A-Level Mathematics

Year 2

~ Pure Mathematics ~

F U N C T I O N S
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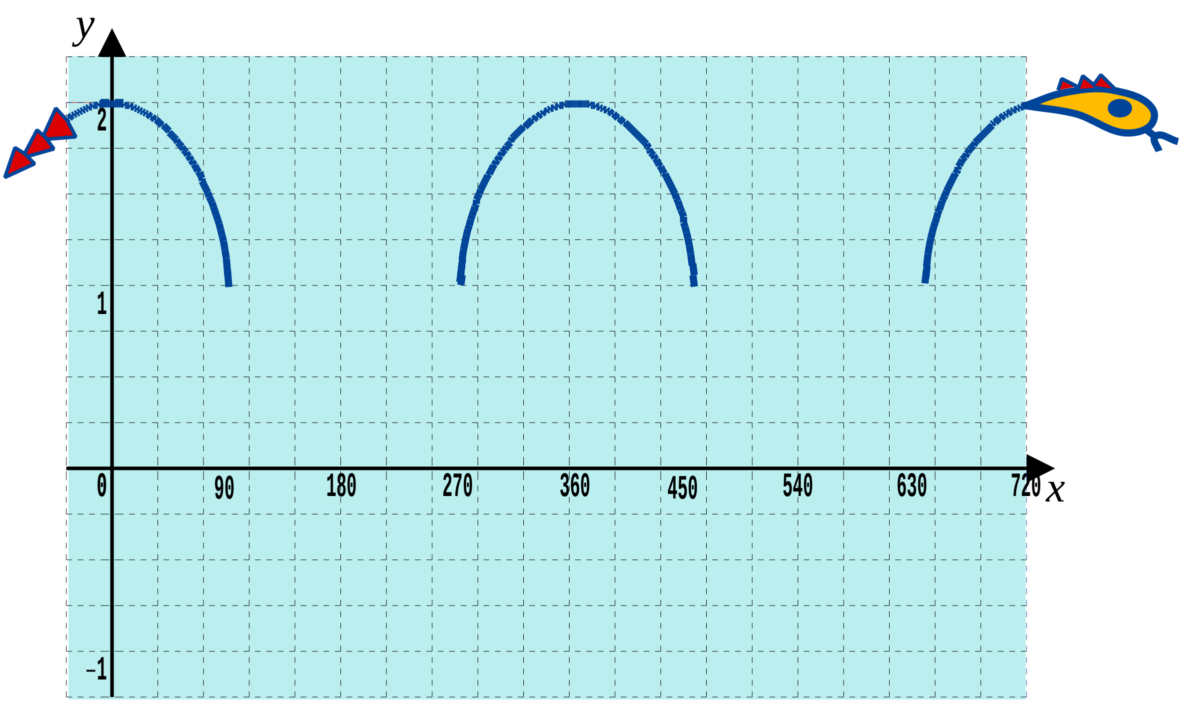


The Loch Ness Monster Function

$$y = 1 + \sqrt{\cos x}$$

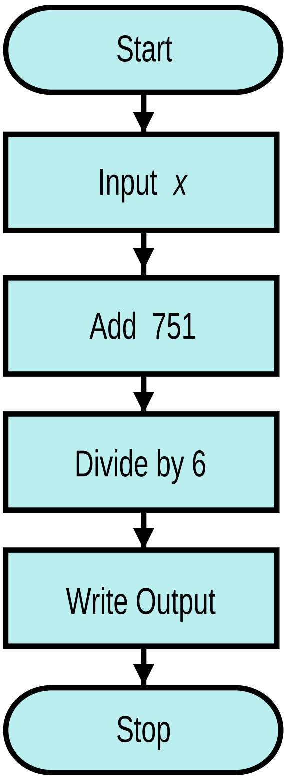
F U N C T I O N S I I

Lesson 1

1.1 What is a function ?

A basic electronic calculator is a good example of a function machine.
A single number is input, buttons are pressed, and a single number output results.

Here is a flow chart of a function.



If the number 149 is input, what is the output ?

[1 mark]

The function, f, described by the flow chart is;

$$f(x) = \frac{x + 751}{6}$$

Determine the value of, $f(-151)$

[1 mark]

1.2 Domain and Range

The set of numbers allowed into a function is referred to as the function's ***domain***.
Typically, this will be as big as possible, to maximise the function's usefulness.
The set of numbers that can come out of a function is the function's ***range***.
Often, a function's domain is given.
Sometimes, but not often, a function's range is also given.

1.2.1 Example

$$\underbrace{f(x) = \frac{12}{6-x},}_{\text{function}} \quad \underbrace{x \in \mathbb{R},\ x \neq 6,}_{\substack{\text{domain} \\ \text{(numbers allowed in)}}} \quad \underbrace{f(x) \in \mathbb{R},\ f(x) \neq 0}_{\substack{\text{range} \\ \text{(numbers coming out)}}}$$

In this example, all real numbers except 6 may be used as the input number.
The 6 is excluded because is would result in a division by zero which is not defined.
All real numbers are possible outputs, except no input will give 0 as output.

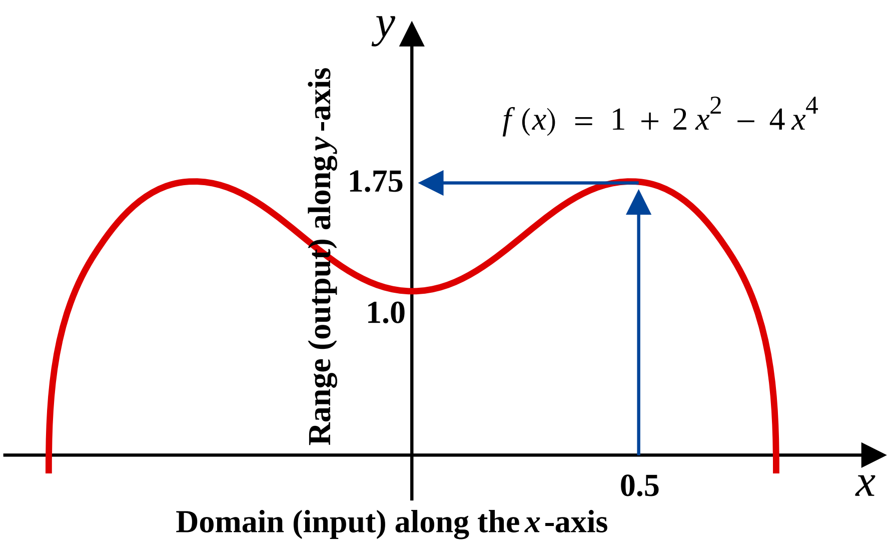
1.2.2 Example

$$f(x) = \sqrt{x-4}, \qquad x \in \mathbb{R},\ x \geqslant 4, \qquad f(x) = \mathbb{R},\ f(x) \geqslant 0$$

In this example, all real numbers greater than or equal to 4 may be input.
Numbers less than 4 are excluded to avoid the square rooting of a negative number.
All real numbers greater than or equal to 0 are possible outputs.

1.3 Graphs of functions

A graph, often a quick sketch of $y = f(x)$, is a good visualisation of a function.
The input numbers, the domain, are along the x-axis.
The output numbers, the range, are along the y-axis.



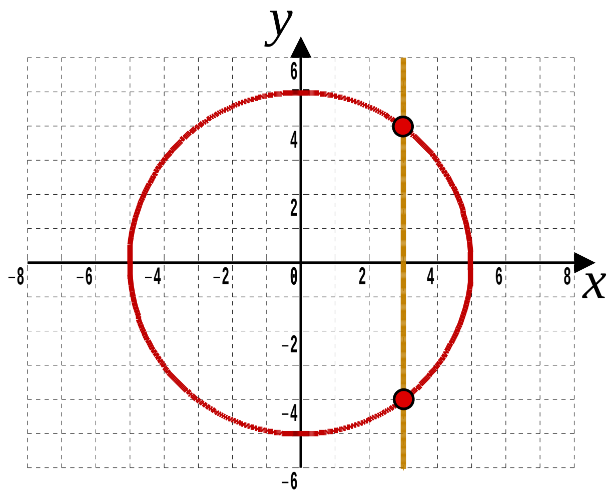
The sketch shows 0.5 going into the function and 1.75 coming out.

1.4 Is it a Function ?

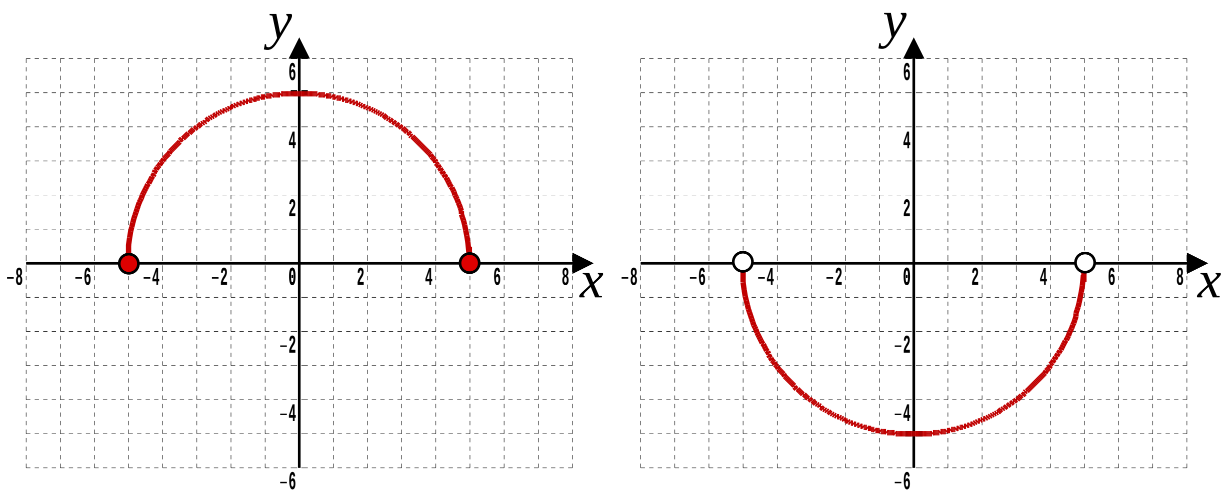
When looking at the graph of a piece of algebra, to be a function no vertical line must intersect the graph more than once.

1.4.1 Example

No entire circle can be a function because a vertical line can be found that intersects the curve more than once.



The circle $x^2 + y^2 = 5^2$ is not a function because a vertical line, for example $x = 3$, can be found that cuts its graph more than once.



However, that circle can be broken up into two separate pieces each of which is a function. Notice that the end points have been allocated to the piece depicted on the left hand graph but they could both have been allocated to the right hand graph or one given to each. In breaking up the circle, it has been made clear where they went. Here are the two semicircle functions, graphed above;

$$f(x) = \sqrt{5^2 - x^2} \qquad x \in \mathbb{R}, \quad -5 \leqslant x \leqslant 5 \qquad \text{(Left hand graph)}$$

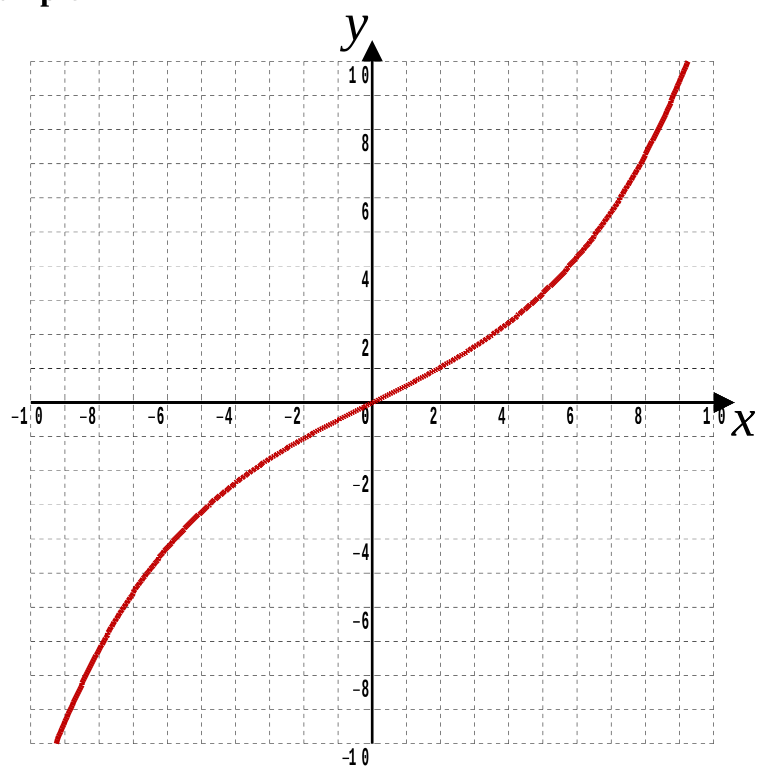
$$g(x) = -\sqrt{5^2 - x^2} \qquad x \in \mathbb{R}, \quad -5 < x < 5 \qquad \text{(Right hand graph)}$$

Notice how the endpoint allocation is captured by the inequalities of the domains.

1.5 One-to-One functions

A function is one-to-one if every input has a unique output, and vice-versa.
On the graph of the function, every horizontal line will cut the graph exactly once.

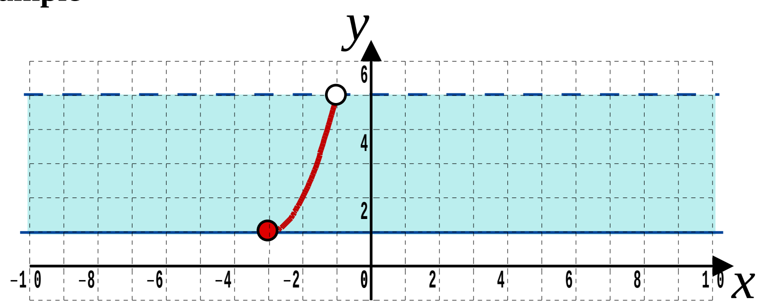
1.5.1 Example



The graph is of the function, $f(x) = e^{\frac{x}{4}} - e^{-\frac{x}{4}} \quad x \in \mathbb{R}$

The function is one-to-one because a horizontal line, no matter where it is drawn, will intersect the graph of this function exactly once.

1.5.2 Example



The graph is of the function,

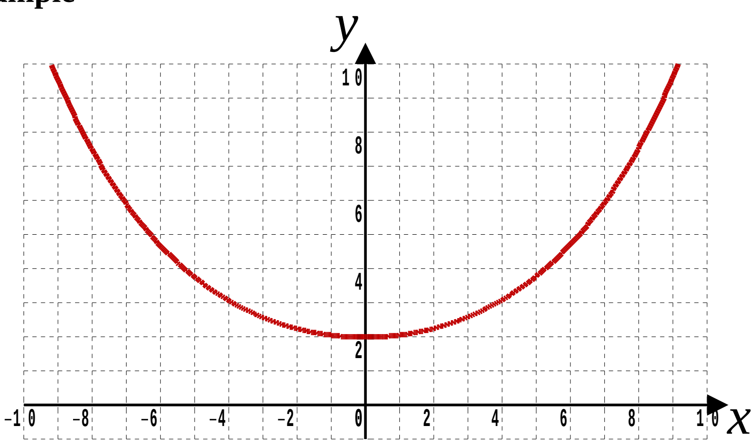
$f(x) = (x+3)^2 + 1, \ x \in \mathbb{R}, \ -3 \leqslant x < -1, \ f(x) \in \mathbb{R}, \ 1 \leqslant f(x) < 5$

The function is one-to-one because a horizontal line, no matter where it is drawn, will intersect the graph of this function exactly once, within the range (shaded blue) of the function.

1.6 Many-to-One Functions

A function is many-to-one if more than one input maps to the same output. Graphically, a horizontal line can be found that will cut the plot of the function more than once.

1.6.1 Example



The graph is of the function, $f(x) = e^{\frac{x}{4}} + e^{-\frac{x}{4}} \quad x \in \mathbb{R}$

The function is many-to-one because a horizontal line can be found that will intersect, the graph of this function more than once.

Add your own example of such a line to the graph above.

[1 mark]

1. 7 Breaking Up Functions

Mathematicians like one-to-one functions because they have many special properties. For example, only a one-to-one function has an inverse.

(This key result will be explored later, in lesson 7)

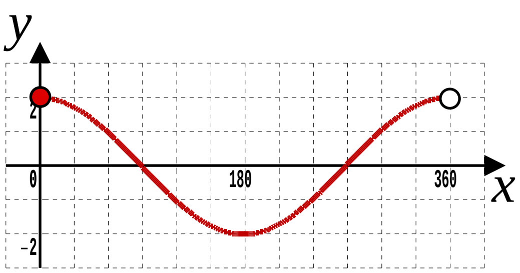
The desire to have one-to-one functions results in many-to-one functions being broken up into pieces, each of which is one-to-one.

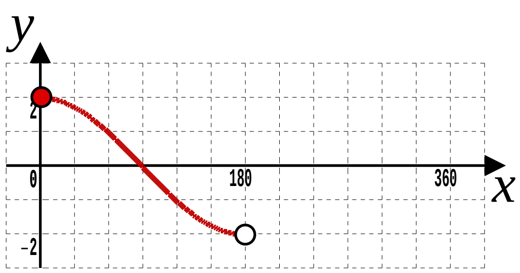
1.7.1 Example

Consider the function, $f(x) = cos\, x, \ x \in \mathbb{R}, \ 0 \leqslant x < 360°$

The graph of $y = f(x)$ shown left, reveals that this function in many-to-one.

The right hand graph shown the one-to-one piece that a calculator uses to give values for the inverse cosine function, *arccos x*





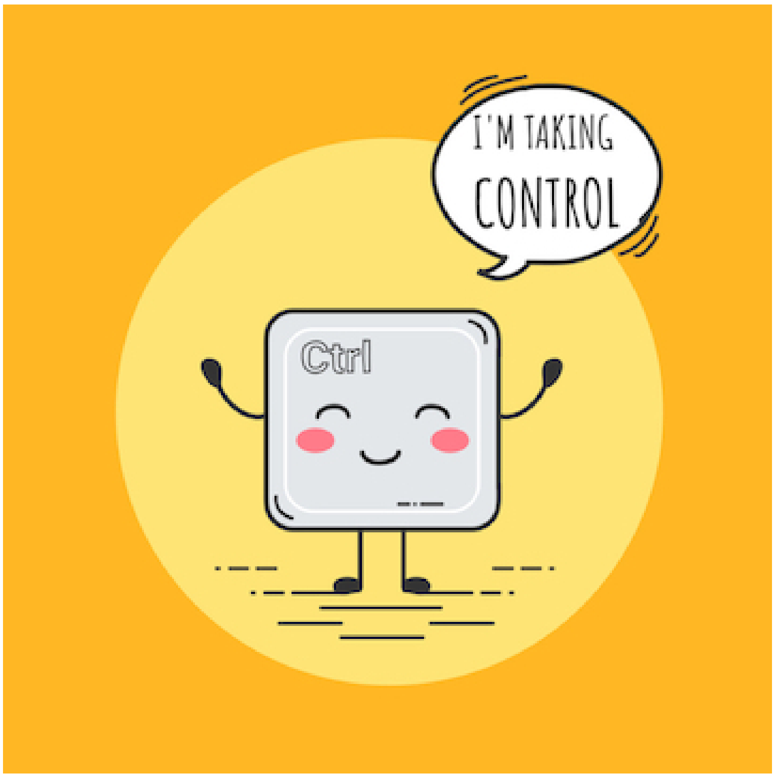
The right hand one-to-one graph is $g(x) = cos\, x, \ x \in \mathbb{R}, \ 0 \leqslant x < 180°$

1.8 The Bigger Picture

The topic of Functions may seem technical and awkward at first. The driving force behind thinking about them is this pedantic way is summed up by “Taking Control”. Being able to specify a domain allows mathematicians to shut out confusions caused by such things as “division by zero” and the “taking of negative square roots”.
(Later on, negative square roots get let back in but under tight control !)
As a further example, rather than always letting all real numbers in, there are situations where only the integers are to be worked with.

The simple statement in the domain, $x \in \mathbb{Z}$, shuts out all but the integers.

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$



1.9 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 60

Question 1

(i) Sketch the graph of the squaring function, $f(x) = x^2, \; x \in \mathbb{R}$

[2 marks]

(ii) Show that the squaring function is many-to-one by adding a horizontal line to your sketch that cuts the graph twice.

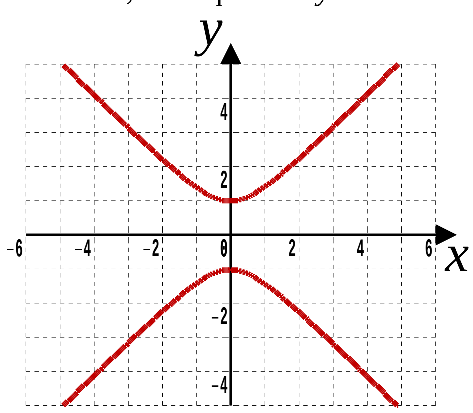
[1 mark]

(iii) What is the range of the squaring function ?

[2 marks]

Question 2

The hyperbola, graphed below, has equation $y^2 - x^2 = 1$



Explain why this cannot be the graph of a function.

[2 marks]

Question 3

State a value of x that must be excluded from the domain of the following function;

$$f(x) = \frac{1}{(1 + x)^2}$$

Give a reason for your answer.

[2 marks]

Question 4

(i) Sketch the graph of $f(x) = \cos x, \quad x \in \mathbb{R}, \quad 0 \leqslant x < 360°$

[3 marks]

(ii) Show that the function is many-to-one by adding a horizontal line to your sketch that cuts the graph more than once.

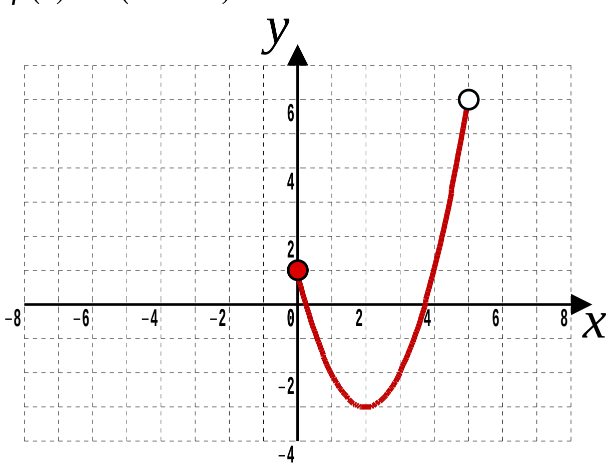
[1 mark]

(iii) What is the range of the cosine function ?

[2 marks]

Question 5

The graph is of $f(x) = (x - 2)^2 - 3$ but with a restricted domain.



Taking care over any inequalities, write down the restricted domain of this function.

[2 marks]

Question 6

(i) Sketch the graph of $f(x) = tan\ x$ over the interval $0° \leqslant x \leqslant 360°$

[2 marks]

(ii) Show that the tangent function is many-to-one by adding a to your sketch a horizontal line that cuts the graph more than once.

[1 mark]

(iii) State the values of x that must be excluded from the domain of $f(x) = tan\ x$ over the interval $0° \leqslant x \leqslant 360°$
Give a reason for your answer.

[3 marks]

Question 7

(i) Sketch the cubing function, $f(x) = x^3, \quad x \in \mathbb{R}$

[2 marks]

(ii) Is the cubing function many-to-one or one-to-one ?
Give a reason for your answer.

[2 marks]

Question 8

Here is the description of a hybrid function made from two pieces.

$$f(x) = \begin{cases} \sqrt{x} & 0 \leqslant x < 1 \\ 2 - x & 1 \leqslant x \leqslant 4 \end{cases}$$

This has domain $x \in \mathbb{R},\ 0 \leqslant x \leqslant 4$

(i) Sketch the graph of $f(x)$

[3 marks]

(ii) Show that $f(x)$ is many-to-one by adding a horizontal line to your sketch that cuts the graph more than once.

[1 mark]

(iii) State the range of $f(x)$

[3 marks]

Question 9

State the range of the function

$$f(x) = 3 + \cos x, \quad x \in \mathbb{R}$$

(A sketch may help)

[3 marks]

Question 10

Consider the function, $f(x) = x^2 - 2x + 9, \qquad x \in \mathbb{R}$

(i) Express $f(x)$ in the completed square form.

That is, in the form $f(x) = (x - a)^2 + b$

for some constants a and b which are to be determined.

[2 marks]

(ii) Hence, or otherwise, sketch the graph of $f(x)$ paying particular attention to the coordinates of the minimum point.

[2 marks]

(iii) Hence, or otherwise, state the range of the function.

[2 marks]

Question 11

$$f(x) = \sqrt{1 - 4x}$$

(i) State the domain of $f(x)$, assuming it is as large as possible.

[2 marks]

(ii) State the range of $f(x)$.

[2 marks]

Question 12

$$f(x) = (x - 2)^2 + 5, \qquad x \in \mathbb{R}, \qquad 1 \leqslant x \leqslant 5$$

(i) Sketch $f(x)$ paying particular attention to the vertex of the parabola and the points at either end of the closed interval on which it is defined.

[3 marks]

(ii) State the range of $f(x)$

[2 marks]

(iii) Solve, $f(x) = 9$

[2 marks]

Question 13

A-Level Examination Question from January 2019, paper C3, Q3(a) (Edexcel)

The function f is defined by,

$$f : x \rightarrow 2x^2 + 3kx + k^2 \quad x \in \mathbb{R}, \quad -4k \leqslant x \leqslant 0$$

where k is a positive constant.

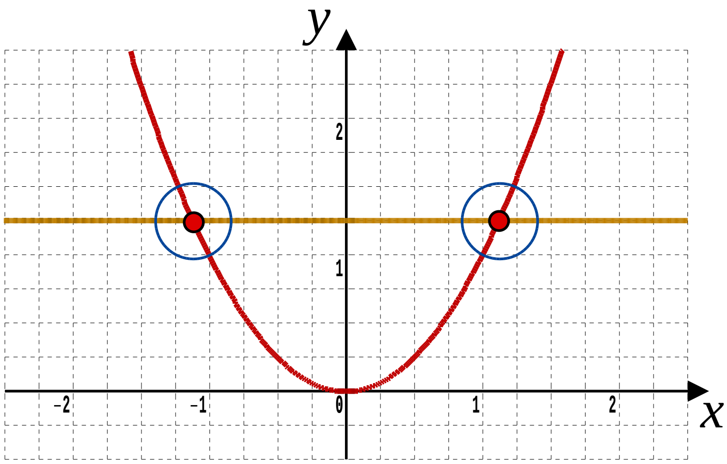
Find, in terms of k, the range of f

[6 marks]

1.10 Answers

1.10.1 Solutions to Exercise 1.9

Answer 1
(i), (ii)



$y = f(x)$ (red) with a horizontal line (gold) cutting $f(x)$ more than once

[2, 1 marks]

(iii) Range: $f(x) \in \mathbb{R},\ f(x) \geqslant 0$
"All real numbers that are non-negative"

[2 marks]

Answer 2
Every vertical line that can be drawn that cuts the hyperbola more than once.
To be a function, every vertical line (within the function's domain) cuts the graph of the function precisely once.

[2 marks]

Answer 3

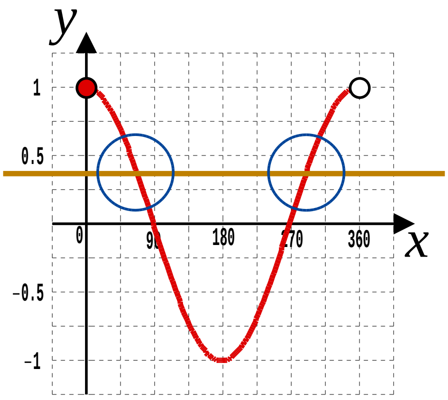
Excluded from the domain: $x = -1$

Reason : If included it would cause a division by zero.

[2 marks]

Answer 4

(i), (ii)



$y = f(x)$ (red) with a horizontal line (gold) cutting $f(x)$ more than once

[3, 1 marks]

(iii) Range: $f(x) \in \mathbb{R}, \quad -1 \leqslant f(x) \leqslant 1$

"All real numbers between – 1 and 1 inclusive"

[2 marks]

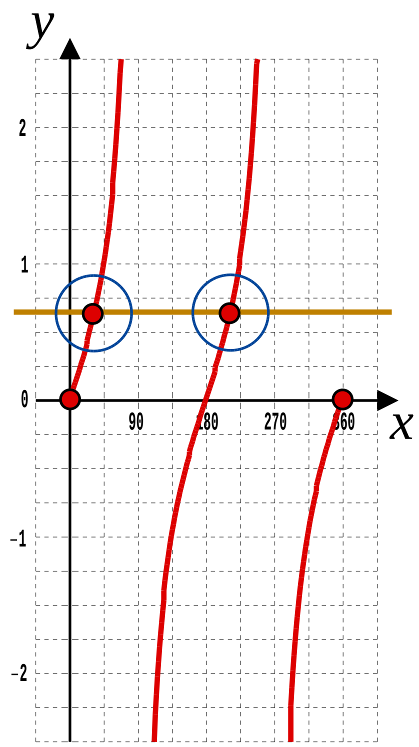
Answer 5

Domain: $x \in \mathbb{R}, \quad 0 \leqslant x < 5$

[2 marks]

Answer 6

(i), (ii)



$y = f(x)$ (red) with a horizontal line (gold) cutting $f(x)$ more than once

[2, 1 marks]

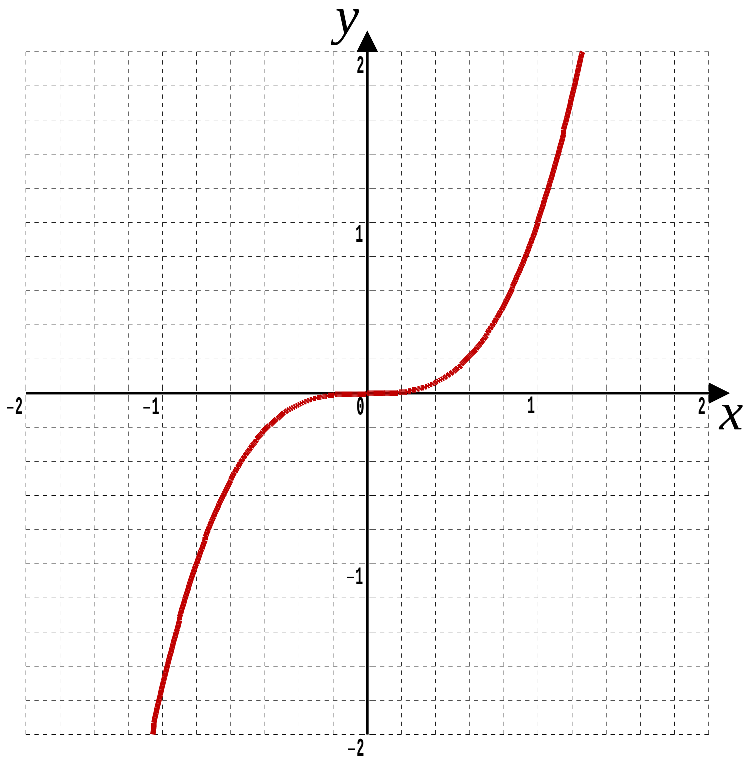
(iii) Domain: $x \in \mathbb{R},\ 0 \leqslant x \leqslant 360°,\ x \neq 90°,\ 270°$

$tan\,x = \dfrac{sin\,x}{cos\,x}$ but $cos\,x = 0$ when $x = 90°$ or $x = 270°$

The need to exclude 90° and 270° from the domain of *tan x* is thus, once again, arising from the need to avoid division by zero.

[3 marks]

Answer 7
(i)



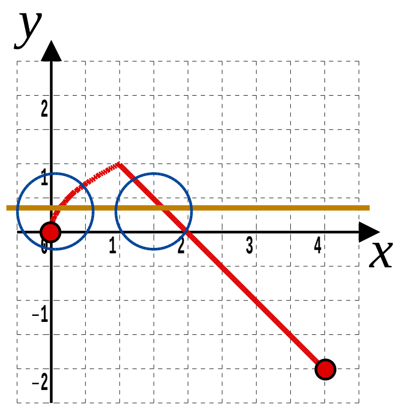
$y \; = \; f\,(x)$ (red). No horizontal line cuts $f\,(x)$ more than once

[2 marks]

(ii) The cubing function is one-to-one.

[2 marks]

Answer 8
(i), (ii)



$y \; = \; f\,(x)$ (red) with a horizontal line (gold) cutting $f\,(x)$ more than once

[3, 1 marks]

(iii) Range: $f\,(x) \in \mathbb{R}, \quad -\,2 \leqslant f\,(x) \leqslant 1$

[3 marks]

Answer 9

Range: $f(x) \in \mathbb{R},\ 2 \leqslant f(x) \leqslant 4$

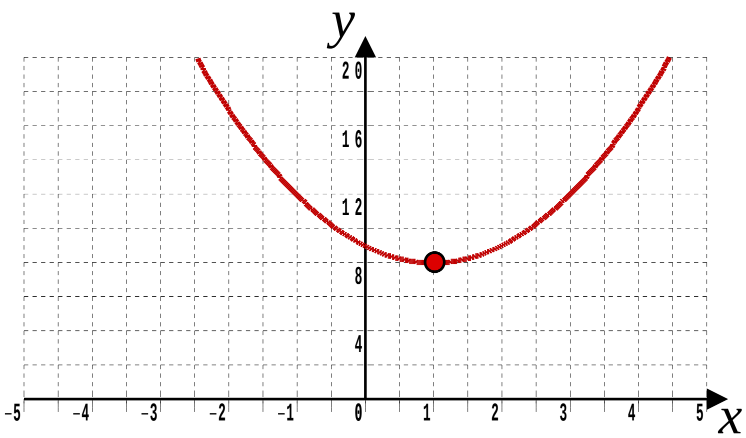
[3 marks]

Answer 10

(i) $f(x) = (x - 1)^2 + 8$ That is, $a = 1$ and $b = 8$

[2 marks]

(ii)

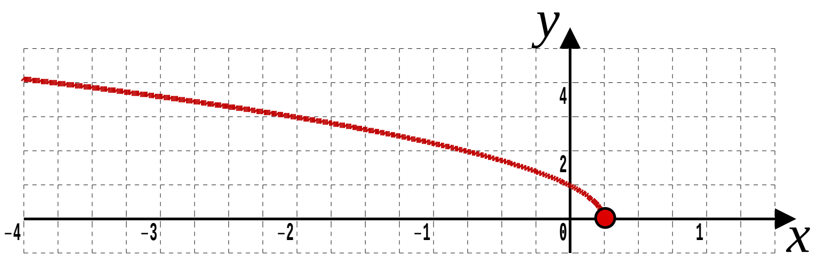


[2 marks]

(iii) Range: $f(x) \in \mathbb{R},\ f(x) \geqslant 8$

[2 marks]

Answer 11



(i) Domain: $x \in \mathbb{R},\ x \leqslant \dfrac{1}{4}$

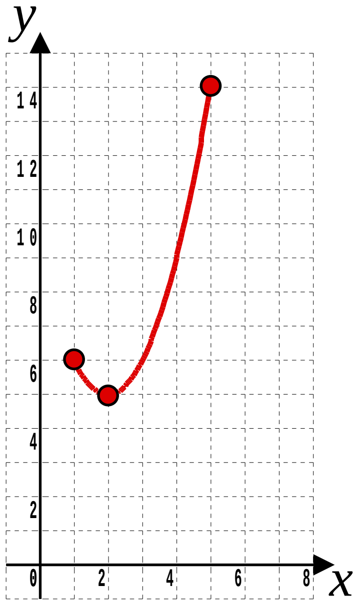
[2 marks]

(ii) Range: $f(x) \in \mathbb{R},\ f(x) \geqslant 0$

[2 marks]

Answer 12

(i)



[3 marks]

(ii) Range: $f(x) \in \mathbb{R}, \quad 5 \leqslant f(x) \leqslant 14$

[2 marks]

(iii)

$$f(x) = 9$$
$$(x - 2)^2 + 5 = 9$$
$$x^2 - 4x + 4 + 5 = 9$$
$$x^2 - 4x = 0$$
$$x(x - 4) = 0$$

Either $x = 0$ (rejected) or $x = 4$ (the only solution).

[2 marks]

Answer 13
Here are two methods of finding the local minimum point.

Method 1 : Completing The Square

$$
\begin{aligned}
f(x) &= 2x^2 + 3kx + k^2 \\
&= \left[\, 2x^2 + 3kx \,\right] + k^2 \\
&= 2\left[\, x^2 + \frac{3k}{2}x \,\right] + k^2 \\
&= 2\left[\, \left(x + \frac{3k}{4}\right)^2 - \frac{9k^2}{16} \,\right] + k^2 \\
&= 2\left(x + \frac{3k}{4}\right)^2 - \frac{9k^2}{8} + \frac{8k^2}{8} \\
&= 2\left(x + \frac{3k}{4}\right)^2 - \frac{k^2}{8}
\end{aligned}
$$

$$\therefore \text{ Minimum is } \left(-\frac{3k}{4}, -\frac{k^2}{8}\right)$$

Method 2 : Differentiation

$$
\begin{aligned}
f(x) &= 2x^2 + 3kx + k^2 \\
f'(x) &= 4x + 3k \\
\text{At the minimum point, } f'(x) &= 0 \\
4x + 3k &= 0 \\
x &= -\frac{3k}{4} \\
f\left(-\frac{3k}{4}\right) &= 2\left(-\frac{3k}{4}\right)^2 + 3k\left(-\frac{3k}{4}\right) + k^2 \\
&= \frac{9k^2}{8} - \frac{9k^2}{4} + k^2 \\
&= \frac{9k^2 - 18k^2 + 8k^2}{8} \\
&= -\frac{k^2}{8}
\end{aligned}
$$

$$\therefore \text{ Minimum is } \left(-\frac{3k}{4}, -\frac{k^2}{8}\right)$$

As $-\frac{3k}{4}$ lies in the domain $(-4k \leqslant x \leqslant 0)$ the $-\frac{k^2}{8}$ is the minimum value.

Each endpoint of the segment of curve now needs to be determined to see which is the maximum value of the function.

For the endpoint with an x value of $-4k$,

$$\begin{aligned} f\left(-4k\right) &= 2(-4k)^2 + 3k(-4k) + k^2 \\ &= 32k^2 - 12k^2 + k^2 \\ &= 21k^2 \end{aligned}$$

For the endpoint with an x value of 0,

$$\begin{aligned} f(0) &= 2 \times 0^2 + 3k \times 0 + k^2 \\ &= k^2 \end{aligned}$$

So the maximum value is $21k^2$

The range is thus $f(x) \in \mathbb{R},\ -\frac{k^2}{8} \leqslant f(x) \leqslant 21k^2$

[6 marks]

2.1 The Modulus Function (Part 1)

$|x|$ means "*magnitude of x*" although mathematicians also say "*modulus of x*".

When the modulus function applies to a positive quantity is leaves it alone.
When applied to a negative quantity it makes it positive.

For example, $|45| = 45$

$|-62| = 62$

2.2 Modulus Sketching Rule 1

To sketch $y = |f(x)|$ **Bounce negative x**

◇ Sketch $y = f(x)$ using a dashed line for points below the x-axis.

◇ Reflect any part of the curve below the x-axis in the x-axis.

