8

(a) Express $3\cos\theta + 4\sin\theta$ in the form $R\cos(\theta - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < 90°$

[4 marks]

(b) Hence find the maximum value of $3\cos\theta + 4\sin\theta$ and the smallest positive value of θ for which this maximum occurs.

[3 marks]

The temperature, $f(t)$, of a warehouse is modelled using the equation

$$f(t) = 10 + 3\cos(15t) + 4\sin(15t)$$

where t is the time in hours from midday and $0 \leqslant t < 24$

(c) Calculate the minimum temperature of the warehouse as given by this model.

[2 marks]

(d) Find the value of t when this minimum temperature occurs.

[3 marks]

Question 4

(a) Express

$$5\,sin^2\,\theta \;-\; 3\,cos^2\,\theta \;+\; 6\,sin\,\theta\,cos\,\theta$$

in the form

$$a\,sin\,2\theta \;+\; b\,cos\,2\theta \;+\; c$$

where a, b and c are constants to be found.

[3 marks]

(b) Hence find the maximum and minimum values of

$$5\,sin^2\,\theta \;-\; 3\,cos^2\,\theta \;+\; 6\,sin\,\theta\,cos\,\theta$$

[3 marks]

(c) Solve $5\,sin^2\,\theta \;-\; 3\,cos^2\,\theta \;+\; 6\,sin\,\theta\,cos\,\theta \;=\; -1$

for $0 \;\leqslant\; \theta \;\leqslant\; 180$, rounding your answers to 1 decimal place.

[4 marks]

8.4 Answers

8.4.1 Solution to 8.2 Example (Teaching Video)

$$R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

which is required to be $4\cos\theta + 3\sin\theta$

$$\therefore R\cos\alpha = 4 \quad \text{and} \quad R\sin\alpha = 3$$

Solving these two equations simultaneously by division,

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{4}$$

$$\alpha = \arctan\left(\frac{3}{4}\right)$$

$$\alpha = 36.9°$$

$$R = \sqrt{4^2 + 3^2}$$

$$= 5$$

Conclusion :

$$4\cos\theta + 3\sin\theta = 5\cos(\theta - 36.9°)$$

[4 marks]

8.4.2 Solutions to 8.3 Exercise

Answer 1

(i)

$$R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

which is required to be $15\cos\theta + 36\sin\theta$

$$\therefore R\cos\alpha = 15 \quad \text{and} \quad R\sin\alpha = 36$$

Solving these two equations simultaneously by division,

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{36}{15}$$

$$\alpha = \arctan\left(\frac{12}{5}\right)$$

$$\alpha = 67.4°$$

$$R = \sqrt{15^2 + 36^2}$$

$$= 39$$

Conclusion :

$$15\cos\theta + 36\sin\theta = 39\cos(\theta - 67.2°)$$

[3 marks]

(ii)

$$15\cos\theta + 36\sin\theta = 13$$

$$39\cos(\theta - 67.4°) = 13$$

$$\cos(\theta - 67.4°) = \frac{1}{3}$$

$$\theta - 67.4 = 70.5, \ 289.5$$

$$\theta = 137.9°, \ 356.9°$$

[3 marks]

Answer 2

(a)

$$R\cos(\theta + \alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

which is required to be $12\cos\theta - 4\sin\theta$

$$\therefore R\cos\alpha = 12 \quad \text{and} \quad R\sin\alpha = 4$$

Solving these two equations simultaneously by division,

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{4}{12}$$

$$\alpha = arctan\left(\frac{1}{3}\right)$$

$$\alpha = 18.43°$$

$$R = \sqrt{12^2 + 4^2}$$

$$= 4\sqrt{10} \text{ (As a decimal, 12.6)}$$

Conclusion :

$$12\cos\theta - 4\sin\theta = 4\sqrt{10}\cos(\theta + 18.43°)$$

[4 marks]

(b)

$$12\cos\theta - 4\sin\theta = 7$$

$$4\sqrt{10}\cos(\theta + 18.43°) = 7$$

$$\cos(\theta + 18.43°) = 0.5534$$

$$\theta + 18.43 = 56.40, \; 303.60$$

$$\theta = 38°, \; 285.2°$$

[5 marks]

