8.3 Answers

8.3.1 Solution to 8.1 Example

(i)

$$h(x) = \frac{5x + 2}{3x + 4}$$

$$y = \frac{5x + 2}{3x + 4}$$

swap the input and output

$$x = \frac{5y + 2}{3y + 4}$$

$$x(3y + 4) = 5y + 2$$

$$3xy + 4x = 5y + 2$$

$$3xy - 5y = -4x + 2$$

$$y(3x - 5) = -4x + 2$$

$$y = \frac{-4x + 2}{3x - 5}$$

$$h^{-1}(x) = \frac{-4x + 2}{3x - 5}$$

[4 marks]

(ii)

$$h^{-1}(x) \text{ has domain} \qquad x \in \mathbb{R}, \quad x \neq \frac{5}{3}$$

[1 mark]

8.3.2 Solutions to 8.2 Exercise

Answer 1

(i)

$$m^{-1}(x) = \frac{-3x + 1}{2x - 7}$$

[4 marks]

(ii)

$$m^{-1}(x) \text{ has domain} \qquad x \in \mathbb{R}, \quad x \neq \frac{7}{2}$$

[1 mark]

Answer 2

$$f(x) = \frac{ax + b}{cx + d}$$

$$y = \frac{ax + b}{cx + d}$$

swap input and output

$$x = \frac{ay + b}{cy + d}$$

$$x(cy + d) = ay + b$$

$$cxy + dx = ay + b$$

$$cxy - ay = -dx + b$$

$$y(cx - a) = -dx + b$$

$$y = \frac{-dx + b}{cx - a}$$

$$f^{-1}(x) = \frac{-dx + b}{cx - a} \qquad x \in \mathbb{R}, \quad x \neq \frac{a}{c}$$

[6 marks]

Answer 3

(i)

$$s(x) = \frac{2x - 1}{x - 2} \qquad x \neq 2$$

$s(x)$ is a Möbius function with,

$$a = 2, \qquad b = -1, \qquad c = 1, \qquad d = -2$$

The inverse is

$$s^{-1}(x) = \frac{-dx + b}{cx - a}$$

$$= \frac{-(-2)x + (-1)}{(1)x - (2)}$$

$$= \frac{2x - 1}{x - 2}$$

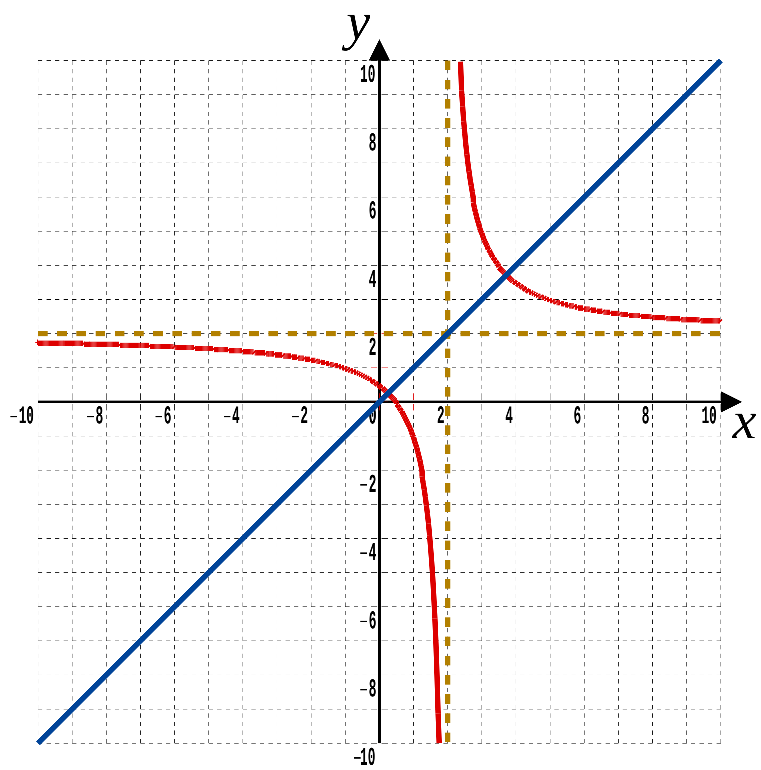
[2 marks]

(ii) The expression for the inverse is the same as the original function

$$s(x) = s^{-1}(x)$$

[1 mark]

(iii) Graph of inverse proportion with asymptotes at $x = 2$ and $y = 2$



[3 marks]

(iv) The line $y = x$ is a line of mirror symmetry

The graph of the function is symmetrical in $y = x$

[2 marks]

Answer 4

(i) There is one asymptote along the *y*-axis, which has equation $x = 0$

and another asymptote along the *x*-axis, which has equation $y = 0$

[2 marks]