2.3 Example

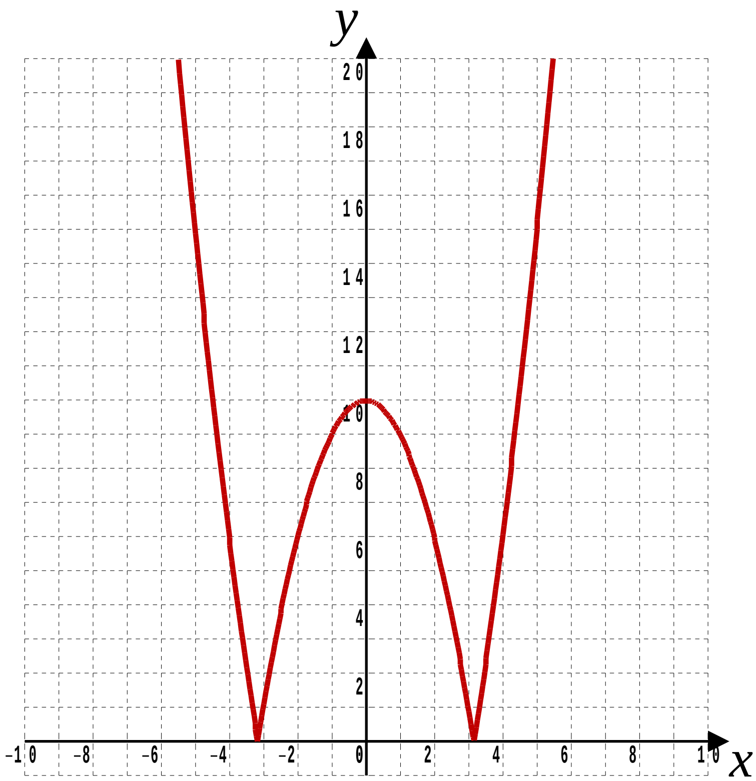
Sketch the curve,

(i) $y = x^2 - 4$ **(ii)** $y = |x^2 - 4|$

[2, 2 marks]

2.4 Solving Modulus Equations: An Example

The graph is of the function, $f(x) = |x^2 - 10|, \quad x \in \mathbb{R}$



(i) From the graph, write down all the solutions to the equation $f(x) = 6$

[2 marks]

(ii) Obtain the part (i) solutions using algebra.

[4 marks]

(iii) To the graph add a plot of the function $g(x) = x + 10$

[1 mark]

(iv) Estimate, graphically, the solutions to the equation;

$$f(x) = g(x)$$

That is, $\left| x^2 - 10 \right| = x + 10$

[2 marks]

(v) Solve the equation $f(x) = g(x)$ using algebra.

[4 marks]

2.5 Watch Out !

Extraneous Solutions
When solving modulus equations it is possible to generate extra solutions and so a check that each solution satisfies the original equation is always advised.

2.6 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 40

Question 1

Solve these equations,

(i) $|x - 4| = 3, \quad x \in \mathbb{R}$

[2 marks]

(ii) $|x - 7| = 15, \quad x \in \mathbb{R}$

[2 marks]

(iii) $|9 - x| = 2, \quad x \in \mathbb{R}$

[2 marks]

Question 2

(i) Sketch the graph of $y = |\sin x| \quad x \in \mathbb{R}, \quad 0° \leqslant x \leqslant 360°$

[2 marks]

(ii) Hence, or otherwise, solve the equation;

$$|\sin x| = 0.5, \quad x \in \mathbb{R}, \quad 0° \leqslant x \leqslant 360°$$

[2 marks]

Question 3
A function *f* is defined by;

$$f \ : \ x \rightarrow |x - 2| - 3, \qquad x \in \mathbb{R}$$

Solve the equation, $f(x) = 1$

[3 marks]

Question 4
Find the exact values of *x* for which ;

$$\left| \frac{2}{x - 3} \right| = 3$$

[3 marks]

Question 5
Solve the equation;

$$2 - |x + 1| = \frac{1}{2} x$$

[3 marks]

Question 6
Find the exact solutions of the equation;

$$4 - x^2 = |2x - 1|$$

[5 marks]

Question 7

Solve the equation;

$$\left| |x - 4| - 2 \right| = 1$$

[5 marks]

Question 8

(i) Explain why the graph of a function of the form

$$y = f(\,|x|\,)$$

has mirror symmetry in the *y*-axis.

[2 marks]

(ii) Sketch the graph of;

$$y = sin(\,|x|\,) \qquad x \in \mathbb{R}, \qquad -360° \leqslant x \leqslant 360°$$

[2 marks]

(iii) Hence, or otherwise, write down the solutions to the equation;

$$sin(\,|x|\,) = \frac{\sqrt{3}}{2} \qquad x \in \mathbb{R}, \qquad -360° \leqslant x \leqslant 360°$$

[2 marks]

Question 9

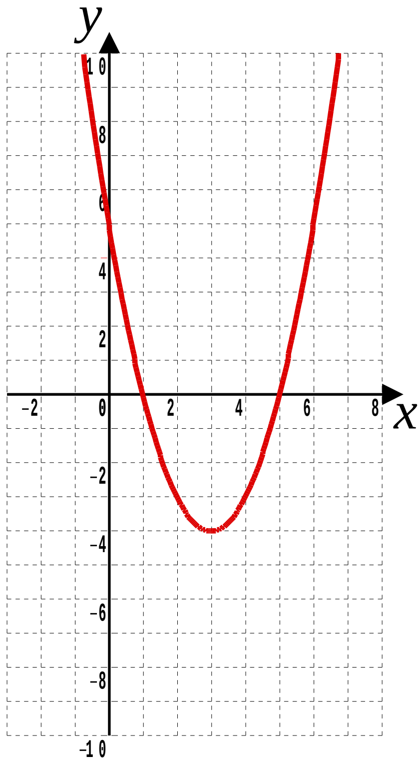
Solve the equation;

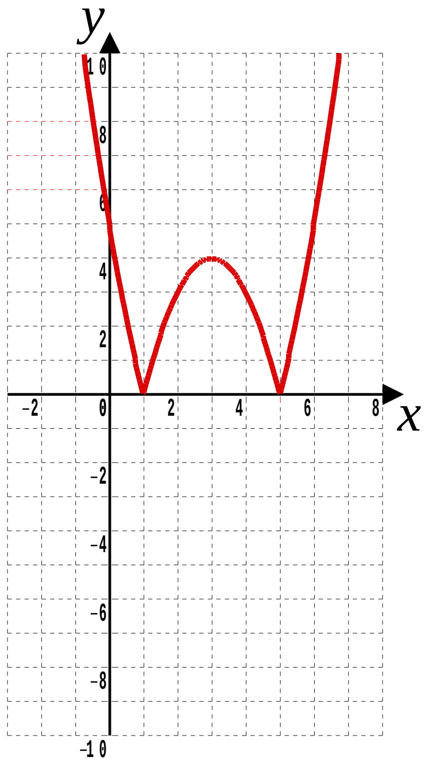
$$\left| x^2 - \pi \right| = x + \pi$$

[5 marks]

2.7 Answers

2.7.1 Solution to 2.3 Example

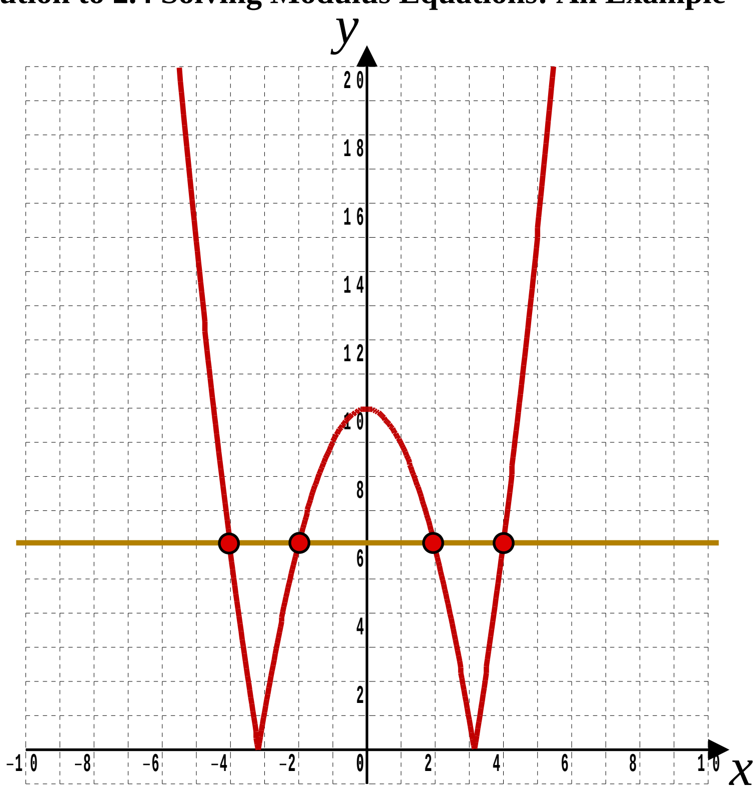




$y = x^2 - 4$ $\qquad$ $y = |x^2 - 4|$

[2, 2 marks]

2.7.2 Solution to 2.4 Solving Modulus Equations: An Example



(i) From the graph provided (red) with the addition of the line $y = 6$ (gold) the solutions are $x = \pm 4, \; x = \pm 2$

[2 marks]

(ii) Strategy: Replace the modulus with ordinary brackets and consider the two possible signs in front of those brackets.

$$+\left(x^2 - 10\right) = 6 \qquad\qquad -\left(x^2 - 10\right) = 6$$

$$x^2 = 16 \qquad\qquad -x^2 = -4$$

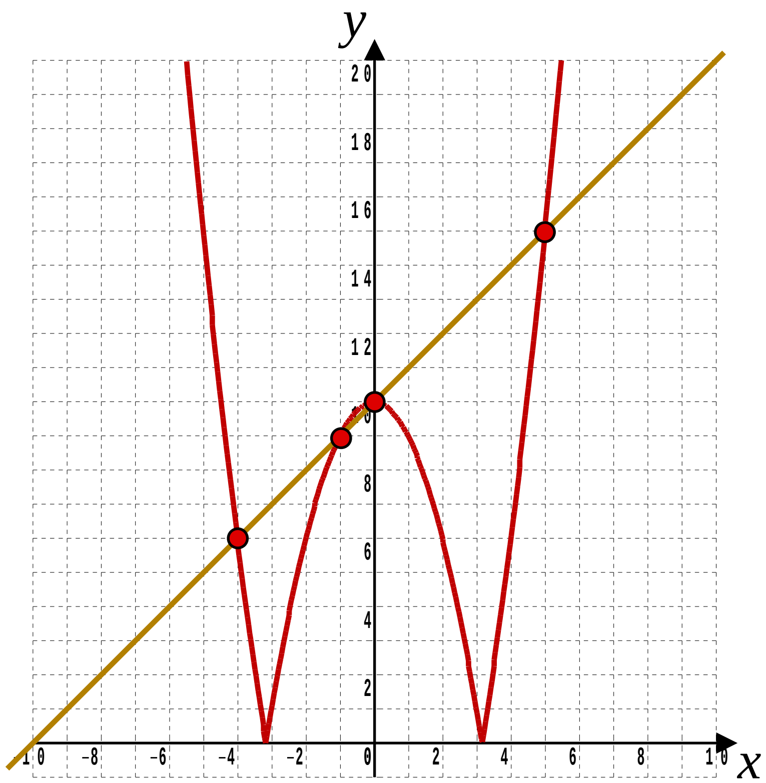
$$x = \pm 4 \qquad\qquad x^2 = 4$$

$$x = \pm 2$$

$$x \in \{\pm 2, \pm 4\}$$

[4 marks]

(iii)



[1 mark]

(iv) Part (iii) graph gives, $x = -4, \; x = -1, \; x = 0, \; x = 5$

[2 marks]

(v)

$$\left|x^2 - 10\right| = x + 10$$

$$+\left(x^2 - 10\right) = x + 10 \qquad\qquad -\left(x^2 - 10\right) = x + 10$$

$$x^2 - x - 20 = 0 \qquad\qquad x^2 + x = 0$$

$$(x + 4)(x - 5) = 0 \qquad\qquad x(x + 1) = 0$$

$$x = -4 \text{ or } x = 5 \qquad \text{or} \qquad x = 0 \text{ or } x = -1$$

[4 marks]

2.7.2 Solution to 2.6 Exercise

Answer 1

(i) $|x - 4| = 3, \quad x \in \mathbb{R}$

$$-(x - 4) = 3 \qquad\qquad +(x - 4) = 3$$
$$-x + 4 = 3 \qquad\qquad x = 7$$
$$x = 1$$
$$x \in \{1,\ 7\}$$

[2 marks]

(ii) $|x - 7| = 15, \quad x \in \mathbb{R}$

$$-(x - 7) = 15 \qquad\qquad +(x - 7) = 15$$
$$-x + 7 = 15 \qquad\qquad x = 22$$
$$x = -8$$
$$x \in \{-8,\ 22\}$$

[2 marks]

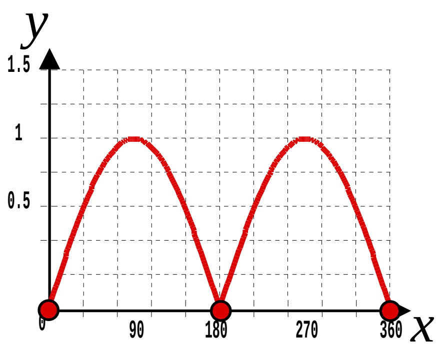
(iii) $|9 - x| = 2, \quad x \in \mathbb{R}$

$$-(9 - x) = 2 \qquad\qquad +(9 - x) = 2$$
$$-9 + x = 2 \qquad\qquad -x = -7$$
$$x = 11 \qquad\qquad x = 7$$
$$x \in \{7,\ 11\}$$

[2 marks]

Answer 2

(i)



[2 marks]

(ii) $-(\sin x) = 0.5 \qquad\qquad +(\sin x) = 0.5$

$$\sin x = -0.5 \qquad\qquad x = 30,\ 150$$
$$x = 210,\ 330$$
$$x \in \{30,\ 150,\ 210,\ 330\}$$

[2 marks]

Answer 3

$$|x - 2| - 3 = 1$$

$$-(x - 2) - 3 = 1 \qquad\qquad +(x - 2) - 3 = 1$$

$$-x + 2 - 3 = 1 \qquad\qquad x = 6$$

$$x = -2$$

$$x \in \{-2,\ 6\}$$

[3 marks]

Answer 4

$$\left|\frac{2}{x - 3}\right| = 3$$

$$-\left(\frac{2}{x - 3}\right) = 3 \qquad\qquad +\left(\frac{2}{x - 3}\right) = 3$$

$$-2 = 3(x - 3) \qquad\qquad 2 = 3(x - 3)$$

$$-2 = 3x - 9 \qquad\qquad 2 = 3x - 9$$

$$3x = 7 \qquad\qquad 3x = 11$$

$$x = \frac{7}{3} \qquad\qquad x = \frac{11}{3}$$

$$x \in \left\{\frac{7}{3},\ \frac{11}{3}\right\}$$

[3 marks]

Answer 5

$$2 - |x + 1| = \frac{1}{2}x$$

$$2 - (x + 1) = \frac{1}{2}x \qquad\qquad 2 + (x + 1) = \frac{1}{2}x$$

$$4 - 2x - 2 = x \qquad\qquad 4 + 2x + 2 = x$$

$$2 = 3x \qquad\qquad x = -6$$

$$x = \frac{2}{3}$$

$$x \in \left\{-6,\ \frac{2}{3}\right\}$$

[3 marks]

Answer 6

$$4 - x^2 = |2x - 1|$$

Case 1 :

$$4 - x^2 = -(2x - 1)$$

$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

$$x = -1 \text{ or } x = 3$$

Case 2 :

$$4 - x^2 = +(2x - 1)$$

$$x^2 + 2x - 5 = 0$$

$$x = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times (-5)}}{2}$$

$$= \frac{-2 \pm \sqrt{24}}{2}$$

$$= -1 \pm \sqrt{6}$$ exact solutions asked for, but,

as approximations, $x = -3.449$ or $x = 1.449$

Combining cases,

$$x \in \{-1 \pm \sqrt{6}, -1, 3\}$$

[5 marks]

Answer 7

$$||x - 4| - 2| = 1$$

Case 1 :

$$-(|x - 4| - 2) = 1$$

$$-|x - 4| + 2 = 1$$

$$|x - 4| = 1$$

$$x = 3 \text{ or } x = 5$$

Case 2 :

$$(|x - 4| - 2) = 1$$

$$|x - 4| = 3$$

$$x = 1 \text{ or } x = 7$$

Combining cases,

$$x \in \{1, 3, 5, 7\}$$

[5 marks]

Answer 8

(i) Let a be a given positive value of x, in which case,

$$f \mid -a \mid \;=\; f \mid a \mid$$

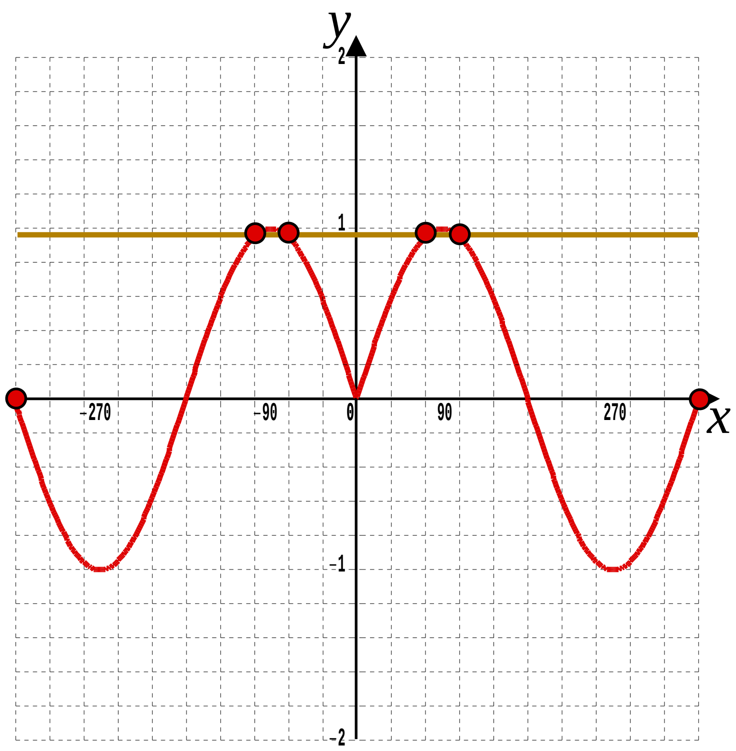
which has mirror symmetry in the y-axis.

An Even Function

In general if $f\,(-\,a) \;=\; f\,(a)$
then the function is described as being "even"
and has mirror symmetry in the *y-axis*

[2 marks]

(ii)



[2 marks]

(iii) $x \;\in\; \{-\,120,\; -\,60,\;\; 60,\;\; 120\;\}$

[2 marks]

Extra :

An Odd Function

In general if $f\,(-\,a) \;=\; -\,f\,(a)$
then the function is described as being "odd"
and has half-turn rotational symmetry about the origin

Answer 9

$$\left| x^2 - \pi \right| = x + \pi$$

Case 1 :

$$-\left(x^2 - \pi \right) = x + \pi$$
$$-x^2 + \pi = x + \pi$$
$$x^2 + x = 0$$
$$x\ (x + 1) = 0$$
$$x = -1 \text{ or } x = 0$$

Case 2 :

$$+\left(x^2 - \pi \right) = x + \pi$$
$$x^2 - x - 2\pi = 0$$
$$x = \frac{1 \pm \sqrt{1 - 4 \times 1 \times (-2\pi)}}{2}$$
$$= \frac{1 \pm \sqrt{1 + 8\pi}}{2}$$
$$= -2.056 \text{ or } 3.056$$

Combining cases,

$$x \in \{-2.056, -1, 0, 3.056\}$$

[5 marks]

3.1 Transformation Of Graphs (Part 1)

All mathematicians have a knowledge of the shape of many basic graphs. Attached to each of these graphs is the algebraic description of that graph.

It's useful to be able to start with one of these basic graphs, transform it, and then write down the algebra that describes the new graph. Alternatively, given a new piece of algebra, spot if it's the transformation of a basic graph.

The transformations to be considered include

- ◇ Translations
- ◇ Reflections
- ◇ Stretches parallel to the x-axis
- ◇ Stretches parallel to the y-axis

Note : If stretches parallel to the x-axis and y-axis of equal scale factor are applied to a graph then the result is an enlargement.

3.2 An Example Utilising Existing Knowledge

A circle, centre (0, 0) and radius r has equation;

$$x^2 + y^2 = r^2$$

A circle, centre (4, 7) and radius r has equation;

$$(x - 4)^2 + (y - 7)^2 = r^2$$

The general principal is this;

◇ Replacing all occurrences of x with $(x - a)$ in ***any*** equation translates the graph of the equation $\begin{pmatrix} a \\ 0 \end{pmatrix}$

◇ Replacing all occurrences of y with $(y - b)$ in ***any*** equation translates the graph of the equation $\begin{pmatrix} 0 \\ b \end{pmatrix}$

For the example of the circle with centre (a, b) and radius r, the equation is thus;

$$(x - a)^2 + (y - b)^2 = r^2$$

which was used in the Year 1 part of the A-level course but without viewing it from this broader perspective.

Note : A circle is not a function because a vertical line can be found that cuts its graph in more than one point but these results apply ***all*** equations.

3.3 An Inverse Proportion Example

Consider this inverse proportion function;

$$f(x) = \frac{12}{x} \qquad x \in \mathbb{R},\ x \neq 0, \qquad f(x) \in \mathbb{R},\ f(x) \neq 0$$

(a) Sketch the graph of $y = f(x)$

[2 marks]

(b) Sketch the graph of the equation,

$$y - 3 = \frac{12}{x + 5}$$

[2 marks]

(c) Consider the function given by;

$$g(x) = \frac{12}{x + 5} + 3$$

(i) State the domain of $g(x)$

[1 mark]

(ii) State the range of $g(x)$

[1 mark]

3.4 What is an Asymptote ?

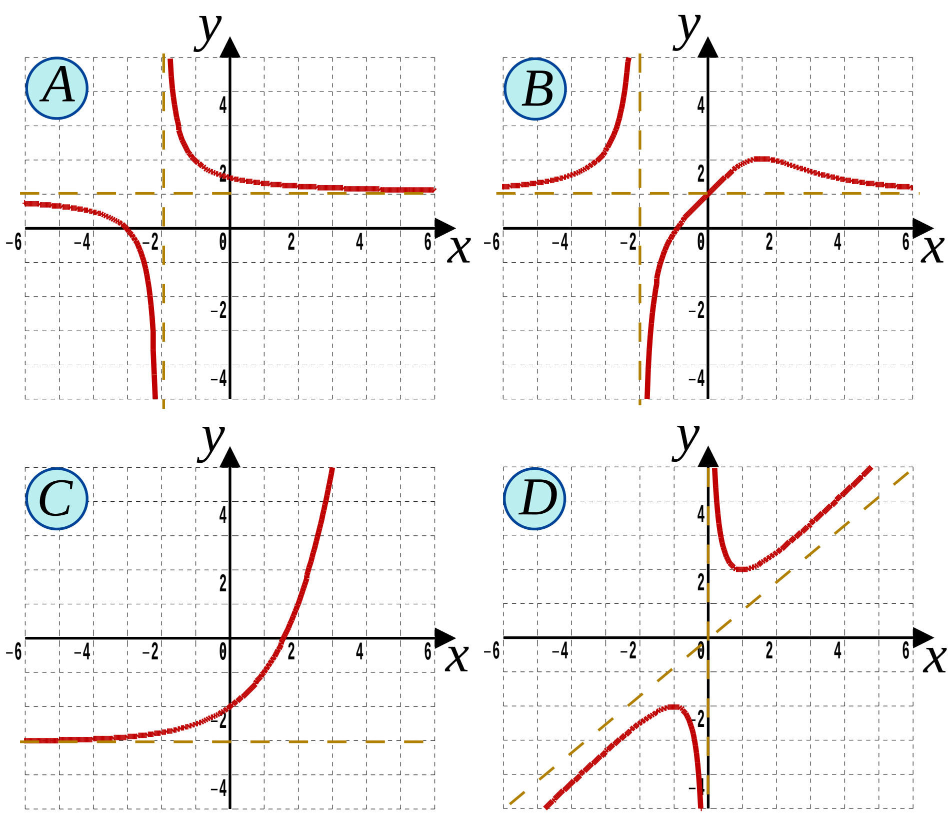
In the inverse proportion example the vertical and horizontal asymptotes were particularly useful when it came to translating the graph.

A top tip when a graph with asymptotes has to be translated is to use the asymptotes. However, what exactly is an asymptote ?

Definition : Asymptote

An asymptote of a curve is a straight broken line such that the distance between the curve and the line approaches zero as one, or both, of the *x* or *y* coordinates tends to plus infinity or minus infinity.

3.4.1 Examples



$$A \;:\; y - 1 = \frac{1}{x+2} \quad : \text{ Vertical asymptote } x = -2, \text{ Horizontal asymptote } y = 1$$

$$B \;:\; y - 1 = \frac{8x}{x^3+8} \quad : \text{ Vertical asymptote } x = -2, \text{ Horizontal asymptote } y = 1$$

$$C \;:\; y + 3 = 2^x \quad : \text{ No vertical asymptote , Horizontal asymptote } y = -3$$

$$D \;:\; y = x + \frac{1}{x} \quad : \text{ Vertical asymptote } x = 0, \text{ Oblique asymptote } y = x$$

3.5 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 50

Question 1

Consider the parabola;

$$f(x) = x^2 \qquad x \in \mathbb{R}$$

(a) Sketch the graph of $y = f(x)$

[1 mark]

(b) Sketch the graph of the equation,

$$y + 7 = (x + 4)^2$$

[2 marks]

(c) Consider now the function given by;

$$g(x) = (x + 4)^2 - 7$$

(i) State the domain of $g(x)$

[1 mark]

(ii) State the range of $g(x)$

[1 mark]

Question 2

Consider the positive square root curve;

$$f(x) = \sqrt{x} \qquad x \in \mathbb{R},\ x \geqslant 0$$

(a) Sketch the graph of $y = f(x)$

[2 marks]

(b) Sketch the graph of the equation,

$$y + 1 = \sqrt{x + 2}$$

[2 marks]

(c) Consider now the function given by;

$$g(x) = \sqrt{x + 2} - 1$$

(i) State the domain of $g(x)$

[1 mark]

(ii) State the range of $g(x)$

[1 mark]

Question 3

Consider the trigonometric curve;

$$f(x) = \sin x \qquad x \in \mathbb{R}, \qquad -90° \leqslant x \leqslant 360°$$

(i) Sketch the graph of $f(x)$ over the specified domain.

[1 mark]

(ii) Sketch the graph of the function,

$$g(x) = \sin(x + 90°) \qquad x \in \mathbb{R}, \qquad -90° \leqslant x \leqslant 360°$$

[2 marks]

(iii) A interesting relationship between the sine and cosine functions should be suggested by your part (b) sketch. State what this might be.

[1 mark]

Question 4

Sketch the graph of $y = | 2x + 3 |$ clearly indicating all axis contact points.

[3 marks]

Question 5

Consider “The Loch Ness Monster Function” given by;

$$f(x) = 1 + \sqrt{\cos x}$$

Clearly, care is needed over which values of x are in the domain of this function, as attempting to take negative square roots is to be avoided.

(i) Sketch the cosine function over an interval that includes $0° \leqslant x \leqslant 720°$. Along the x-axis mark: 0°, 90°, 180°, 270°, 360°, 450°, 540°, 630°, 720°.

[2 marks]

(ii) Using your part (a) sketch, and working over the interval $0° \leqslant x \leqslant 720°$, state two intervals for which the cosine function is negative.

[2 marks]

(iii) With parts (a) and (b) in mind, sketch the function $f(x)$. Along the x-axis mark: 0°, 90°, 180°, 270°, 360°, 450°, 540°, 630°, 720°.

[2 marks]

(iv) Why is this function known as “The Lock Ness Monster Function” ?

[1 mark]

Question 6
Solve the equation

$$|2x - 4| = x$$

[3 marks]

Question 7
Sketch the function

$$f(x) = |x^3 - x|$$

[3 marks]

Question 8
Solve the equation

$$|3x + 2| = 2 - x$$

[3 marks]

Question 9

Consider the function

$$f(x) = -\sqrt{x} \qquad x \in \mathbb{R},\ x > 0$$

(a) Sketch the graph of $y = f(x)$

[2 marks]

(b) Sketch the graph of the equation,

$$y - 4 = -\sqrt{x + 7}$$

[3 marks]

(c) Consider now the function given by;

$$g(x) = 4 - \sqrt{x + 7}$$

(i) State the domain of $g(x)$

[1 mark]

(ii) State the range of $g(x)$

[1 mark]

Question 10

A-Level Examination Question from June 2019, Paper 1, Q5 (Edexcel)

$$f(x) = 2x^2 + 4x + 9, \qquad x \in \mathbb{R}$$

(a) Write $f(x)$ in the form $a(x + b)^2 + c$, where a, b and c are integers.

[3 marks]

(b) Sketch the curve with equation $y = f(x)$ showing any points of intersection with the coordinate axes and the coordinates of any turning point.

[3 marks]

(c) **(i)** Describe fully the transformation that maps the curve with equation $y = f(x)$ onto the curve with equation $y = g(x)$ where,

$$g(x) = 2(x - 2)^2 + 4x - 3, \qquad x \in \mathbb{R}$$

[2 marks]

(ii) Find the range of the function

$$h(x) = \frac{21}{2x^2 + 4x + 9} \qquad x \in \mathbb{R}$$

[2 marks]

Question 11

$$f(x) = (x - 3)^2 + 7, \qquad x \in \mathbb{R}$$

(i) Sketch $f(x)$ paying particular attention to the vertex of the parabola.

[2 marks]

(ii) State the range of $f(x)$

[1 mark]

(iii) Solve

$$f(x) = 2x$$

[3 marks]

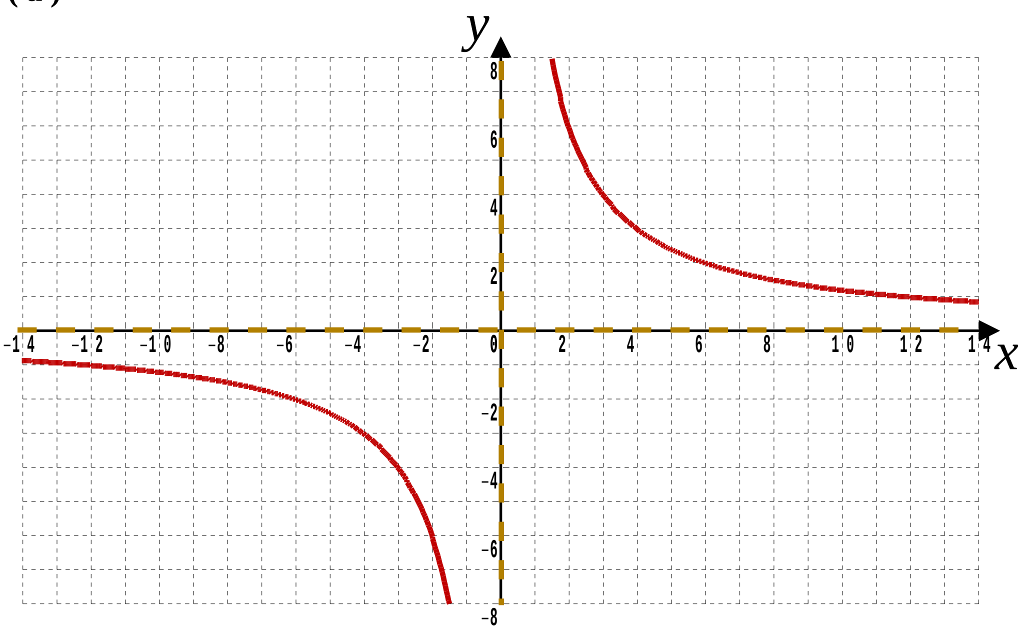
(iv) With the assistance of your part (c) answer, describe the relationship between $f(x)$ and the line $y = 2x$

[3 marks]

3.6 Answers

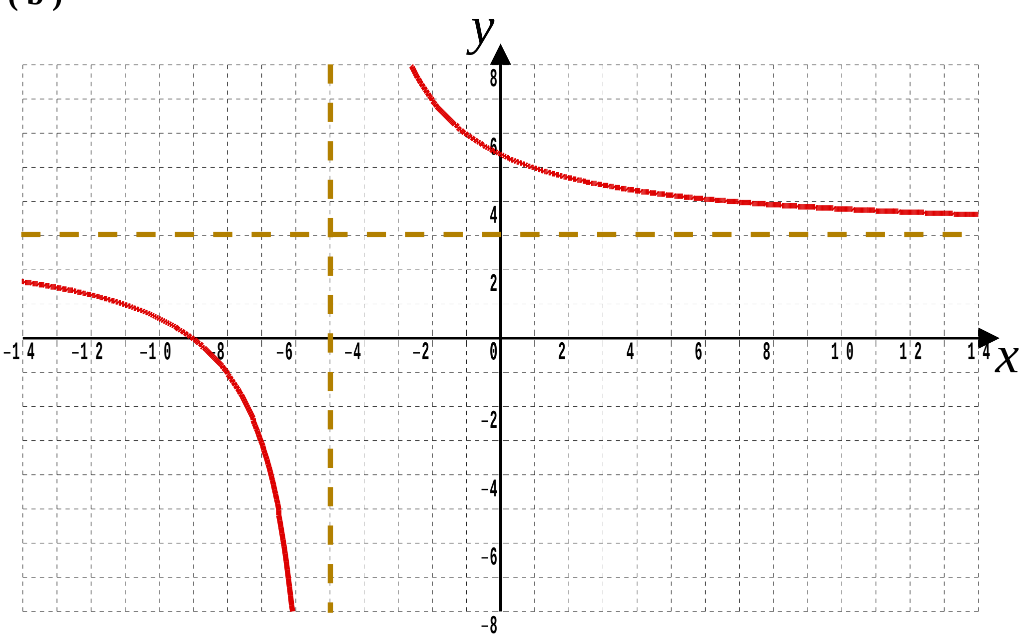
3.6.1 Solution to 3.3 An Inverse Proportion Example

(a)



[2 marks]

(b)



[2 marks]

(c) **(i)** Domain: $x \in \mathbb{R}, \; x \neq -5$

[1 mark]

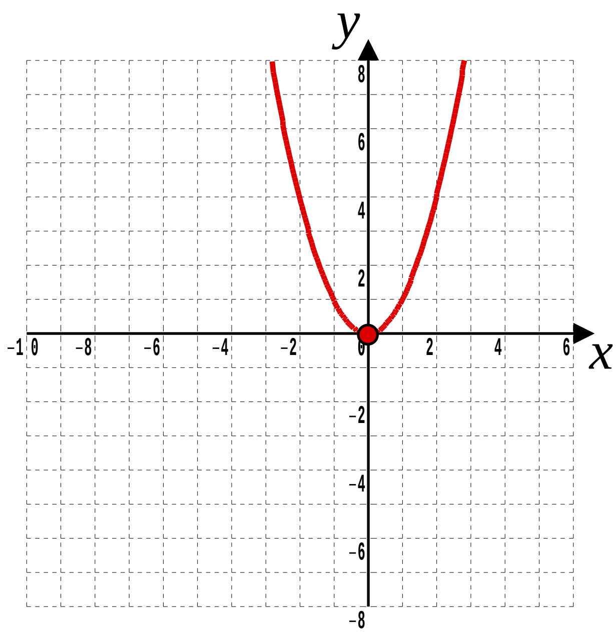
(ii) Range: $g(x) \in \mathbb{R}, \; g(x) \neq 3$

[1 mark]

3.6.2 Solution to 3.5 Exercise

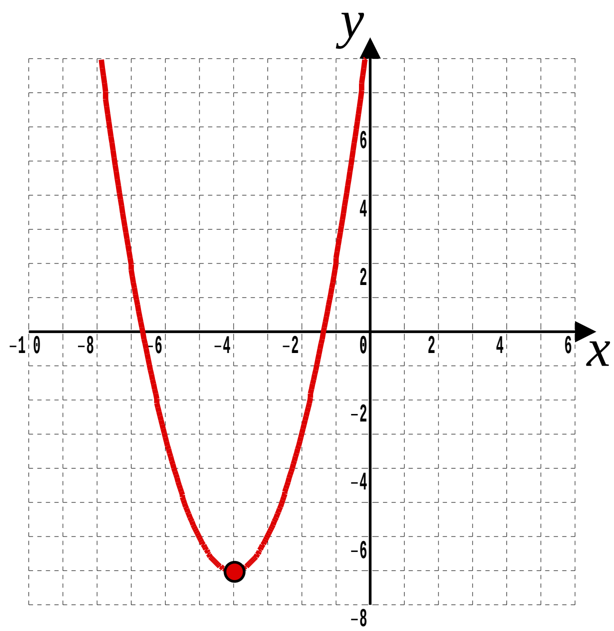
Answer 1

(a)



[1 mark]

(b)



[2 marks]

(c) **(i)** Domain: $x \in \mathbb{R}$

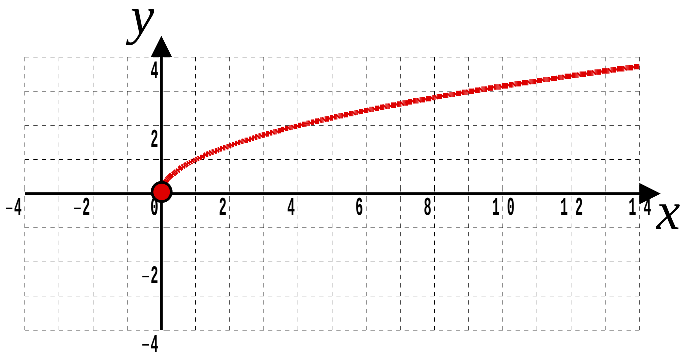
[1 mark]

(ii) Range: $g(x) \in \mathbb{R}, \quad g(x) \geqslant -7$

[1 mark]

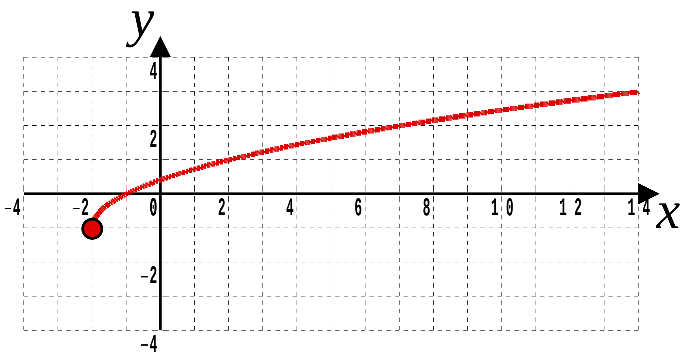
Answer 2

(a)



[2 marks]

(b)



[2 marks]

(c) **(i)** Domain: $x \in \mathbb{R},\ x \geqslant -2$

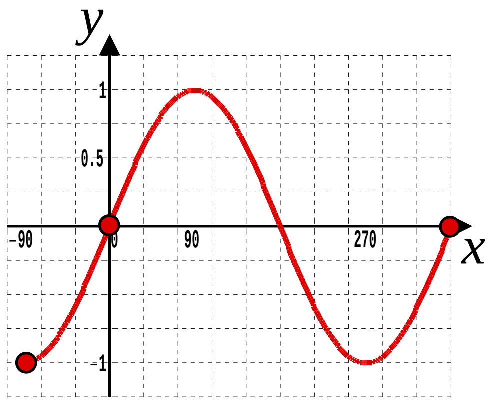
[1 mark]

(ii) Range: $g(x) \in \mathbb{R},\ g(x) \geqslant -1$

[1 mark]

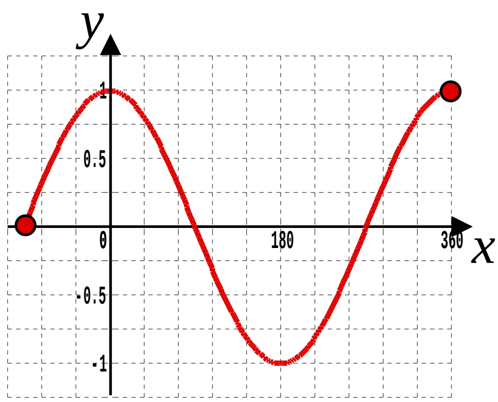
Answer 3

(i)



[1 mark]

(ii)



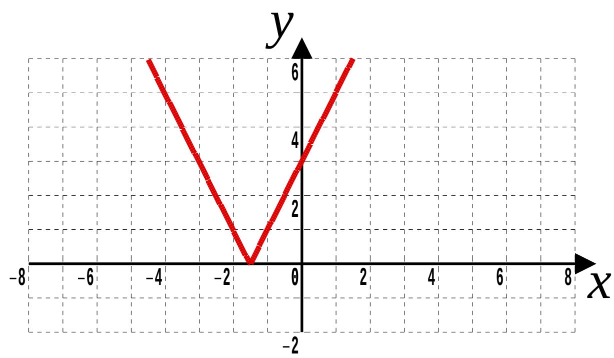
[2 marks]

(iii)

$$\sin\ (\ x\ +\ 90^\circ\ )\ =\ \cos\ (x)$$

[1 mark]

Answer 4



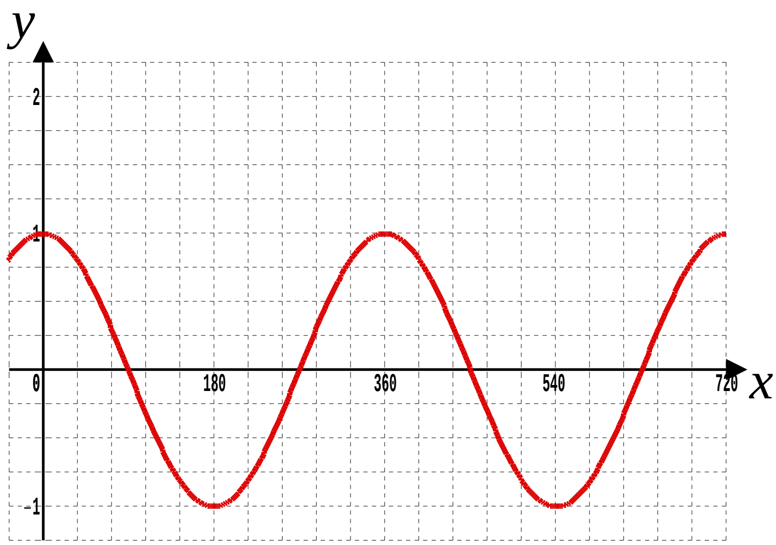
Contact points are,

$$\left(-\frac{3}{2},\ 0\right),\quad (\ 0,\ \ 3\ )$$

[3 marks]

Answer 5

(i)

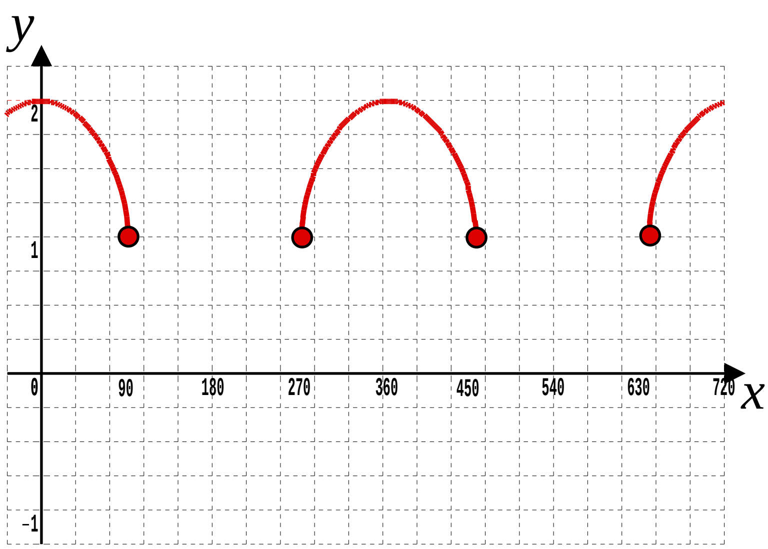


[2 marks]

(ii) Negative for the intervals: $90° \leqslant x \leqslant 270°$ $450° \leqslant x \leqslant 630$

[2 marks]

(iii)



[2 marks]

(iv) It's said that if you scroll far enough to the right you'll get to the monster's head, and far enough to the left will arrive at Nessie's tail.
In spite of some claiming to have seen these body parts there is no hard evidence of such sightings (see this topic's cover).