(c) **(i)** $-4\sqrt{10}$

[1 mark]

(ii)

$$\cos(\theta + 18.43) = -1$$

$$\theta + 18.43 = 180$$

$$\theta = 161.57°$$

[2 marks]

Answer 3

(a)

$$R\,cos\,(\,\theta\,-\,\alpha\,)\;=\;R\,cos\,\theta\;cos\,\alpha\;+\;R\,sin\,\theta\;sin\,\alpha$$

which is required to be $3\,cos\,\theta\;+\;4\,sin\,\theta$

$$\therefore\;R\,cos\,\alpha\;=\;3\quad\text{and}\quad R\,sin\,\alpha\;=\;4$$

Solving these two equations simultaneously by division,

$$\frac{R\,sin\,\alpha}{R\,cos\,\alpha}\;=\;\frac{4}{3}$$

$$\alpha\;=\;arctan\left(\frac{4}{3}\right)$$

$$\alpha\;=\;53.13°$$

$$R\;=\;\sqrt{3^2\;+\;4^2}$$

$$=\;5$$

Conclusion :

$$3\,cos\,\theta\;+\;4\,sin\,\theta\;=\;5\,cos\,(\,\theta\;-\;53.13°\,)$$

[4 marks]

(b) Maximum value is 5 which occurs when,

$$cos\,(\,\theta\;-\;53.13\,)\;=\;1$$

$$\theta\;-\;53.13\;=\;0$$

$$\theta\;=\;53.13°$$

[3 marks]

(c) $f\,(t)\;=\;10\;+\;3\,cos\;(15t)\;\;+\;4\,sin\;(15t)$

$\;=\;10\;+\;5\,cos\,(\,15t\;-\;53.13\,)$

$\therefore$ The minimum temperature of the warehouse is 5 (degrees Centigrade, perhaps)

[2 marks]

(d) This minimum will occur when, $cos\,(\,15t\;-\;53.13\,)\;=\;-\,1$

$$15t\;-\;53.13\;=\;180$$

$$15t\;=\;233.13$$

$$t\;=\;15.54$$

This is 15 hours and 32 minutes after midday which is 3.32 am the following morning

[3 marks]

Answer 4

(a) The immediate problem is the different number of $sin^2\,\theta$ and $cos^2\,\theta$

They can be evened up as follows,

$$5\,sin^2\,\theta \;-\; 3\,cos^2\,\theta \;+\; 6\,sin\,\theta\,cos\,\theta$$

$$=\; 4\,sin^2\,\theta \;+\; sin^2\,\theta \;-\; 3\,cos^2\,\theta \;+\; 6\,sin\,\theta\,cos\,\theta$$

$$=\; 4\,sin^2\,\theta \;+\; (\,1 \;-\; cos^2\,\theta\,) \;-\; 3\,cos^2\,\theta \;+\; 6\,sin\,\theta\,cos\,\theta$$

$$=\; (-\,4)\,(\,cos^2\,\theta \;-\; sin^2\,\theta\,) \;+\; 1 \;+\; 3\;(\,2\,sin\,\theta\,cos\,\theta\,)$$

$$=\; (-\,4)\;cos\,2\theta \;+\; 1 \;+\; 3 \times sin\,2\theta$$

$$=\; 3\,sin\,2\theta \;-\; 4\,cos\,2\theta \;+\; 1$$

i.e. $a \;=\; 3,\;\; b \;=\; -\,4$ and $c \;=\; 1$

[3 marks]

(b)

$$R\,sin\,(\,2\theta \;-\; \alpha\,) \;=\; R\,sin\,2\theta\,cos\,\alpha \;-\; R\,cos\,2\theta\,sin\,\alpha$$

which is required to be $3\,sin\,2\theta \;-\; 4\,cos\,2\theta$

$$\therefore\; R\,cos\,\alpha \;=\; 3 \quad \text{and} \quad R\,sin\,\alpha \;=\; 4$$

