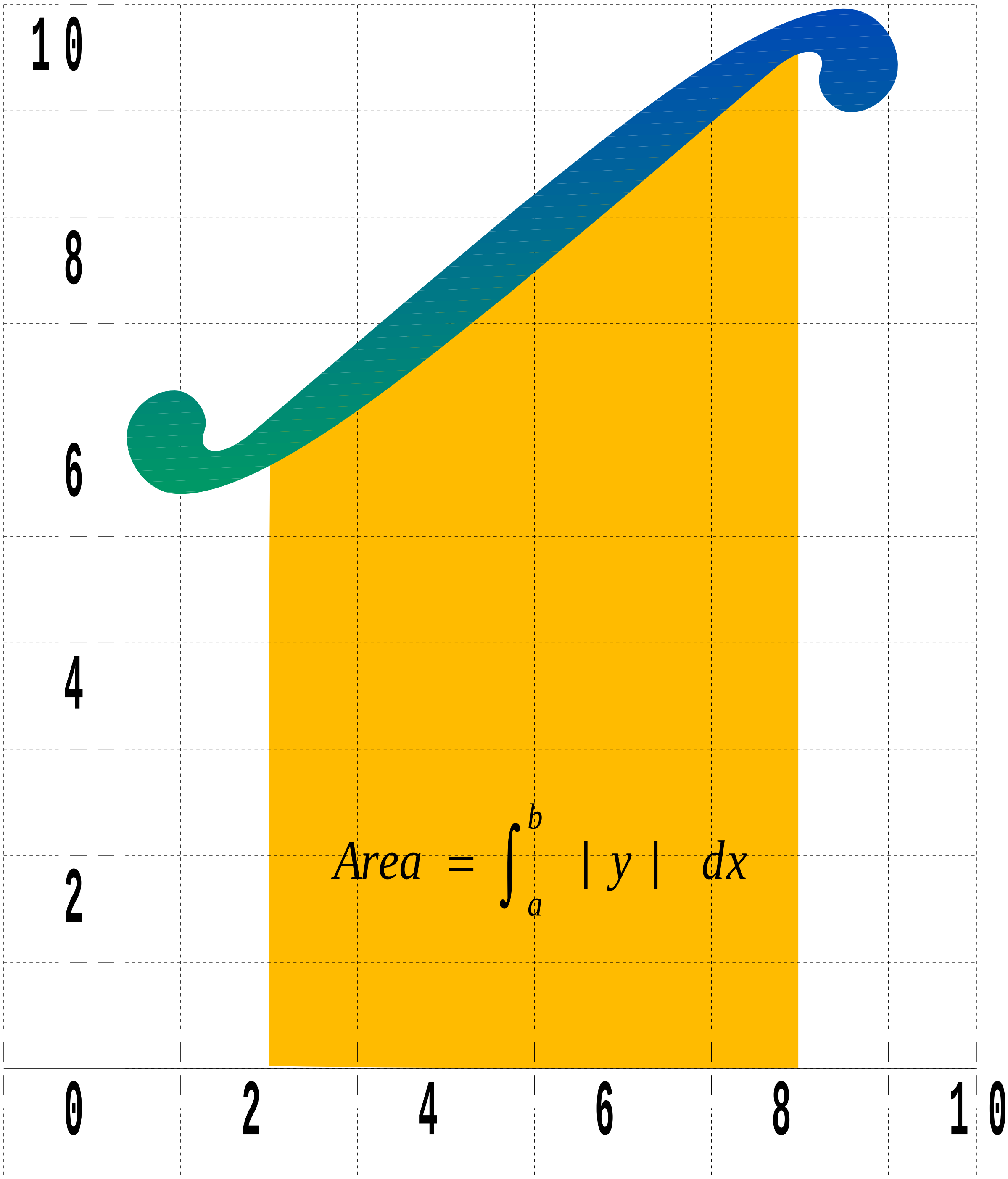
INTEGRATION



A-Level Pure Mathematics (Year 1)
Additional Mathematics

I N T E G R A T I O N   I

Lesson 1

A-Level Pure Mathematics, Year 1
Additional Mathematics
Integration I

1.1 The Fundamental Theorem Of Calculus

In the early 1600s there were two big unsolved problems in mathematics to do with curves.

