10.2 Answers

A-Level Pure Mathematics, Year 2
Functions II

Answer 1

(i)

$$fg(x) = f\big(\ln(x+1)\big)$$
$$= e^{2\ln(x+1)} + 4$$
$$= e^{\ln(x+1)^2} + 4$$
$$= (x+1)^2 + 4 \quad * * *$$
$$= x^2 + 2x + 1 + 4$$
$$= x^2 + 2x + 5 \leftarrow \text{Full Marks}$$

Completing the square to get the range

$$= [x^2 + 2x] + 5$$
$$= [(x+1)^2 - 1] + 5$$
$$= (x+1)^2 + 4 \quad * * *$$

$\therefore$ The range is $fg(x) \in \mathbb{R}, \ fg(x) \geqslant 4$

[4 marks]

(ii)

$$fg(x) = 85$$
$$x^2 + 2x + 5 = 85$$
$$x^2 + 2x - 80 = 0$$
$$(x+10)(x-8) = 0$$

Either $x = -10$ or $x = 8$

But $x = -10$ cannot be put into $g(x)$ and so is not a valid solution.

$\therefore$ $x = 8$ is the only solution

[3 marks]

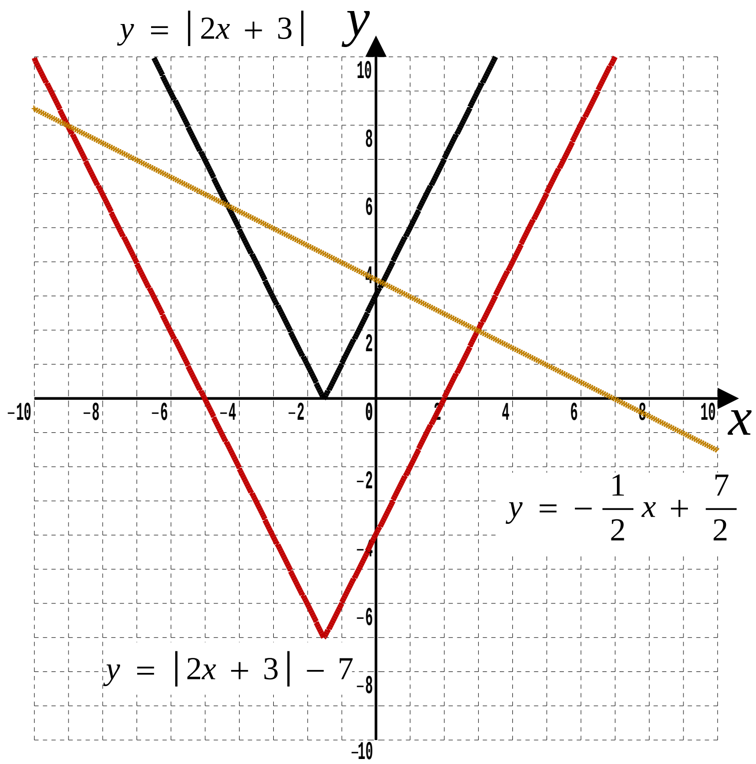
Answer 2

(i) Black line on the graph : $y = |2x + 3|$

It touches the x-axis when $y = 0 \Rightarrow \left(-\dfrac{3}{2},\ 0\right)$

and crosses the y-axis at $(0,\ 3)$

[3 marks]



(ii) The graph of $g(x)$ is $f(x)$ translated $\begin{pmatrix} 0 \\ -7 \end{pmatrix}$ to give the red line on the graph.

It crosses the y-axis at $(0, -4)$ and the x-axis when $|2x + 3| - 7 = 0$

That's when $+(2x + 3) = 7 \Rightarrow x = 2 \Rightarrow (2,\ 0)$

or when $-(2x + 3) = 7 \Rightarrow x = -5 \Rightarrow (-5,\ 0)$

[3 marks]