[1 mark]

Answer 6

$$|2x - 4| = x$$

$$-(2x - 4) = x \qquad\qquad +(2x - 4) = x$$

$$-2x + 4 = x \qquad\qquad x = 4$$

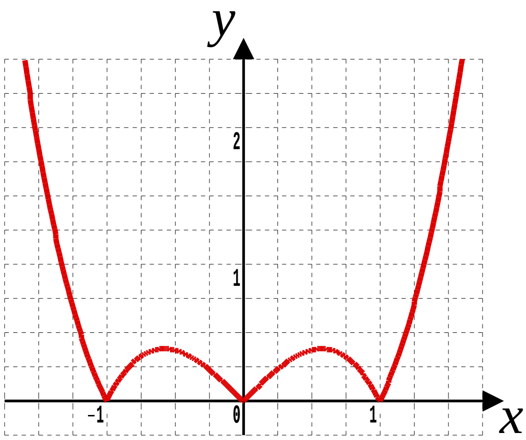
$$4 = 3x$$

$$x = \frac{4}{3}$$

$$x \in \left\{\frac{4}{3},\ 4\right\}$$

[3 marks]

Answer 7



[3 marks]

Answer 8

$$|3x + 2| = 2 - x$$

$$-(3x + 2) = 2 - x \qquad\qquad +(3x + 2) = 2 - x$$

$$-3x - 2 = 2 - x \qquad\qquad 4x = 0$$

$$-2x = 4 \qquad\qquad x = 0$$

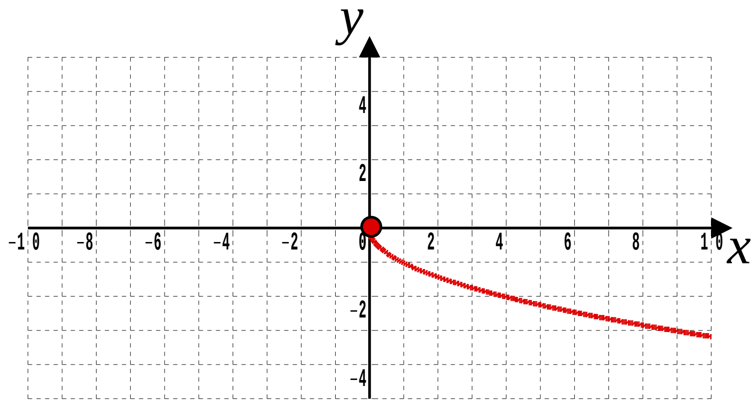
$$x = -2$$

$$x \in \{-2,\ 0\}$$

[3 marks]

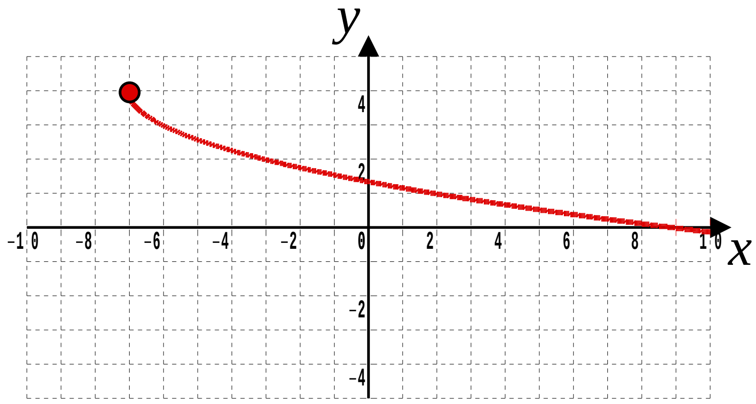
Answer 9

(a)



[2 marks]

(b)



[3 marks]

(c) **(i)** Domain: $x \in \mathbb{R},\ x \geqslant -7$

[1 mark]

(ii) Range: $g(x) \in \mathbb{R},\ g(x) \leqslant 4$

[1 mark]

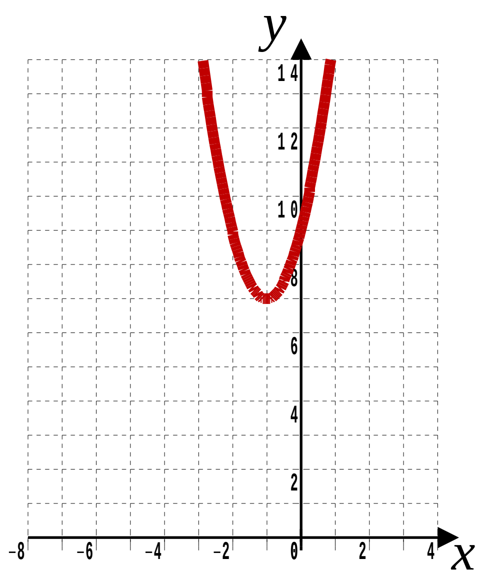
Answer 10

(a)

$$f(x) = 2x^2 + 4x + 9$$
$$= 2\left[x^2 + 2x\right] + 9$$
$$= 2\left[(x+1)^2 - 1\right] + 9$$
$$= 2(x+1)^2 + 7$$

[3 marks]

(b)



[3 marks]

(c) **(i)**

$$g(x) = 2(x-2)^2 + 4x - 3$$
$$= 2(x-2)^2 + 4x - 8 + 8 - 3$$
$$= 2(x-2)^2 + 4(x-2) + 5$$
$$= 2(x-2)^2 + 4(x-2) + 9 - 9 + 5$$
$$= 2(x-2)^2 + 4(x-2) + 9 - 4$$
$$g(x) + 4 = 2(x-2)^2 + 4(x-2) + 9$$

Compare this with $f(x) = 2x^2 + 4x + 9$

$\therefore\ f(x)$ translated $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ will give $g(x)$

[2 marks]

(ii) $h(x) = \dfrac{21}{2x^2 + 4x + 9} = \dfrac{21}{2(x+1)^2 + 7}$

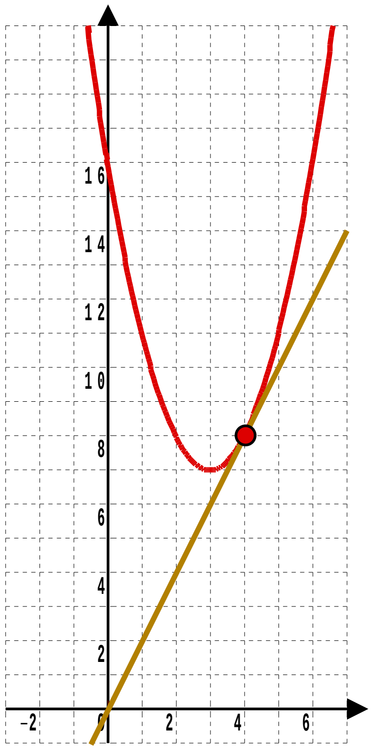
$$\therefore\ h(x)_{MAX} = \frac{21}{7} = 3$$

Range : $0 < h(x) \leqslant 3$

[2 marks]

Answer 11

(i)



[2 marks]

(ii) Range: $f(x) \in \mathbb{R}, \;\; f(x) \geqslant 7$

[1 mark]

(iii)

$$(x - 3)^2 + 7 = 2x$$
$$x^2 - 8x + 16 = 0$$
$$(x - 4)^2 = 0$$
$$x = 4 \;\; \text{(twice)}$$

[3 marks]

(iv) The line $y = 2x$ is a tangent to the parabola at (4, 8)

[3 marks]

3E.1 Rational Functions and Asymptotes

The asymptotes of rational functions can be of three types; horizontal, vertical or oblique. Behind the scenes, as with differentiation, the theory of limits is at play. However, as with differentiation, a set of straight forward rules finds the asymptotes without having to battle through tedious technicalities.

Definition : A Rational Function...

...is of the form $\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.

In such functions the degree of the two polynomials, $P(x)$ and $Q(x)$ is of crucial importance when determining the asymptotes.
With each polynomial arranged with the powers of x in decreasing order the leading term is then the first term.
The coefficient of the leading term of $P(x)$ is p
The coefficient of the leading term of $Q(x)$ is q

That is,
$$P(x) = p\,x^{deg(P(x))} + \textit{other lower power terms}$$
$$Q(x) = q\,x^{deg(Q(x))} + \textit{other lower power terms}$$

Test for Horizontal Asymptotes

Compare the degree of the numerator $P(x)$ with that of the denominator $Q(x)$;

If $deg(P(x)) > deg(Q(x))$ then there is no horizontal asymptote.

If $deg(P(x)) = deg(Q(x))$ then $y = \dfrac{p}{q}$ is a horizontal asymptote.

If $deg(P(x)) < deg(Q(x))$ then the x-axis is a horizontal asymptote.

Test for Vertical Asymptotes

Set the denominator, $Q(x)$, equal to zero and solve.

The solutions are the equations of vertical asymptotes.
(Unless the numerator has an identical solution in which case there is a point hole in the curve for that value of x)

Test for Oblique Asymptotes

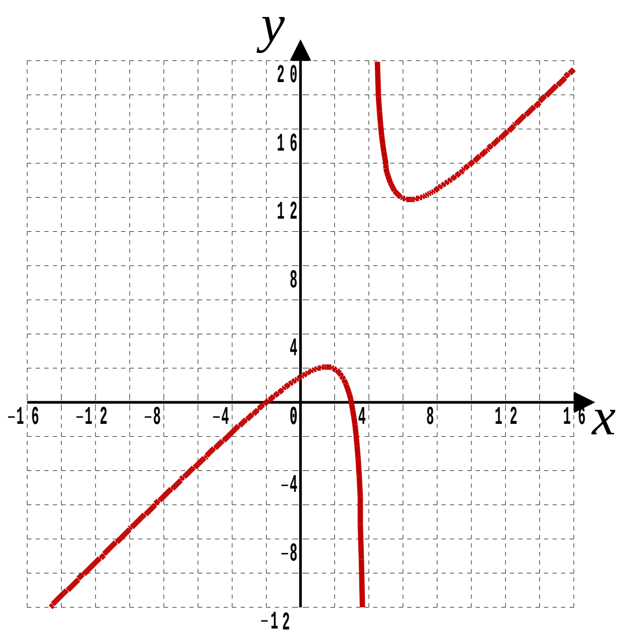
If $deg(P(x)) - deg(Q(x)) = 1$ then there exists an oblique asymptote.

Polynomial division of $P(x)$ by $Q(x)$ gives its equation.
(Any remainder from the division is ignored)

3E.2 Example

Find all asymptotes to the curve, $y = \dfrac{x^2 - x - 6}{x - 4}$

Once found, carefully add them to the graph of the curve presented below.



A full solution is presented on the following page.

[6 marks]

3E.3 Solution

$$P(x) = x^2 - x - 6 \ : \ deg(P(x)) = 2$$

$$Q(x) = x - 4 \ : \ deg(Q(x)) = 1$$

Testing for horizontal asymptotes;

As $deg(P(x)) > deg(Q(x))$ there is no horizontal asymptote.

Testing for vertical asymptotes;

$$Q(x) = 0$$
$$x - 4 = 0$$
$$x = 4$$

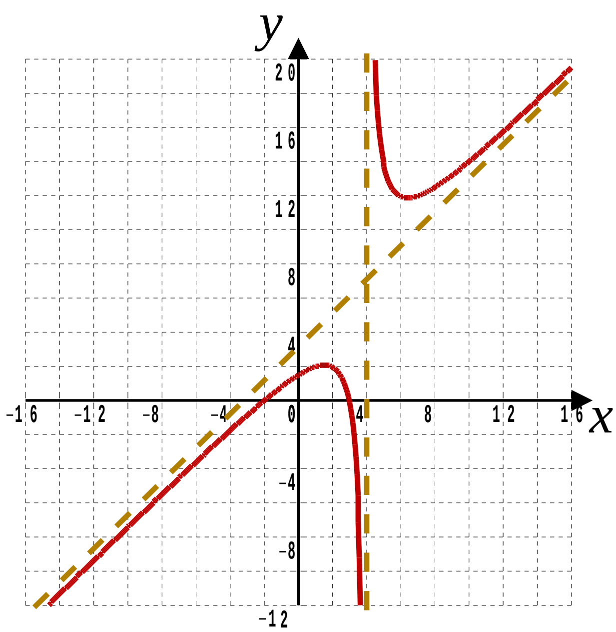
There is a vertical asymptote with equation, $x = 4$

Testing for oblique asymptotes;

As $deg(P(x)) - deg(Q(x)) = 1$ there exists an oblique asymptote.

$$\begin{array}{r|l} & \phantom{x^2 -{}} x + 3 \\ \cline{2-2} x - 4 & x^2 - x - 6 \\ & \underline{x^2 - 4x} \\ & \phantom{x^2 -{}} 3x - 6 \\ & \phantom{x^2 -{}} \underline{3x - 12} \\ & \phantom{x^2 - 3x - {}} 6 \ \leftarrow \text{Ignore !} \end{array}$$

There is an oblique asymptote with equation $y = x + 3$



[6 marks]

3E.4 Exercise

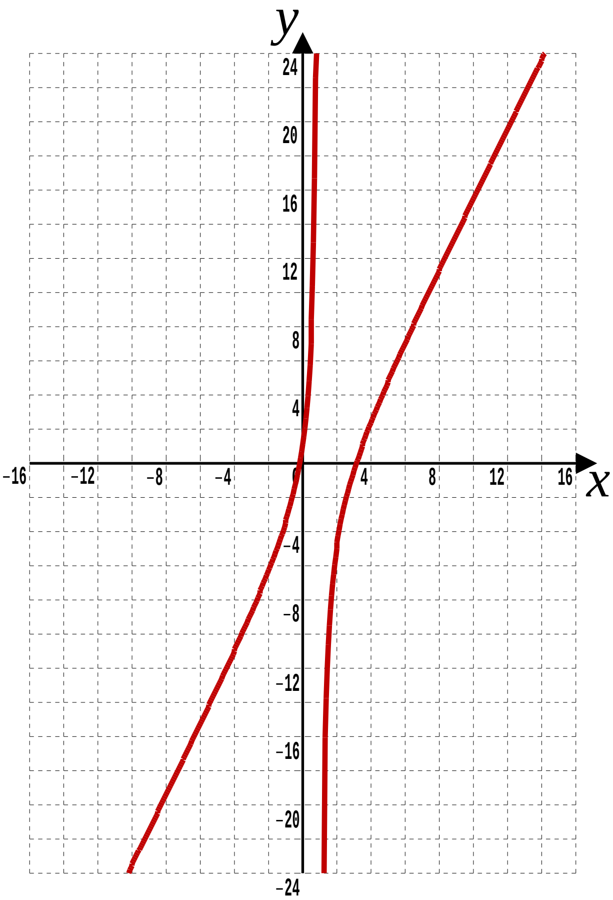
Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 18

Question 1

Find all asymptotes to the curve, $y = \dfrac{2x^2 - 6x - 1}{x - 1}$

Once found, carefully add them to the graph of the curve presented below.

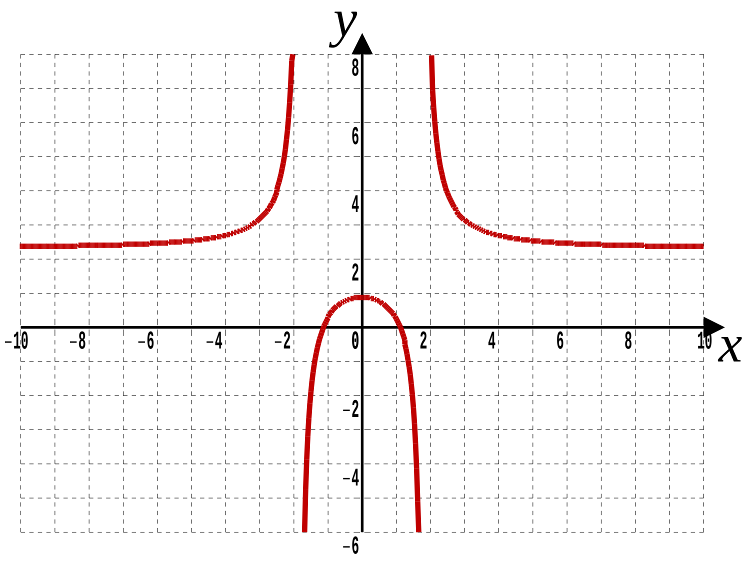


[6 marks]

Question 2

Find all asymptotes to the curve, $y = \dfrac{7x^2 - 9}{3x^2 - 10}$

Once found, carefully add them to the graph of the curve presented below.

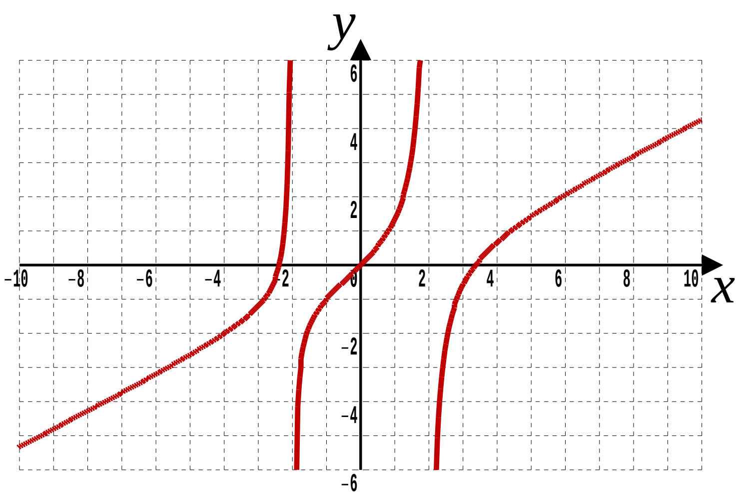


[6 marks]

Question 3

Find all asymptotes to the curve, $y = \dfrac{x^3 - x^2 - 8}{2x^2 - 8}$

Once found, carefully add them to the graph of the curve presented below.



[6 marks]

3E.4 Answers

3E.4.1 Solutions to 3E.3 Exercise

Answer 1

$$P(x) = 2x^2 - 6x - 1 \;:\; deg(P(x)) = 2$$

$$Q(x) = x - 1 \;:\; deg(Q(x)) = 1$$

Testing for horizontal asymptotes;

As $deg(P(x)) > deg(Q(x))$ there is no horizontal asymptote.

Testing for vertical asymptotes;

$$Q(x) = 0$$

$$x - 1 = 0$$

$$x = 1$$

There is a vertical asymptote with equation, $x = 1$

Testing for oblique asymptotes;

As $deg(P(x)) - deg(Q(x)) = 1$ there exists an oblique asymptote.

$$\begin{array}{r|rrr}
 & & 2x & -\ 4 \\ \cline{2-4}
x - 1 & 2x^2 & -\ 6x & -\ 1 \\
 & 2x^2 & -\ 2x & \\ \cline{2-3}
 & & -\ 4x & -\ 1 \\
 & & -\ 4x & -\ 4 \\ \cline{3-4}
 & & & 3 \leftarrow \text{Ignore !}
\end{array}$$

There is an oblique asymptote with equation $y = 2x - 4$

[6 marks]

Answer 2

$$P(x) = 7x^2 - 9 \ : \ deg(P(x)) = 2$$

$$Q(x) = 3x^2 - 10 \ : \ deg(Q(x)) = 2$$

Testing for horizontal asymptotes;

As $deg(P(x)) = deg(Q(x))$ there is a horizontal asymptote.

It's equation is, $y = \dfrac{7}{3}$

Testing for vertical asymptotes;

$$Q(x) = 0$$

$$3x^2 - 10 = 0$$

$$x^2 = \frac{10}{3}$$

$$x = \pm\sqrt{\frac{10}{3}}$$

There are two vertical asymptotes with equations, $x = \pm\sqrt{\dfrac{10}{3}}$

Testing for oblique asymptotes;

As $deg(P(x)) - deg(Q(x)) \neq 1$ there is no oblique asymptote.

[6 marks]

Answer 3

$$P(x) = x^3 - x^2 - 8 \ : \ deg(P(x)) = 3$$

$$Q(x) = 2x^2 - 8 \ : \ deg(Q(x)) = 2$$

Testing for horizontal asymptotes;

As $deg(P(x)) > deg(Q(x))$ there is no horizontal asymptote.

Testing for vertical asymptotes;

$$Q(x) = 0$$

$$2x^2 - 8 = 0$$

$$x^2 = 4$$

There are two vertical asymptotes with equations, $x = \pm 2$

Testing for oblique asymptotes;

As $deg(P(x)) - deg(Q(x)) = 1$ there exists an oblique asymptote.

$$\begin{array}{r|l}
 & \frac{1}{2}x - \frac{1}{2} \\
\hline
2x^2 + 0x - 8 & x^3 - x^2 + 0x - 8 \\
 & \underline{x^3 - 0x^2 - 4x} \\
 & \quad - x^2 + 4x - 8 \\
 & \quad \underline{- x^2 + 0x + 4} \\
 & \qquad 4x - 12 \ \leftarrow \text{Ignore !}
\end{array}$$

There is an oblique asymptote with equation $y = \frac{1}{2}x - \frac{1}{2}$

[6 marks]

4.1 The Modulus Function (Part 2)

The fact that the modulus function can take any part of an equation's graph that is below the *x*-axis, and reflects it in the *x*-axis was studied previously. Here is the rule that was used;

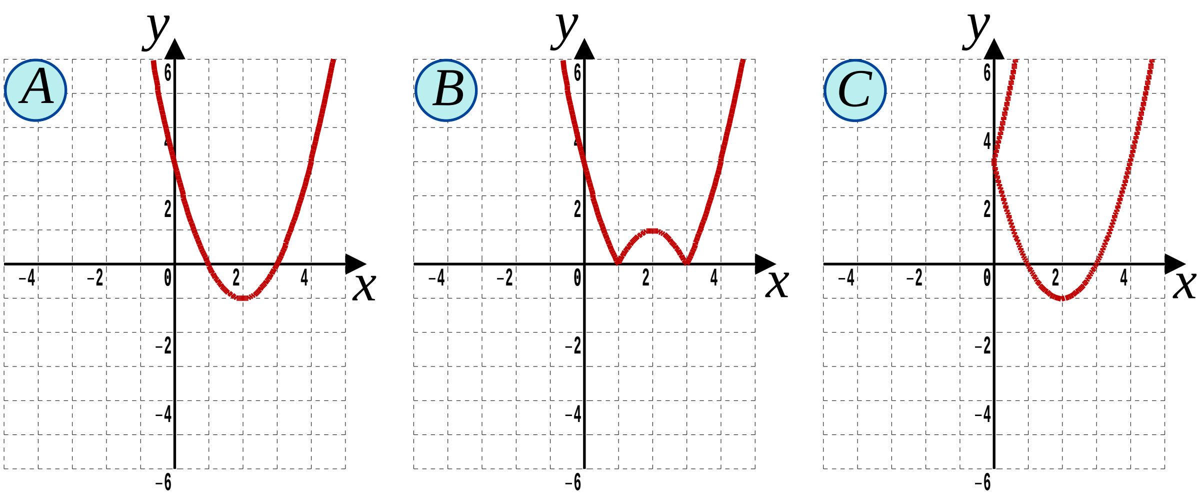
Modulus Sketching Rule 1

To sketch $y = |\,f(x)\,|$ **Bounce negative** x

◇ Sketch $y = f(x)$ using a dashed line for points below the *x*-axis.

◇ Reflect any part of the curve below the *x*-axis in the *x*-axis.

For example, graph *A* shows the parabola with quadratic equation $y = x^2 - 4x + 3$ and graph *B* shows the effect of modulising graph *A*: That is, $y = |x^2 - 4x + 3|$



4.1.1 Example

Find the equation that will yield graph *C*, where the "bounce" is against the *y*-axis.

[6 marks]

4.2 By Hand

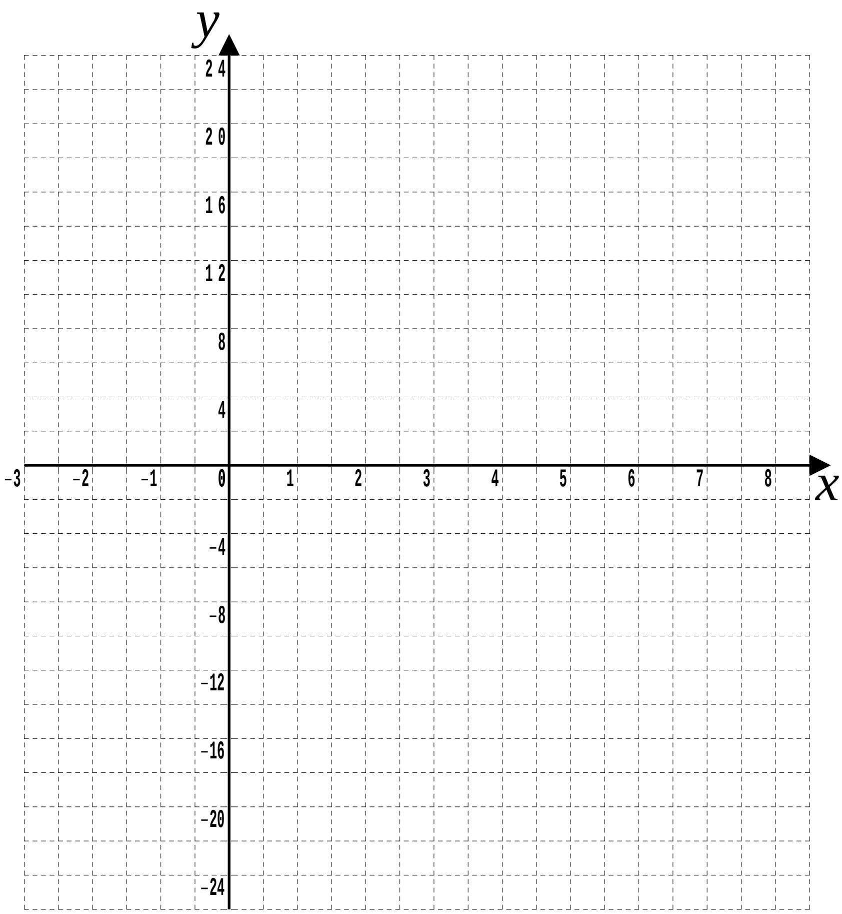
Faced with an unfamiliar modulus question it may be best to go back to basics by drawing up a table and plotting points to get a feel of how it is behaving. With such plotting, keeping one's wits sharp is essential !

4.2.1 Example

Use the table below to assist in plotting the following equation;

$$|y + 1| = |(x - 2)^2|$$

x	-3	-2	-1	0	1	2	3	4	5	6	7
y											



[6 marks]

4.3 A Subtle Variation

Typically, an examination question will ask for two separate graphs of

◇ $y = | f(x) |$

and ◇ $y = f(| x |)$

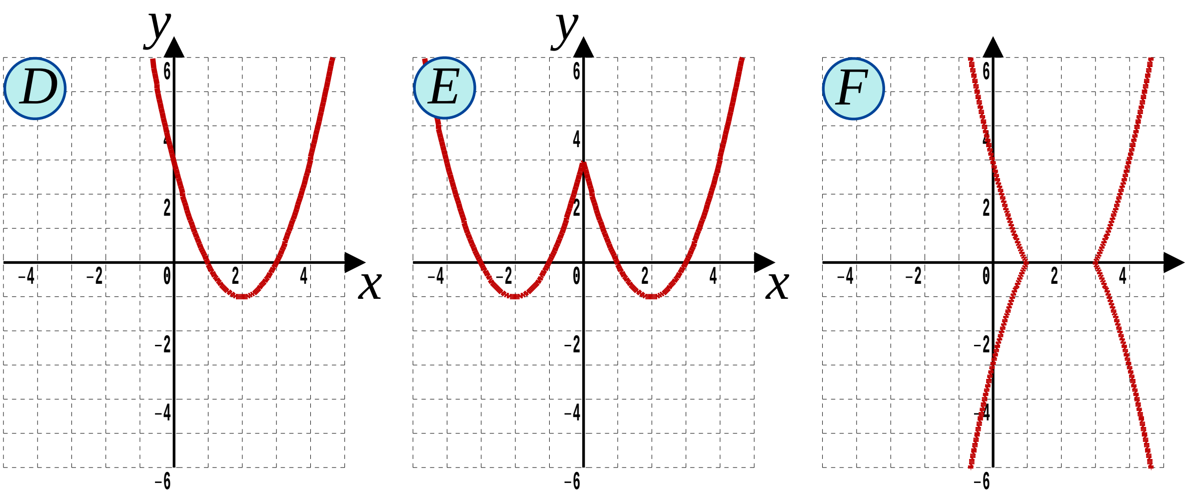
Modulus Sketching Rule 2

To sketch $y = f(| x |)$ **Overwrite negative *x***

◇ Sketch the curve of $y = f(x)$ for $x \geqslant 0$

◇ Reflect the parts of the curve to the right of the *y*-axis in the *y*-axis.

Graph *D* shows the parabola with quadratic equation $y = x^2 - 4x + 3$ and graph *E* shows the effect of modulising each individual *x*. That is, $y = |x^2| - 4|x| + 3$ Notice that the information to the left of the *y*-axis of the original graph has been overwritten, whilst information to the right has been “duplicated” in a reflected form.



4.3.1 Example

Find the equation that will yield graph *F*, where the “mirror” is in the *x*-axis.

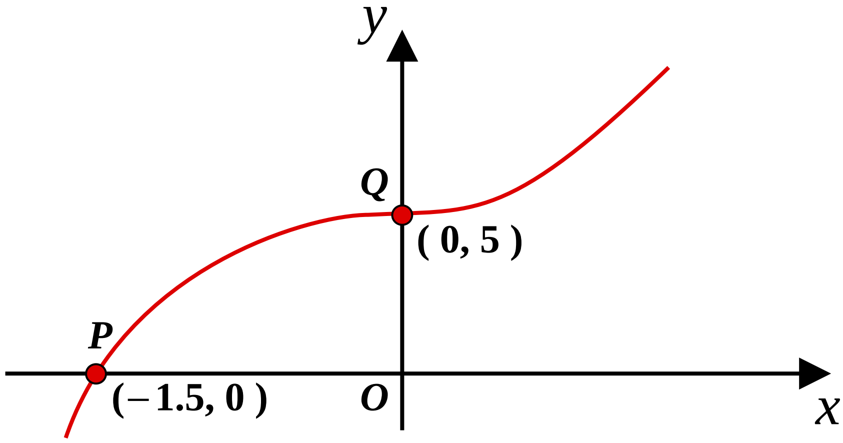
[2 marks]

4.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available: 100

Question 1

A-Level Examination Question from June 2012, paper C3, Q4(a)(b) (Edexcel)



The curve $y = f(x)$ passes through the points $P(-1.5,\ 0)$ and $Q(0,\ 5)$ as shown.
On separate diagrams, sketch the curve with equation,

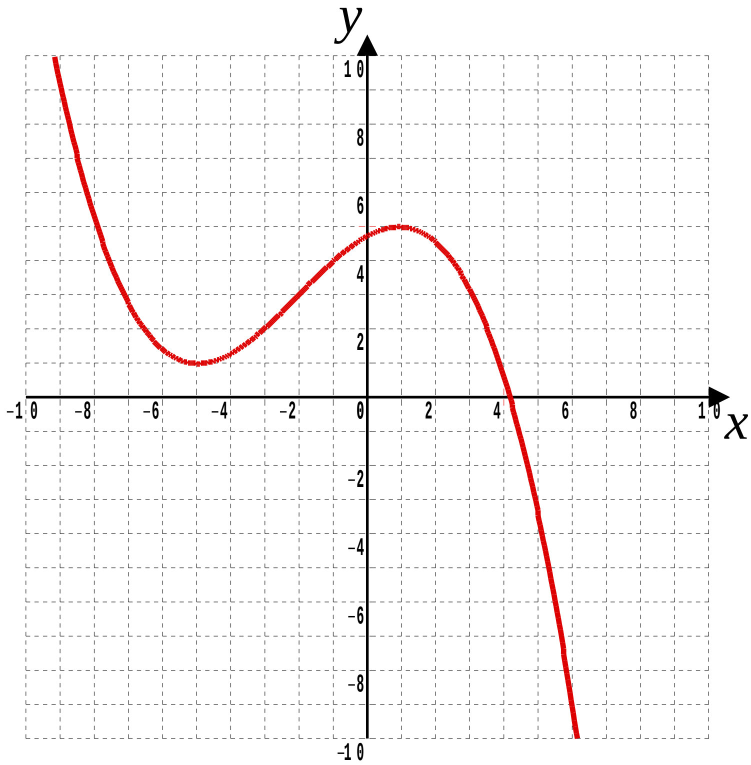
(a) $y = |f(x)|$

(b) $y = f(|x|)$

Indicate clearly on each sketch the coordinates of the points at which the curve crosses or meets the axes.

[2, 2 marks]

Question 2



The diagram shows the graph of $y = f(x)$

The point (1, 5) is the maximum turning point of the graph.

The point (– 5, 1) is the minimum turning point of the graph.

Sketch, on separate diagrams, the graphs of

(i) $y = | f(x) |$ **(ii)** $y = f(| x |)$

Show on each graph the coordinates of any turning points.

[2, 3 marks]

Question 3

$$h(x) = 2(x - 3)^2 - 8, \qquad x \in \mathbb{R}$$

(a) For each of the following, draw a sketch.
Label any turning points and axes intercepts.

(i) $y = h(x)$ **(ii)** $y = |h(x)|$ **(iii)** $y = h(|x|)$

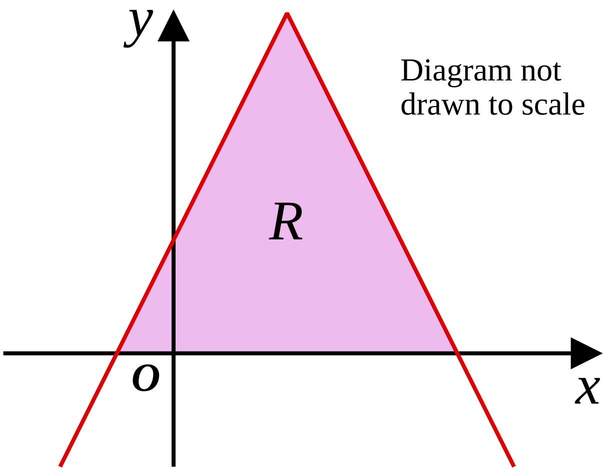
[3, 3, 3 marks]

(b) For what values of k will the equation $h(|x|) = k$ have four solutions ?

[2 marks]

Question 4

A-Level Mock Paper 2 from 2019, Q5 (Edexcel)



The graph is part of the equation $y = f(x)$ where $f(x) = 7 - |3x - 5|,\ x \in \mathbb{R}$

The finite region R, shown shaded, is bounded by the graph $y = f(x)$ and the x-axis.