- The first was to find the gradient of a curve
- The second was to find the area under a curve

It came as a stunning revelation to find that these two problems were different aspects of the same mathematical process. Each was the reverse of the other. To emphasise the importance of this connection it is known under the grand title of;

The Fundamental Theorem of Calculus

The process of integration (finding areas under curves)
is the inverse of
the process of differentiation (finding gradients of curves)

1.2 Backwards Differentiation

Generally speaking, the process of differentiation is easier than integration and often an integration answer is checked by differentiation to make sure that the integration answer is then turned back into its question.

The following rule is key,

Definite Integration Power Rule

$$\text{If} \quad y = x^n \quad \text{then} \quad \int_a^b y \; dx = \left[\frac{x^{n+1}}{n+1} \right]_a^b$$

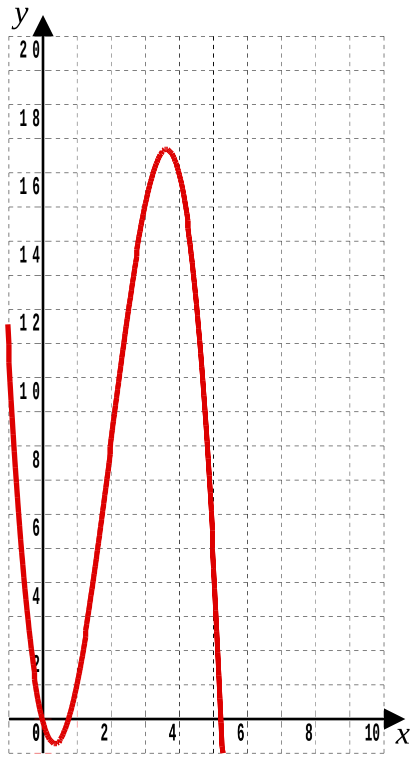
1.3 A First Integration Example

Teaching Video : http://www.NumberWonder.co.uk/v9043/1.mp4

<= The video will talk through the example on the next page

Find the area between the curve with equation $y = 6x^2 - x^3 - 4x$ and the x-axis from $x = 1$ to $x = 4$

Begin by shading in the area to be found on the following graph;



Translated into mathematics, the question becomes;

$$Area = \int_1^4 6x^2 - x^3 - 4x \; dx$$

$$= \left[\frac{6x^3}{3} - \frac{x^4}{4} - \frac{4x^2}{2} \right]_1^4$$

$$= \left[2x^3 - \frac{x^4}{4} - 2x^2 \right]_1^4$$

$$= \left[2 \times 4^3 - \frac{4^4}{4} - 2 \times 4^2 \right] - \left[2 \times 1^3 - \frac{1^4}{4} - 2 \times 1^2 \right]$$

$$= [\,128 - 64 - 32\,] - \left[2 - \frac{1}{4} - 2 \right]$$

$$= [\,32\,] - \left[-\frac{1}{4} \right]$$

$$= 32.25$$

1.4 Definite and Indefinite Integration

◊ ***Definite*** Integration questions are those that involving limits.
A calculator can be used to check the numerical answer.

◊ ***Indefinite*** Integrations are the algebra part only, without any limits.

1.5 Integrating a Isolated Number

An isolated number is a number that is not multiplying any *x*.

When integrated it integrates to that number of *x*.

So, for example, suppose the following isolated 1 is to be integrated,

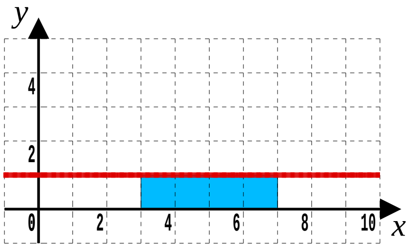
$$\int_{3}^{7} 1 \; dx$$

☞ It integrates to 1*x* or just *x*

$$\int_{3}^{7} 1 \; dx = [\,x\,]_{3}^{7}$$

$$= [\,7\,] - [\,3\,]$$

$$= 4$$



The area graph is a rectangle of height 1 and width 4 which obviously has an area of 4

1.6 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Show that the following integral has a value of 237

$$\int_{2}^{5} 6x^2 + 1 \ \ dx$$

[3 marks]