(iii)

$$+(2x + 3) - 7 = -\frac{1}{2}x + \frac{7}{2} \qquad -(2x + 3) - 7 = -\frac{1}{2}x + \frac{7}{2}$$

$$4x - 8 = -x + 7 \qquad -4x - 20 = -x + 7$$

$$5x = 15 \qquad -3x = 27$$

$$x = 3 \Rightarrow (3,\ 2) \qquad x = -9 \Rightarrow (-9,\ 8)$$

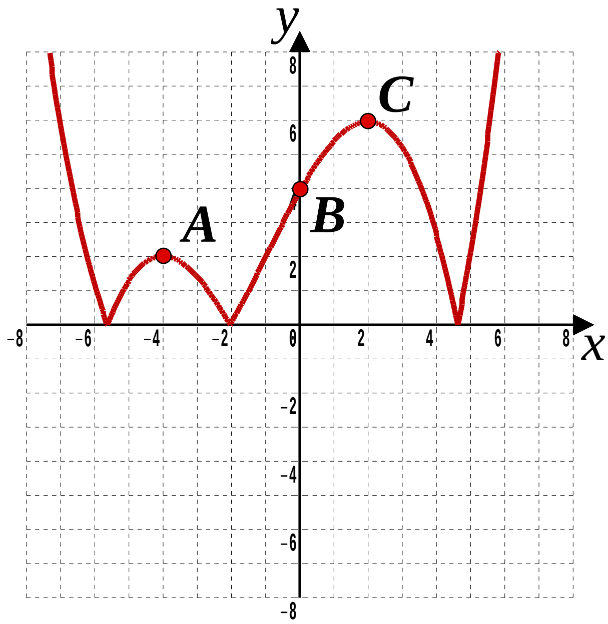
[4 marks]

(iv) Gold line of $y = -\dfrac{1}{2}x + \dfrac{7}{2}$ added to graph.

[1 mark]

Answer 3

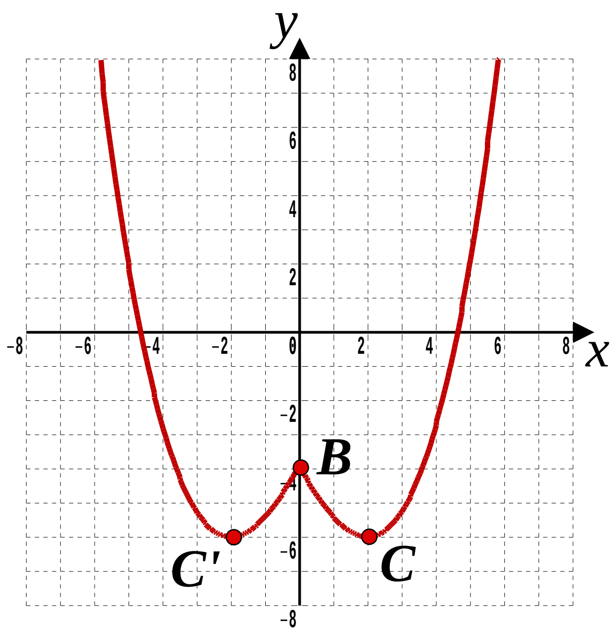
(i)



$A(-4,\ 2),\ B(0,\ 4),\ C(2,\ 6)$

[3 marks]

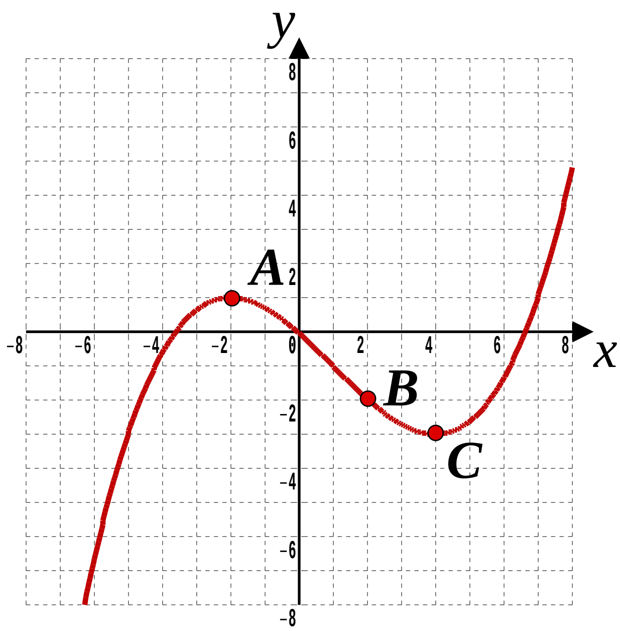
(ii)



$C'(-2,\ -6),\ B(0,\ -4),\ C(2,\ -6)$

[3 marks]

(iii)