(a) Find the area of R, giving your answer in its simplest form.

[4 marks]

The equation $7 - |3x - 5| = k$ where k is a constant has two distinct real solutions.

(b) Write down the range of possible values for k

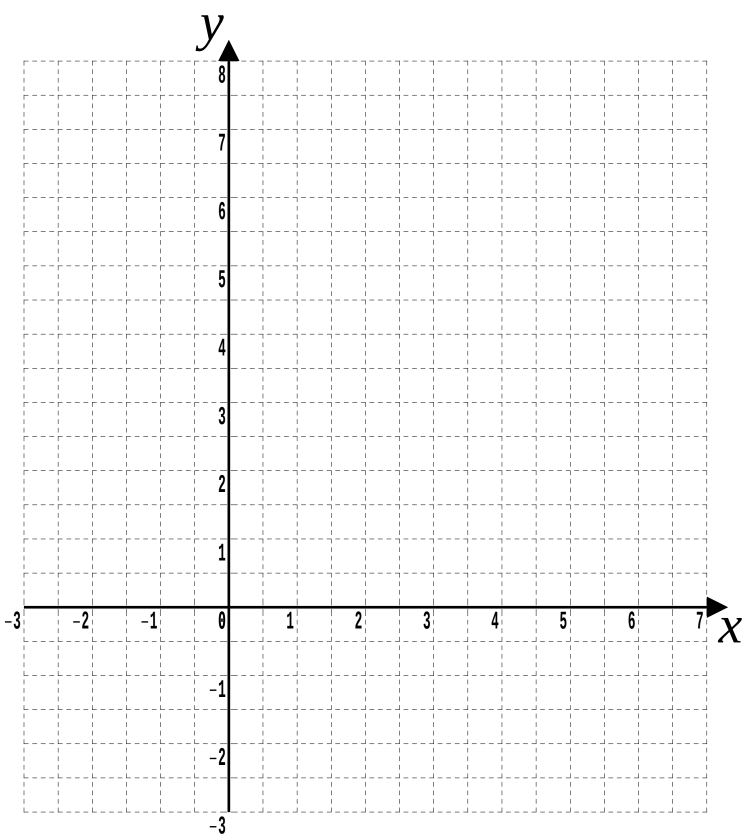
[1 mark]

Question 5

(i) Use the table below to assist in plotting the following equation;

$$|y - 3| = |x - 2|$$

x	-3	-2	-1	0	1	2	3	4	5	6	7
y											



[5 marks]

(ii) Given that the following equation has exactly one real solution, find k;

$$|y - 3| - |x - 2| = k$$

[1 mark]

(iii) Given that the following equation has exactly one real solution, find c;

$$|y - 3| - |x - 2| = \frac{1}{4}x + c$$

[1 mark]

Question 6

$$f(x) = |x|^2 - 4|x| + 1 \qquad x \in \mathbb{R}$$

(a) Given that x is a real number, which of the following are true ?

(i) $|x|^2 = x^2$ **(ii)** $|x^2| = x^2$ **(iii)** $|x^2| = |x|^2$

[3 marks]

(b) Sketch the curve, $y = f(x)$, marking on the coordinates of any local minima or maxima.

[3 marks]

(c) Using algebra, solve the equation, $f(x) = -2$

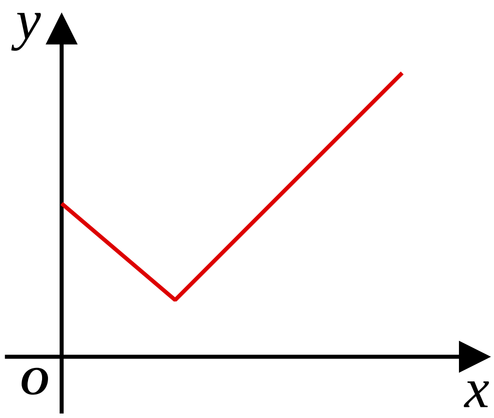
[3 marks]

(d) If $f(x) = k$, where k is a constant, has exactly two solutions, state the possible values of k

[3 marks]

Question 7

A-Level Sample Assessment Materials from 2017, Paper 2, Q11 (Edexcel)



The graph is a sketch of $y = f(x)$ where $f(x) = 2|3 - x| + 5, \quad x \geqslant 0$

(a) State the range of f

[1 mark]

(b) Solve the equation, $f(x) = \frac{1}{2}x + 30$

[3 marks]

Given that the equation $f(x) = k$, where k is a constant, has two distinct roots,

(c) state the set of possible values for k

[2 marks]

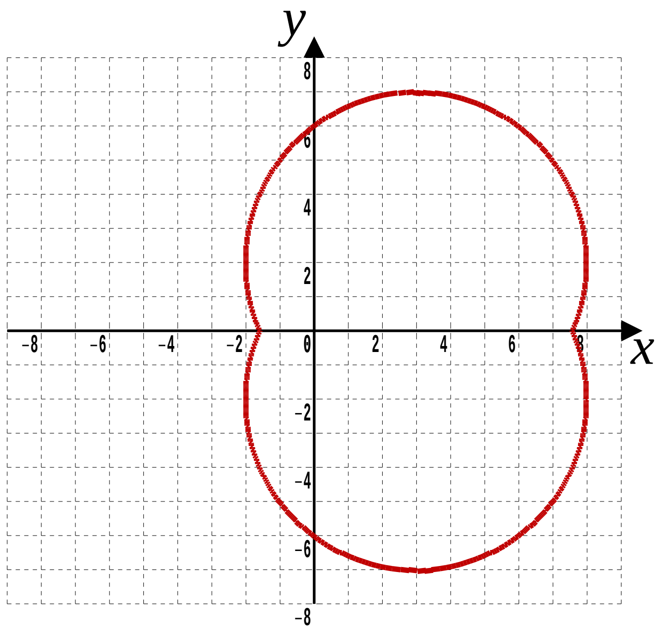
Question 8

A circle centre $(3,\ 2)$ and radius 5 has equation $(x - 3)^2 + (y - 2)^2 = 25$

This equation has been manipulated in various ways and the modulus function applied to obtain the following graphs on a computer capable of processing the $\pm$ sign in an input equation.

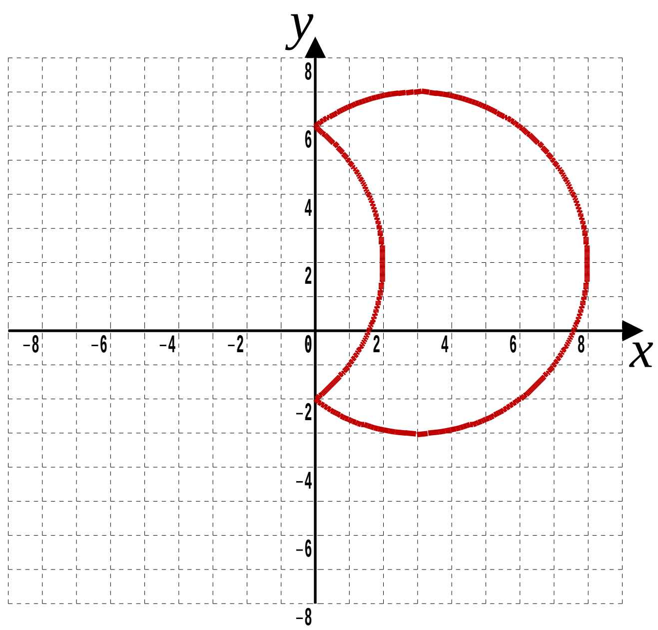
Underneath each graph write the equation used to produce the graph.

(i)



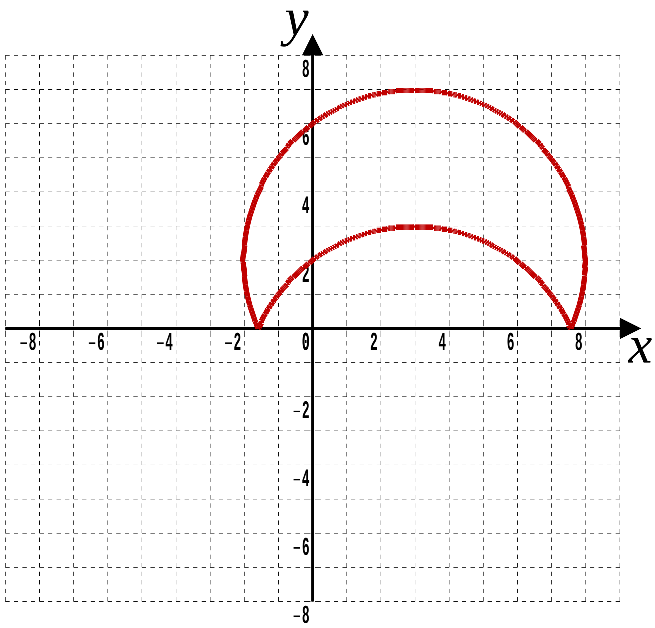
[3 marks]

(ii)



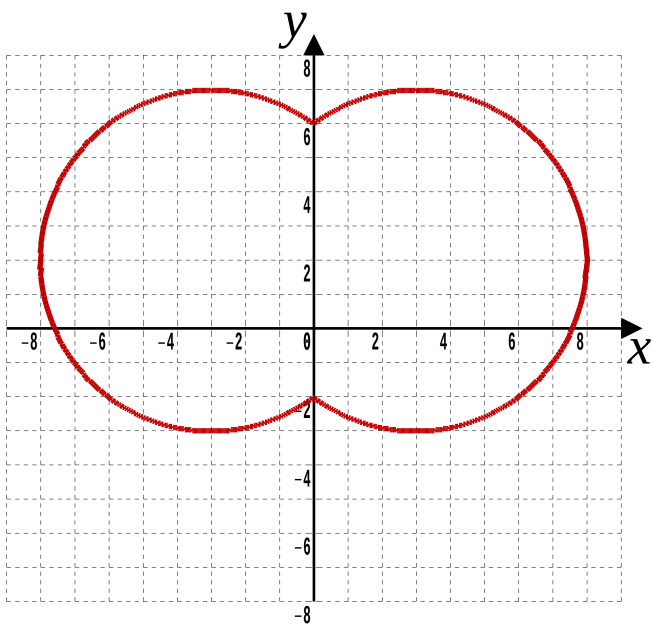
[3 marks]

(iii)



[3 marks]

(iv)



[3 marks]

If you have a graphics calculator you may like to see if you can get it to plot each of these curves. Where a ± is involved you may have to plot the equation with the plus separately to the equation with a minus.

Question 9

$$f(x) = 3x^2 - 12x + 7, \qquad x \in \mathbb{R}$$

(a) Write $f(x)$ in the form $a(x+b)^2 + c$, where a, b and c are integers.

[3 marks]

(b) Sketch the curve with equation $y = f(x)$ showing any points of intersection with the coordinate axes and the coordinates of any turning point.

[3 marks]

(c) **(i)** Describe fully the transformation that maps the curve with equation $y = f(x)$ onto the curve with equation $y = g(x)$ where,

$$g(x) = 3(x-1)^2 - 12x + 9, \qquad x \in \mathbb{R}$$

[2 marks]

(ii) Find the range of the function

$$h(x) = \frac{20}{3x^2 - 12x + 7} \qquad x \in \mathbb{R}$$

[2 marks]

Question 10

A-Level Examination Question from January 2014, Paper C3, Q6 (Edexcel)

Given that a and b are constants and that $0 < a < b$,

(a) on separate diagrams, sketch the graph with equation

(i) $y = |2x + a|$ **(ii)** $y = |2x + a| - b$

Show on each sketch the coordinates of each point at which the graph crosses or meets the axes.

[6 marks]

(b) Solve, for x, the equation

$$|2x + a| - b = \frac{1}{3}x$$

giving any answers in terms of a and b

[4 marks]

Question 11

The functions f and g are defined as,

$$f(x) = 4a^2 - x^2, \quad x \in \mathbb{R}$$

$$g(x) = |4x - a|, \quad x \in \mathbb{R}$$

where a is a constant, such that $a \geqslant 1$

(i) Sketch the graph of $f(x)$ and the graph of $g(x)$ on the same diagram.

The sketch must include the coordinates of any points where each of the graphs meets the coordinate axes.

[6 marks]

(ii) Find, in exact form where appropriate, the solutions of the equation,

$$4 - x^2 = |4x - 1|$$

[4 marks]

Question 12

Consider the function

$$f(x) = \frac{\pi}{180}x - 5 + \sin x, \qquad -300^\circ \leqslant x \leqslant 600$$

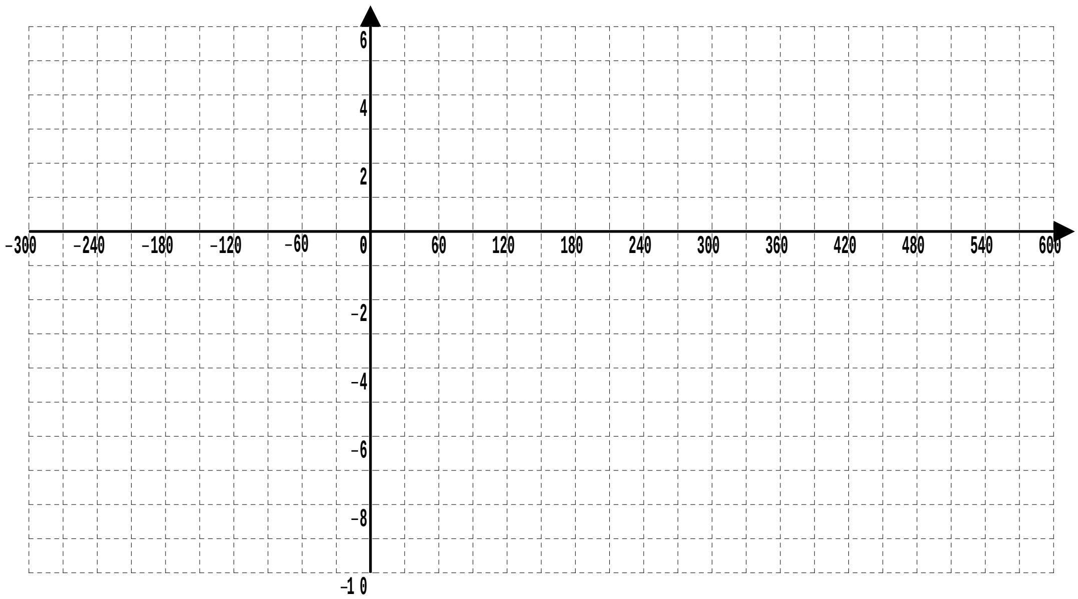
(a) Complete the following table,

x	– 300	– 240	– 180	– 120	– 60	0	60	120
y								

x	180	240	300	360	420	480	540	600
y								

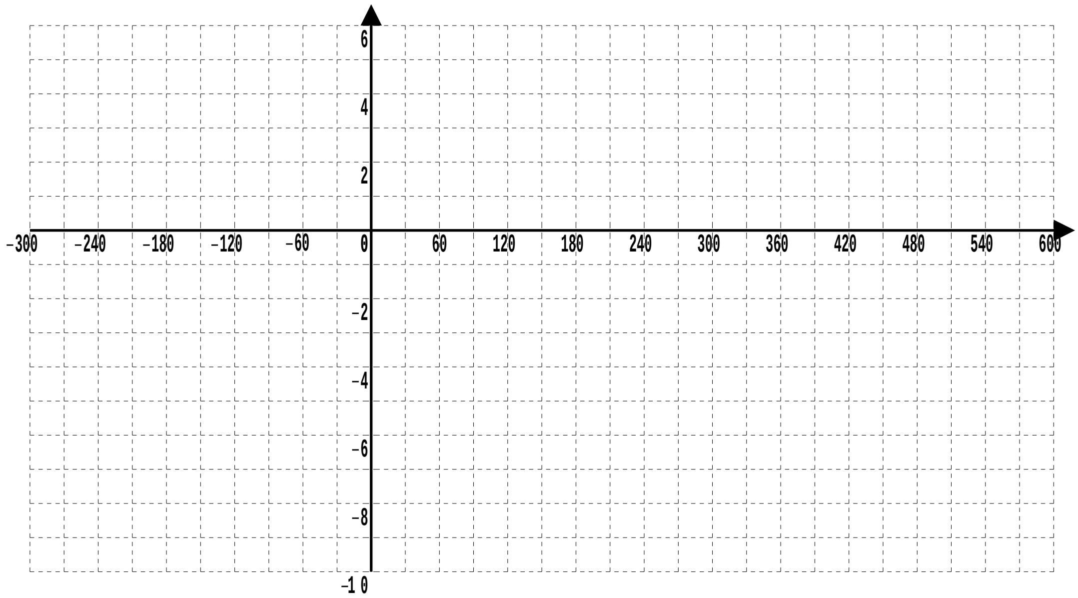
[3 marks]

(b) **(i)** On the grid below plot the graph of $y = f(x)$



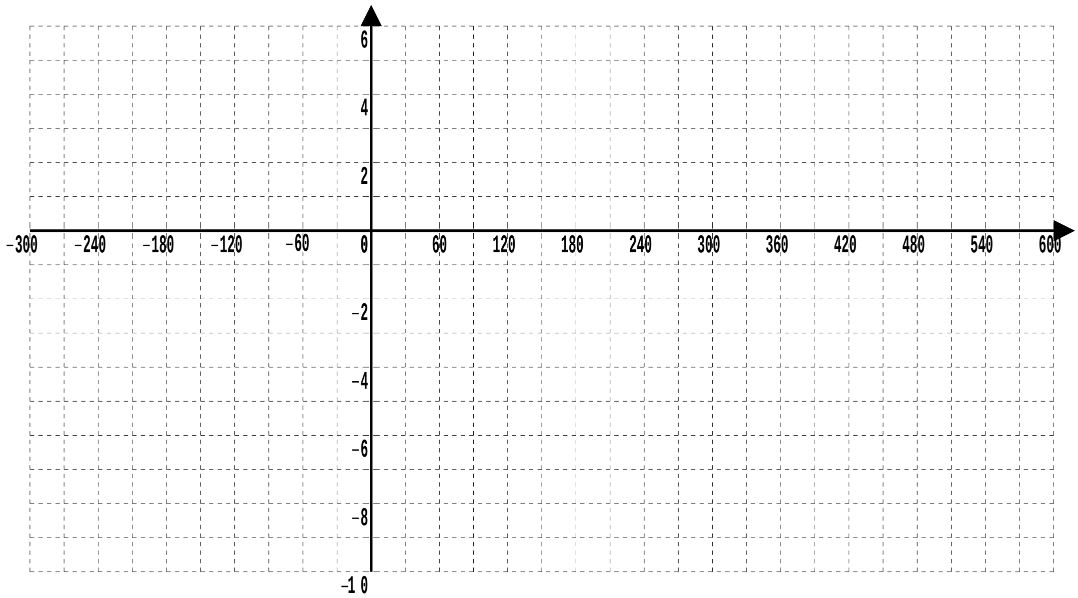
[2 marks]

(ii) On the grid below plot the graph of $y = |f(x)|$



[2 marks]

(iii) On the grid below plot the graph of $y = f(|x|)$



[2 marks]

4.5 Answers

4.5.1 Solution to 4.1.1 Example

All sorts of guesses can be made, most of which will have other effects !
The logical way to proceed is as follows.

There will exist a modulus sketching rule similar to rule 1;

Modulus Sketching Rule 1b

To sketch $x = | f(y) |$ **Bounce negative y**

◇ Sketch $x = f(y)$ using a dashed line for points left of the y-axis.

◇ Reflect any part of the curve left of the y-axis in the y-axis.

$$y = x^2 - 4x + 3$$
$$y = (x - 2)^2 + 1$$
$$(x - 2)^2 = y - 1$$

Square root both sides

$$x - 2 = \pm\sqrt{y - 1}$$
$$x = 2 \pm \sqrt{y - 1}$$

$\therefore$ To plot curve C use $x = \left| 2 \pm \sqrt{y - 1} \right|$

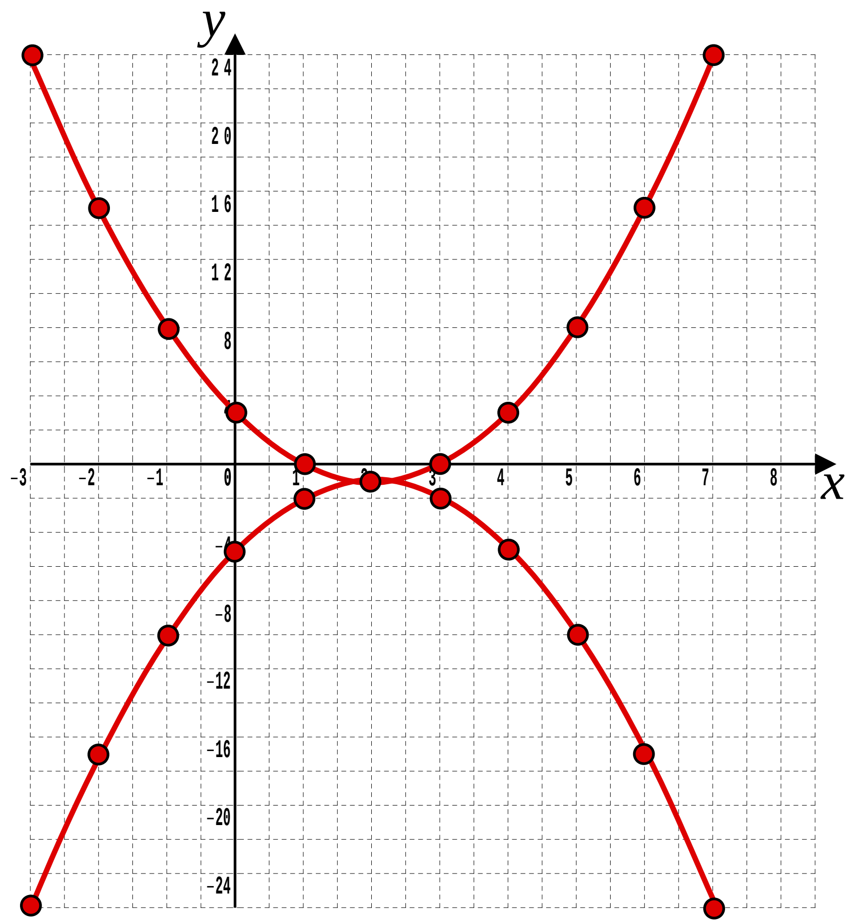
To plot this (depending on the sophistication of your plotter) you may have to plot the two curves separately, $x = \left| 2 + \sqrt{y - 1} \right|$, $x = \left| 2 - \sqrt{y - 1} \right|$

[6 marks]

4.5.2 Solution to 4.2.1 Example

$$|y + 1| = |(x - 2)^2|$$

x	-3	-2	-1	0	1	2	3	4	5	6	7
y	24 -26	15 -17	8 -10	3 -5	0 -2	-1 -1	0 -2	3 -5	8 -10	15 -17	24 -26



Geometrically, this is both the “original” curve and its reflection in the line, $y = -1$

[6 marks]

4.5.3 Solution to 4.3.1 Example

Previously...

$$y = x^2 - 4x + 3 \iff x = 2 \pm \sqrt{y - 1}$$

This time each individual *y* needs to be modulised,

$$\therefore \text{ To plot curve } C \text{ use } x = 2 \pm \sqrt{|y| - 1}$$

Modulus Sketching Rule 2b

To sketch $x = f(|y|)$ **Overwrite negative *y***

◇ Sketch the curve of $x = f(y)$ for $y \geqslant 0$

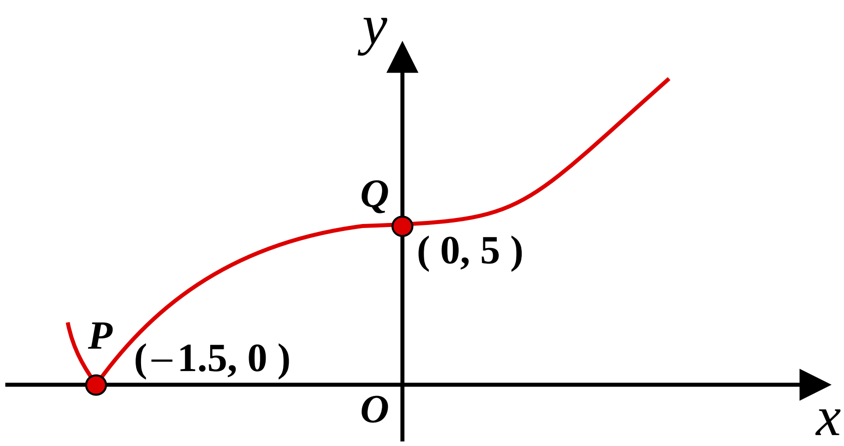
◇ Reflect the parts of the curve above the *x*-axis in the *x*-axis.

[2 marks]

4.5.4 Solution to 4.4 Exercise

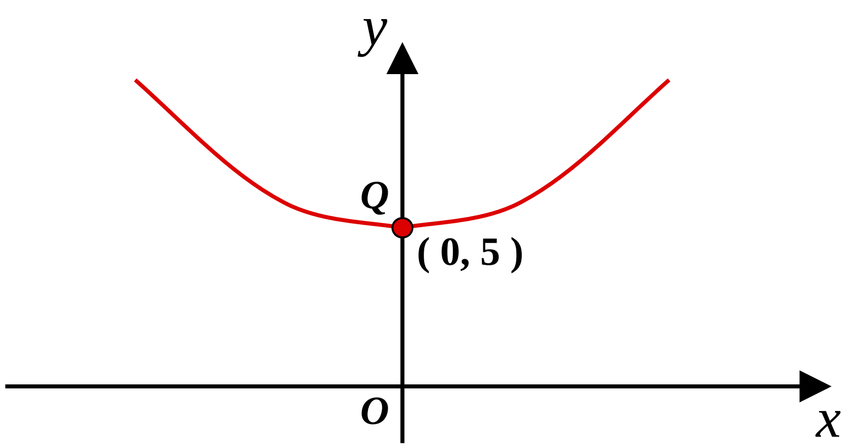
Answer 1

(a)



[2 marks]

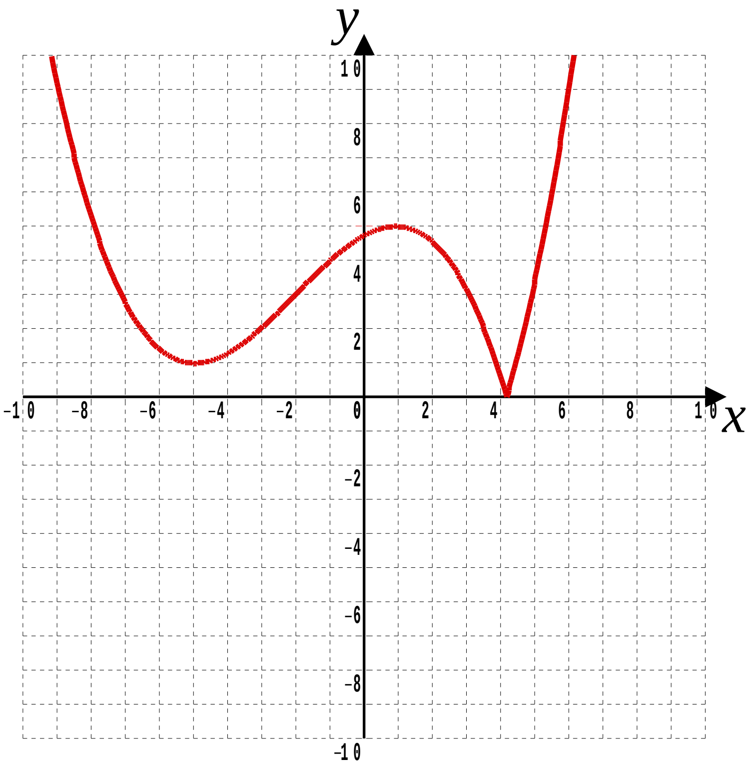
(b)



[2 marks]

Answer 2

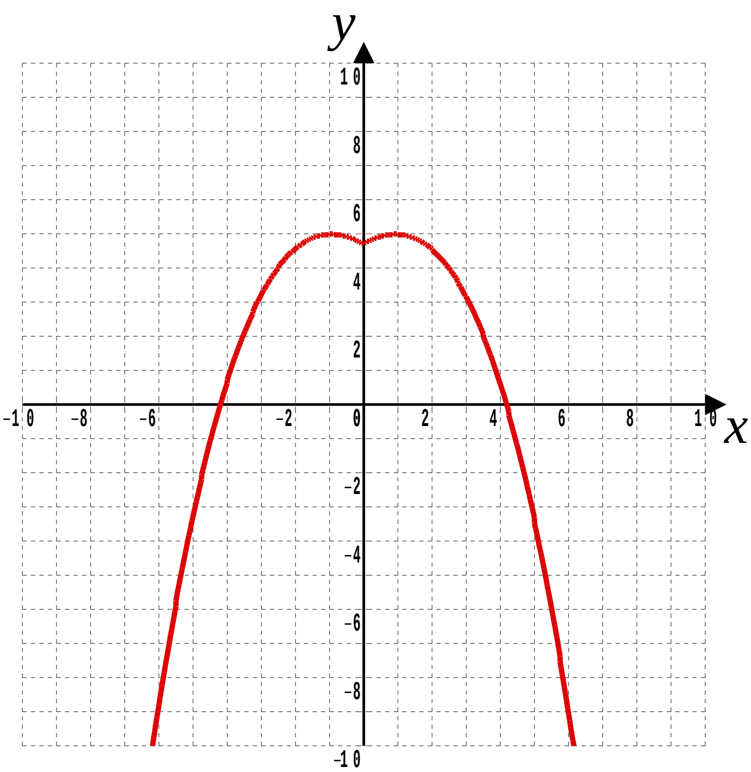
(i)



(1, 5) is a maximum and (– 5, 1) is a minimum turning point.

[2 marks]

(ii)



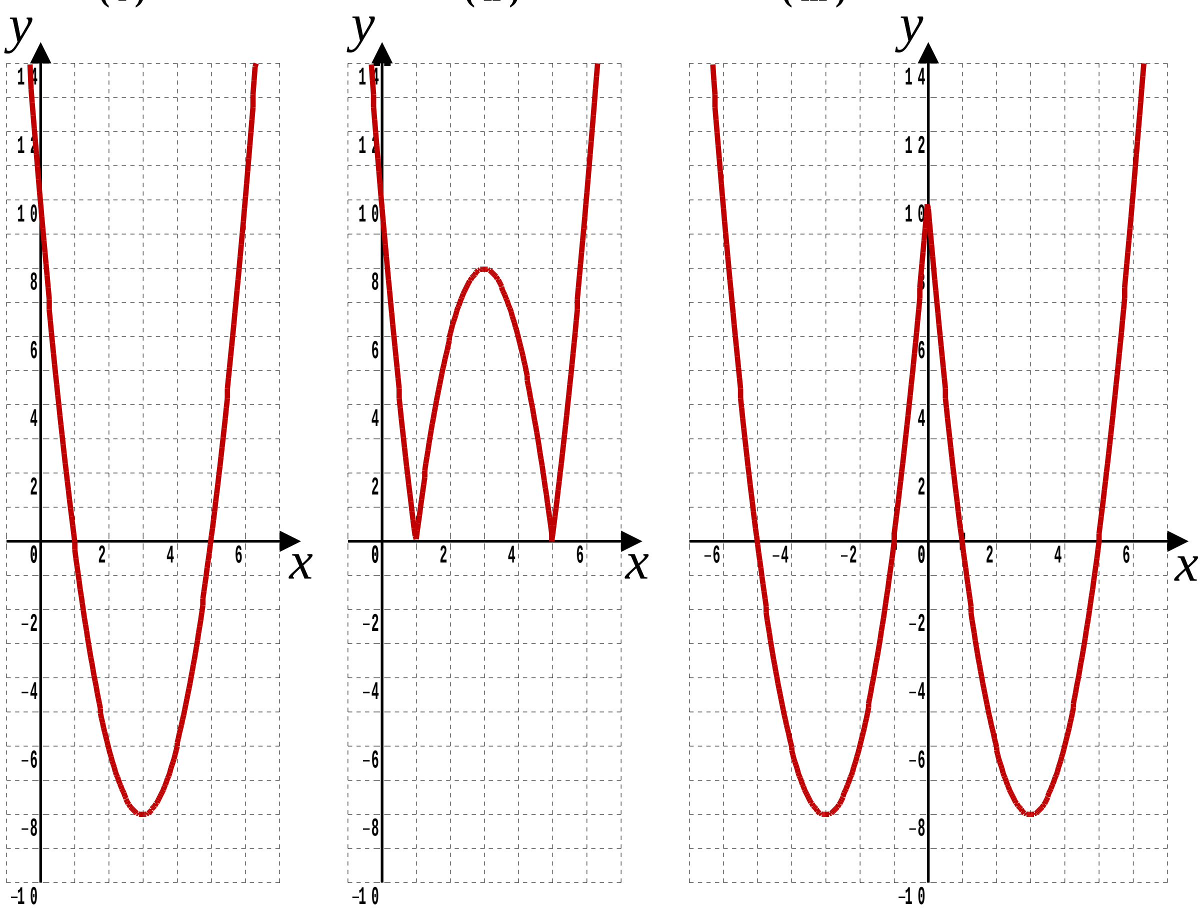
Two maximum turning points: One at (1, 5), the other at (– 1, 5)

[3 marks]

Answer 3

(a)

(i) **(ii)** **(iii)**



To work out x-axis intercepts of original equation:

$$2(x-3)^2-8=0$$

$$(x-3)^2-4=0$$

$$(x-3)^2=4$$

$$x-3=\pm 2$$

$$x=1 \text{ or } x=5$$

[3, 3, 3 marks]

(b) $-8<k<10$

[2 marks]

Answer 4

(a) Finding the line equations of the sides of the triangle, (other than the *x*-axis);

$$y = 7 - (3x - 5) \qquad\qquad y = 7 + (3x - 5)$$

$$y = 7 - 3x + 5 \qquad\qquad y = 7 + 3x - 5$$

$$y = -3x + 12 \qquad\qquad y = 3x + 2$$

Finding where each intersects the *x*-axis;

$$-3x + 12 = 0 \qquad\qquad 3x + 2 = 0$$

$$x = 4 \qquad\qquad x = -\frac{2}{3}$$

So the base of the triangle is of length $\frac{14}{3}$

Finding the apex of triangle, where lines intersect,

$$3x + 2 = -3x + 12$$

$$6x = 10$$

$$x = \frac{5}{3} \qquad \therefore\ y = 3 \times \frac{5}{3} + 2$$

$$= 7$$

Finding the area *R;*

$$Area\ \triangle = \frac{1}{2} \times \frac{14}{3} \times 7$$

$$= \frac{49}{3} \qquad \left(= 16\frac{1}{3}\right)$$

[4 marks]

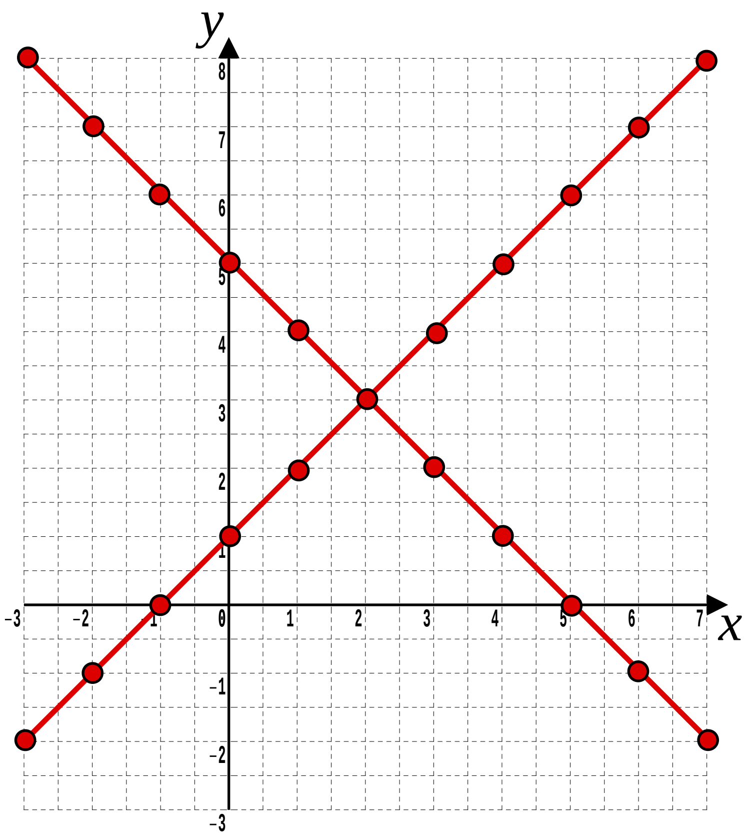
(b) $k < 7$

[1 mark]

Answer 5

(i) $$|y - 3| = |x - 2|$$

x	-3	-2	-1	0	1	2	3	4	5	6	7
y	8 -2	7 -1	6 0	5 1	4 2	3 3	2 4	1 5	0 6	-1 7	-2 8



[5 marks]

(ii) $k = 3$

[1 mark]

(iii) $c = \dfrac{5}{2}$

[1 mark]

Answer 6

(a) **(i)** True **(ii)** True **(iii)** True

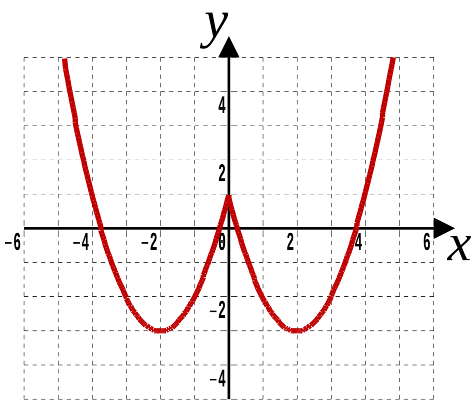
[3 marks]

(b)

$$f(x) = |x|^2 - 4|x| + 1$$
$$= x^2 - 4|x| + 1 \quad \text{(Making use of part (a))}$$

Case 1 : $y = x^2 - 4x + 1$ for $x \geqslant 0$
$$= (x - 2)^2 - 4 + 1$$
$$= (x - 2)^2 - 3 \qquad \therefore \text{ Minimum at } (2, -3)$$

Case 2 : $y = x^2 + 4x + 1$ for $x < 0$
$$= (x + 2)^2 - 4 + 1$$
$$= (x + 2)^2 - 3 \qquad \therefore \text{ Minimum at } (-2, -3)$$



[3 marks]

(c)

$$f(x) = -2$$
$$x^2 - 4|x| + 1 = -2 \quad \text{(Making use of part(a))}$$

Case 1 : $x^2 - 4x + 1 = -2$ for $x \geqslant 0$
$$x^2 - 4x + 3 = 0$$
$$(x - 1)(x - 3) = 0 \qquad \therefore x = 1 \text{ or } x = 3$$

Case 2 : $x^2 + 4x + 1 = -2$ for $x < 0$
$$x^2 + 4x + 3 = 0$$
$$(x + 1)(x + 3) = 0 \qquad \therefore x = -1 \text{ or } x = -3$$

Overall, $x \in \{\pm 1, \pm 3\}$

[3 marks]

(d) $k > 1$ or $k = -3$

[3 marks]

Answer 7

$$f(x) = 2|3 - x| + 5$$

(a) Range : $f(x) \in \mathbb{R},\ x \geqslant 5$

[1 mark]

(b)

$$2|3 - x| + 5 = \frac{1}{2}x + 30$$

$$-2(3 - x) + 5 = \frac{1}{2}x + 30 \qquad 2(3 - x) + 5 = \frac{1}{2}x + 30$$

$$-6 + 2x + 5 = \frac{1}{2}x + 30 \qquad 6 - 2x + 5 = \frac{1}{2}x + 30$$

$$\frac{3}{2}x = 31 \qquad -\frac{5}{2}x = 19$$

$$x = \frac{62}{3} \qquad x = -\frac{38}{5}$$

However, $-\dfrac{38}{5}$ is an extraneous solution.