Solving these two equations simultaneously by division,

$$\frac{R\,sin\,\alpha}{R\,cos\,\alpha} \;=\; \frac{4}{3}$$

$$\alpha \;=\; arctan\left(\frac{4}{3}\right)$$

$$\alpha \;=\; 53.13°$$

$$R \;=\; \sqrt{3^2 \;+\; 4^2}$$

$$=\; 5$$

Conclusion :

$$3\,sin\,2\theta \;-\; 4\,cos\,2\theta \;=\; 5\,sin\,(\,2\theta \;-\; 53.13°\,)$$

$$5\,sin^2\,\theta \;-\; 3\,cos^2\,\theta \;+\; 6\,sin\,\theta\,cos\,\theta \;=\; 3\,sin\,2\theta \;-\; 4\,cos\,2\theta \;+\; 1$$

$$=\; 5\,sin\,(\,2\theta \;-\; 53.13°\,) \;+\; 1$$

which will have a maximum value of 6 and a minimum value of $(-\,4)$

[3 marks]

(c)

$$5\,sin^2\,\theta \;-\; 3\,cos^2\,\theta \;+\; 6\,sin\,\theta\,cos\,\theta \;=\; -\,1$$

$$3\,sin\,2\theta \;-\; 4\,cos\,2\theta \;+\; 1 \;=\; -\,1$$

$$5\,sin\,(\,2\theta \;-\; 53.13°\,) \;+\; 1 \;=\; -\,1$$

$$5\,sin\,(\,2\theta \;-\; 53.13°\,) \;=\; -\,2$$

$$sin\,(\,2\theta \;-\; 53.13°\,) \;=\; -\,\frac{2}{5}$$

$$2\theta \;-\; 53.13 \;=\; -\,23.58,\;\; 203.58$$

$$2\theta \;=\; 29.55,\;\; 256.71$$

$$\theta \;=\; 14.8°,\;\; 128.4°$$

[4 marks]

Lesson 9

A-Level Pure Mathematics : Year 2
Trigonometric Identities

9.1 Revision

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

(i) Find the four solutions to the following equation,

$$\sin\theta = \frac{\sqrt{3}}{2} \qquad \text{for} \quad 0° \leqslant \theta \leqslant 720°$$

[4 marks]

(ii) Hence, or otherwise, solve,

$$\sin(2\theta + 10°) = \frac{\sqrt{3}}{2} \quad , \qquad 0° \leqslant \theta \leqslant 360°$$

[2 marks]

Question 2

Prove the identity,

$$(\cos\theta + \sin\theta)(\cos\theta - \sin\theta) + 1 = 2\cos^2\theta$$

[5 marks]

Question 3

Here is a table of exact trigonometric values that most mathematicians know “off by heart”;

Exact values table:

	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Not Defined

Show, using the formula for $\tan(A + B)$ and the table of exact trigonometric values that

$$\tan 75° = \frac{3 + \sqrt{3}}{3 - \sqrt{3}}$$

Rationalise the denominator to further show that for some integer values of *a* and *b*,

$$\tan 75° = a + b\sqrt{3}$$

Clearly state the value of *a* and the value of *b*.

[6 marks]

Question 4

Prove the identity

$$csc^2 \theta + sec^2 \theta = csc^2 \theta \ sec^2 \theta$$

[5 marks]

Question 5

(i) Express $5\,sin\,x + 12\,cos\,x$ in the form $R\,sin\,(x + \alpha)$,
where $R > 0$ and α is the smallest positive angle possible.
In your answer express α in degrees and accurate to one decimal place.

(ii) Hence, or otherwise, solve the following equation over $0° \leqslant x \leqslant 360°$

$$5\,sin\,x + 12\,cos\,x = 9$$

[8 marks]

Question 6

The depth of water, D metres, in a harbour on a particular day is modelled by the formula

$$D = 7 + 3\,sin\ (30t)° \qquad 0 \leqslant t \leqslant 24$$

where t is the number of hours after midnight.

(a) High tide is the time at which the depth of water is at it's greatest.
On the day being modelled there are two high tides.
Find the times of these high tides.