Question 2

Find the value of,

$$\int_{1}^{3} 24x^3 - 6x \ \ dx$$

[3 marks]

Question 3

Find the area between $y = 36\,x^2$ and the x-axis from $x = 3$ to $x = 5$

[3 marks]

Question 4

Show that the following integral has a value of 3.2

$$\int_1^2 x^4 - 3 \;\; dx$$

[3 marks]

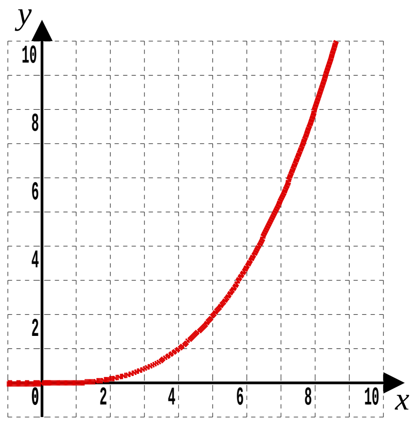
Question 5

This question is about finding the area between the curve

$$y = \frac{x^3}{64}$$

and the x-axis from $x = 4$ to $x = 8$

(i) Shade in the area to be found on the following graph of the curve;



[1 mark]

(ii) Show by integration that the area specified is 15 exactly

[3 marks]

Question 6
By first expanding the brackets, evaluate,

$$\int_2^3 (3x + 2)(x + 1)\ dx$$

[4 marks]

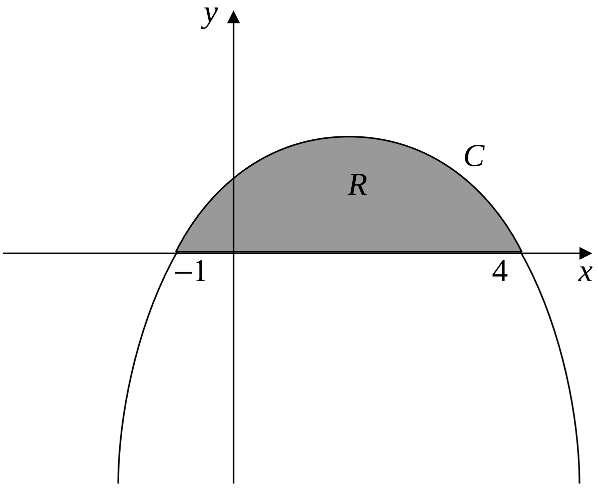
Question 7
Additional Mathematics Examination Question from June 2006, Q1 (OCR)
Find

$$\int_1^3 \left(x^2 + 3\right)\ dx$$

[4 marks]

Question 8

A-Level Examination Question from January 2009, Paper C2, Q2 (Edexcel)



The diagram shows part of the curve C with equation $y = (1 + x)(4 - x)$

The curve intersects the x-axis at $x = -1$ and $x = 4$
The region R, shown shaded, is bounded by C and the x-axis.

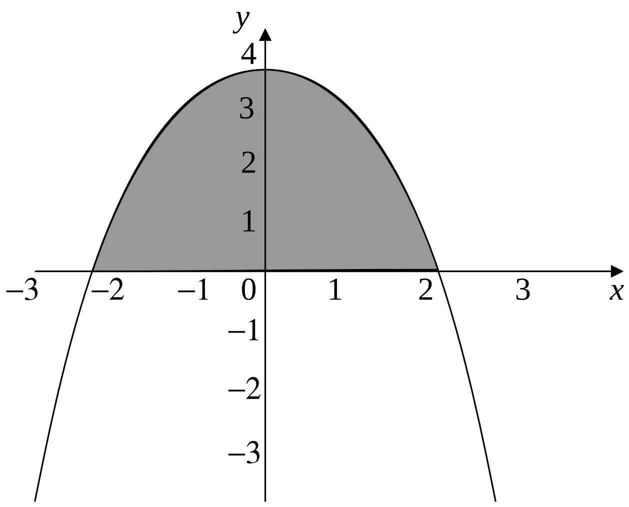
Use calculus to find the exact area of R.