$\therefore$ The only solution is $\dfrac{62}{3}$

[3 marks]

(c) $5 < k \leqslant 11$

[2 marks]

Answer 8
(i)
Modulus Sketching Rule 2b

To sketch $x = f\,(\,|\,y\,|\,)$ **Overwrite negative *y***

◇ Sketch the curve of $x = f\,(y)$ for $y \geqslant 0$
◇ Reflect the parts of the curve above the *x*-axis in the *x*-axis

---

$$\therefore\ x = 3 \pm \sqrt{25 - (\,|y| - 2\,)^2}$$

[3 marks]

(ii)
Modulus Sketching Rule 1b

To sketch $x = \left|\, f\,(y) \,\right|$ **Bounce negative *y***

◇ Sketch $x = f\,(\mathrm{y})$ using a dashed line for points left of the *y*-axis.
◇ Reflect any part of the curve left of the *y*-axis in the *y*-axis.

$$\therefore\ x = \left|\, 3 \pm \sqrt{25 - (\,y - 2\,)^2}\, \right|$$

[3 marks]

(iii)
Modulus Sketching Rule 1

To sketch $y = \left|\, f\,(x) \,\right|$ **Bounce negative *x***

◇ Sketch $y = f\,(x)$ using a dashed line for points below the *x*-axis.
◇ Reflect any part of the curve below the *x*-axis in the *x*-axis.

$$y = \left|\, 2 \pm \sqrt{25 - (\,x - 3\,)^2}\, \right|$$

[3 marks]

(iv)
Modulus Sketching Rule 2

To sketch $y = f\,(\,|\,x\,|\,)$ **Overwrite negative *x***

◇ Sketch the curve of $y = f\,(x)$ for $x \geqslant 0$
◇ Reflect the parts of the curve to the right of the *y*-axis in the *y*-axis.

$$y = 2 \pm \sqrt{25 - (\,|x| - 3\,)^2}$$

[3 marks]

Answer 9

(a)

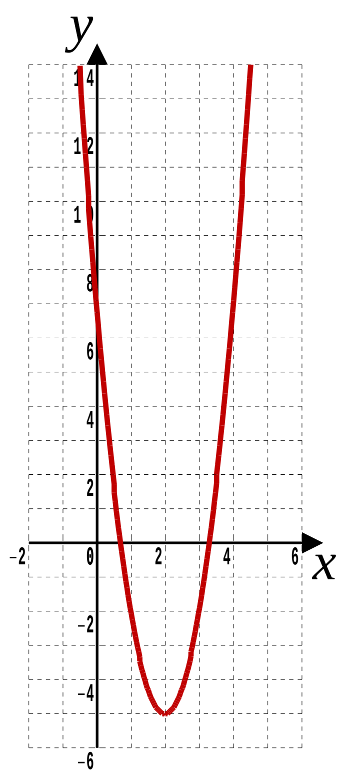
$$\begin{aligned} f(x) &= 3x^2 - 12x + 7 \\ &= 3\left[x^2 - 4x\right] + 7 \\ &= 3\left[(x-2)^2 - 4\right] + 7 \\ &= 3(x-2)^2 - 12 + 7 \\ &= 3(x-2)^2 - 5 \end{aligned}$$

[3 marks]

(b) To obtain the *x*-axis crossing points,

$$\begin{aligned} 3(x-2)^2 - 5 &= 0 \\ 3(x-2)^2 &= 5 \\ (x-2)^2 &= \frac{5}{3} \\ x - 2 &= \pm\sqrt{\frac{5}{3}} \\ x &= 2 \pm \frac{\sqrt{15}}{3} \end{aligned}$$

(Approx: $x = 0.709$ $x = 3.291$)



[3 marks]

(c) **(i)**

$$g(x) = 3(x-1)^2 - 12x + 9$$
$$= 3(x-1)^2 - 12x + 12 - 12 + 9$$
$$= 3(x-1)^2 - 12(x-1) - 3$$
$$= 3(x-1)^2 - 12(x-1) + 7 - 7 - 3$$
$$= 3(x-1)^2 - 12(x-1) + 7 - 10$$
$$g(x) + 10 = 3(x-1)^2 - 12(x-1) + 7$$

Compare this with $f(x) = 3x^2 - 12x + 7$

$\therefore\ f(x)$ translated $\begin{pmatrix} 1 \\ -10 \end{pmatrix}$ will give $g(x)$

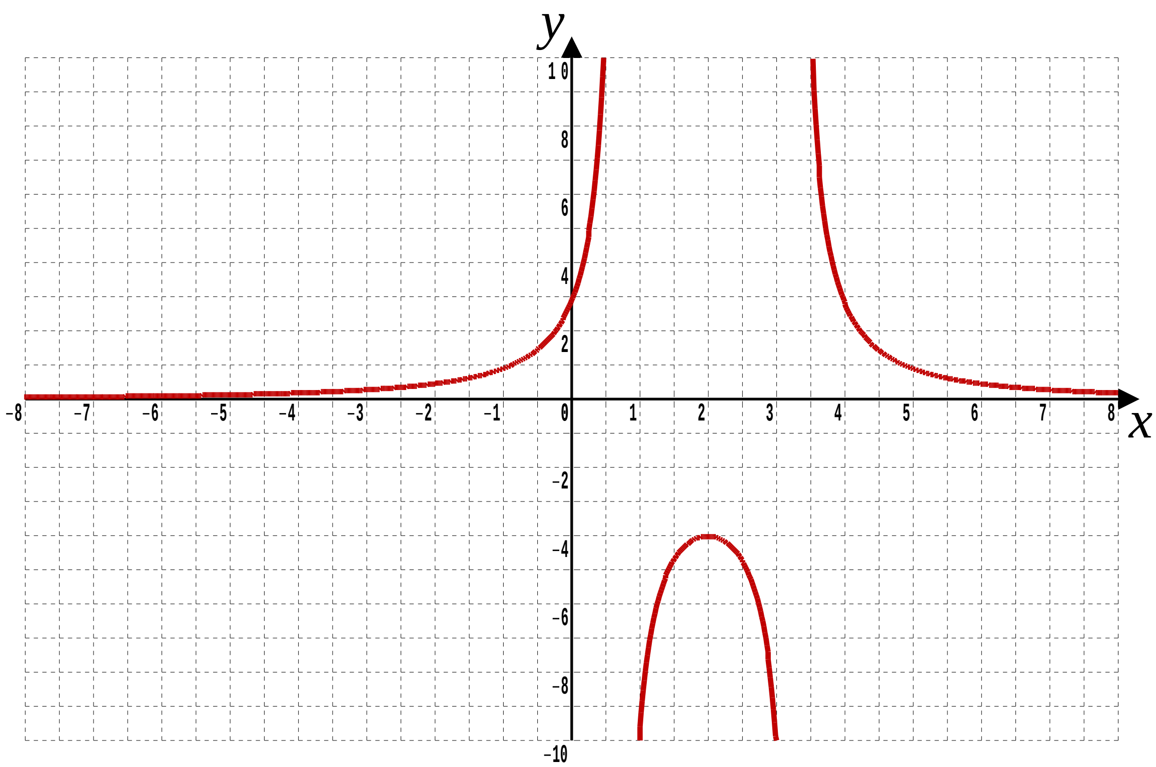
[2 marks]

(ii) This is tricky but the graph below helps with the understanding;

$$h(x) = \frac{20}{3x^2 - 12x + 7} = \frac{20}{3(x-2)^2 - 5}$$

$$h(2) = \frac{20}{-5} = -4 \quad \text{(A local maximum)}$$

Range : $h(x) < -4,\ h(x) > 0$

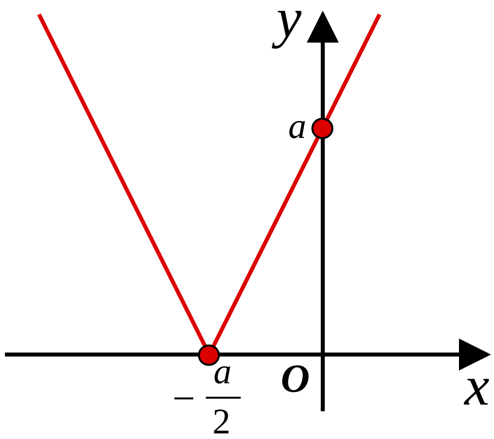


[2 marks]

Answer 10

(a) **(i)** Working out the x-axis "touch";

$$|2x + a| = 0$$
$$2x + a = 0$$
$$x = -\frac{a}{2}$$



[2 marks]

(ii) Working out the x-axis crossing points;

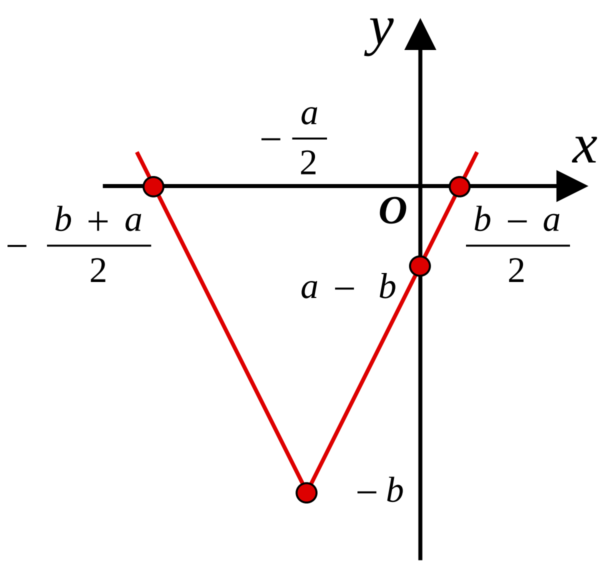
$$|2x + a| - b = 0$$
$$|2x + a| = b$$

Case 1: $2x + a = b$

$$2x = b - a$$
$$x = \frac{b - a}{2}$$

Case 2: $-2x - a = b$

$$2x + a = -b$$
$$2x = -b - a$$
$$x = -\frac{b + a}{2}$$



[4 marks]

(b)

$$|2x + a| - b = \frac{1}{3}x$$

$$\text{Case 1: } 2x + a - b = \frac{1}{3}x$$

$$\frac{6x}{3} - \frac{x}{3} = b - a$$

$$5x = 3(b - a)$$

$$x = \frac{3}{5}(b - a)$$

$$\text{Case 2: } -2x - a - b = \frac{1}{3}x$$

$$-a - b = \frac{x}{3} + \frac{6x}{3}$$

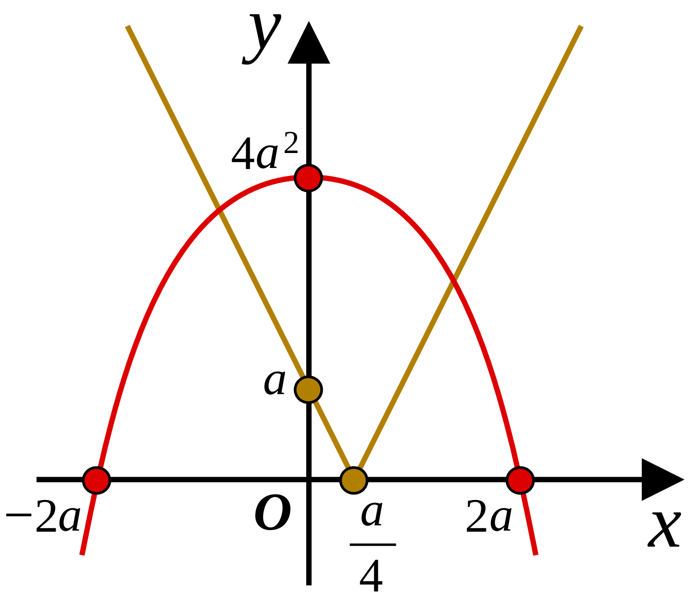
$$7x = -3(a + b)$$

$$x = -\frac{3}{7}(a + b)$$

[4 marks]

Answer 11

(i)



Red: $f(x) = 4a^2 - x^2$ Gold: $g(x) = |4x - a|$

[6 marks]

(ii)

$$4 - x^2 = |4x - 1|$$

$$4 - x^2 = 4x - 1 \qquad\qquad 4 - x^2 = -4x + 1$$

$$x^2 + 4x - 5 = 0 \qquad\qquad x^2 - 4x - 3 = 0$$

$$(x + 5)(x - 1) = 0 \qquad\qquad (x - 2)^2 - 7 = 0$$

$$x = -5 \text{ or } x = 1 \qquad\qquad x = 2 \pm \sqrt{7}$$

Two of these proposed solutions are extraneous.

The only valid solutions are, $x = 1$ or $2 + \sqrt{7}$

[4 marks]

Answer 12

$$f(x) = \frac{\pi}{180}x - 5 + \sin x \qquad 300° \leqslant x \leqslant 600°$$

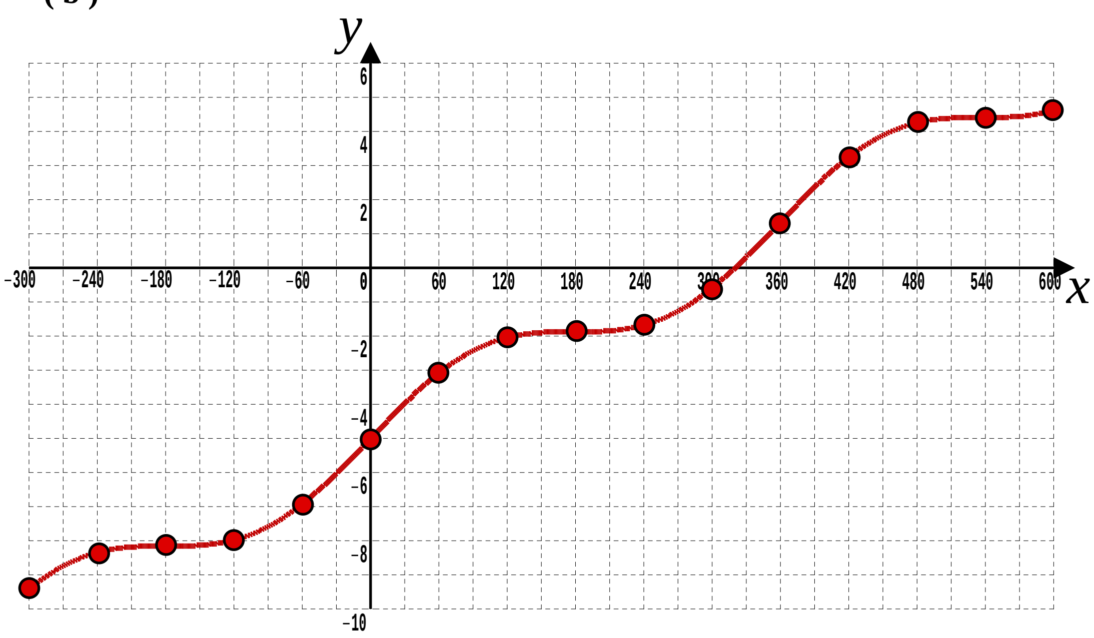
(a)

x	– 300	– 240	– 180	– 120	– 60	0	60	120
y	– 9.37	– 8.32	– 8.14	– 7.96	– 6.91	– 5	– 3.09	– 2.04

x	180	240	300	360	420	480	540	600
y	– 1.86	– 1.68	– 0.630	1.28	3.20	4.24	4.42	4.61

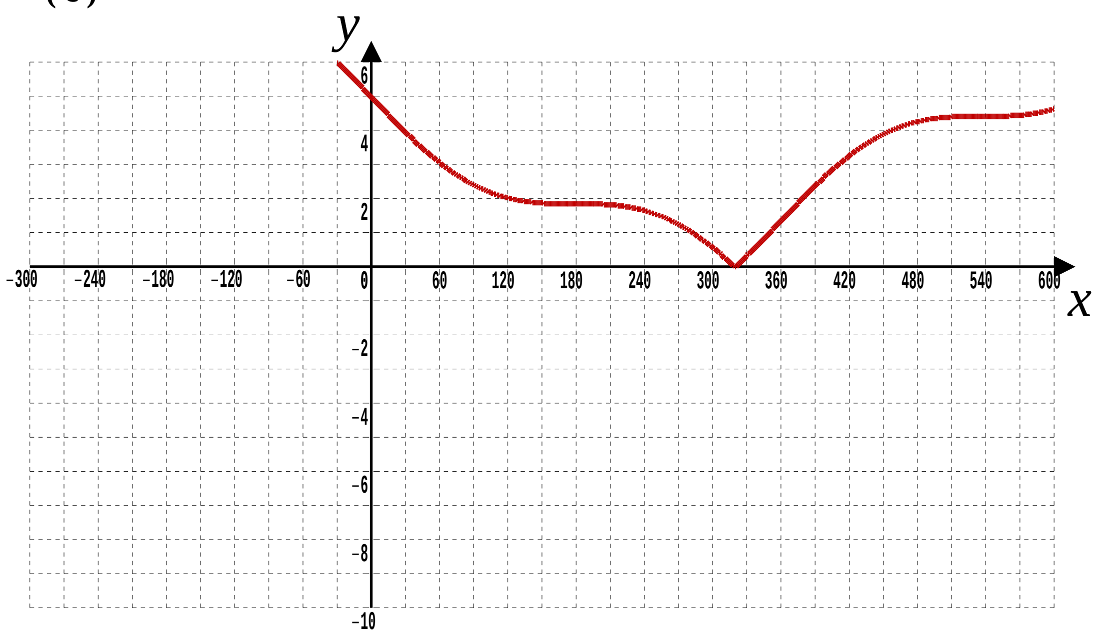
[3 marks]

(b)



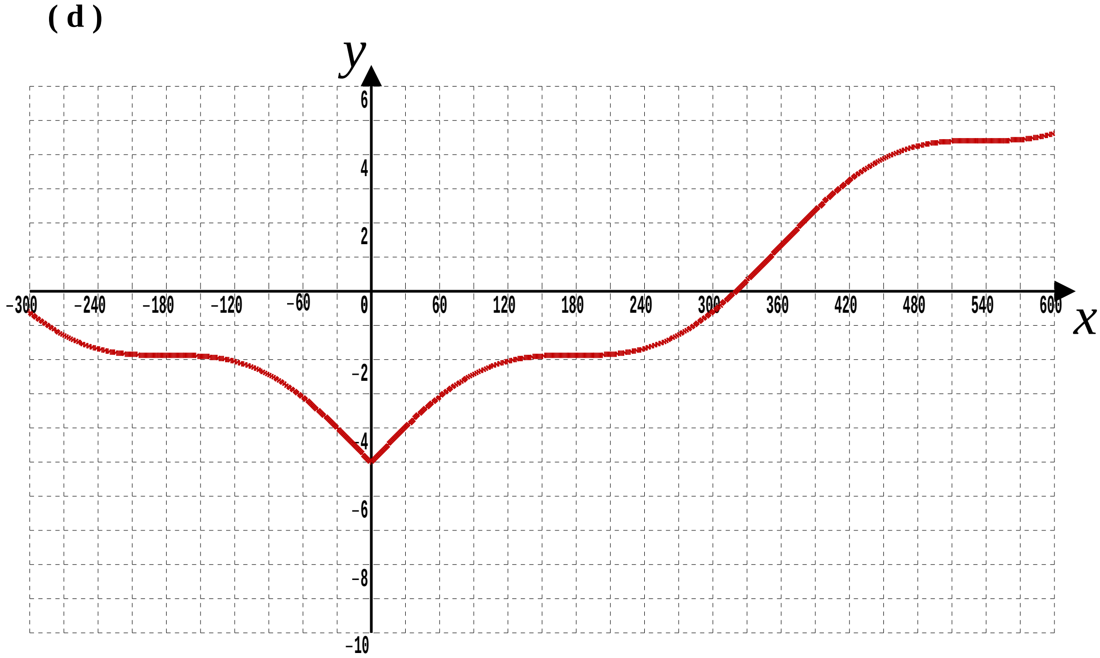
[2 marks]

(c)



[2 marks]

(d)



[2 marks]

5.1 Transformation Of Graphs (Part 2)

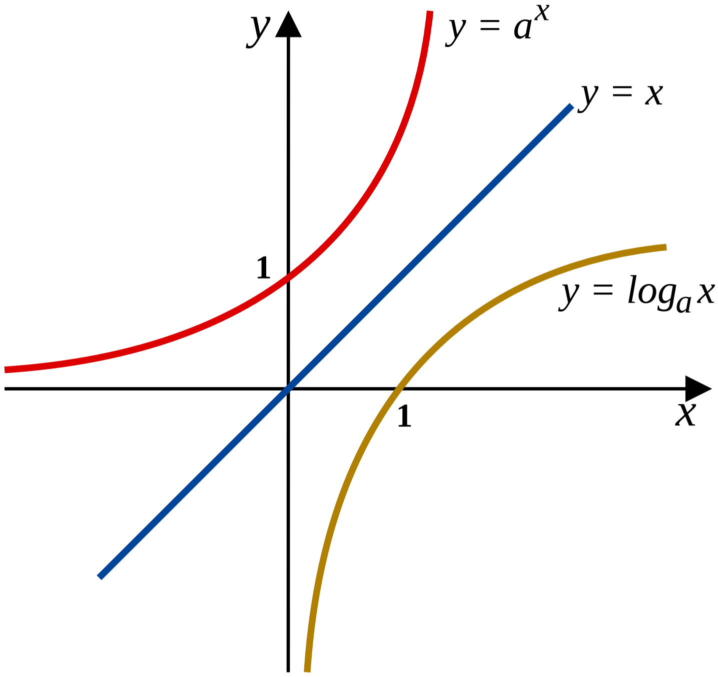
Replace all occurrences of...	...with	Effect on graph (Transformation)
x	$(x - a)$	Translation $\begin{pmatrix} a \\ 0 \end{pmatrix}$
y	$(y - b)$	Translation $\begin{pmatrix} 0 \\ b \end{pmatrix}$
x	$(-x)$	Reflection in the y-axis
y	$(-y)$	Reflection in the x-axis
x	(cx)	Stretch parallel to the x-axis with scale factor $\frac{1}{c}$
y	(dy)	Stretch parallel to the y-axis with scale factor $\frac{1}{d}$

- ◇ Replacing all occurrences of x with y AND all occurrences of y with x causes reflection in the line $y = x$.
- ◇ Reflecting the graph of a one-to-one function in the line $y = x$ gives the graph of the inverse function.

5.2 Example #1

If $f(x) = a^x$ then $f^{-1}(x) = log_a x, \quad x > 0$

When plotted on a common graph each is a reflection of the other in the line $y = x$



5.3 Example #2

Sketch on separate diagrams the graph of the following four related equations, each time, stating the range of the corresponding function.

(i) $y = (x - 3)^2 - 4$ **(ii)** $y = \left| (x - 3)^2 - 4 \right|$

(iii) $y + 2 = \left| (x - 3)^2 - 4 \right|$ **(iv)** $2 - y = \left| (x - 3)^2 - 4 \right|$

[3, 3, 3, 3 marks]

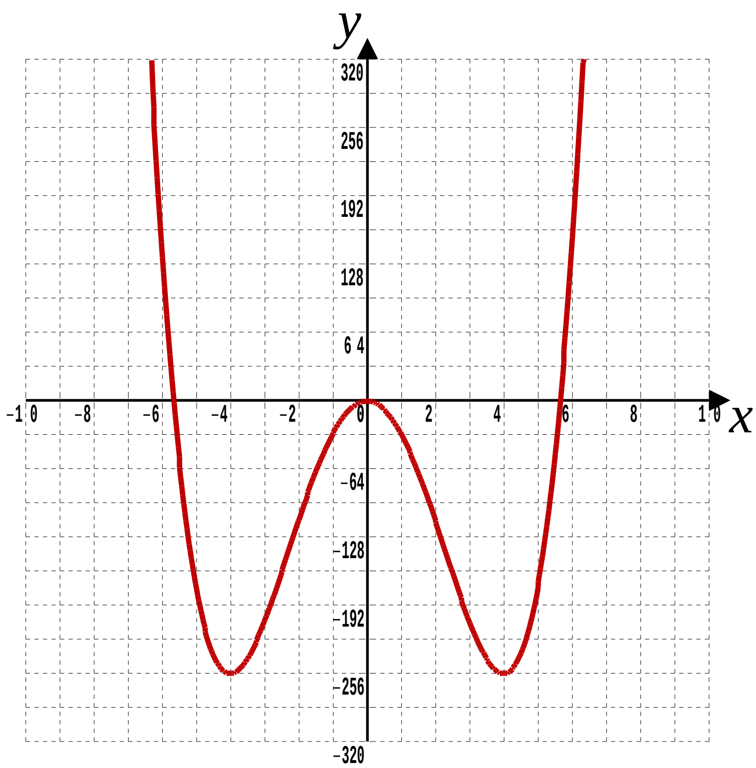
5.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 46

Question 1

The function $y = x^4 - 32x^2$ is graphed below.



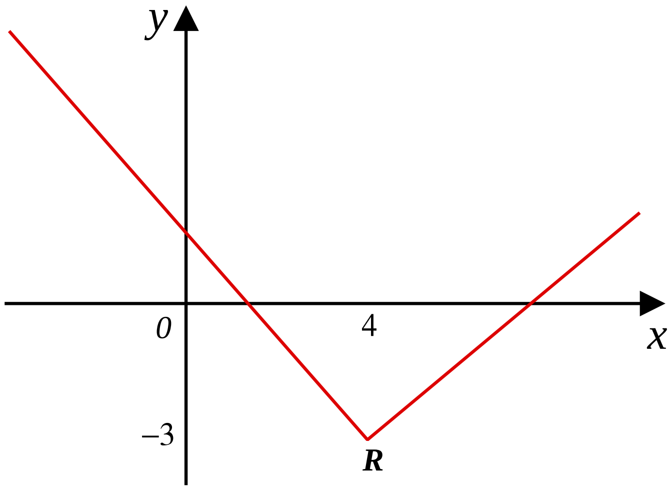
Paying attention to where the three turning points lie, sketch the related curves,

(i) $y = |x^4 - 32x^2|$ **(ii)** $y = |(x-2)^4 - 32(x-2)^2| + 32$

[2, 3 marks]

Question 2

The graph of a function, $y = f(x)$, $x \in \mathbb{R}$, consists of two line segments that meet at the point $R(4, -3)$



Sketch, on two separate diagrams, the graphs of,

(a) $y = 2f(x + 4)$

(b) $y = |f(-x)|$

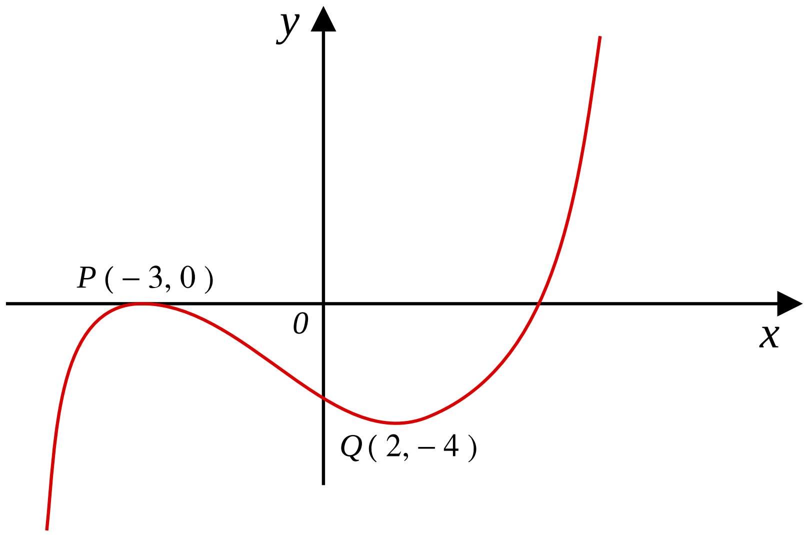
On each graph, show the coordinates of the point corresponding to R

[3, 3 marks]

Question 3

The graph of a function, $y = f(x)$, $x \in \mathbb{R}$, has two turning points.

One is at $P(-3,\ 0)$ and the other is at $Q(2, -4)$



Sketch, on two separate diagrams, the graphs of,

(a) $y = 3f(x + 2)$

(b) $y = |\, f(x) \,|$

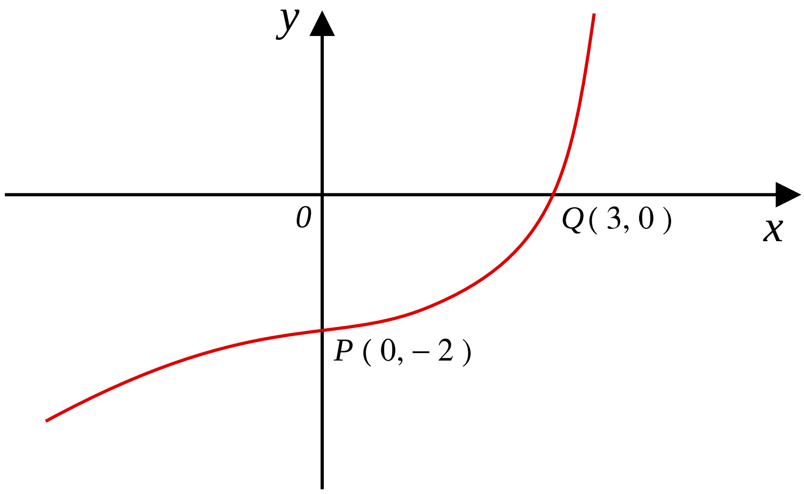
On each graph, show the coordinates of any turning points.

[3, 3 marks]

Question 4

The graph of a function, $y = f(x)$, passes through $P(0, -2)$ and $Q(3, 0)$

Furthermore, $f(x)$ is an increasing function.



Sketch, on three separate diagrams, the graphs of,

(a) $y = |\, f(x)\, |$

(b) $y = f^{-1}(x)$

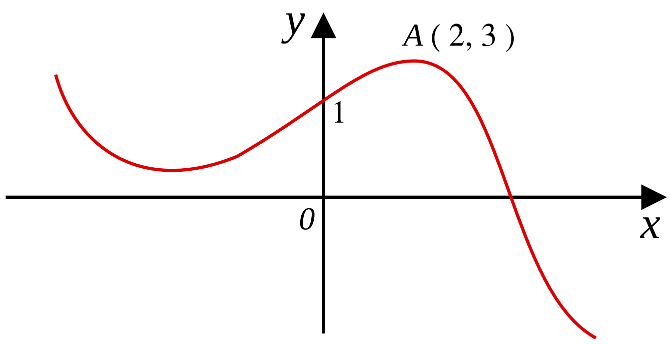
(c) $y = \dfrac{1}{2} f(3x)$

On each graph, show the coordinates of points where contact is made with axes.

[3, 3, 3 marks]

Question 5

The graph of a function, $y = f(x)$, $x \in \mathbb{R}$, intercepts the y-axis at $(0, 1)$ and has a local maximum at $A(2, 3)$, as shown.



Sketch, on three separate diagrams, the graphs of,

(a) $y = f(-x) + 1$

(b) $y = f(x + 2) + 3$

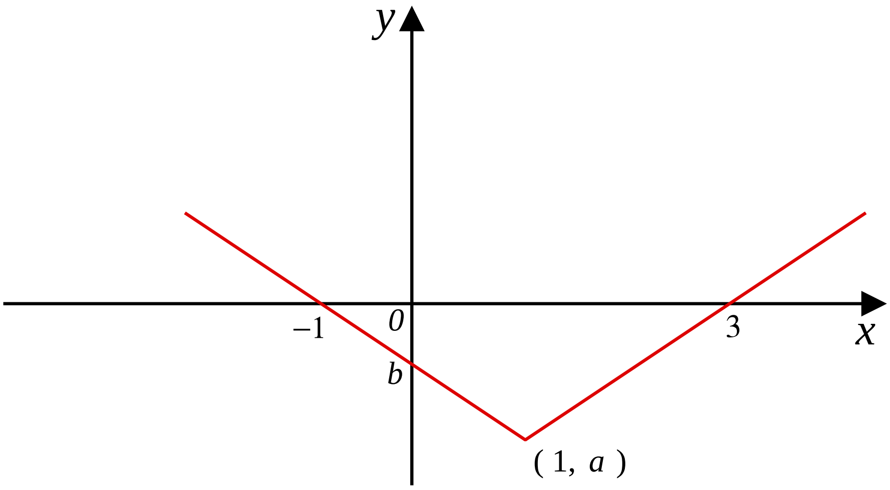
(c) $y = 2f(2x)$

On each sketch, show the coordinates of the point at which it intersects the y-axis and the coordinates of the point to which A is transformed.

[3, 3, 3 marks]

Question 6

The diagram shows part of the graph of $y = f(x)$, $x \in \mathbb{R}$



The graph consists of two line segments that meet at the point $(1, a)$, $a < 0$

One line meets the *x*-axis at $(3, 0)$

The other line meets the *x*-axis at $(-1, 0)$ and the *y*-axis at $(0, b)$, $b < 0$

In separate diagrams, sketch the graph with equation,

(a) $y = f(x + 1)$

(b) $y = f(|x|)$

Indicate on each sketch the coordinates of any points of intersection with the axes.

[2, 3 marks]

Given that

$$f(x) = |x - 1| - 2$$

find

(c) the value of a and the value of b

[2 marks]

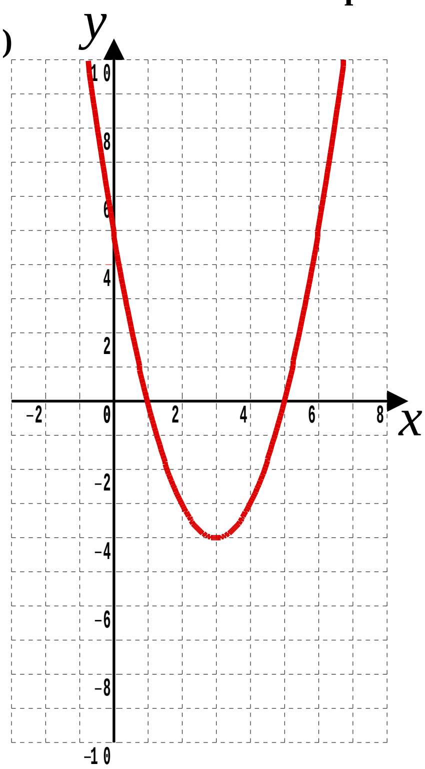
(d) the value of x for which $f(x) = 5x$

[4 marks]

5.5 Answers

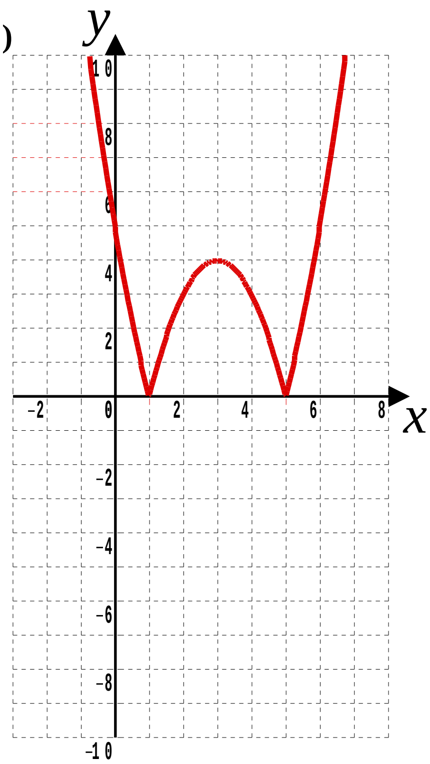
5.5.1 Solution to the 5.3 Example

(a)



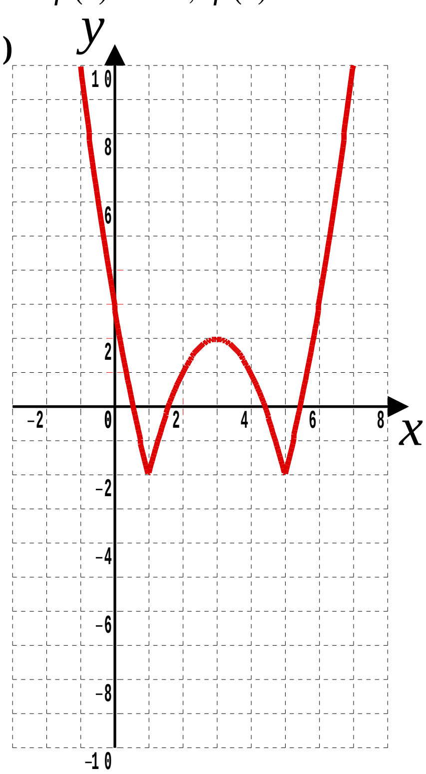
$f(x) \in \mathbb{R},\ f(x) \geqslant -4$

(b)



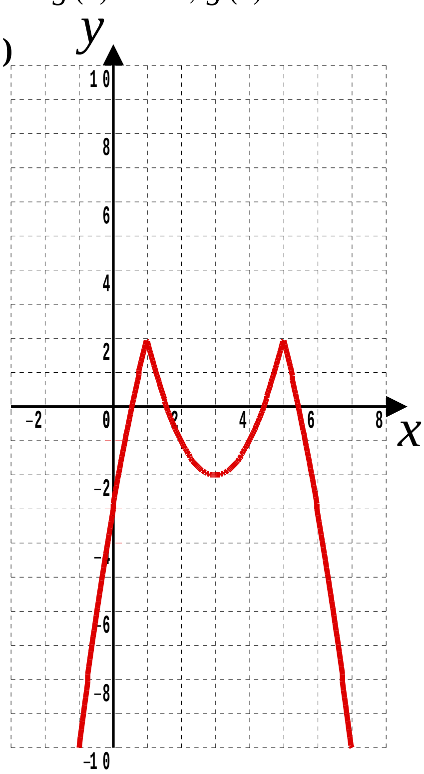
$g(x) \in \mathbb{R},\ g(x) \geqslant 0$

(c)



$h(x) \in \mathbb{R},\ h(x) \geqslant -2$

(d)



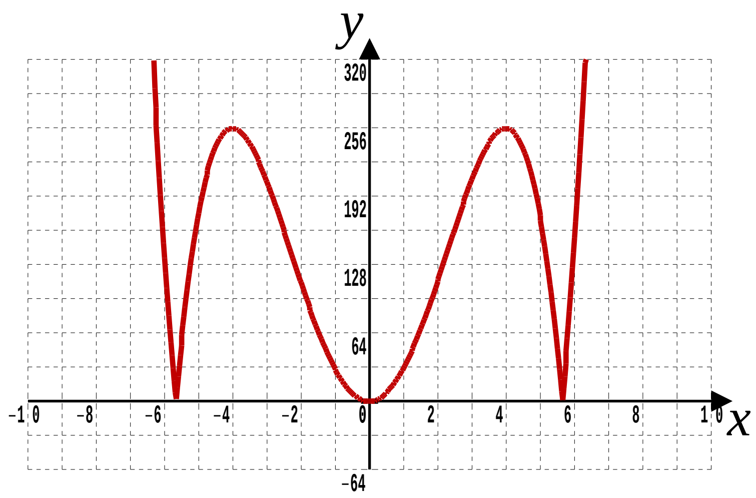
$k(x) \in \mathbb{R},\ k(x) \leqslant 2$