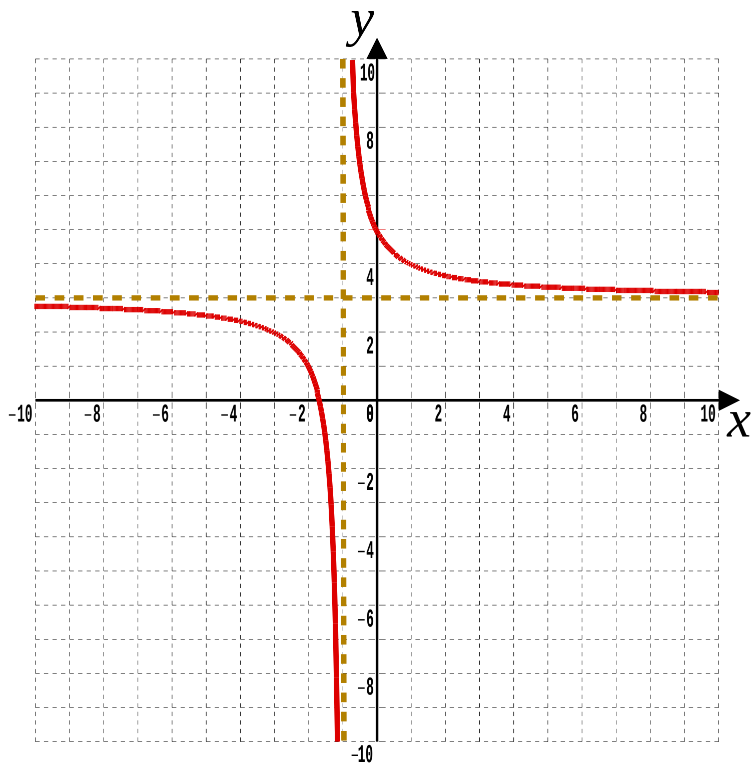
(ii) $h(x)$ is $f(x)$ translated $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ giving asymptotes $x = -1, y = 3$

[2 marks]

(iii) The domain of $h(x)$ must exclude $x = -1$ to avoid a division by zero.

[1 mark]

(iv)



[3 marks]

(v)

$$h(x) = \frac{2}{x+1} + 3$$
$$= \frac{2}{x+1} + \frac{3(x+1)}{x+1}$$
$$= \frac{3x+5}{x+1}$$

which is in the form of a Möbius transformation

[3 marks]

(vi) $$h^{-1}(x) = \frac{-x+5}{x-3} \qquad x \in \mathbb{R}, \quad x \neq 3$$

[4 marks]

Answer 5

(a) Three methods (below) are given to determine that the given function has a horizontal asymptote with equation $y = 3$ from which it can be deduced that the range of $g(x)$ is $g(x) \in \mathbb{R},\ g(x) < 3$

Methods 1 and 2

$$\begin{aligned} y &= \frac{3x - 4}{x - 3} \\ &= \frac{3x - 9 + 9 - 4}{x - 3} \\ &= \frac{3x - 9}{x - 3} + \frac{5}{x - 3} \\ &= \frac{3(x - 3)}{x - 3} + \frac{5}{x - 3} \\ &= 3 + \frac{5}{x - 3} \end{aligned}$$

Method 1

$y - 3 = \dfrac{5}{x - 3}$

This is $y = \dfrac{5}{x}$ under translation $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and so the asymptote along the x-axis moves to $y = 3$

Method 2

As $x \to \pm\infty$, $\dfrac{5}{x - 3} \to 0$ and $y \to 3$

$\therefore\ y = 3$ is a horizontal asymptote

Method 3

Testing for horizontal asymptotes; (See Homework 3)

$P(x) = 3x - 4 \ :\ deg(P(x)) = 1,\ p = 3$

$Q(x) = x - 3 \ :\ deg(Q(x)) = 1,\ q = 1$

As $deg(P(x)) = deg(Q(x))$. $y = \dfrac{p}{q}$ is a horizontal asymptote.

That is, $y = 3$ is a horizontal asymptote.