[5 marks]

Question 9

Additional Mathematics Examination Question from June 2004, Q2 (OCR)

The curve shown is part of the graph of $y = 4 - x^2$

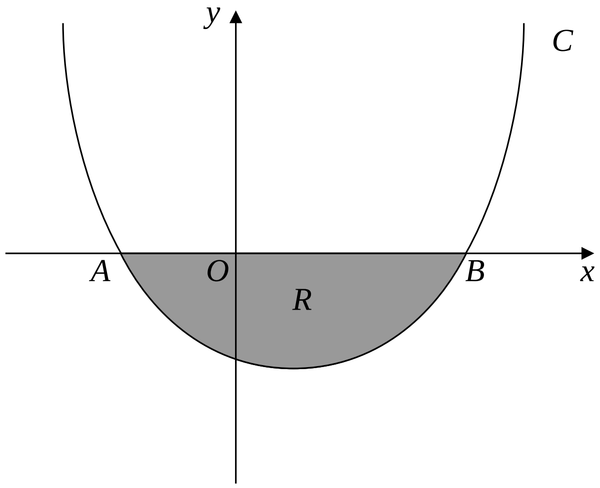


Calculate the area of the shaded region between this curve and the *x*-axis, giving your answer as an exact fraction.

[4 marks]

Question 10

A-Level Examination Question from January 2011, Paper C2, Q4 (Edexcel)



The diagram shows a sketch of part of the curve C with equation

$$y = (x + 1)(x - 5)$$

The curve crosses the x-axis at the points A and B.

(a) Write down the x-coordinates of A and B.

[1 mark]

The finite region R, shown shaded, is bounded by C and the x-axis.

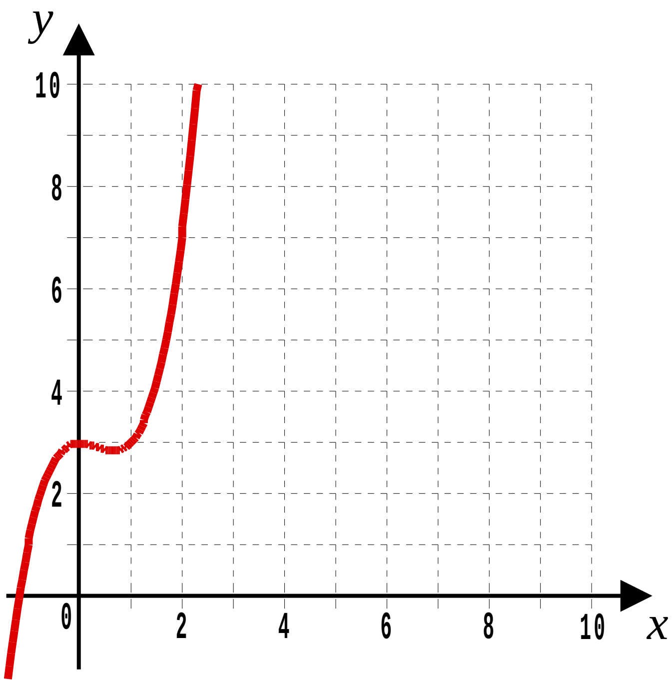
(b) Use integration to find the area of R.

[6 marks]

Question 11

Find the area between $y = x^3 - x^2 + 3$ and the x-axis from $x = 1$ to $x = 2$

(i) Begin by shading in the area to be found on the following graph.



[1 mark]

(ii) Find by integration the exact area of the specified region.

[4 marks]

Question 12

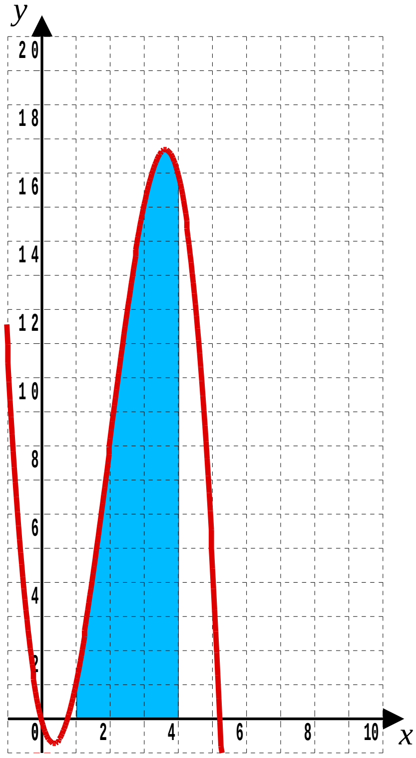
Show that,

$$\int_1^5 \frac{1}{x^3}\ dx = \frac{12}{25}$$

HINT : $\frac{1}{x^3} = x^{-3}$

[5 marks]