$A(-2,\ 1),\ B(2,-2),\ C(4,-3)$

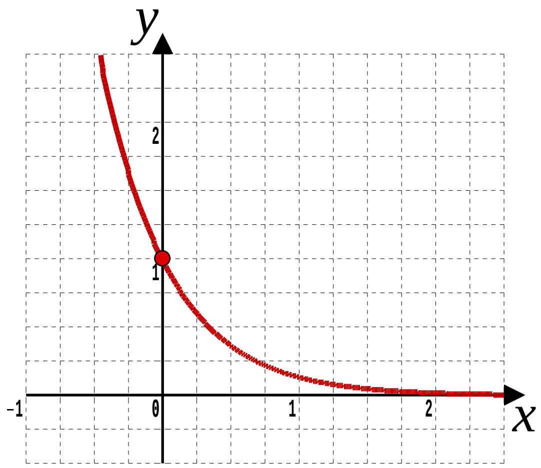
Note : The equation I used to (minor fudge) draw the original graph is

$$y = 0.2225\left(\frac{x^3}{3} + x^2 - 8x\right) - 3.9$$

[4 marks]

Answer 4

(a)

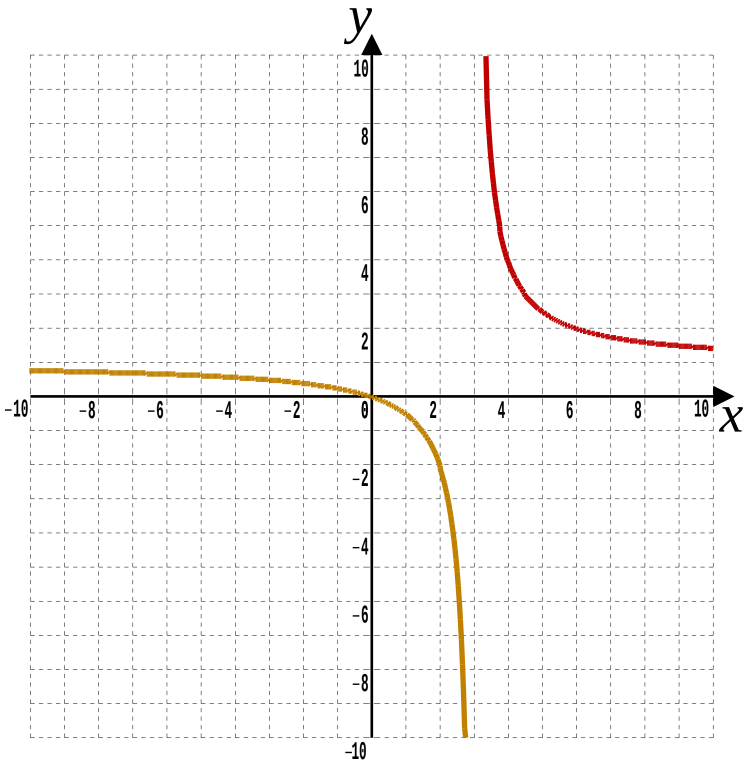


There is only one axis crossing point.

It's on the y-axis at $(\ 0,\ \ 1\ )$

[2 marks]

(b)



The graph of $g\,(x)$ for $x\ >\ 3$ is in red

(The gold part is excluded from the domain)

There is a vertical asymptote at $x\ =\ 3,$ and a horizontal one at $y\ =\ 1$

$\therefore$ The range is, $g\,(x)\ \in\ \mathbb{R},\ \ g\,(x)\ >\ 1$

[2 marks]

(c)

$$g(x) = \frac{x}{x - 3} \qquad x > 3$$

$$y = \frac{x}{x - 3}$$

Swapping $x \Leftrightarrow y$ because input $\Leftrightarrow$ output

$$x = \frac{y}{y - 3}$$

$$x(y - 3) = y$$

$$xy - 3x = y$$

$$xy - y = 3x$$

$$y(x - 1) = 3x$$

$$y = \frac{3x}{x - 1}$$

$$g^{-1}(x) = \frac{3x}{x - 1}$$

The domain of the inverse function is the range of the original function

So the domain is $g^{-1}(x) \in \mathbb{R}, \quad g^{-1}(x) > 1$

[4 marks]

(d)

$$fg(x) = 3$$

$$f\left(\frac{x}{x - 3} \right) = 3$$

$$exp\left(-2 \times \left(\frac{x}{x - 3} \right)\right) = 3$$

take the *ln* of both sides

$$-\left(\frac{2x}{x - 3} \right) = ln\,3$$

$$\frac{2x}{3 - x} = ln\,3$$

$$2x = (3 - x)\ ln\,3$$

$$2x = 3\,ln\,3 - x\,ln\,3$$

$$2x + x\,ln3 = 3\,ln\,3$$

$$x(2 + ln\,3) = 3\,ln\,3$$

$$x = \frac{3\,ln\,3}{2 + ln\,3}$$

[4 marks]