[2 marks]

(b) A boat enters the harbour at Noon.
It must leave before the depth of the water in the harbour falls below 6 metres.
By what time, to the nearest minute, must the boat have left the harbour ?

[8 marks]

Question 7

(i) By expressing 3θ in the form $2\theta + \theta$, prove that,

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

(ii) Hence, or otherwise, solve the equation,

$$\cos 3\theta + \cos\theta = 0 \quad \text{for} \quad 0° \leqslant \theta \leqslant 360°$$

[10 marks]

9.2 Answers

Answer 1

(i) 60°, 120°, 420°, 480°

[4 marks]

(ii) $2\theta + 10 = 60,\ 120,\ 420,\ 480$

$$2\theta = 50,\ 110,\ 410,\ 470$$

$$\theta = 25°,\ 55°,\ 205°,\ 235°$$

[2 marks]

Answer 2

$$\begin{aligned}
\text{LHS} &= (\cos\theta + \sin\theta)(\cos\theta - \sin\theta) + 1 \\
&= \cos^2\theta - \sin^2\theta + 1 \\
&= \cos^2\theta - \left(1 - \cos^2\theta\right) + 1 \\
&= \cos^2\theta - 1 + \cos^2\theta + 1 \\
&= 2\cos^2\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[5 marks]

Answer 3

$$\begin{aligned}
\text{LHS} &= \tan 75 \\
&= \tan(45 + 30) \\
&= \frac{\tan 45 + \tan 30}{1 - \tan 45 \tan 30} \\
&= \frac{1 + \frac{\sqrt{3}}{3}}{1 - 1 \times \frac{\sqrt{3}}{3}} \times \frac{3}{3} \\
&= \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \qquad \square \\
&= \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \times \frac{3 + \sqrt{3}}{3 + \sqrt{3}} \\
&= \frac{9 + 6\sqrt{3} + 3}{9 - 3} \\
&= \frac{12 + 6\sqrt{3}}{6} \\
&= 2 + \sqrt{3}
\end{aligned}$$

i.e. $a = 2$ and $b = 1$

[6 marks]

Answer 4

$$\begin{aligned}
\text{LHS} &= csc^2\,\theta + sec^2\,\theta \\
&= \frac{1}{sin^2\,\theta} + \frac{1}{cos^2\,\theta} \\
&= \frac{cos^2\,\theta + sin^2\,\theta}{sin^2\,\theta\,cos^2\,\theta} \\
&= \frac{1}{sin^2\,\theta\,cos^2\,\theta} \\
&= \frac{1}{sin^2\,\theta} \times \frac{1}{cos^2\,\theta} \\
&= csc^2\,\theta\,sec^2\,\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[5 marks]

Answer 5

(i) $R\,sin\,(\,\theta + \alpha\,) = R\,sin\,\theta\,cos\,\alpha + R\,cos\,\theta\,sin\,\alpha$

which is required to be $5\,sin\,\theta + 12\,cos\,\theta$

$$\therefore\ R\,cos\,\alpha = 5 \quad \text{and} \quad R\,sin\,\alpha = 12$$

Solving these two equations simultaneously by division,

$$\frac{R\,sin\,\alpha}{R\,cos\,\alpha} = \frac{12}{5}$$

$$\alpha = arctan\left(\frac{12}{5}\right)$$

$$\alpha = 67.4°$$

$$\begin{aligned}
R &= \sqrt{5^2 + 12^2} \\
&= 13
\end{aligned}$$

Conclusion :

$$5\,sin\,\theta + 12\,cos\,\theta = 13\,sin\,(\,\theta + 67.4°\,)$$

[4 marks]

(ii)

$$\begin{aligned}
5\,sin\,\theta + 12\,cos\,\theta &= 9 \\
13\,sin\,(\,\theta + 67.4\,) &= 9 \\
sin\,(\,\theta + 67.4\,) &= 0.6923 \\
\theta + 67.4 &= 43.81,\ 136.19,\ 403.81 \\
\theta &= 68.79°,\ 336.41°
\end{aligned}$$

[4 marks]