1.7.1 Answer (1.3 Integration Example)



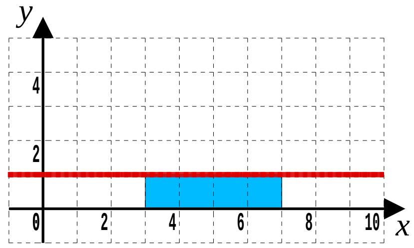
Remainder of solution already provided.

1.7.2 Answer (1.5 Integrating an Isolated Number)

$$\int_3^7 1 \; dx = [\, x \,]_3^7$$

$$= [\, 7 \,] - [\, 3 \,]$$

$$= 4$$



Graph is a rectangle of height 1 and width 4 which obviously has an area of 4.

1.7.3 Answers (1.6 Exercise)

Answer 1

$$\int_2^5 6x^2 + 1 \; dx = \left[\frac{6x^3}{3} + x \right]_2^5$$

$$= \left[2x^3 + x \right]_2^5$$

$$= [2 \times 125 + 5] - [2 \times 8 + 2]$$

$$= [255] - [18]$$

$$= 237 \qquad \square$$

[3 marks]

Answer 2

$$\int_1^3 24x^3 - 6x \; dx = \left[\frac{24x^4}{4} - \frac{6x^2}{2} \right]_1^3$$

$$= \left[6x^4 - 3x^2 \right]_1^3$$

$$= [6 \times 81 - 3 \times 9] - [6 \times 1 - 3 \times 1]$$

$$= [459] - [3]$$

$$= 456$$

[3 marks]

Answer 3

$$\int_3^5 36x^2 \; dx = \left[\frac{36x^3}{3} \right]_3^5$$

$$= \left[12x^3 \right]_3^5$$

$$= [12 \times 125] - [12 \times 27]$$

$$= [1500] - [324]$$

$$= 1176$$

[3 marks]

Answer 4

$$\int_1^2 x^4 - 3 \; dx = \left[\frac{x^5}{5} - 3x \right]_1^2$$

$$= \left[\frac{32}{5} - 3 \times 2 \right] - \left[\frac{1}{5} - 3 \times 1 \right]$$

$$= [\, 6.4 - 6 \,] - [\, 0.2 - 3 \,]$$

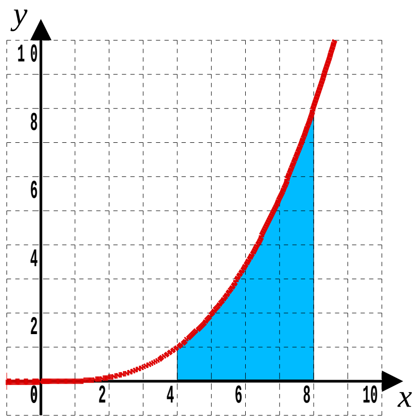
$$= [\, 0.4 \,] - [\, -2.8 \,]$$

$$= 3.2$$ □

[3 marks]

Answer 5

(i)



[1 mark]

(ii)

$$\int_4^8 \frac{x^3}{64} \; dx = \frac{1}{64} \int_4^8 x^3 \, dx$$

$$= \frac{1}{64} \left[\frac{x^4}{4} \right]_4^8$$

$$= \frac{1}{64} \left\{ \left[\frac{8^4}{4} \right] - \left[\frac{4^4}{4} \right] \right\}$$

$$= \frac{1}{64} \{ \, [1024] - [\, 64 \,] \}$$

$$= \frac{1}{64} \{ \, 960 \, \}$$

$$= 15$$ □

[3 marks]