[3, 3, 3, 3 marks]

5.5.2 Solution to the 5.4 Exercise

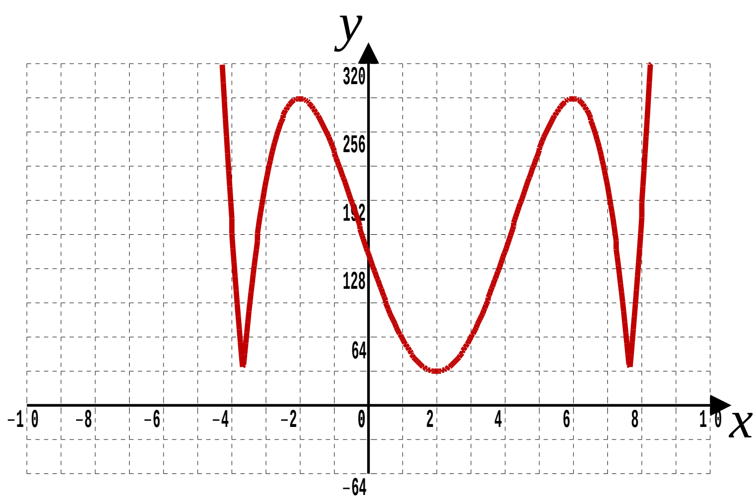
Answer 1

(i)



[2 marks]

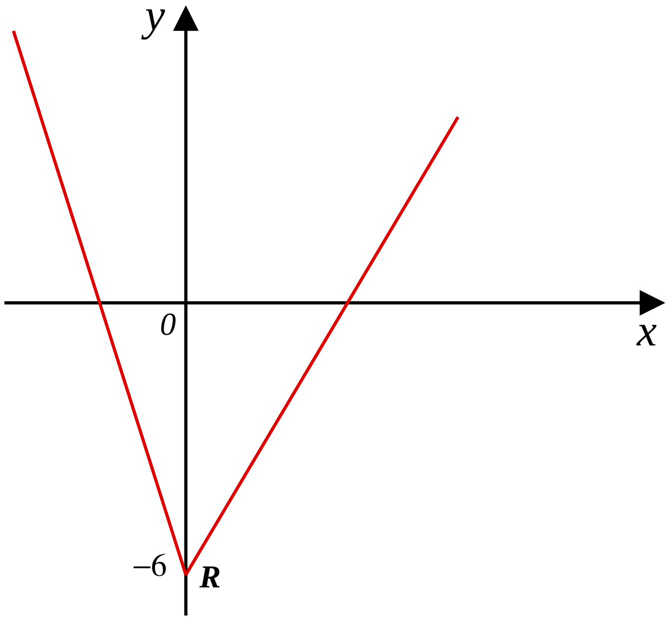
(ii)



[3 marks]

Answer 2

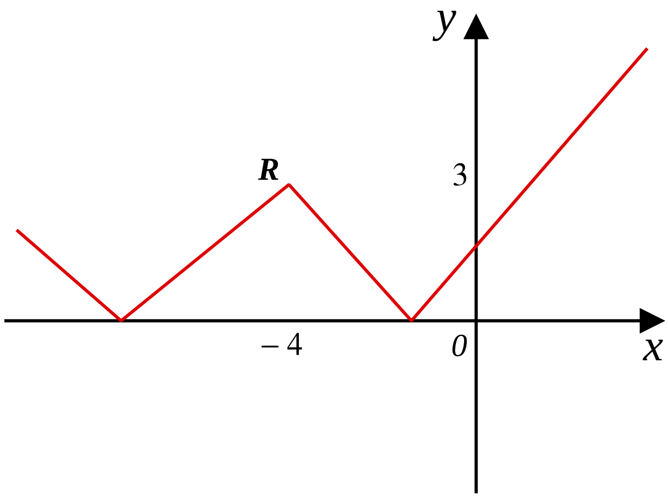
(a)



$R(0, -6)$

[3 marks]

(b)

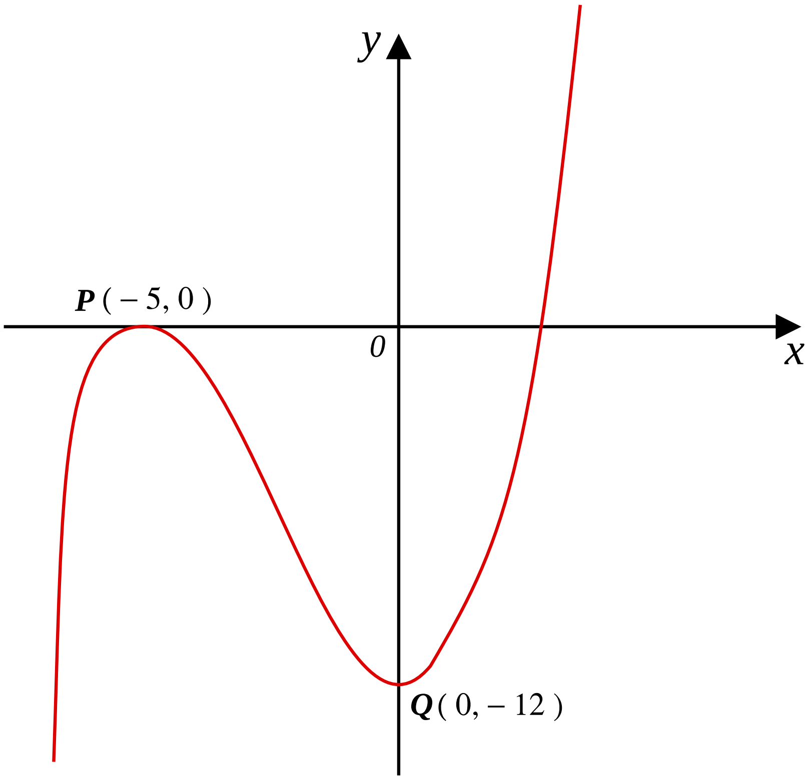


$R(-4, 3)$

[3 marks]

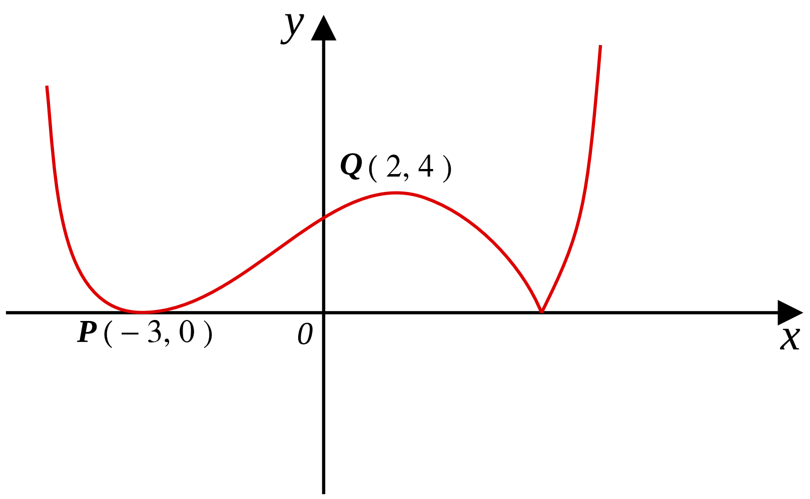
Answer 3

(a)



[3 marks]

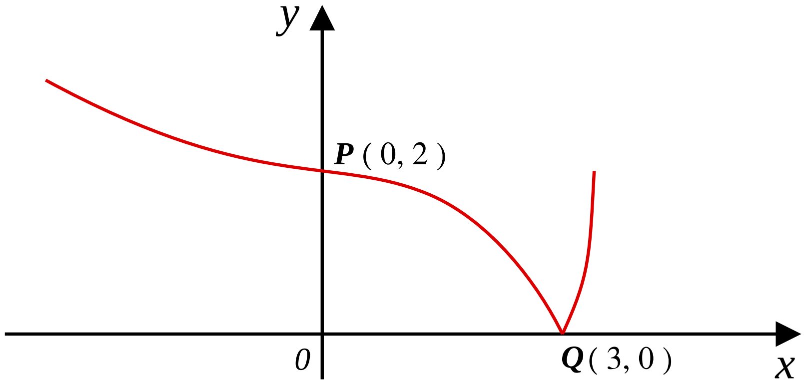
(b)



[3 marks]

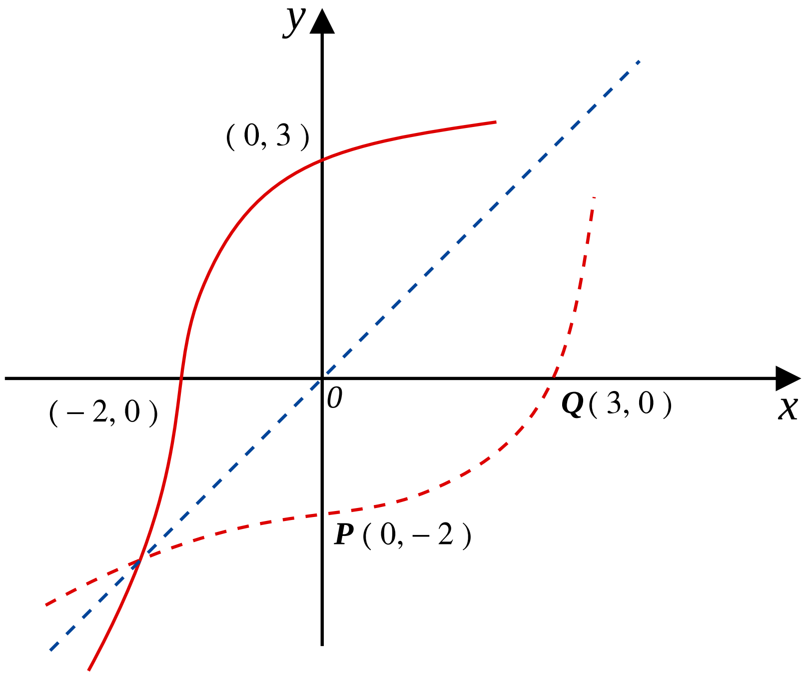
Answer 4

(a)



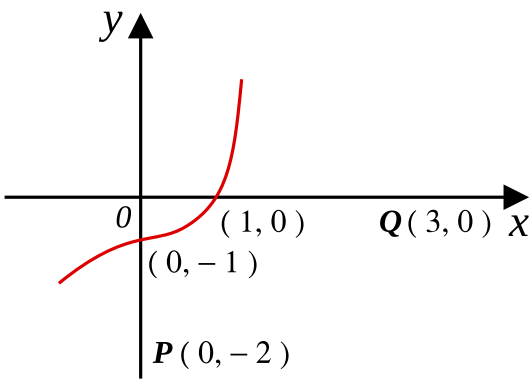
[3 marks]

(b)



[3 marks]

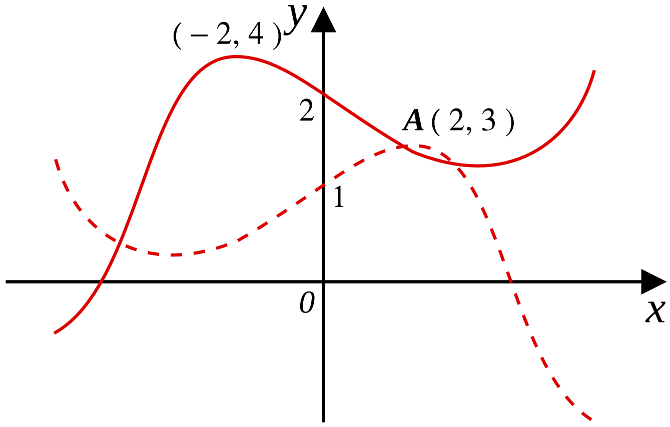
(c)



[3 marks]

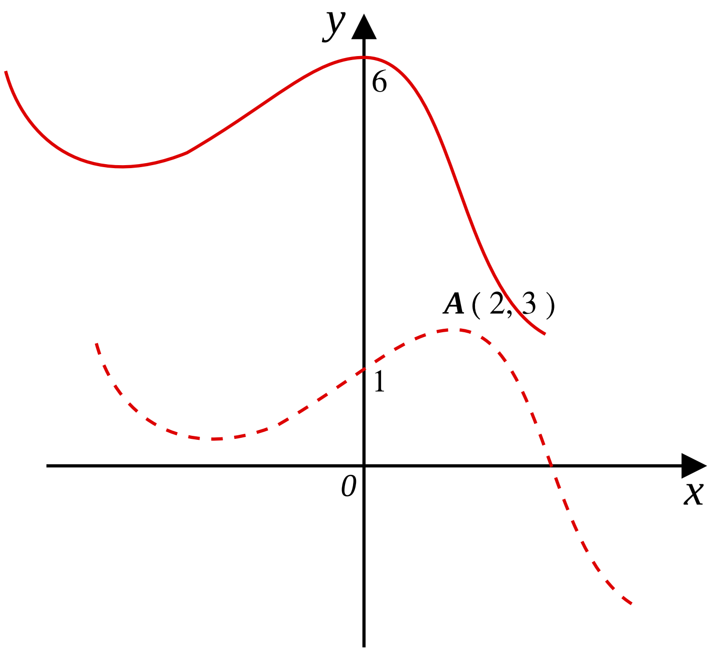
Answer 5

(a)



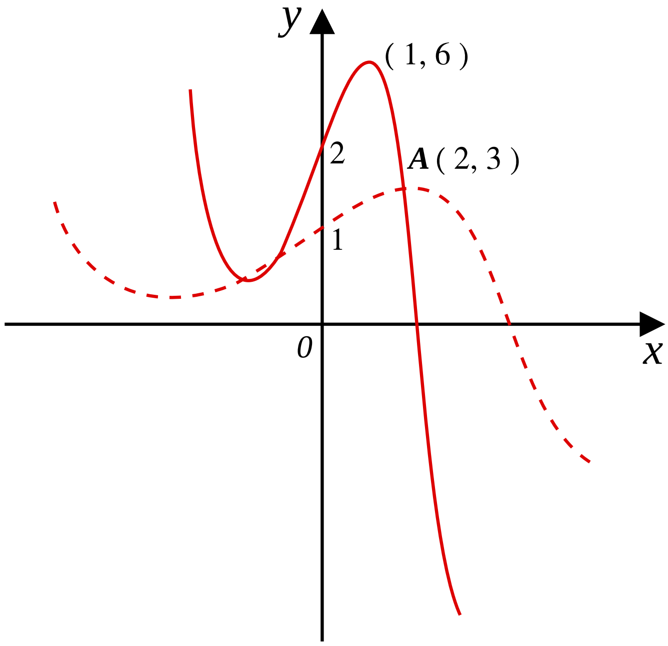
[3 marks]

(b)



[3 marks]

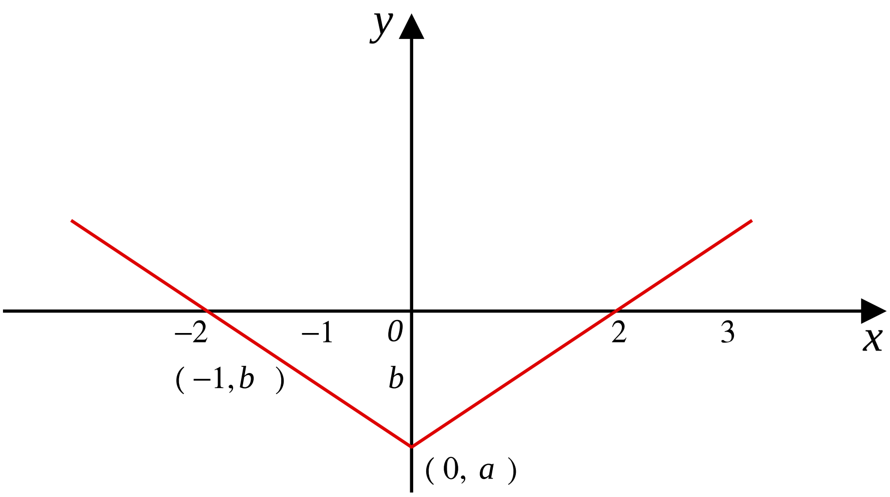
(c)



[3 marks]

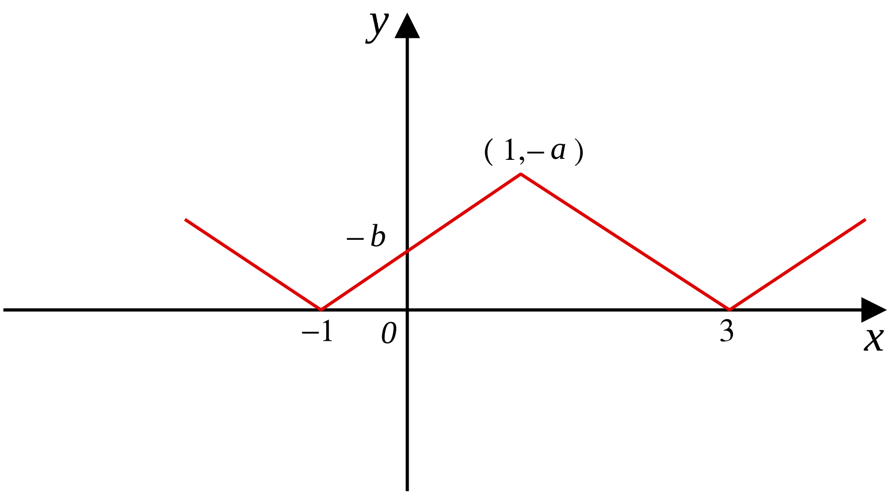
Answer 6

(a)



[2 marks]

(b)



[3 marks]

(c) $a = -2, \quad b = 1$

[2 marks]

(d)

$$-(x-1)-2 = 5x \qquad\qquad +(x-1)-2 = 5x$$

$$-x+1-2 = 5x \qquad\qquad x-1-2 = 5x$$

$$-1 = 6x \qquad\qquad -3 = 4x$$

$$x = -\frac{1}{6} \qquad\qquad x = -\frac{3}{4}$$

From the graph, it's obvious that only one of these solutions is going to satisfy the original equation.
By substitution the only answer is,

$$x = -\frac{1}{6}$$

[4 marks]

6.1 Composite Functions

The topic of Composite Functions is covered at GCSE.
At that level, the emphasis is on numerical problems, with the algebraic treatment kept simple and straightforward. For A Level, the emphasis is on the algebra along with concern about domains and ranges.

Keep in mind that *fg(x)* means *eff the already gee'd x*

6.2 Example

Let *s* and *t* be the functions

$$s(x) = (x + 1)^2 \qquad x \in \mathbb{R}$$

$$t(x) = x - 2 \qquad x \in \mathbb{R}$$

(a) Determine each of the following,

(i) $st(8)$ **(ii)** $ts(8)$

(iii) $ts(x)$ **(iv)** $st(x)$

[4 marks]

(b) Find the unique value of x such that

$$ts(x) = st(x)$$

[2 marks]

Notice that, in general, $st(x) \neq ts(x)$

6.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 60

Question 1

Given that, $f(x) = 3x + 2, \quad x \in \mathbb{R}$ and $g(x) = 5x - 4, \quad x \in \mathbb{R}$

Find an expression for $gf(x)$ that does not contain any brackets.

[2 marks]

Question 2

Given that, $f(x) = 7x - 5, \quad x \in \mathbb{R}$ and $g(x) = 10 - x, \quad x \in \mathbb{R}$

Find an expression for $fg(x)$ that does not contain any brackets.

[2 marks]

Question 3

Given that, $f(x) = x^2 + x, \quad x \in \mathbb{R}$ and $g(x) = x + 3, \quad x \in \mathbb{R}$

(i) Find an expression for $fg(x)$ that does not contain any brackets.

[2 marks]

(ii) Find the two values of x such that, $fg(x) = 0$

[2 marks]

Question 4

Given that, $f(x) = 4x - 1, \quad x \in \mathbb{R}$ and $g(x) = x^2 + 1, \quad x \in \mathbb{R}$

Determine the following, giving answers free of any brackets;

(i) $f(3)$

(ii) $g(4)$

(iii) $fg(1)$

(iv) $gf(2)$

(v) $fg(x)$

(vi) State the range of your part **(v)** composite function

[1, 1, 2, 2, 3, 1 marks]

Question 5

$$f(x) = \frac{12}{x + 5} \qquad x \in \mathbb{R},\ x \neq -5$$

$$g(x) = 6x - 5 \qquad x \in \mathbb{R}$$

Find a simplified expression for $fg(x)$ that does not contain any brackets.

[2 marks]

Question 6

$$s(x) = x^2 + x, \qquad x \in \mathbb{R}$$

$$t(x) = x - 2, \qquad x \in \mathbb{R}$$

(a) Determine;

(i) $s(5)$ **(ii)** $sss(1)$ **(iii)** $ttt(1)$

(iv) $ts(10)$ **(v)** $st(3)$ **(vi)** $ts(-1)$

[3 marks]

(b) Determine $s\,t\,(x)$ writing your answer without any brackets.

[2 marks]

(c) Find the two values of x for which $st(x) = 0$

[2 marks]

Question 7

$$f(x) = \sqrt{x} + 3 \qquad x \in \mathbb{R},\ x \geqslant 0$$

$$g(x) = x + 2 \qquad x \in \mathbb{R}$$

(i) Find $fg(x)$

[2 marks]

(ii) State the domain of $fg(x)$ given that it should be as large as possible

[1 mark]

(iii) State the corresponding range of $fg(x)$

[1 mark]

Question 8

$$f(x) = x^2 - 1 \qquad x \in \mathbb{R}$$

Solve the equation,

$$ff(x) = 0$$

[4 marks]

Question 9

$$f(x) = x^2 + 8 \qquad x \in \mathbb{R}$$

$$g(x) = 2x - 5 \qquad x \in \mathbb{R}$$

Solve the equation

$$fg(x) = gf(x)$$

giving your solutions as exact surds.

[5 marks]

Question 10

Let two functions, *m* and *n*, be;

$$m(x) = 10x - 9, \qquad x \in \mathbb{R}, \qquad x \geqslant 0.9$$

$$n(x) = 100 - \sqrt{x} \qquad x \in \mathbb{R}, \qquad 0 \leqslant x \leqslant 9820.81$$

Determine each of the following, giving bracket free answers;

(i) $n(64)$

(ii) $m(3)$

(iii) $mn(9)$

(iv) $nm(9)$

(v) $mn(x)$

(vi) $nm(x)$

[8 marks]

(vii) Function *m* is only defined on domain $x \geqslant 0.9$

To see why, calculate $m(0)$ then $nm(0)$

[2 marks]

(viii) Having had to restrict the domain of *m* so that *nm* exists, the domain of *n* has to then also be restricted so that its output can be fed into *m*. Calculate, $n(9820.8)$ and explain why any input greater than the 9820.81 would cause a problem.

[2 marks]

Observation : *fg*, can be formed only if the range of *g* is a subset of the domain of *f*.

Question 11

Let m and n be the functions;

$$m(x) = 9x - 5 \qquad x \in \mathbb{R}$$

$$n(x) = \sqrt{x - 7} \qquad x \in \mathbb{R},\ x \geqslant 7,$$

Evaluate each of the following;

(i) $mn(8)$

(ii) $mn(56)$

(iii) $mn\left(z^2 + 7\right)$

(iv) $mn\left(4z^2 + 7\right)$

[8 marks]

6.4 Answers

6.4.1 Solutions to 6.2 Example

(i) 49 **(ii)** 79

[1, 1 marks]

(iii) $(x + 1)^2 - 2$ **(iv)** $((x - 2) + 1)^2$

[1, 1 marks]

(v)

$$\begin{aligned} (x + 1)^2 - 2 &= ((x - 2) + 1)^2 \\ x^2 + 2x + 1 - 2 &= (x - 1)^2 \\ x^2 + 2x - 1 &= x^2 - 2x + 1 \\ 4x &= 2 \\ x &= \frac{1}{2} \end{aligned}$$

[2 marks]

6.4.2 Solutions to 6.4 Exercise

Answer 1

$gf(x) = 15x + 6$

Answer 2

$fg(x) = 65 - 7x$

[2, 2 marks]

Answer 3

(i) $x^2 + 7x + 12$ **(ii)** $x \in \{-3, -4\}$

[2, 2 marks]

Answer 4

(i) 11 **(ii)** 17

[1, 1 marks]

(iii) 7 **(iv)** 50

[2, 2 marks]

(v) $4x^2 + 3$ **(vi)** $fg(x) \in \mathbb{R}, \ fg(x) \geqslant 3$

[3, 1 marks]

Answer 5

$\frac{2}{x}$

[2 marks]

Answer 6

(a) **(i)** 30 **(ii)** 42 **(iii)** -5

(iv) 108 **(v)** 2 **(vi)** -2

[3 marks]

(b) $x^2 - 3x + 2$ **(c)** $x \in \{1, 2\}$

[2, 2 marks]

Answer 7

(i) $fg(x) = \sqrt{x + 2} + 3$

[2 marks]

(ii) $x \in \mathbb{R}, x \geqslant -2$

[1 mark]

(iii) $fg(x) \in \mathbb{R},\ fg(x) \geqslant 3$

[1 mark]

Answer 8

$$\begin{aligned} ff(x) &= 0 \\ \left(x^2 - 1\right)^2 - 1 &= 0 \\ x^4 - 2x^2 &= 0 \\ x^2\left(x^2 - 2\right) &= 0 \\ x &\in \left\{0, \pm\sqrt{2}\right\} \end{aligned}$$

[4 marks]

Answer 9

$$\begin{aligned} fg(x) &= gf(x) \\ f(2x - 5) &= g\left(x^2 + 8\right) \\ (2x - 5)^2 + 8 &= 2\left(x^2 + 8\right) - 5 \\ 4x^2 - 20x + 25 + 8 &= 2x^2 + 16 - 5 \\ 2x^2 - 20x + 22 &= 0 \\ x^2 - 10x + 11 &= 0 \\ x &= \frac{10 \pm \sqrt{10^2 - 4 \times 1 \times 11}}{2} \\ &= \frac{10 \pm \sqrt{56}}{2} \\ &= 5 \pm \sqrt{14} \end{aligned}$$

[5 marks]

Answer 10

(i) 92 **(ii)** 21

[1, 1 marks]

(iii) 961 **(iv)** 91

[1, 1 marks]

(v) $mn(x) = 991 - 10\sqrt{x}$ **(vi)** $100 - \sqrt{10x - 9}$

[2, 2 marks]

(vii) $m(0) = -9$

$nm(0) = n(-9)$ but then are attempting to square root a negative number

[2 marks]

(viii) $n(9820.81) = 0.9$ and $m(0.9) = 0$

If $x > 9820.81$ then input into m is $\leqslant 0.9$

That is, outside domain of m

[2 marks]

Answer 11

(i) 4 **(ii)** 58

[2, 2 marks]

(iii)

$$\begin{aligned} mn\left(z^2 + 7\right) &= m\left(\sqrt{z^2 + 7 - 7}\right) \\ &= m(z) \\ &= 9z - 5 \end{aligned}$$

[2 marks]

(iv)

$$\begin{aligned} mn\left(4z^2 + 7\right) &= m\left(\sqrt{4z^2 + 7 - 7}\right) \\ &= m(2z) \\ &= 18z - 5 \end{aligned}$$

[2 marks]

7.1 Inverse Functions

Inverse functions have been met before at GCSE level (See Functions I).
Here is a question by way of recalling some of that previous knowledge;

(i) Write down the function described by the following diagram,

$$x \longrightarrow \boxed{\times 5} \longrightarrow \boxed{+3} \longrightarrow \boxed{\div 4} \longrightarrow f(x)$$

[2 marks]

(ii) Write down the inverse function $f^{-1}(x)$

[2 marks]

7.2 A More Advanced Example

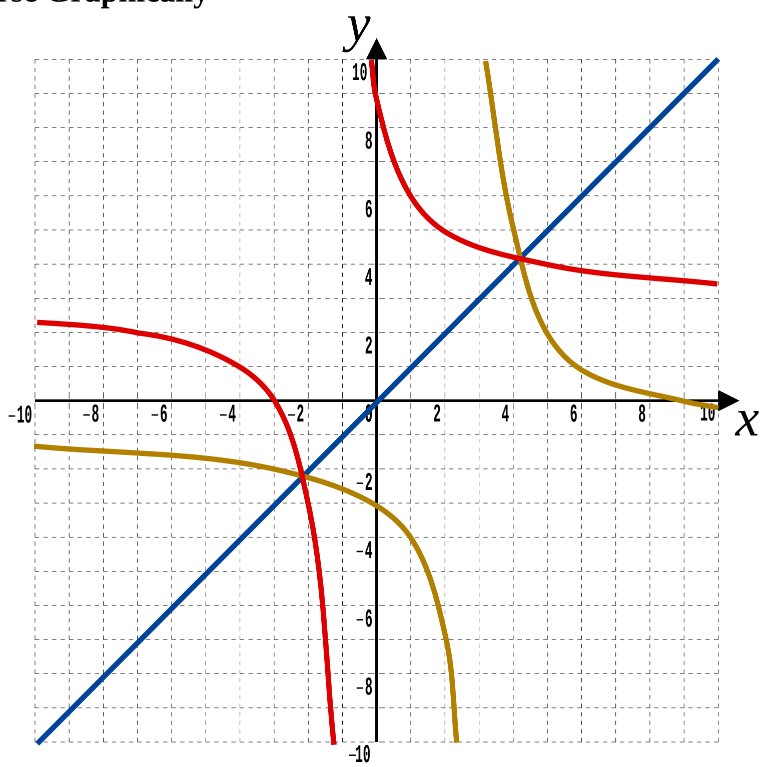
For the one-to-one function,

$$f(x) = \frac{6}{x+1} + 3$$

find $f^{-1}(x)$

[3 marks]

7.3 Inverse Graphically



Red: $f(x)$ Gold: $f^{-1}(x)$

(a) For the function, $f(x) = \dfrac{6}{x+1} + 3$

(i) Write down the domain:

[1 mark]

(ii) Write down the range:

[1 mark]

(b) For the inverse function,

$$f^{-1}(x) = \frac{6}{x-3} - 1$$

(i) Write down the domain:

[1 mark]

(ii) Write down the range:

[1 mark]

General Observations

- Geometrically, the inverse of a function is a reflection in the line $y = x$
- The domain of the function became the range of the inverse
- The range of the function became the domain of the inverse

Graphing the Function

The function, $f(x) = \dfrac{6}{x+1} + 3$ is better thought of as, $(y-3) = \dfrac{6}{(x+1)}$,

because the relationship to the graph of inverse proportion, $y = \dfrac{1}{x}$, is then obvious.

Focussing on the two asymptotes, one along the x-axis, the other along the y, it's then deduced that replacing x with $(x+1)$ will translate the y-axis asymptote to $x = -1$ and that replacing the y with $(y-3)$ will translate the x-axis asymptote to $y = 3$

Keeping in mind the fact that, for example,

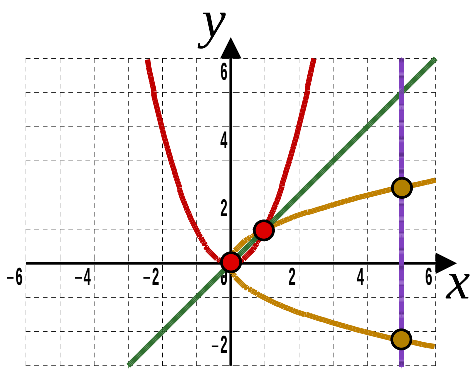
$y = \dfrac{1}{x}$, $\quad y = \dfrac{6}{x}$, $\quad y = \dfrac{12}{x}$, and indeed $y = \dfrac{k}{x}$ where k is a constant,

all have asymptotes along the x and y axis explains why the 6 has no effect on where the asymptotes of the transformed graph lie.

7.4 Existence of an Inverse

To have an inverse, a function must be one-to-one

The graph below shows (in red) the many-to-one function $f(x) = x^2$, $x \in \mathbb{R}$ and also (in gold) its reflection in the line $y = x$. However the reflection is not a function as a vertical line can be drawn that cuts it more than once.



If the inverse of a many-to-one function is sought, it must be broken up into pieces each of which is one-to-one and then, separately, the inverse of each piece found.

So, for example, to find the inverse of the many-to-one function $f(x) = x^2$ requires breaking it into two separate pieces,

$$g(x) = x^2, \qquad x \geqslant 0$$

$$h(x) = x^2 \qquad x < 0$$

and separately finding the inverse of each piece;

$$g^{-1}(x) = \sqrt{x} \qquad x \geqslant 0$$

$$h^{-1}(x) = -\sqrt{x} \quad x \geqslant 0$$

7.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available: 50

Question 1

Consider the following diagram,

$$x \rightarrow \boxed{+\ 2} \rightarrow \boxed{\div\ 3} \rightarrow \boxed{\times\ 8} \rightarrow f(x)$$

(a) Write down the function described by the diagram

[1 mark]

(b) Find the inverse function in two different ways,

(i) By looking at the diagram and working through it 'backwards'
(As in section 7.1)

[2 marks]

(ii) By algebraic manipulation.
(As in section 7.2)

[2 marks]

Question 2

Using a method of your choice, find the inverse of the function

$$g(x) = 8x + 3$$

[2 marks]

Question 3

Using a method of your choice, find the inverse of the function

$$h(x) = \frac{x}{5} - 4$$

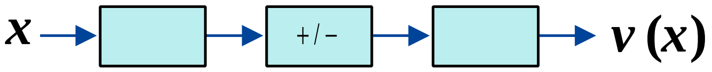
[2 marks]

Question 4

The function $v(x)$ is described by,

$$v(x) = 11 - 6x$$

A flow diagram for the function, shown below, includes the “change sign” instruction, which reverses the sign of whatever is fed into it.



(a) Complete the flow diagram by filling in the two empty boxes with appropriate instructions.

[2 marks]

(b) Find the inverse function in two different ways,

(i) By looking at the diagram and working through it 'backwards' (As in section 7.1)

[2 marks]

(ii) By algebraic manipulation (As in section 7.2)

[2 marks]

Question 5

Using a method of your choice, find the inverse of the function

$$u(x) = \frac{3x}{5} + 4$$

[3 marks]

Question 6

Consider the following diagram,

$$x \longrightarrow \boxed{\sqrt{\ }} \longrightarrow \boxed{\times 4} \longrightarrow \boxed{-\,7} \longrightarrow h\,(x)$$

(a) **(i)** Write down the function described by the diagram

[1 mark]

(ii) What is the domain of the function ?

[1 mark]

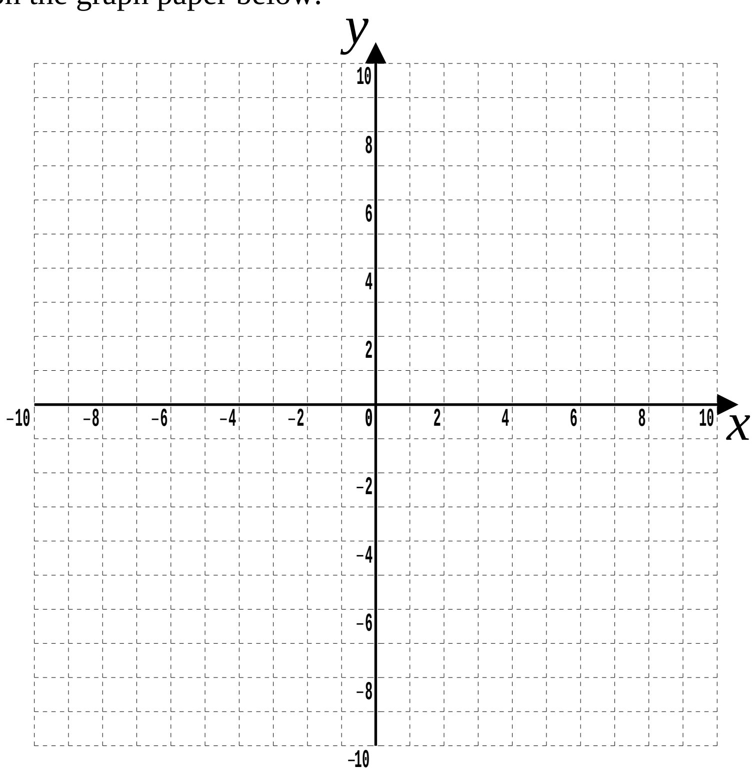
(b) Find the inverse function.

[3 marks]

(c) What is the domain of the inverse function ?

[1 mark]

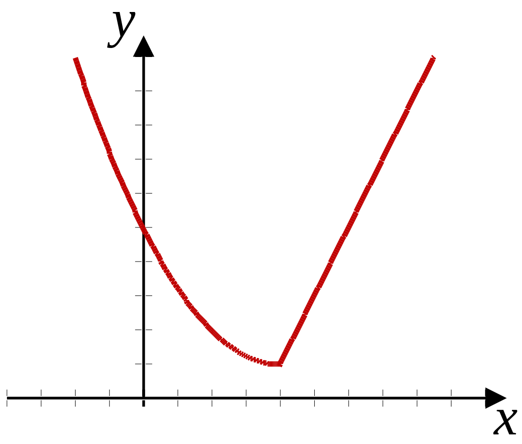
(d) Provide a fairly accurate plot of both the original function and the inverse on the graph paper below.



[4 marks]

Question 7

A-Level Examination Question from June 2019, Paper 2, Q6 (Edexcel)



The diagram shows a sketch of $y = g(x)$, where,

$$g(x) = \begin{cases} (x-2)^2 + 1 & x \leqslant 2 \\ 4x - 7 & x > 2 \end{cases}$$

(a) Find the value of $gg(0)$

[2 marks]

(b) Find all values of x for which $g(x) > 28$

[4 marks]

The function h is defined by, $h(x) = (x-2)^2 + 1, \quad x \leqslant 2$

(c) Explain why h has an inverse but g does not.

[1 mark]

(d) Solve the equation $h^{-1}(x) = -\dfrac{1}{2}$

[3 marks]

Question 8

A-Level Examination Question from June 2018, Paper C34, Q5 (Edexcel)

(i) The functions *f* and *g* are defined by

$$f \ : \ x \ \rightarrow \ e^{2x} \ - \ 5, \qquad x \ \in \ \mathbb{R}$$

$$g \ : \ x \ \rightarrow \ ln\,(\ 3x \ - \ 1 \ ), \qquad x \ \in \ \mathbb{R}, \ x \ > \ \frac{1}{3}$$

(a) Find f^{-1} and state its domain.

[3 marks]

(b) Find $fg\,(3)$, giving your answer in its simplest form.

[2 marks]

(ii) **(a)** Sketch the graph with equation $y \ = \ |\,4x \ - \ a\,|$ where *a* is a positive constant. State the coordinates of each point where the graph cuts or meets the coordinate axes.

[2 marks]

Given that, $|\,4x \ - \ a\,| \ = \ 9a$, where *a* is a positive constant,

(b) find the possible values of,

$$|x - 6a| + 3|x|$$

giving your answers, in terms of a, in their simplest form.