Answer 6

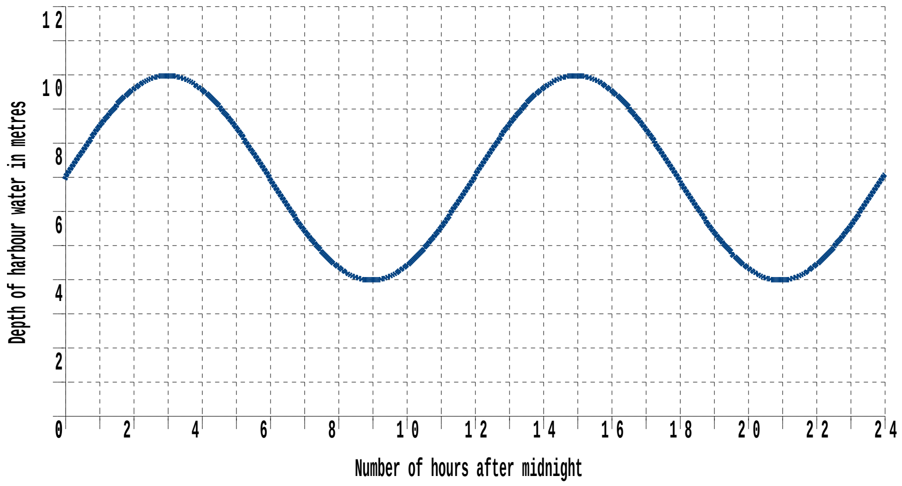
(a) For D to be a maximum, $sin\,(30t) = 1$

$$30t = 90, \; 450, \; ...$$

$$t = 3, \; 15, \; ...$$

i.e. High tide at 3 am and 3 pm

[2 marks]



(b) To find the critical times,

$$7 + 3\,sin\,(30t) = 6$$

$$3\,sin\,(30t) = -1$$

$$sin\,(30t) = -\frac{1}{3}$$

$$30t = 199.5, \; 340.5, \; 559.5, \; 700.5$$

$$t = 6.65, \; 11.35, \; 18.65, \; 23.35 \text{ hours}$$

The boat must leave before 18 hours, 39 minutes

i.e. Before 6.39 pm

[8 marks]

Answer 7
(i)

$$
\begin{aligned}
\text{LHS} &= \cos 3\theta \\
&= \cos\ (2\theta + \theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (\cos^2\theta - \sin^2\theta) \cos\theta - 2\sin\theta\cos\theta\sin\theta \\
&= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta \\
&= \cos^3\theta - 3\sin^2\theta\cos\theta \\
&= \cos^3\theta - 3\left(1 - \cos^2\theta \right)\cos\theta \\
&= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]

(ii)

$$
\begin{aligned}
\cos 3\theta + \cos\theta &= 0 \\
4\cos^3\theta - 3\cos\theta + \cos\theta &= 0 \\
4\cos^3\theta - 2\cos\theta &= 0 \\
2\cos\theta\left(2\cos^2\theta - 1 \right) &= 0
\end{aligned}
$$

Either $2\cos\theta = 0$ or $2\cos^2\theta - 1 = 0$

$\theta = 90°,\ 270°$ $\qquad$ $\cos^2\theta = \dfrac{1}{2}$

$$\cos\theta = \pm\frac{\sqrt{2}}{2}$$

$$\theta = 45°,\ 135°,\ 225°,\ 315°$$

$\therefore$ Solutions are $45°,\ 90°,\ 135°,\ 225°,\ 270°,\ 315°$

[5 marks]

Lesson 10

A-Level Pure Mathematics : Year 2
Trigonometric Identities

10.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

(i) Find the four solutions to the following equation,

$$\cos\theta = \frac{\sqrt{3}}{2} \qquad \text{for} \quad 0° \leqslant \theta \leqslant 720°$$

[4 marks]

(ii) Hence, or otherwise, solve,

$$\cos(2\theta + 15°) = \frac{\sqrt{3}}{2} \quad , \qquad 0° \leqslant \theta \leqslant 360°$$

[2 marks]

Question 2

Prove the identity,

$$4\sin^2\theta\cos^2\theta + \cos^2 2\theta = 1$$

[5 marks]

Question 3

Here is a table of exact trigonometric values that most mathematicians know “off by heart”;

Exact values table:

	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Not Defined

Show, using the formula for $\tan(A - B)$ and the table of exact trigonometric values that

$$\tan 15° = \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

Rationalise the denominator to further show that for some integer values of a and b,

$$\tan 15° = a + b\sqrt{3}$$

Clearly state the value of a and the value of b.