Answer 6

$$\int_2^3 (3x+2)(x+1)\ dx$$

$$= \int_2^3 3x^2 + 5x + 2\ dx$$

$$= \left[\frac{3x^3}{3} + \frac{5x^2}{2} + 2x \right]_2^3$$

$$= \left[x^3 + \frac{5x^2}{2} + 2x \right]_2^3$$

$$= \left[27 + \frac{5 \times 9}{2} + 2 \times 3 \right] - \left[8 + \frac{5 \times 4}{2} + 2 \times 2 \right]$$

$$= \left[\frac{111}{2} \right] - \left[\frac{44}{2} \right]$$

$$= 33\frac{1}{2}$$

[4 marks]

Answer 7

$$\int_1^3 x^2 + 3\ dx = \left[\frac{x^3}{3} + 3x \right]_1^3$$

$$= [9 + 9] - \left[\frac{1}{3} + 3 \right]$$

$$= \left[\frac{54}{3} \right] - \left[\frac{10}{3} \right]$$

$$= \frac{44}{3}$$

$$= 14\frac{2}{3}$$

[4 marks]

Answer 8

$$\int_{-1}^4 (1+x)(4-x)\ dx = \int_{-1}^4 4 + 3x - x^2\ dx$$

$$= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3} \right]_{-1}^4$$

$$= \left[16 + 24 - \frac{64}{3} \right] - \left[-4 + \frac{3}{2} + \frac{1}{3} \right]$$

$$= \left[\frac{125}{6} \right] = 20\frac{5}{6}$$

[5 marks]

Answer 9

As the curve has mirror symmetry in the *y*-axis, the area to be found is given by

$$2 \times \int_0^2 4 - x^2 \; dx = 2 \times \left[4x - \frac{x^3}{3} \right]_0^2$$

$$= 2 \times \left\{ \left[\frac{24}{3} - \frac{8}{3} \right] - [\, 0 \,] \right\}$$

$$= 2 \times \frac{16}{3}$$

$$= \frac{32}{3}$$

$$= 10\frac{2}{3}$$

[4 marks]

Answer 10

(a) − 1 and 5

[1 mark]

(b) $$\int_{-1}^{5} (\, x + 1\,)(\, x - 5\,) \; dx = \int_{-1}^{5} x^2 - 4x - 5 \; dx$$

$$= \left[\frac{x^3}{3} - 2\,x^2 - 5x \right]_{-1}^{5}$$

$$= \left[\frac{125}{3} - 50 - 25 \right] - \left[-\frac{1}{3} - 2 + 5 \right]$$

$$= -36$$

$$\therefore \; Area = 36$$

[6 marks]

Answer 11

$$\int_1^2 x^3 - x^2 + 3 \; dx = \left[\frac{x^4}{4} - \frac{x^3}{3} + 3x \right]_1^2$$

$$= \left[\frac{2^4}{4} - \frac{2^3}{3} + 3 \times 2 \right] - \left[\frac{1^4}{4} - \frac{1^3}{3} + 3 \times 1 \right]$$

$$= \left[4 - \frac{8}{3} + 6 \right] - \left[\frac{1}{4} - \frac{1}{3} + 3 \right]$$

$$= \frac{48 - 32 + 72 - 3 + 4 - 36}{12}$$

$$= \frac{124 - 71}{12}$$

$$= \frac{53}{12} = 4\frac{5}{12}$$

[1, 4 marks]

Answer 12

$$\int_1^5 \frac{1}{x^3}\ dx = \int_1^5 x^{-3}\ dx$$

$$= \left[\frac{x^{-2}}{-2} \right]_1^5$$

$$= \left[-\frac{1}{2x^2} \right]_1^5$$

$$= \left[-\frac{1}{50} \right] - \left[-\frac{1}{2} \right]$$

$$= \left[\frac{-1 + 25}{50} \right]$$

$$= \frac{24}{50}$$

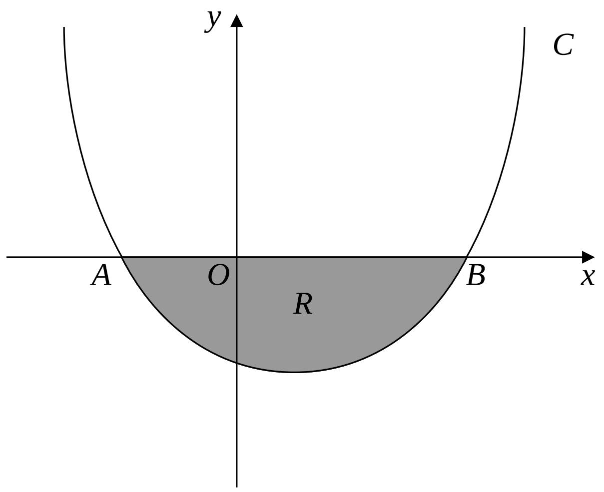
$$= \frac{12}{25}$$ ☐

[5 marks]