[5 marks]

7.6 Answers

7.6.1 Solution to 7.1 Example

(i) $f(x) = \dfrac{5x + 3}{4}$

[2 marks]

(ii) $f^{-1}(x) = \dfrac{4x - 3}{5}$

[2 marks]

7.6.2 Solution to 7.2 Example

$$f(x) = \frac{6}{x + 1} + 3$$

$$y = \frac{6}{x + 1} + 3$$

swap input with output

$$x = \frac{6}{y + 1} + 3$$

$$x - 3 = \frac{6}{y + 1}$$

$$\frac{y + 1}{6} = \frac{1}{x - 3}$$

$$y + 1 = \frac{6}{x - 3}$$

$$f^{-1}(x) = \frac{6}{x - 3} - 1 \qquad x \neq 3$$

[3 marks]

7.6.3 Solution to 7.3 Example

(i)

Domain: $x \in \mathbb{R}, \quad x \neq -1$

[1 mark]

Range: $f(x) \in \mathbb{R}, \quad f(x) \neq 3$

[1 mark]

(ii)

Domain: $x \in \mathbb{R}, \quad x \neq 3$

[1 mark]

Range: $f^{-1}(x) \in \mathbb{R}, \quad f^{-1}(x) \neq 3$

[1 mark]

7.6.4 Solutions to 7.5 Exercise

Question 1

(a)

$$f(x) = \frac{x + 2}{3} \times 8$$
$$= \frac{8x + 16}{3}$$

[1 mark]

(b) **(i)**

$$f^{-1}(x) = \frac{x}{8} \times 3 - 2$$
$$= \frac{3x}{8} - 2$$

[2 marks]

(ii)

$$f(x) = \frac{8x + 16}{3}$$
$$y = \frac{8x + 16}{3}$$

swap input with output

$$x = \frac{8y + 16}{3}$$
$$3x = 8y + 16$$
$$3x - 16 = 8y$$
$$y = \frac{3x - 16}{8}$$
$$f^{-1}(x) = \frac{3x}{8} - 2 \qquad \text{as before}$$

[2 marks]

Question 2

$$g^{-1}(x) = \frac{x - 3}{8}$$

[2 marks]

Question 3

$$h^{-1}(x) = 5x + 20$$

[2 marks]

Question 4

(a)

$$x \longrightarrow \boxed{\times\ 6} \longrightarrow \boxed{+/-} \longrightarrow \boxed{+\ 11} \longrightarrow v\,(x)$$

[2 marks]

(b) **(i)**

$$v^{-1}(x) = \frac{-\,(\,x - 11\,)}{6}$$

$$= \frac{11 - x}{6}$$

[2 marks]

(ii)

$$v\,(x) = 11 - 6x$$

$$y = 11 - 6x$$

swap input and output

$$x = 11 - 6y$$

$$6y = 11 - x$$

$$y = \frac{11 - x}{6}$$

$$f^{-1}(x) = \frac{11 - x}{6} \qquad \text{as before}$$

[2 marks]

Question 5

$$u^{-1}(x) = \frac{5\,(\,x - 4\,)}{3}$$

[3 marks]

Question 6

(a) **(i)** $h(x) = 4\sqrt{x} - 7$

[1 mark]

(ii) $x \in \mathbb{R}, \quad x \geqslant 0$

[1 mark]

(b)

$$h(x) = 4\sqrt{x} - 7$$

$$y = 4\sqrt{x} - 7$$

swap input and output

$$x = 4\sqrt{y} - 7$$

$$x + 7 = 4\sqrt{y}$$

$$\sqrt{y} = \frac{x+7}{4}$$

$$y = \left(\frac{x+7}{4}\right)^2$$

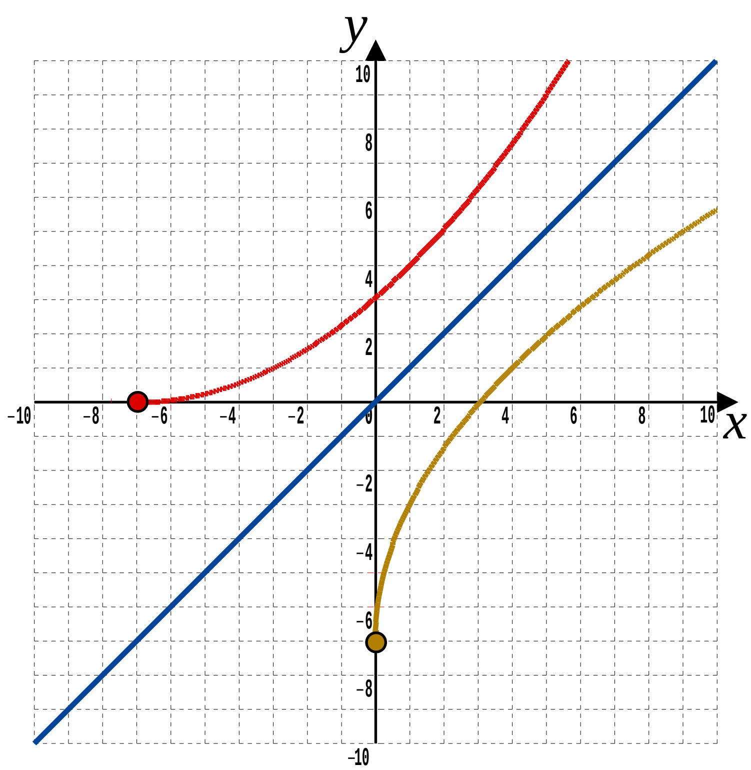
$$h^{-1}(x) = \left(\frac{x+7}{4}\right)^2$$

[3 marks]

(c) $x \in \mathbb{R}, \quad x > 0$

[1 mark]

(d)



[4 marks]

Answer 7

(a) $gg(0) = g(5) = 13$

[2 marks]

(b) For $x \leqslant 2$, the possible critical values are given by,

$$(x - 2)^2 + 1 = 28$$

$$x - 2 = \pm\sqrt{27}$$

$$x = 2 \pm 3\sqrt{3}$$

Solution for $x \leqslant 2$ is $x \leqslant 2 - 3\sqrt{3}$

For $x > 2$, the possible critical value are given by,

$$4x - 7 = 28$$

$$4x = 35$$

$$x = \frac{35}{4}$$

Solution for $x > 2$ is $x > \frac{35}{4}$

[4 marks]

(c) Only a function that is one-to-one can have an inverse.

$g(x)$ is a many-to-one function and so does not have an inverse.

$f(x)$ over its domain is a one-to-one function and so has an inverse.

[1 mark]

(d)

$$y = (x - 2)^2 + 1$$

$$(x - 2)^2 = y - 1$$

$$x = 2 \pm \sqrt{y - 1}$$

Only valid solution for $x \leqslant 2$ is $h^{-1}(x) = 2 - \sqrt{x - 1}$

$$\therefore 2 - \sqrt{x - 1} = -\frac{1}{2}$$

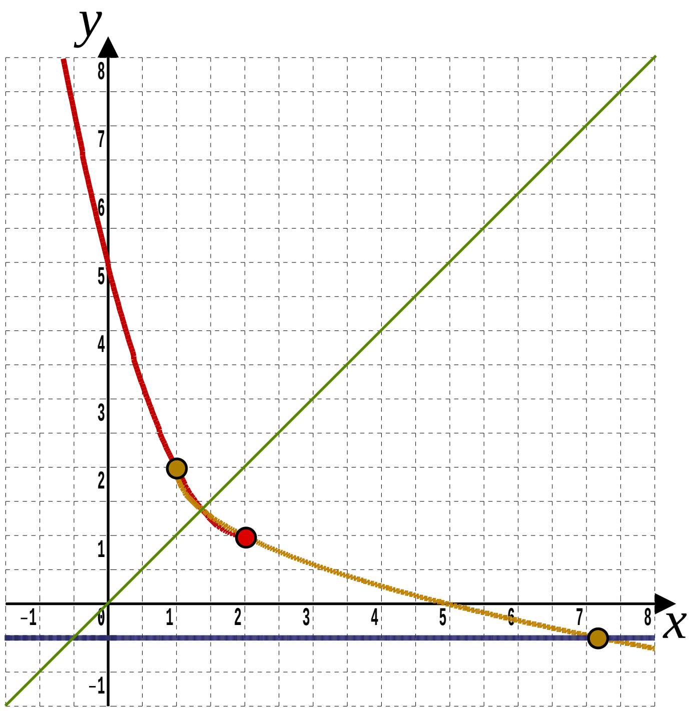
$$\sqrt{x - 1} = \frac{5}{2}$$

$$x - 1 = \frac{25}{4}$$

$$x = \frac{29}{4}$$

(See the graph on the following page)

[3 marks]



The red curve is the original function, the gold its inverse.
The inverse function has a value of – 0.5 (the blue line) when x = 7.25

Answer 8

(i) **(a)**

$$f(x) = e^{2x} - 5$$

$$y = e^{2x} - 5$$

$$y + 5 = e^{2x}$$

$$ln(y + 5) = 2x$$

$$x = \frac{1}{2} ln(y + 5)$$

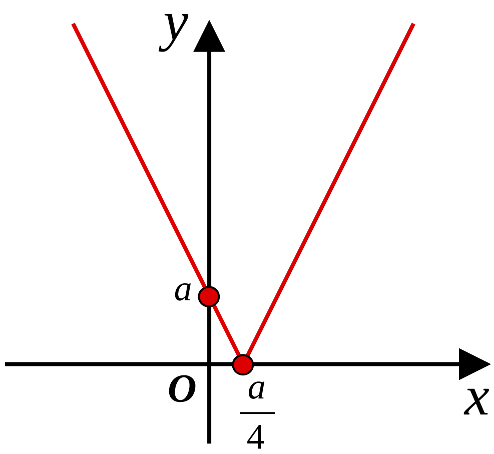
$$\therefore f^{-1}(x) = ln\left(\sqrt{x + 5}\right), \qquad x \in \mathbb{R},\ x > -5$$

[3 marks]

(b) $$fg(3) = f\left(ln\,8\right) = e^{2\,ln\,8} - 5 = e^{ln\,64} - 5 = 64 - 5 = 59$$

[2 marks]

(ii) **(a)**



[2 marks]

(b) There are two cases to consider,

$$4x - a = 9a \qquad\qquad -4x + a = 9a$$
$$4x = 10a \qquad\qquad -4x = 8a$$
$$x = \frac{5a}{2} \qquad\qquad x = -2a$$

For $x = \frac{5a}{2}$, $$|x - 6a| + 3|x| = \left|\frac{5a}{2} - 6a\right| + 3\left|\frac{5a}{2}\right| = \left|-\frac{7a}{2}\right| + \frac{15a}{2} = 11a$$

For $x = -2a$, $$|x - 6a| + 3|x| = |-2a - 6a| + 3|-2a| = 8a + 6a = 14a$$

[5 marks]

8.1 The Möbius Function

A Möbius function is of the form,

$$f(x) = \frac{ax + b}{cx + d} \qquad x \in \mathbb{R}, \ ad - bc \neq 0, \quad x \neq -\frac{d}{c}$$

where a, b, c and d are constants.

Finding the inverse of this one-to-one function requires a method for dealing with the fact that the the x appears in two different places, in both numerator and denominator.

Example

(i) Find the inverse of,

$$h(x) = \frac{5x + 2}{3x + 4} \qquad x \in \mathbb{R}, \quad x \neq -\frac{4}{3}$$

[4 marks]

(ii) State the domain of the inverse function.

[1 mark]

8.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available: 54

Question 1

(i) Find the inverse of, $m(x) = \frac{7x + 1}{2x + 3} \quad x \in \mathbb{R}, \quad x \neq -\frac{3}{2}$

[4 marks]

(ii) State the domain of the inverse function

[1 mark]

Question 2

Prove that the inverse of

$$f(x) = \frac{ax + b}{cx + d} \qquad x \in \mathbb{R}, \ ad - bc \neq 0, \quad x \neq -\frac{d}{c}$$

where a, b, c and d are constants, is

$$f^{-1}(x) = \frac{-dx + b}{cx - a} \qquad x \in \mathbb{R}, \quad x \neq \frac{a}{c}$$

[6 marks]

Question 3

(i) Find the inverse of, $s(x) = \dfrac{2x - 1}{x - 2} \quad x \neq 2$

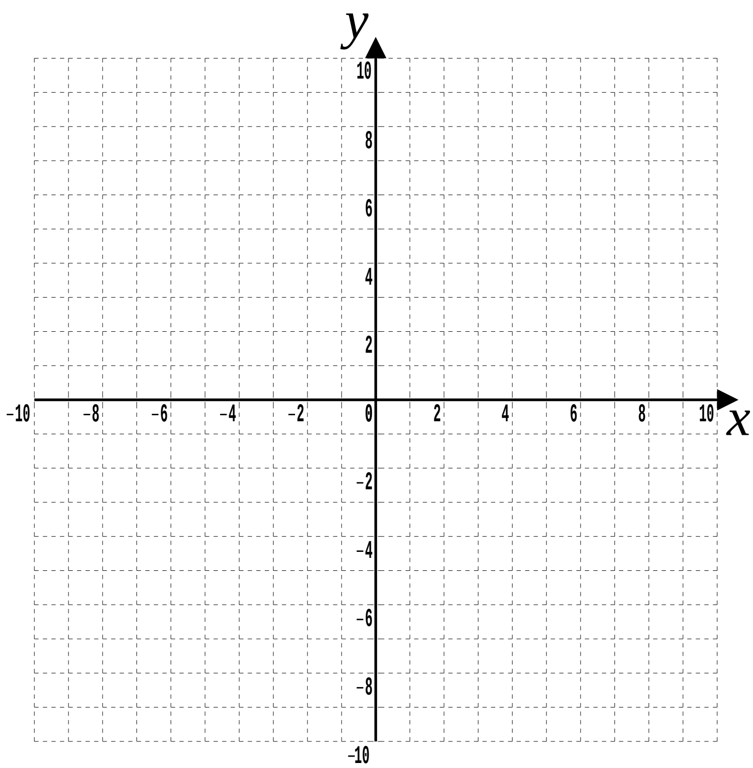
by using the result proven in question 2

[2 marks]

(ii) This is an example of a "self inverse" function.
Explain what this means.

[1 mark]

(iii) Plot an accurate graph of $s(x)$ and also the line $y = x$
Mark on the horizontal and the vertical asymptotes of $s(x)$



[3 marks]

(iv) What property do the graphs of all self-inverse functions have ?

[2 marks]

Question 4

(i) Write down the equations of the two asymptotes of the function,

$$f(x) = \frac{2}{x} \qquad x \in \mathbb{R}, \quad x \neq 0$$

[2 marks]

(ii) With your part (i) answer in mind, write down the equations of the two asymptotes of the graph of the function,

$$h(x) = \frac{2}{x+1} + 3$$

[2 marks]

(iii) What number must be excluded from the domain of $h(x)$?

[1 mark]

(iv) Sketch the graph of $h(x)$
(No need for an accurate graph, just the essential shape and the asymptotes)

[3 marks]

(v) Show that $h(x)$ is a Möbius transformation by writing it in the form,

$$f(x) = \frac{ax + b}{cx + d} \qquad x \in \mathbb{R}, \; ad - bc \neq 0, \quad x \neq -\frac{d}{c}$$

where a, b, c, and d are constants the values of which you have determined.

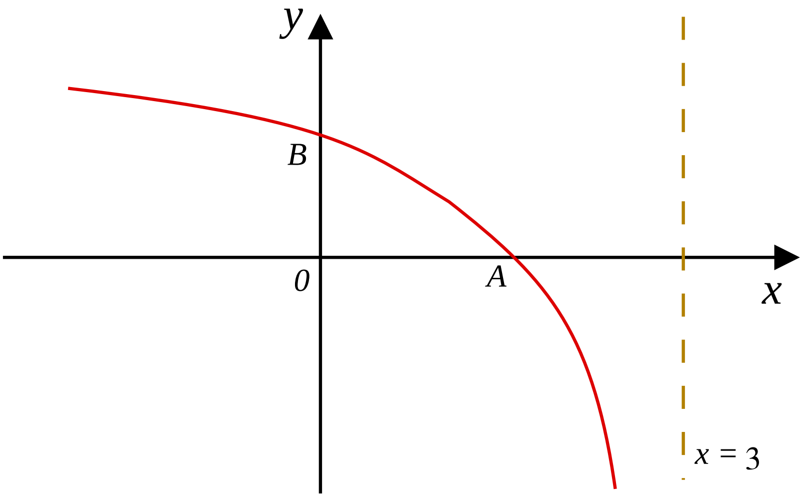
[3 marks]

(vi) Find $h^{-1}(x)$, stating its domain.

[4 marks]

Question 5

A-Level Examination Question from November 2018, Paper C34, Q10 (Edexcel)



The sketch is of the graph with equation $y = g(x)$, where,

$$g(x) = \frac{3x - 4}{x - 3}, \quad x \in \mathbb{R},\ x < 3$$

The graph cuts the x-axis at the point A and the y-axis at the point B, as shown.

(a) State the range of g

[1 mark]

(b) State the coordinates of,

(i) point A

[1 mark]

(ii) point B

[1 mark]

(c) Find $gg(x)$ in its simplest form.

[3 marks]

(d) Sketch the graph with equation $y = |g(x)|$

On your sketch, show the coordinates of each point at which the graph meets or cuts the axes and state the equation of each asymptote.

[3 marks]

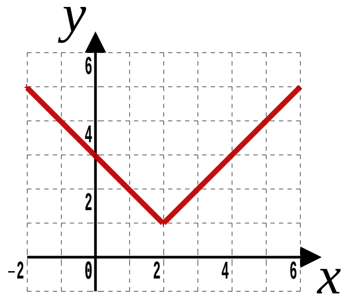
(e) Find the exact solution of the equation $|g(x)| = 8$

[3 marks]

Question 6

$$f(x) = |x - 2| + 1,\ x \in \mathbb{R}$$

The function $f(x)$ has a "critical value" when $|x - 2| = 0$ in the sense that there will be a "drastic change" in the smoothness of the graph at $x = 2$



This function can also be written in hybrid form as, $f(x) = \left\{ \begin{array}{ll} x - 1 & x \geqslant 2 \\ -x + 3 & x < 2 \end{array} \right\}$

Consider now the following function,

$$g(x) = |x + 1| + |2x + 1| - |x - 2|$$

Determine the critical values of $g(x)$, sketch $g(x)$ and express $g(x)$ in hybrid form.

[8 marks]

8.3 Answers

8.3.1 Solution to 8.1 Example

(i)

$$h(x) = \frac{5x + 2}{3x + 4}$$

$$y = \frac{5x + 2}{3x + 4}$$

swap the input and output

$$x = \frac{5y + 2}{3y + 4}$$

$$x(3y + 4) = 5y + 2$$

$$3xy + 4x = 5y + 2$$

$$3xy - 5y = -4x + 2$$

$$y(3x - 5) = -4x + 2$$

$$y = \frac{-4x + 2}{3x - 5}$$

$$h^{-1}(x) = \frac{-4x + 2}{3x - 5}$$

[4 marks]

(ii)

$h^{-1}(x)$ has domain $\quad x \in \mathbb{R}, \ x \neq \frac{5}{3}$

[1 mark]

8.3.2 Solutions to 8.2 Exercise

Answer 1

(i)

$$m^{-1}(x) = \frac{-3x + 1}{2x - 7}$$

[4 marks]

(ii)

$m^{-1}(x)$ has domain $\quad x \in \mathbb{R}, \ x \neq \frac{7}{2}$

[1 mark]

Answer 2

$$f(x) = \frac{ax + b}{cx + d}$$

$$y = \frac{ax + b}{cx + d}$$

swap input and output

$$x = \frac{ay + b}{cy + d}$$

$$x(cy + d) = ay + b$$

$$cxy + dx = ay + b$$

$$cxy - ay = -dx + b$$

$$y(cx - a) = -dx + b$$

$$y = \frac{-dx + b}{cx - a}$$

$$f^{-1}(x) = \frac{-dx + b}{cx - a} \qquad x \in \mathbb{R}, \quad x \neq \frac{a}{c}$$

[6 marks]

Answer 3

(i)

$$s(x) = \frac{2x - 1}{x - 2} \qquad x \neq 2$$

$s(x)$ is a Möbius function with,

$$a = 2, \qquad b = -1, \qquad c = 1, \qquad d = -2$$

The inverse is

$$s^{-1}(x) = \frac{-dx + b}{cx - a}$$

$$= \frac{-(-2)x + (-1)}{(1)x - (2)}$$

$$= \frac{2x - 1}{x - 2}$$

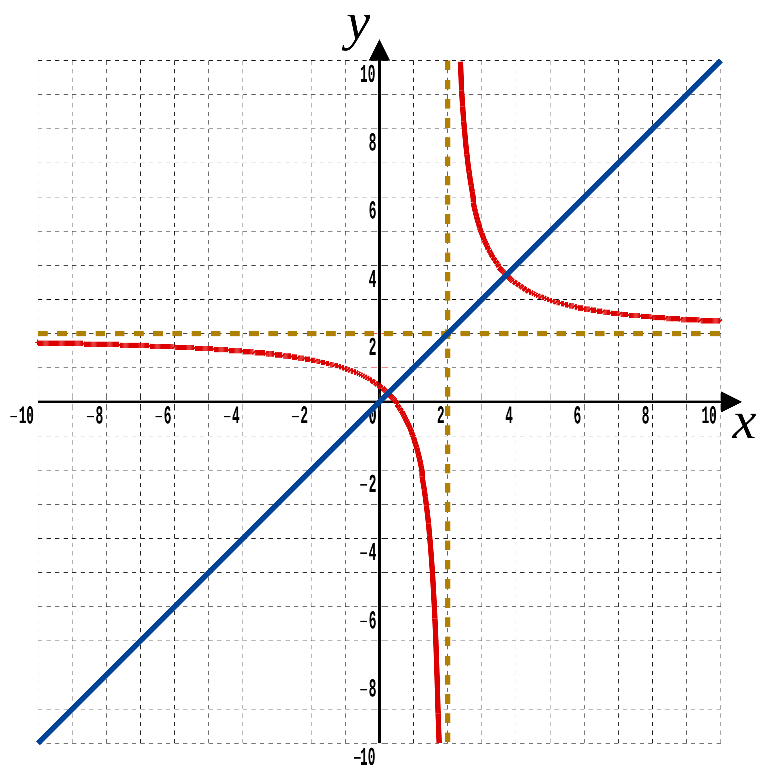
[2 marks]

(ii) The expression for the inverse is the same as the original function

$$s(x) = s^{-1}(x)$$

[1 mark]

(iii) Graph of inverse proportion with asymptotes at $x = 2$ and $y = 2$



[3 marks]

(iv) The line $y = x$ is a line of mirror symmetry

The graph of the function is symmetrical in $y = x$

[2 marks]

Answer 4

(i) There is one asymptote along the y-axis, which has equation $x = 0$

and another asymptote along the x-axis, which has equation $y = 0$

[2 marks]

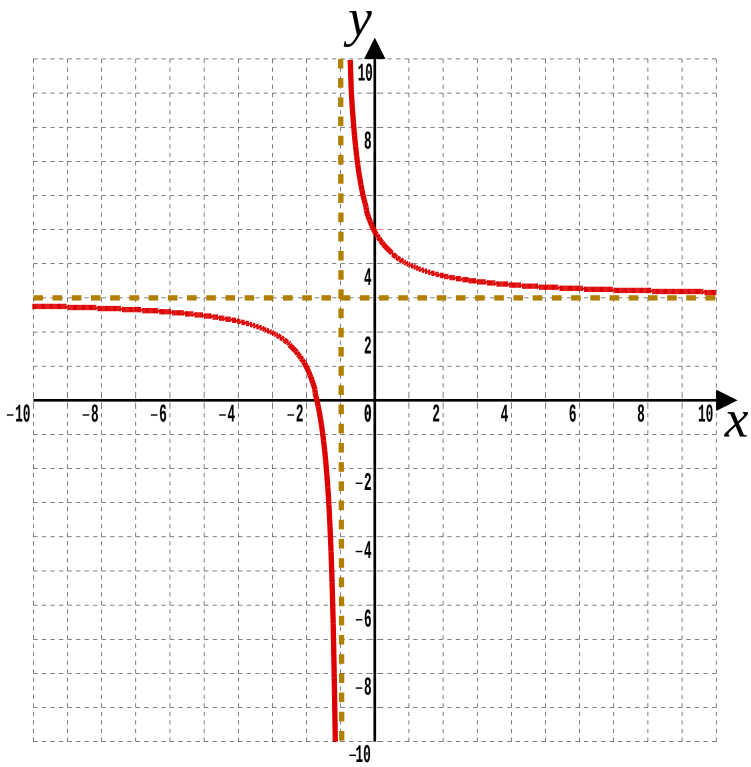
(ii) $h(x)$ is $f(x)$ translated $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ giving asymptotes $x = -1, y = 3$

[2 marks]

(iii) The domain of $h(x)$ must exclude $x = -1$ to avoid a division by zero.

[1 mark]

(iv)



[3 marks]

(v)

$$\begin{aligned} h(x) &= \frac{2}{x+1} + 3 \\ &= \frac{2}{x+1} + \frac{3(x+1)}{x+1} \\ &= \frac{3x+5}{x+1} \end{aligned}$$

which is in the form of a Möbius transformation

[3 marks]

(vi) $h^{-1}(x) = \dfrac{-x+5}{x-3} \qquad x \in \mathbb{R}, \quad x \neq 3$

[4 marks]

Answer 5

(a) Three methods (below) are given to determine that the given function has a horizontal asymptote with equation $y = 3$ from which it can be deduced that the range of $g(x)$ is $g(x) \in \mathbb{R},\ g(x) < 3$

Methods 1 and 2

$$\begin{aligned} y &= \frac{3x - 4}{x - 3} \\ &= \frac{3x - 9 + 9 - 4}{x - 3} \\ &= \frac{3x - 9}{x - 3} + \frac{5}{x - 3} \\ &= \frac{3(x - 3)}{x - 3} + \frac{5}{x - 3} \\ &= 3 + \frac{5}{x - 3} \end{aligned}$$

Method 1

$$y - 3 = \frac{5}{x - 3}$$

This is $y = \dfrac{5}{x}$ under translation $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and so the asymptote along the x-axis moves to $y = 3$

Method 2

As $x \to \pm\infty$, $\dfrac{5}{x - 3} \to 0$ and $y \to 3$

$\therefore\ y = 3$ is a horizontal asymptote

Method 3

Testing for horizontal asymptotes; (See Homework 3)

$P(x) = 3x - 4\ :\ deg(P(x)) = 1,\ p = 3$

$Q(x) = x - 3\ :\ deg(Q(x)) = 1,\ q = 1$

As $deg(P(x)) = deg(Q(x))$. $y = \dfrac{p}{q}$ is a horizontal asymptote.

That is, $y = 3$ is a horizontal asymptote.

[1 mark]

(b) **(i)** When $y = 0$, $\dfrac{3x - 4}{x - 3} = 0$

$$3x - 4 = 0$$

$$x = \frac{4}{3} \qquad \therefore\ A\left(\frac{4}{3},\ 0\right)$$

[1 mark]

(ii) When $x = 0$, $y = \dfrac{-4}{-3}$ $\qquad \therefore\ B\left(0,\ \dfrac{4}{3}\right)$

[1 mark]

(c)

$$gg(x) = g\left(\frac{3x - 4}{x - 3}\right)$$

$$= \frac{3\left(\frac{3x-4}{x-3}\right) - 4}{\left(\frac{3x-4}{x-3}\right) - 3} \times \frac{(x - 3)}{(x - 3)}$$

$$= \frac{3(3x - 4) - 4(x - 3)}{3x - 4 - 3(x - 3)}$$

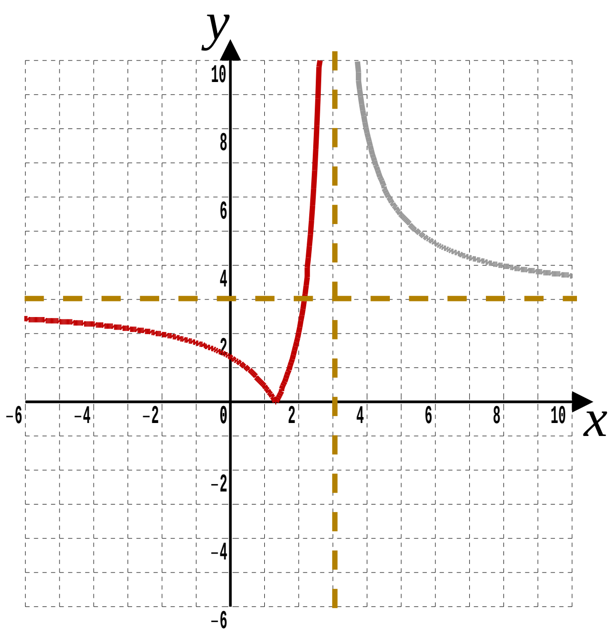
$$= \frac{9x - 12 - 4x + 12}{3x - 4 - 3x + 9}$$

$$= \frac{5x}{5}$$

$$= x$$

[3 marks]

(d)



Equations of asymptotes are $x = 3$, $y = 3$

Axes cutting or meeting points are $\left(\dfrac{4}{3},\ 0\right)$, $\left(0,\ \dfrac{4}{3}\right)$

[3 marks]

(e) $$\left| g(x) \right| = 8$$

$$\frac{3x - 4}{x - 3} = 8 \qquad\qquad -\frac{3x - 4}{x - 3} = 8$$

$$3x - 4 = 8x - 24 \qquad\qquad -3x + 4 = 8x - 24$$

$$5x = 20 \qquad\qquad 11x = 28$$

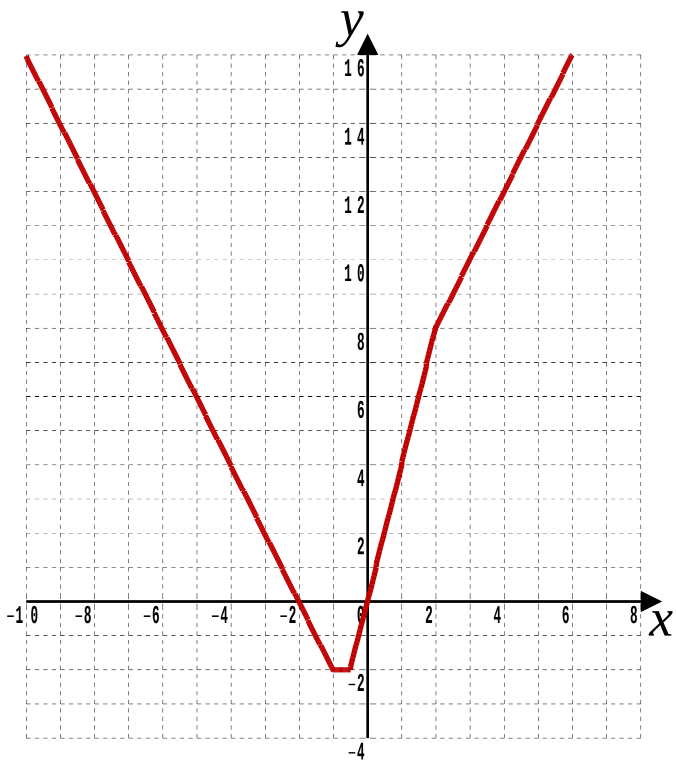
$$x = 4 \text{ (reject)} \qquad\qquad x = \frac{28}{11}$$

The only valid solution is $x = \frac{28}{11}$

[3 marks]

Answer 6

There will be critical values at $x = -1,\ x = -\frac{1}{2},\ x = 2$



$$f(x) = \begin{cases} -2x - 4 & x < -1 \\ -2 & -1 \leqslant x < -\frac{1}{2} \\ 4x & -\frac{1}{2} \leqslant x < 2 \\ 2x + 4 & x \geqslant 2 \end{cases}$$

[8 marks]

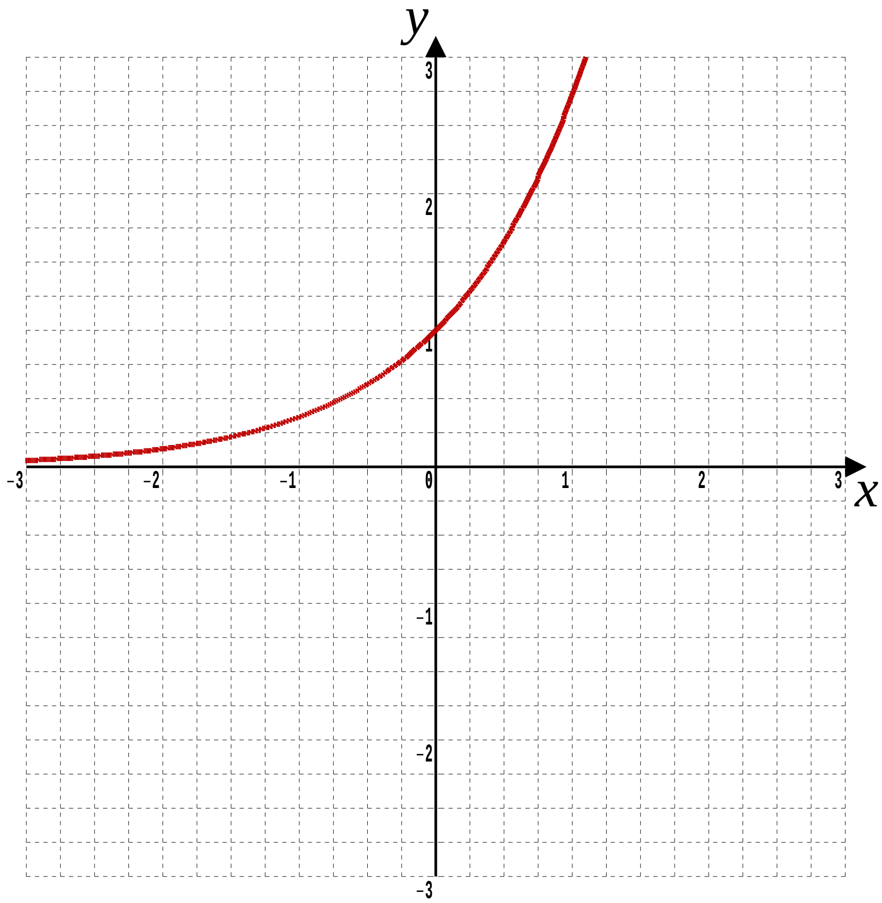
9.1 Revision

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 66

Question 1

This question is about the exponential function $f(x) = e^x,\ x \in \mathbb{R}$ graphed below.



(a) State the range of the function $f(x)$

[1 mark]

(b) To the graph above, add graphs of,

(i) $y = e^x + 1$ **(ii)** $y = x$ **(iii)** $y = \ln x$

[3 marks]

Question 2

Given that $f(x) = e^{3\ln x},\ \ x \in \mathbb{R},\ x > 0,$ solve the equation, $f(x) = 64$

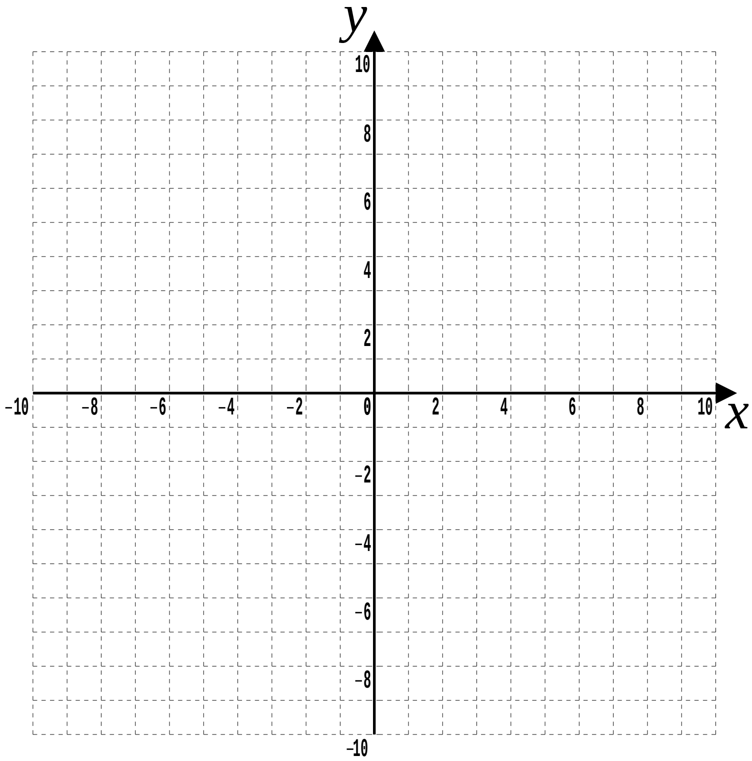
[2 marks]

Question 3

$$f(x) = |3x - 1| \qquad x \in \mathbb{R}$$

(i) Sketch the graph of $y = f(x)$ on the grid below, labelling its vertex and any points of intersection with the coordinate axes.

[3 marks]



$$g(x) = |3x - 1| - 6 \qquad x \in \mathbb{R}$$

(ii) Sketch the graph of $y = g(x)$ on the grid above, labelling its vertex and any points of intersection with the coordinate axes.

[4 marks]

(iii) Using algebra, find the coordinates of the points of intersection of

$$y = |3x - 1| - 6 \qquad \text{and} \qquad y = -\frac{1}{3}x + 3$$

[6 marks]

(iv) Add a line to the graph showing that your part (iii) answers are correct

[1 mark]

Question 4

$$f(x) = ln(x - 4), \quad x \in \mathbb{R},\ x > 4$$

$$g(x) = e^{3x} + 4, \quad x \in \mathbb{R}$$

(i) Find $fg(x)$, expressing the answer in simplified form, and state its range.

[3 marks]

(ii) Solve $fg(x) = 21$

[1 mark]

Question 5

$$p(x) = e^{2x} - 25, \quad x \in \mathbb{R}$$

$$q(x) = ln(x - 3), \quad x \in \mathbb{R},\ x > 3$$

(i) Find $pq(x)$, expressing the answer in simplified form, and state its range.

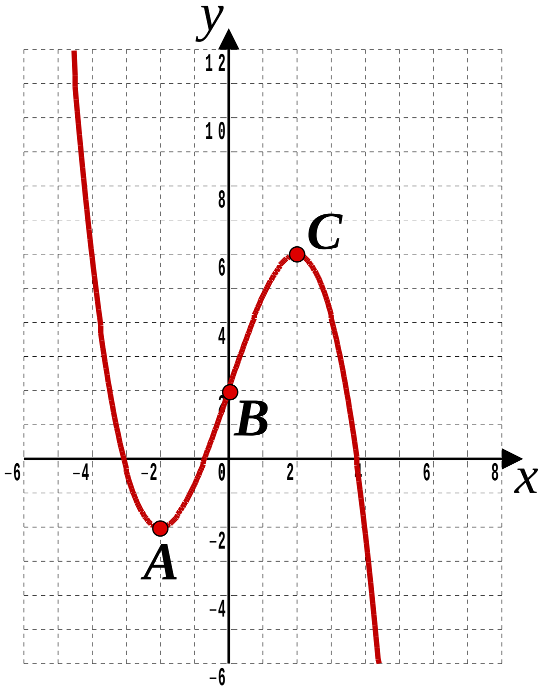
[5 marks]

(ii) Solve $pq(x) = 0$

[2 marks]

Question 6

The graph is of a mystery function $m(x)$



The points $A(-2, -2)$ and $C(2, 6)$ are turning points on the graph which

also passes through the y-axis at $B(0, 2)$

Sketch on separate diagrams the graphs of,

(i) $y = |m(x)|$ **(ii)** $y = m(|x|)$ **(iii)** $y = 2m(x - 2)$

Where possible, label clearly the transformations of the points A, B and C on your diagrams and give their coordinates.

[3, 3, 4 marks]

Question 7

A-Level Examination Question from June 2018, Paper 2, Q1 (Edexcel)

$$g(x) = \frac{2x+5}{x-3} \qquad x \geqslant 5$$

(a) Find $gg(5)$

[2 marks]

(b) State the range of g

[1 mark]

(c) Find $g^{-1}(x)$, stating its domain.