[1 mark]

(b) **(i)** When $y = 0$, $\dfrac{3x - 4}{x - 3} = 0$

$$3x - 4 = 0$$

$$x = \frac{4}{3} \qquad \therefore\ A\left(\frac{4}{3},\ 0\right)$$

[1 mark]

(ii) When $x = 0$, $y = \dfrac{-4}{-3} \qquad \therefore\ B\left(0,\ \dfrac{4}{3}\right)$

[1 mark]

(c)

$$gg(x) = g\left(\frac{3x - 4}{x - 3}\right)$$

$$= \frac{3\left(\frac{3x-4}{x-3}\right) - 4}{\left(\frac{3x-4}{x-3}\right) - 3} \times \frac{(x - 3)}{(x - 3)}$$

$$= \frac{3(3x - 4) - 4(x - 3)}{3x - 4 - 3(x - 3)}$$

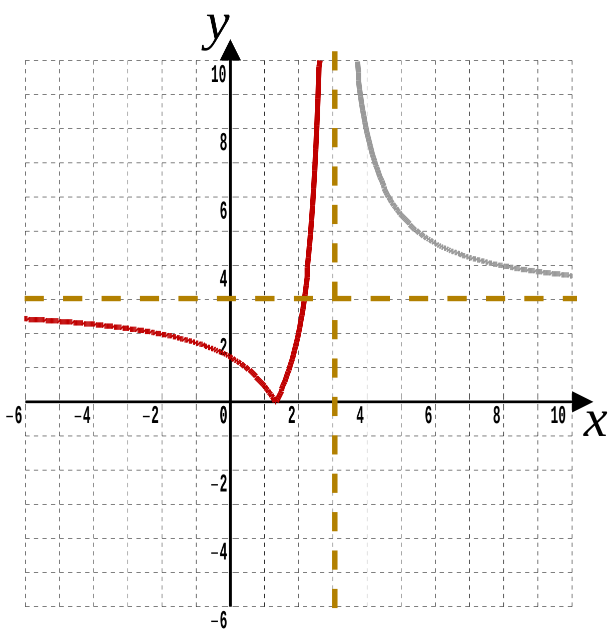
$$= \frac{9x - 12 - 4x + 12}{3x - 4 - 3x + 9}$$

$$= \frac{5x}{5}$$

$$= x$$

[3 marks]

(d)



Equations of asymptotes are $x = 3$, $y = 3$

Axes cutting or meeting points are $\left(\dfrac{4}{3},\ 0\right)$, $\left(0,\ \dfrac{4}{3}\right)$

[3 marks]

(e) $$\left| g(x) \right| = 8$$

$$\frac{3x - 4}{x - 3} = 8 \qquad\qquad -\frac{3x - 4}{x - 3} = 8$$

$$3x - 4 = 8x - 24 \qquad\qquad -3x + 4 = 8x - 24$$

$$5x = 20 \qquad\qquad 11x = 28$$

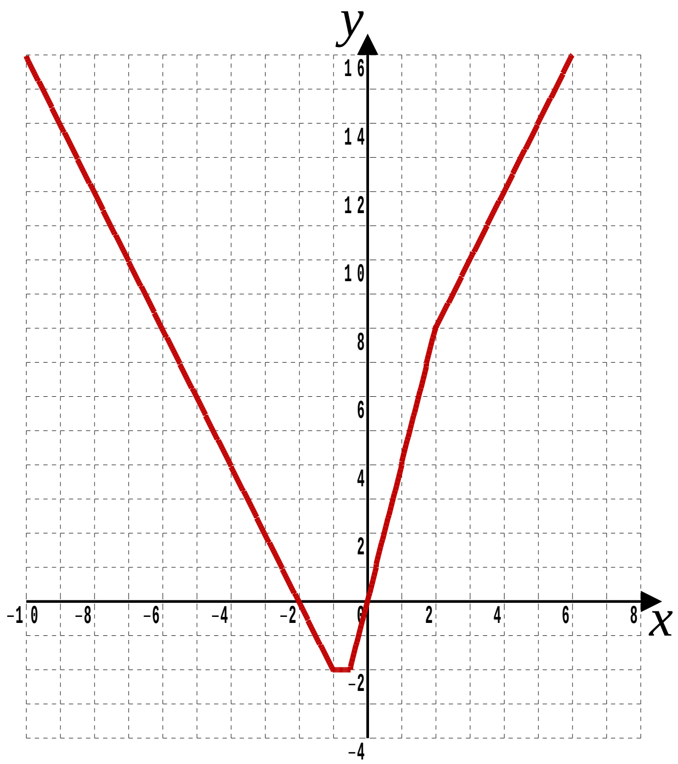
$$x = 4 \text{ (reject)} \qquad\qquad x = \frac{28}{11}$$

The only valid solution is $x = \dfrac{28}{11}$

[3 marks]

Answer 6

There will be critical values at $x = -1, \; x = -\frac{1}{2}, \; x = 2$



$$f(x) = \begin{cases} -2x - 4 & x < -1 \\ -2 & -1 \leqslant x < -\frac{1}{2} \\ 4x & -\frac{1}{2} \leqslant x < 2 \\ 2x + 4 & x \geqslant 2 \end{cases}$$

[8 marks]