[6 marks]

Question 4

Prove the identity

$$\frac{\sin\theta + \cos\theta}{\sin\theta} + \frac{\sin\theta - \cos\theta}{\cos\theta} == 2\csc 2\theta$$

[5 marks]

Question 5

(i) Express $15 \sin x + 8 \cos x$ in the form $R \sin(x + \alpha)$,
where $R > 0$ and α is the smallest positive angle possible.
Give your answer in degrees and accurate to one decimal place.

(ii) Hence, or otherwise, solve the following equation over $0° \leqslant x \leqslant 360°$

$$15 \sin x + 8 \cos x = 13$$

[8 marks]

Question 6

The depth of water, D metres, in a harbour on a particular day is modelled by the formula

$$D = 5 + 3\,cos\ (30t)^{\circ} \qquad 0 \leqslant t \leqslant 24$$

where t is the number of hours after midnight.

(a) Low tide is the time at which the depth of water is at it's least.
On the day being modelled there are two low tides.
Find the times of these low tides.

[2 marks]

(b) A boat enters the harbour at Noon.
It must leave before the depth of the water in the harbour falls below 3 metres.
By what time, to the nearest minute, must the boat have left the harbour ?

[8 marks]

Question 7

(i) By expressing 3θ in the form $2\theta + \theta$, prove that,

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

(ii) Hence, or otherwise, solve the equation,

$$\sin 3\theta - \sin \theta = 0 \qquad \text{for} \quad 0° \leqslant \theta \leqslant 360°$$

[10 marks]

10.2 Answers

Answer 1

(i) 30°, 330°, 390°, 690°

[4 marks]

(ii)

$$2\theta + 15 = 30, \ 330, \ 390, \ 690$$

$$2\theta = 15, \ 315, \ 375, \ 675$$

$$\theta = 7.5°, \ 157.5°, \ 187.5°, \ 337.5°$$

[2 marks]

Answer 2

1st Proof :

$$\begin{aligned} \text{LHS} &= 4\sin^2\theta\cos^2\theta + \cos^2 2\theta \\ &= (2\sin\theta\cos\theta)^2 + \cos^2 2\theta \\ &= \sin^2 2\theta + \cos^2 2\theta \\ &= 1 \\ &= \text{RHS} \qquad \square \end{aligned}$$

2nd Proof :

$$\begin{aligned} \text{LHS} &= 4\sin^2\theta\cos^2\theta + \cos^2 2\theta \\ &= 4\sin^2\theta\cos^2\theta + \left(\cos^2\theta - \sin^2\theta\right)^2 \\ &= 4\sin^2\theta\cos^2\theta + \cos^4\theta - 2\sin^2\theta\cos^2\theta + \sin^4\theta \\ &= \cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta \\ &= \left(\cos^2\theta + \sin^2\theta\right)^2 \\ &= 1^2 \\ &= \text{RHS} \qquad \square \end{aligned}$$

3rd Proof :

$$\begin{aligned} \text{LHS} &= 4\sin^2\theta\cos^2\theta + \cos^2 2\theta \\ &= 4\sin^2\theta\cos^2\theta + \left(\cos^2\theta - \sin^2\theta\right)^2 \\ &= 4\sin^2\theta\cos^2\theta + \left(\cos^2\theta - \left(1 - \cos^2\theta\right)\right)^2 \\ &= 4\sin^2\theta\cos^2\theta + \left(2\cos^2\theta - 1\right)^2 \\ &= 4\sin^2\theta\cos^2\theta + 4\cos^4\theta - 4\cos^2\theta + 1 \\ &= 4\cos^2\theta\left(\sin^2\theta + \cos^2\theta - 1\right) + 1 \\ &= 4\cos^2\theta\left(1 - 1\right) + 1 \\ &= 1 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[5 marks]