[3 marks]

Question 8

A-Level Practice Paper from 2018, Set 2, Paper 1, Q8 (CGP)

Given that, $f^{-1}(x) = \dfrac{2x - 5}{x}, \quad x \neq 0$

$g(x) = \sqrt{2x - k}, \quad x \geqslant \dfrac{k}{2}$, where k is a positive constant

(a) find $fg(x)$, giving your answer in terms of x, and state its domain.

[3 marks]

(b) If $gg(10) = 2$, find the value of k

[3 marks]

Question 9

A-Level Examination Question from November 2017, Paper C34, Q9 (Edexcel)

$$f(x) = 2\ln(x) - 4, \quad x > 0, \quad x \in \mathbb{R}$$

(a) Sketch, on separate diagrams, the curve with equation,

(i) $y = f(x)$

(ii) $y = |f(x)|$

On each diagram, show the coordinates of each point at which the curve meets or cuts the axes.

On each diagram state the equation of the asymptote.

[5 marks]

(b) Find the exact solutions of the equation $|f(x)| = 4$

[4 marks]

$$g(x) = e^{x+5} - 2, \quad x \in \mathbb{R}$$

(c) Find $gf(x)$, giving your answer in its simplest form.

[3 marks]

(d) Hence, or otherwise, state the range of gf

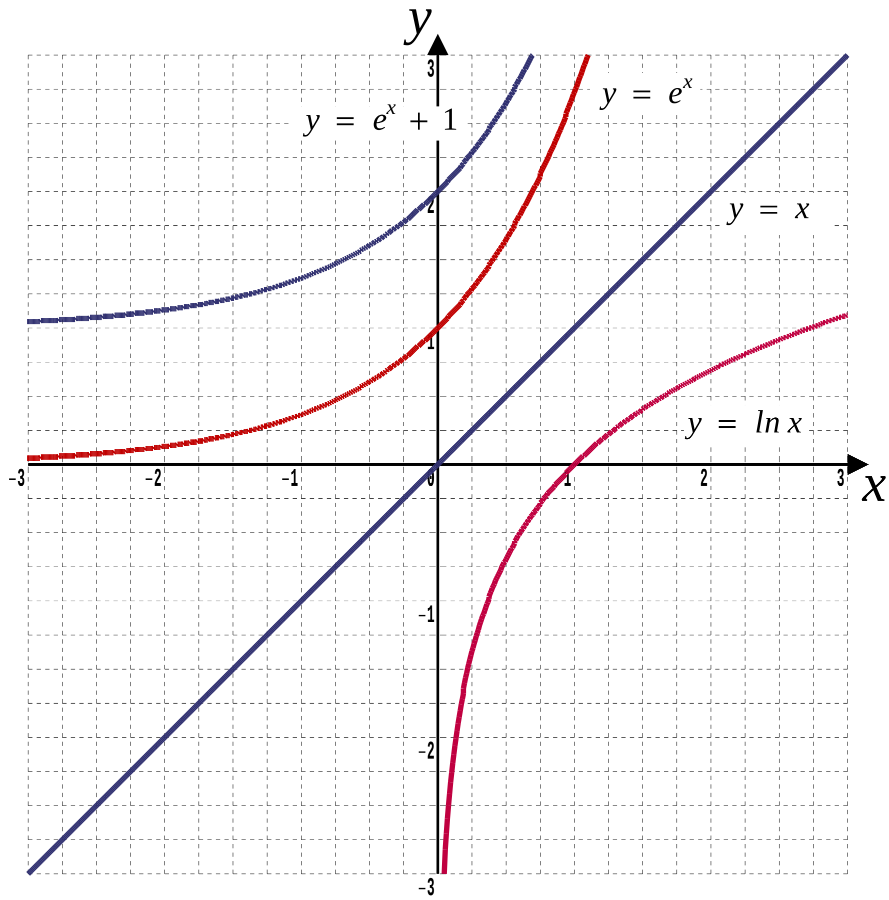
[1 mark]

Answer 1

(a) $f(x) \in \mathbb{R},\ f(x) > 0$

[1 mark]

(b)



[3 marks]

Answer 2

$$f(x) = 64$$
$$e^{3\ln x} = 64$$
$$e^{\ln x^3} = 64$$
$$x^3 = 64$$
$$x = 4$$

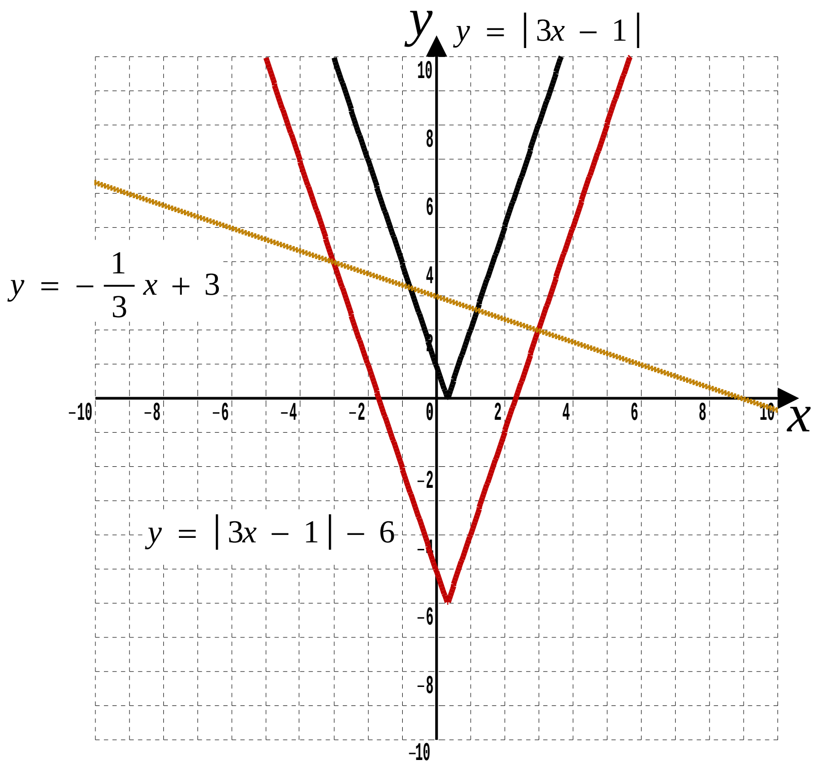
[2 marks]

Answer 3

(i) Black line on graph : $y = |3x - 1|$

It touches the x-axis at $\left(\frac{1}{3},\ 0\right)$ and crosses the y-axis at $(0,\ 1)$

[3 marks]



(ii) The graph of $g(x)$ is $f(x)$ translated $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$ to give the red line on the graph

It crosses the y-axis at $(0, -5)$ and the x-axis when $|3x - 1| - 6 = 0$

That's when $+(3x - 1) = 6 \Rightarrow x = \frac{7}{3} \Rightarrow \left(\frac{7}{3},\ 0\right)$

or when $-(3x - 1) = 6 \Rightarrow x = -\frac{5}{3} \Rightarrow \left(-\frac{5}{3},\ 0\right)$

[4 marks]

(iii)

$$+(3x - 1) - 6 = -\frac{1}{3}x + 3 \qquad -(3x - 1) - 6 = -\frac{1}{3}x + 3$$

$$9x - 21 = -x + 9 \qquad -9x - 15 = -x + 9$$

$$10x = 30 \qquad -8x = 24$$

$$x = 3 \Rightarrow (3,\ 2) \qquad x = -3 \Rightarrow (-3,\ 4)$$

[6 marks]

(iv) Gold line of $y = -\frac{1}{3}x + 3$ added to graph

[1 mark]

Answer 4

(i)

$$fg(x) = f\left(e^{3x} + 4\right)$$
$$= ln\left(e^{3x} + 4 - 4\right)$$
$$= ln\left(e^{3x}\right)$$
$$= 3x$$

Range $fg(x) \in \mathbb{R}$

[3 marks]

(ii)

$$fg(x) = 21$$
$$3x = 21$$
$$x = 7$$

[1 mark]

Answer 5

(i)

$$pq(x) = p\left(ln(x - 3)\right)$$
$$= e^{2\,ln(x-3)} - 25$$
$$= e^{ln(x-3)^2} - 25$$
$$= (x - 3)^2 - 25 \quad * * *$$
$$= x^2 - 6x + 9 - 25$$
$$= x^2 - 6x - 16 \leftarrow \text{Full Marks}$$

Completing the square to get the range

$$= \left[x^2 - 6x\right] - 16$$
$$= \left[(x - 3)^2 - 9\right] - 16$$
$$= (x - 3)^2 - 25 \quad * * *$$

$\therefore$ The range is $pq(x) \in \mathbb{R}, \;\; pq(x) \geqslant -25$

[5 marks]

(ii)

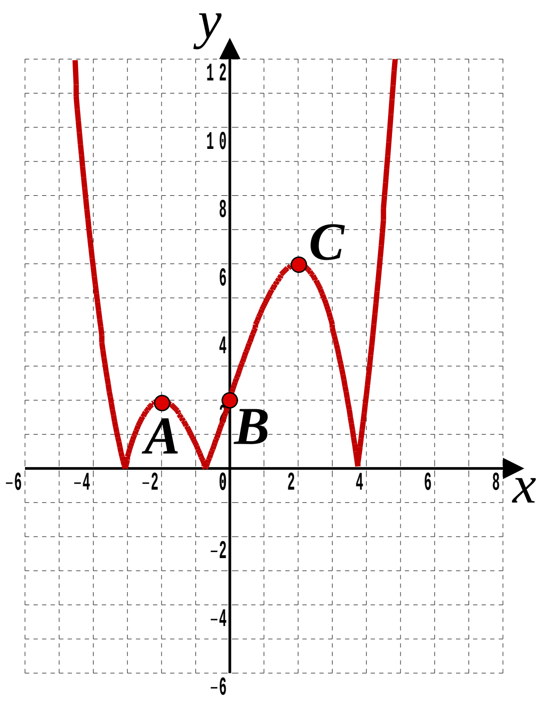
$$pq(x) = 0$$
$$x^2 - 6x - 16 = 0$$
$$(x - 8)(x + 2) = 0$$

Either $x = 8$ or $x = -2$

[2 marks]

Answer 6

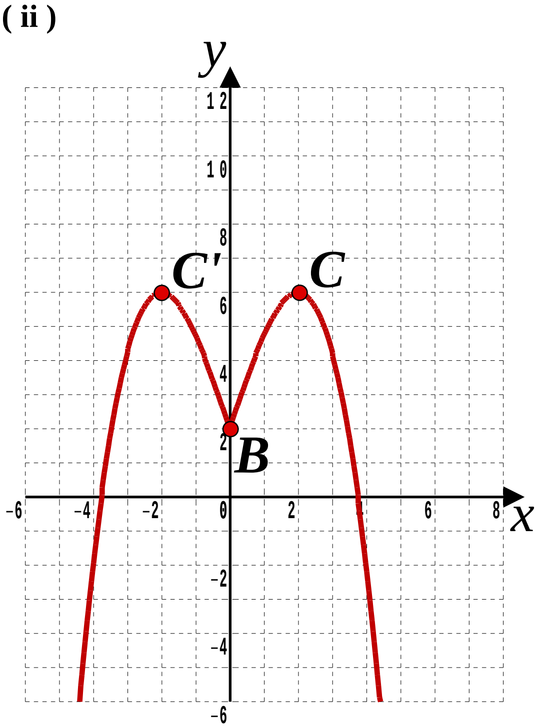
(i)



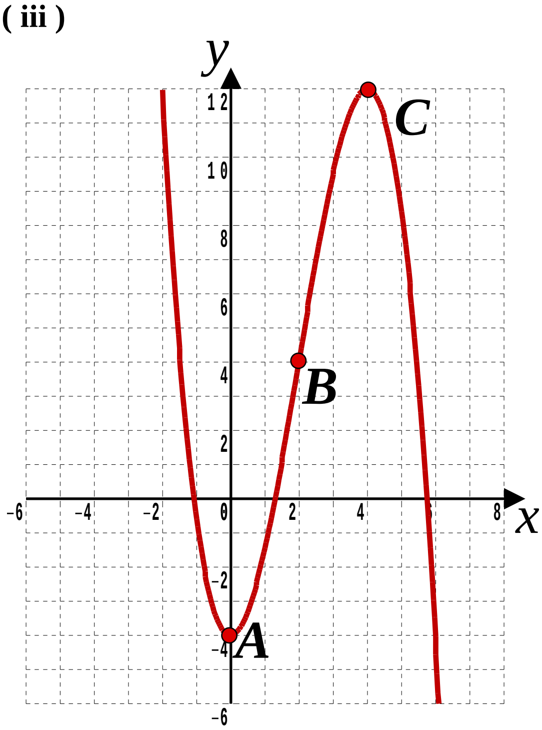
$A(-2,\ 2),\ B(0,\ 2),\ C(2,\ 6)$

[3 marks]

(ii)



(iii)



A is lost, $B(0,\ 2),\ C(2,\ 6),\ C'(-2,\ 6)$

$A(0,\ -4),\ B(2,\ 4),\ C(4,\ 12)$

[3, 4 marks]

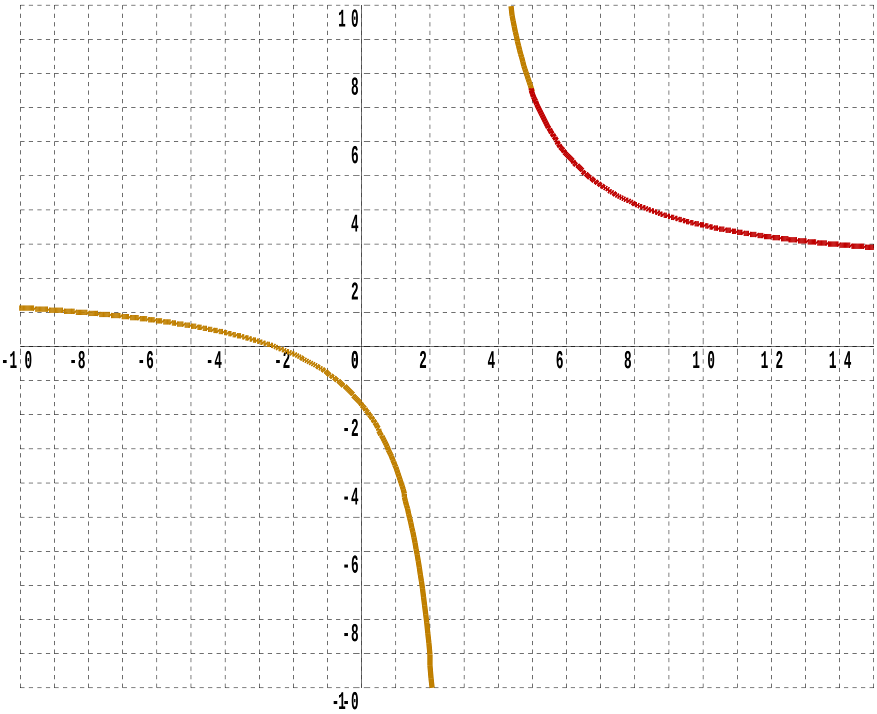
Answer 7

(a)

$$gg(5) = g\left(\frac{2 \times 5 + 5}{5 - 3}\right)$$
$$= g\left(\frac{15}{2}\right)$$
$$= g(7.5)$$
$$= \frac{2 \times 7.5 + 5}{7.5 - 3}$$
$$= \frac{20}{4.5}$$
$$= \frac{40}{9}$$

[2 marks]

(b)



The red line is the part of the function for which $x \geqslant 5$

There is a horizontal asymptote of $y = 2$

So the range is $2 < g(x) \leqslant 7.5$

[1 mark]

(c)

$$g(x) = \frac{2x + 5}{x - 3} \qquad x \geqslant 5$$

$$y = \frac{2x + 5}{x - 3}$$

Swapping $x \Leftrightarrow y$ because input $\Leftrightarrow$ output

$$x = \frac{2y + 5}{y - 3}$$

$$x(y - 3) = 2y + 5$$

$$xy - 3x = 2y + 5$$

$$xy - 2y = 3x + 5$$

$$y(x - 2) = 3x + 5$$

$$y = \frac{3x + 5}{x - 2}$$

$$g^{-1}(x) = \frac{3x + 5}{x - 2}$$

The domain of the inverse function is the range of the original function

So the domain is $2 < x \leqslant 7.5$

[3 marks]

Answer 8

$$f^{-1}(x) = \frac{2x - 5}{x}$$

$$y = \frac{2x - 5}{x}$$

$$x \Leftrightarrow y$$

$$x = \frac{2y - 5}{y}$$

$$xy = 2y - 5$$

$$2y - xy = 5$$

$$y(2 - x) = 5$$

$$y = \frac{5}{2 - x}$$

$$f(x) = \frac{5}{2 - x} \qquad x \neq 2$$

(a)

$$fg(x) = f\left(\sqrt{2x - k}\right)$$

$$= \frac{5}{2 - \sqrt{2x - k}} \qquad x \geqslant \frac{k}{2},\ x \neq 2 + \frac{k}{2}$$

[3 marks]

(b)

$$gg(10) = 2$$

$$g\left(\sqrt{20 - k}\right) = 2$$

$$\sqrt{2\sqrt{20 - k} - k} = 2$$

$$2\sqrt{20 - k} - k = 4$$

$$2\sqrt{20 - k} = 4 + k$$

$$4(20 - k) = 16 + 8k + k^2$$

$$k^2 + 12k - 64 = 0$$

$$(k + 16)(k - 4) = 0$$

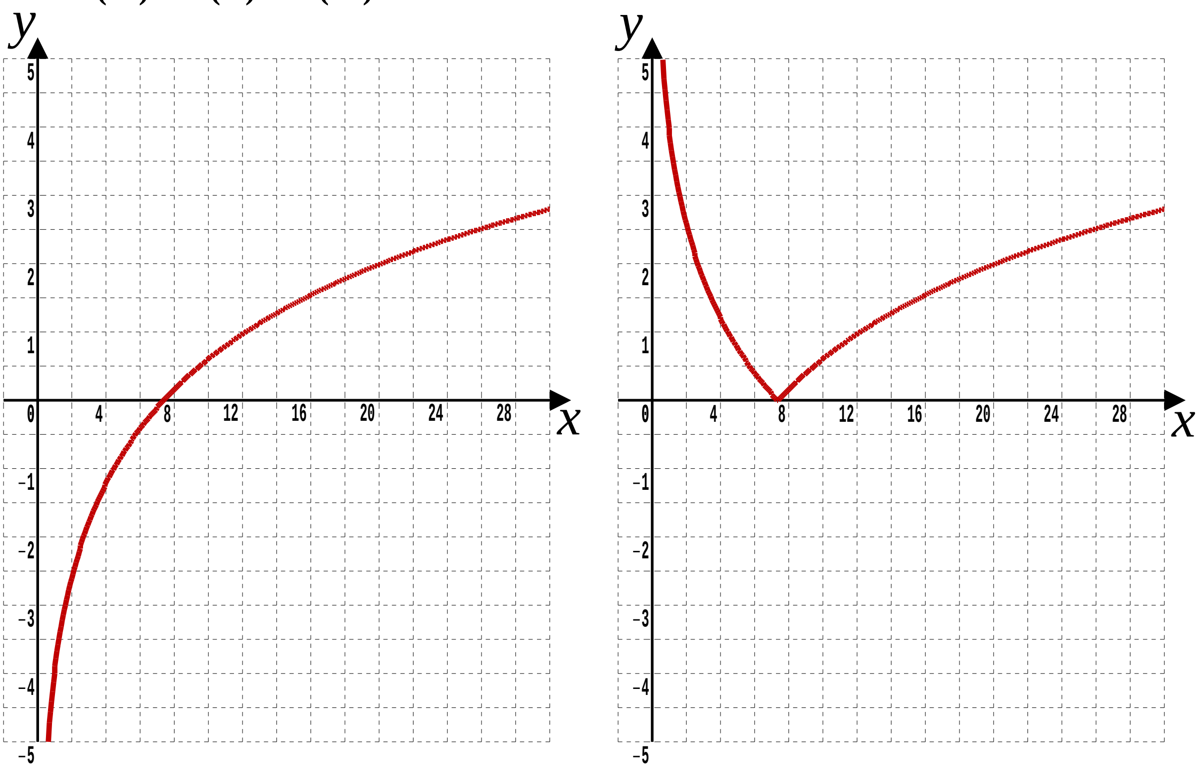
Either $k = -16$ (Reject as told k is positive)

Or $k = 4$

[3 marks]

Answer 9

(a) **(i)** and **(ii)**



x-axis contact, $2\,ln(x) - 4 = 0 \Rightarrow ln(x) = 2 \Rightarrow x = e^2$ (About 7.39)

Asymptote on both graphs is the y-axis which has equation $x = 0$

[5 marks]

(b) $|\,f(x)\,| = 4$

$2\,ln(x) - 4 = 4$ $\qquad$ $-\,2\,ln(x) + 4 = 4$

$ln(x) = 4$ $\qquad$ $ln(x) = 0$

$x = e^4$ (About 54.6) $\qquad$ $x = 1$

[4 marks]

(c)

$$gf(x) = g\big(2\,ln(x) - 4\big)$$
$$= e^{2\,ln\,x - 4 + 5} - 2$$
$$= e^{ln\,x^2 + 1} - 2$$
$$= e\,x^2 - 2$$

[3 marks]

(d) $gf(x) > -2$

[1 mark]

10.1 Test

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

$$f(x) = e^{2x} + 4, \qquad x \in \mathbb{R}$$

$$g(x) = \ln(x + 1), \qquad x \in \mathbb{R},\ x > -1$$

(i) Find $fg(x)$

Present your answer in simplified form and state its range.

[4 marks]

(ii) Solve $fg(x) = 85$

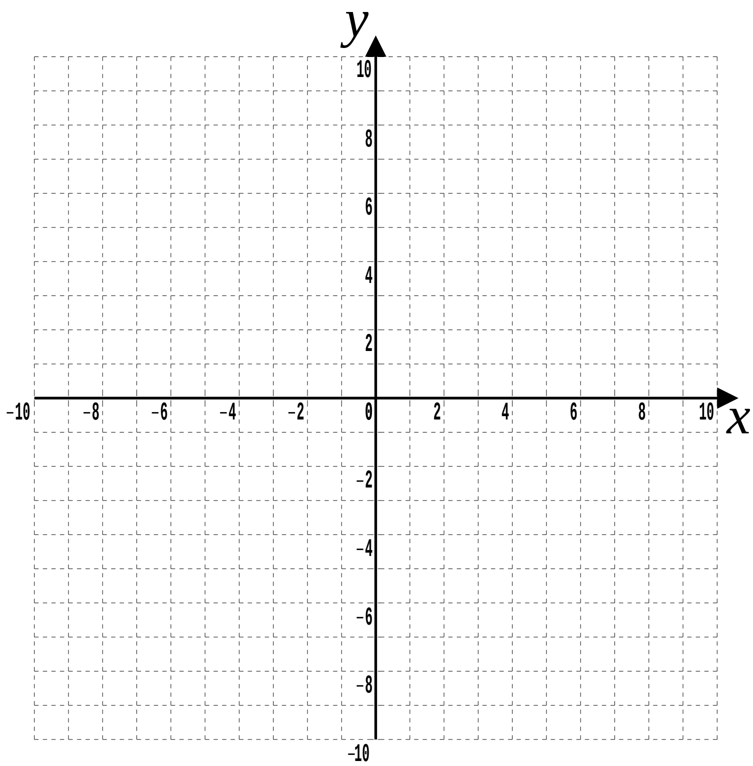
[3 marks]

Question 2

$$f(x) = |2x + 3| \qquad x \in \mathbb{R}$$

(i) Sketch the graph of $y = f(x)$ on the grid below, labelling its vertex and any points of intersection with the coordinate axes.

[3 marks]



$$g(x) = |2x + 3| - 7 \qquad x \in \mathbb{R}$$

(ii) Sketch the graph of $y = g(x)$ on the grid above, labelling its vertex and any points of intersection with the coordinate axes.

[3 marks]

(iii) Using algebra, find the coordinates of the points of intersection of

$$y = |2x + 3| - 7 \qquad \text{and} \qquad y = -\frac{1}{2}x + \frac{7}{2}$$

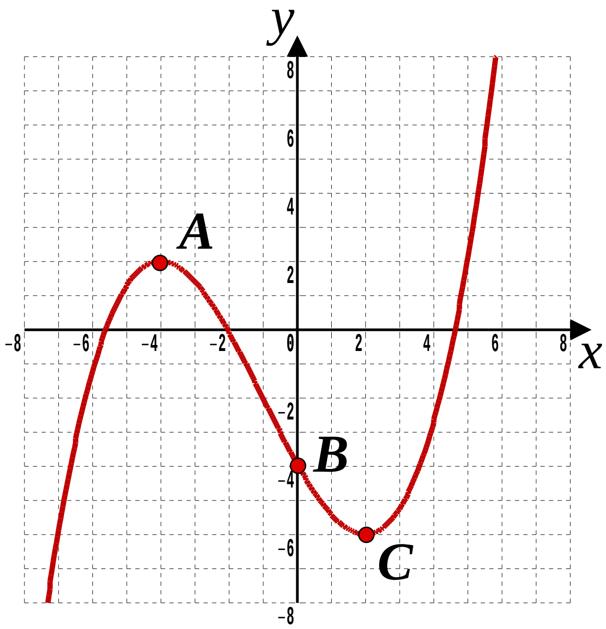
[4 marks]

(iv) Add a line to the graph showing that your part (iii) answers are correct

[1 mark]

Question 3

The graph is of a mystery function $m(x)$



The points $A(-4,\ 2)$ and $C(2, -6)$ are turning points on the graph which also passes through the y-axis at $B(0, -4)$

Sketch on separate diagrams the graphs of,

(i) $y = |m(x)|$ **(ii)** $y = m(|x|)$ **(iii)** $y = \frac{1}{2} m(x-2)$

Where possible, label clearly the transformations of the points *A*, *B* and *C* on your diagrams and give their coordinates.

[3, 3, 4 marks]

Question 4

A-Level Examination Question from January 2018, IAL, Paper C34, Q10 (Edexcel)

It is given that,

$$f(x) = e^{-2x} \qquad x \in \mathbb{R}$$

$$g(x) = \frac{x}{x-3} \qquad x > 3$$

(a) Sketch the graph of $y = f(x)$, showing the coordinates of any points where the graph crosses the axes.

[2 marks]

(b) Find the range of g

[2 marks]

(c) Find $g^{-1}(x)$ stating the domain of g^{-1}

[4 marks]

(d) Using algebra, find the exact value of x for which $fg(x) = 3$

[4 marks]

10.2 Answers

Answer 1

(i)

$$\begin{aligned} fg(x) &= f\left(ln(x+1)\right) \\ &= e^{2\,ln(x+1)} + 4 \\ &= e^{ln(x+1)^2} + 4 \\ &= (x+1)^2 + 4 \quad * * * \\ &= x^2 + 2x + 1 + 4 \\ &= x^2 + 2x + 5 \leftarrow \text{Full Marks} \end{aligned}$$

Completing the square to get the range

$$\begin{aligned} &= \left[x^2 + 2x\right] + 5 \\ &= \left[(x+1)^2 - 1\right] + 5 \\ &= (x+1)^2 + 4 \quad * * * \end{aligned}$$

$\therefore$ The range is $fg(x) \in \mathbb{R},\ fg(x) \geqslant 4$

[4 marks]

(ii)

$$\begin{aligned} fg(x) &= 85 \\ x^2 + 2x + 5 &= 85 \\ x^2 + 2x - 80 &= 0 \\ (x+10)(x-8) &= 0 \end{aligned}$$

Either $x = -10$ or $x = 8$

But $x = -10$ cannot be put into $g(x)$ and so is not a valid solution.

$\therefore\ x = 8$ is the only solution

[3 marks]

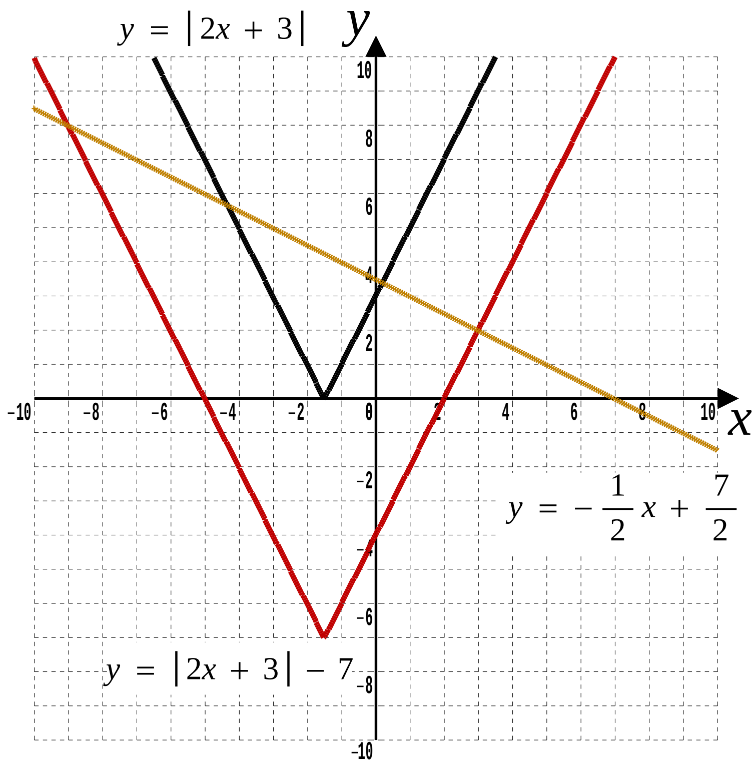
Answer 2

(i) Black line on the graph : $y = |2x + 3|$

It touches the x-axis when $y = 0 \Rightarrow \left(-\frac{3}{2},\ 0\right)$

and crosses the y-axis at $(0,\ 3)$

[3 marks]



(ii) The graph of $g(x)$ is $f(x)$ translated $\begin{pmatrix} 0 \\ -7 \end{pmatrix}$ to give the red line on the graph.

It crosses the y-axis at $(0, -4)$ and the x-axis when $|2x + 3| - 7 = 0$

That's when $+(2x + 3) = 7 \Rightarrow x = 2 \Rightarrow (2,\ 0)$

or when $-(2x + 3) = 7 \Rightarrow x = -5 \Rightarrow (-5,\ 0)$

[3 marks]

(iii)

$$+(2x + 3) - 7 = -\frac{1}{2}x + \frac{7}{2} \qquad -(2x + 3) - 7 = -\frac{1}{2}x + \frac{7}{2}$$

$$4x - 8 = -x + 7 \qquad -4x - 20 = -x + 7$$

$$5x = 15 \qquad -3x = 27$$

$$x = 3 \Rightarrow (3,\ 2) \qquad x = -9 \Rightarrow (-9,\ 8)$$

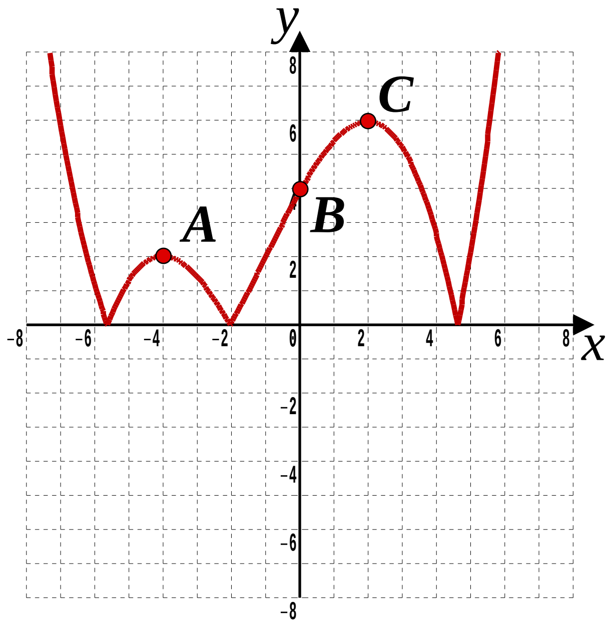
[4 marks]

(iv) Gold line of $y = -\frac{1}{2}x + \frac{7}{2}$ added to graph.

[1 mark]

Answer 3

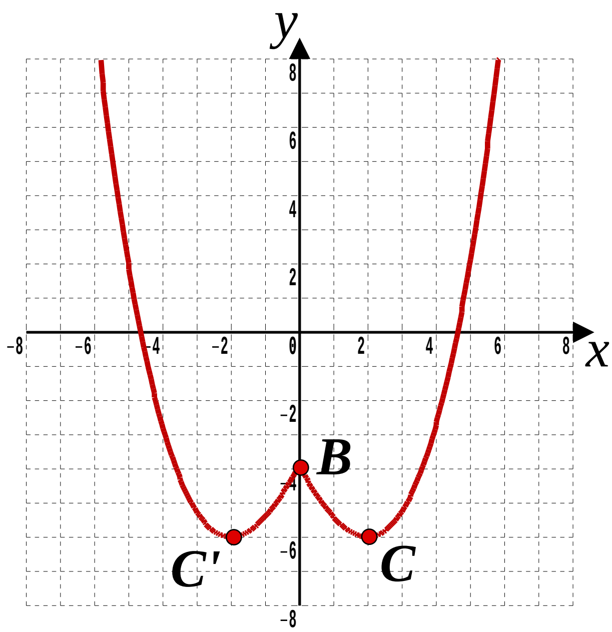
(i)



$A(-4,\ 2)$, $B(0,\ 4)$, $C(2,\ 6)$

[3 marks]

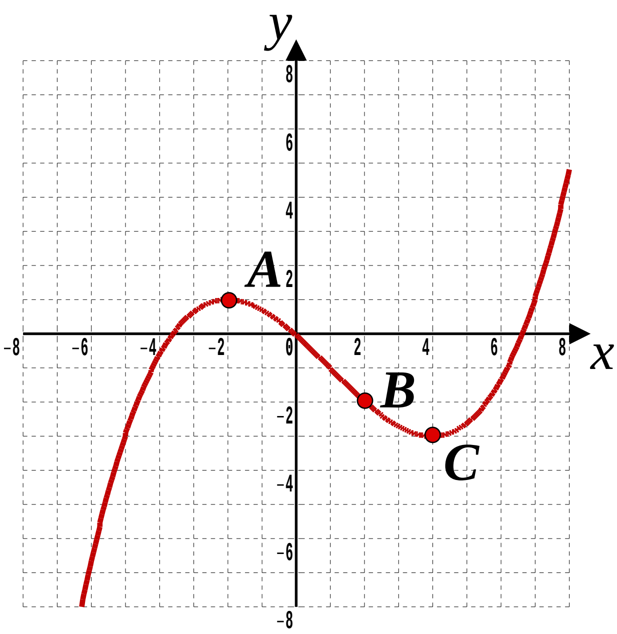
(ii)



$C'(-2, -6)$, $B(0, -4)$, $C(2, -6)$

[3 marks]

(iii)



$A(-2,\ 1),\ B(2, -2),\ C(4, -3)$

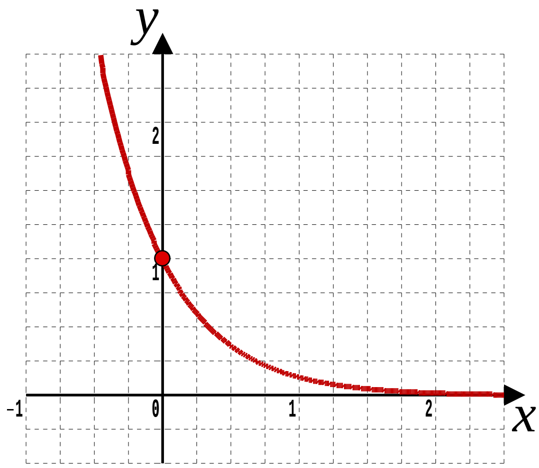
Note : The equation I used to (minor fudge) draw the original graph is

$$y = 0.2225\left(\frac{x^3}{3} + x^2 - 8x\right) - 3.9$$

[4 marks]

Answer 4

(a)

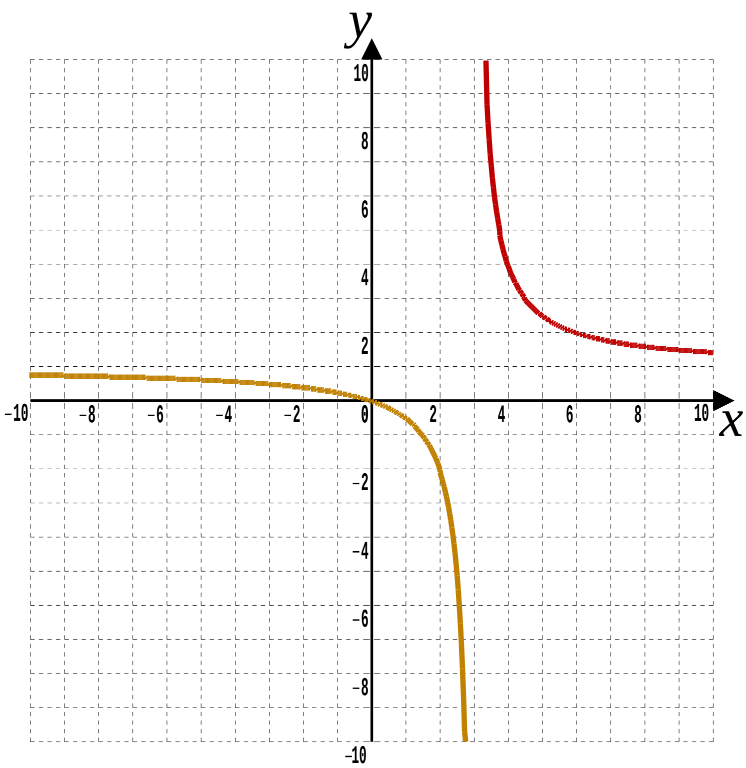


There is only one axis crossing point.

It's on the y-axis at $(\ 0,\ \ 1\ )$

[2 marks]

(b)



The graph of $g\,(x)$ for $x\ >\ 3$ is in red

(The gold part is excluded from the domain)

There is a vertical asymptote at $x\ =\ 3,$ and a horizontal one at $y\ =\ 1$

$\therefore$ The range is, $g\,(x)\ \in\ \mathbb{R},\ \ g\,(x)\ >\ 1$

[2 marks]

(c)

$$g(x) = \frac{x}{x-3} \qquad x > 3$$

$$y = \frac{x}{x-3}$$

Swapping $x \Leftrightarrow y$ because input $\Leftrightarrow$ output

$$x = \frac{y}{y-3}$$

$$x(y-3) = y$$

$$xy - 3x = y$$

$$xy - y = 3x$$

$$y(x-1) = 3x$$

$$y = \frac{3x}{x-1}$$

$$g^{-1}(x) = \frac{3x}{x-1}$$

The domain of the inverse function is the range of the original function

So the domain is $g^{-1}(x) \in \mathbb{R}, \quad g^{-1}(x) > 1$

[4 marks]

(d)

$$fg(x) = 3$$

$$f\left(\frac{x}{x-3}\right) = 3$$

$$exp\left(-2 \times \left(\frac{x}{x-3}\right)\right) = 3$$

take the *ln* of both sides

$$-\left(\frac{2x}{x-3}\right) = ln\,3$$

$$\frac{2x}{3-x} = ln\,3$$

$$2x = (3 - x)\ ln\,3$$

$$2x = 3\,ln\,3 - x\,ln\,3$$

$$2x + x\,ln3 = 3\,ln\,3$$

$$x(2 + ln\,3) = 3\,ln\,3$$

$$x = \frac{3\,ln\,3}{2 + ln\,3}$$

[4 marks]