Answer 3

1st Method

$$\begin{aligned}
\text{LHS} &= \tan 15 \\
&= \tan(45 - 30) \\
&= \frac{\tan 45 - \tan 30}{1 + \tan 45 \tan 30} \\
&= \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \times \frac{\sqrt{3}}{3}} \times \frac{3}{3} \\
&= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \quad \square \\
&= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \times \frac{3 - \sqrt{3}}{3 - \sqrt{3}} \\
&= \frac{9 - 6\sqrt{3} + 3}{9 - 3} \\
&= \frac{12 - 6\sqrt{3}}{6} \\
&= 2 - \sqrt{3}
\end{aligned}$$

i.e. $a = 2$ and $b = -1$

2nd Method

$$\begin{aligned}
\text{LHS} &= \tan 15 \\
&= \tan(60 - 45) \\
&= \frac{\tan 60 - \tan 45}{1 + \tan 60 \tan 45} \\
&= \frac{\sqrt{3} - 1}{1 + \sqrt{3} \times 1} \times \frac{\sqrt{3}}{\sqrt{3}} \\
&= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \quad \square \\
&= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \times \frac{3 - \sqrt{3}}{3 - \sqrt{3}} \\
&= \frac{9 - 6\sqrt{3} + 3}{9 - 3} \\
&= \frac{12 - 6\sqrt{3}}{6} \\
&= 2 - \sqrt{3}
\end{aligned}$$

i.e. $a = 2$ and $b = -1$

[6 marks]

Answer 4

Method 1

$$
\begin{aligned}
\text{LHS} &= \frac{\sin\theta + \cos\theta}{\sin\theta} + \frac{\sin\theta - \cos\theta}{\cos\theta} \\
&= \frac{\sin\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\cos\theta} \\
&= 1 + \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} - 1 \\
&= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \\
&= \frac{\cos\theta}{\sin\theta} \times \frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta} \times \frac{\sin\theta}{\sin\theta} \\
&= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} \\
&= \frac{1}{\sin\theta\ \cos\theta} \times \frac{2}{2} \\
&= \frac{2}{\sin 2\theta} \\
&= 2\csc 2\theta \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

Method 2

$$
\begin{aligned}
\text{LHS} &= \frac{\sin\theta + \cos\theta}{\sin\theta} + \frac{\sin\theta - \cos\theta}{\cos\theta} \\
&= \frac{\sin\theta + \cos\theta}{\sin\theta} \times \frac{\cos\theta}{\cos\theta} + \frac{\sin\theta - \cos\theta}{\cos\theta} \times \frac{\sin\theta}{\sin\theta} \\
&= \frac{\sin\theta\cos\theta + \cos^2\theta + \sin^2\theta - \sin\theta\cos\theta}{\sin\theta\cos\theta} \\
&= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} \\
&= \frac{1}{\sin\theta\ \cos\theta} \times \frac{2}{2} \\
&= \frac{2}{\sin 2\theta} \\
&= 2\csc 2\theta \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]

Answer 5

(i) $$R\,sin\,(\,\theta\,+\,\alpha\,)\;=\;R\,sin\,\theta\,cos\,\alpha\,+\,R\,cos\,\theta\,sin\,\alpha$$

which is required to be $15\,sin\,\theta\,+\,8\,cos\,\theta$

$$\therefore\;R\,cos\,\alpha\;=\;15\quad\text{and}\quad R\,sin\,\alpha\;=\;8$$

Solving these two equations simultaneously by division,

$$\frac{R\,sin\,\alpha}{R\,cos\,\alpha}\;=\;\frac{8}{15}$$

$$\alpha\;=\;arctan\left(\frac{8}{15}\right)$$

$$\alpha\;=\;28.1°$$

$$R\;=\;\sqrt{15^2\,+\,8^2}$$

$$=\;17$$

Conclusion :

$$15\,sin\,\theta\,+\,8\,cos\,\theta\;=\;17\,sin\,(\,\theta\,+\,28.1°\,)$$

[4 marks]

(ii) $$15\,sin\,\theta\,+\,8\,cos\,\theta\;=\;13$$

$$17\,sin\,(\,\theta\,+\,28.1\,)\;=\;13$$

$$sin\,(\,\theta\,+\,28.1\,)\;=\;0.7647$$

$$\theta\,+\,28.1\;=\;49.9,\;\;130.1$$

$$\theta\;=\;21.8°,\;\;102°$$

[4 marks]

Answer 6

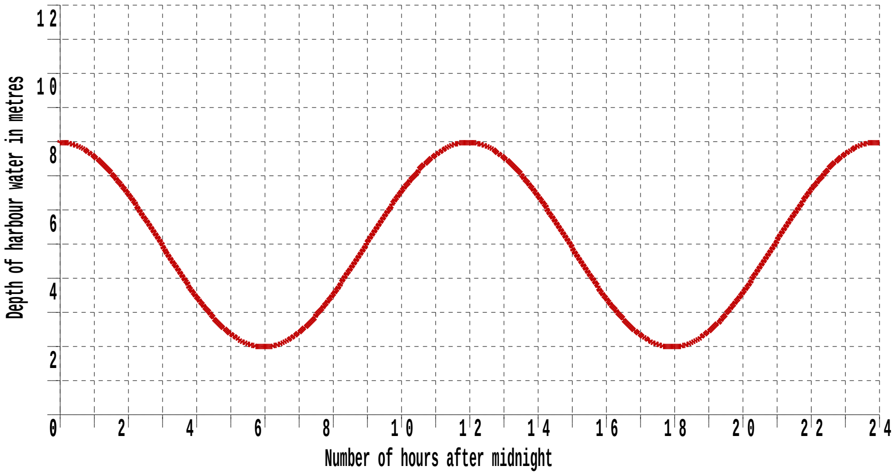
(a) For D to be a minimum, $\cos(30t) = -1$

$$30t = 180, \ 540, \ \ldots$$

$$t = 6, \ 18, \ \ldots$$

i.e. Low tide at 6 am and 6 pm

[2 marks]



(b) To find the critical times,

$$5 + 3\cos(30t) = 3$$

$$3\cos(30t) = -2$$

$$\cos(30t) = -\frac{2}{3}$$

$$30t = 131.8, \ 228.2, \ 491.8, \ 588.2$$

$$t = 4.39, \ 7.61, \ 16.39, \ 19.61 \text{ hours}$$

The boat must leave before 16 hours, 23 minutes

i.e. Before 4.23 pm

[8 marks]

Answer 7

(i)

(a) LHS $= \sin 3\theta$

$$= \sin(2\theta + \theta)$$

$$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$$

$$= 2\sin\theta\cos\theta\cos\theta + \left(\cos^2\theta - \sin^2\theta\right)\sin\theta$$

$$= 2\sin\theta\cos^2\theta + \cos^2\theta\sin\theta - \sin^3\theta$$

$$= 3\sin\theta\cos^2\theta - \sin^3\theta$$

$$= 3\sin\theta\left(1 - \sin^2\theta\right) - \sin^3\theta$$

$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$$

$$= 3\sin\theta - 4\sin^3\theta$$

$$= \text{RHS} \qquad \square$$

[5 marks]

(ii)

$$\sin 3\theta - \sin\theta = 0$$

$$3\sin\theta - 4\sin^3\theta - \sin\theta = 0$$

$$2\sin\theta - 4\sin^3\theta = 0$$

$$2\sin\theta\left(1 - 2\sin^2\theta\right) = 0$$

Either $2\sin\theta = 0$ or $1 - 2\sin^2\theta = 0$

$\theta = 0°, 180°, 360°$ $\qquad \sin^2\theta = \dfrac{1}{2}$

$$\sin\theta = \pm\frac{\sqrt{2}}{2}$$

$$\theta = 45°, 135°, 225°, 315°$$

$\therefore$ Solutions are $0°, 45°, 135°, 180°, 225°, 315°, 360°$

[5 marks]