2.1 When Below the *x*-axis

In Question 10 of Exercise 1.6 we looked at the curve

$$y = (x+1)(x-5)$$



The following two statements are both true;

◊ $\int_{-1}^{5}(x+1)(x-5)\ dx = -36$

◊ $Area = +36$

If a question asks for the value of an integral, simply do the mathematics without worrying about any places where the curve is below the *x*-axis, and give the answer without any alteration to its sign.

If a question asks for an area, you must make any areas under the *x*-axis positive, before summing all positive areas to give the overall area.

2.2 A "Negative Area" Example

Teaching Video : http://www.NumberWonder.co.uk/v9043/2.mp4

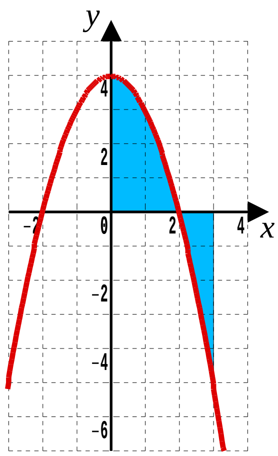
<= The video will talk through the example on the next page

Find the area bounded by $y = 4 - x^2$, the x-axis, and lines $x = 0$ and $x = 3$

$$y = 4 - x^2$$
$$= (2 + x)(2 - x)$$

So curve crosses x-axis at $x = -2$ and $x = 2$

Also, it crosses y-axis at 4



$$\int_0^2 4 - x^2 \, dx = \left[4x - \frac{x^3}{3} \right]_0^2$$
$$= \left[8 - \frac{8}{3} \right] - [0]$$
$$= \left[\frac{24 - 8}{3} \right] - [0]$$
$$= \left[\frac{16}{3} \right] - [0]$$
$$= \frac{16}{3}$$

$$\int_2^3 4 - x^2 \, dx = \left[4x - \frac{x^3}{3} \right]_2^3$$
$$= [12 - 9] - \left[\frac{16}{3} \right]$$
$$= \left[\frac{9}{3} \right] - \left[\frac{16}{3} \right]$$
$$= -\frac{7}{3}$$

$$\therefore Area = \frac{16 + 7}{3} = \frac{23}{3} = 7\frac{2}{3}$$

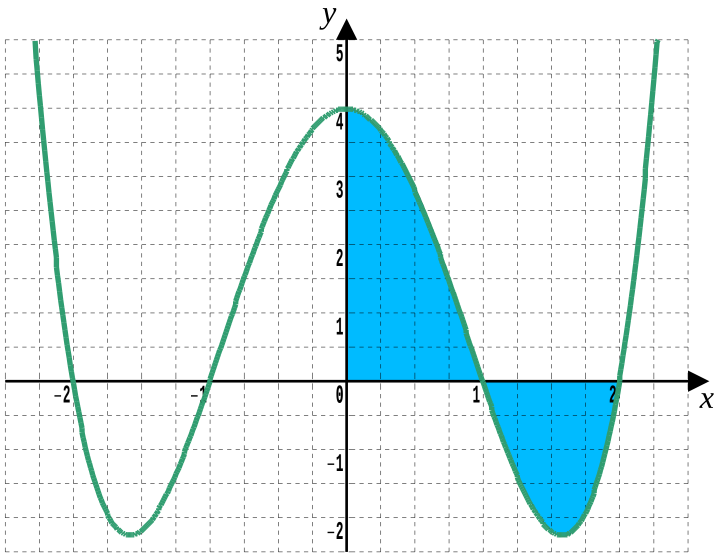
[6 marks]

2.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

The graph is of the quartic curve $y = x^4 - 5x^2 + 4$



(i) Show that $\int_1^2 y \, dx = -\dfrac{22}{15}$

[5 marks]

(ii) Determine the exact value of $\int_0^1 y \, dx$

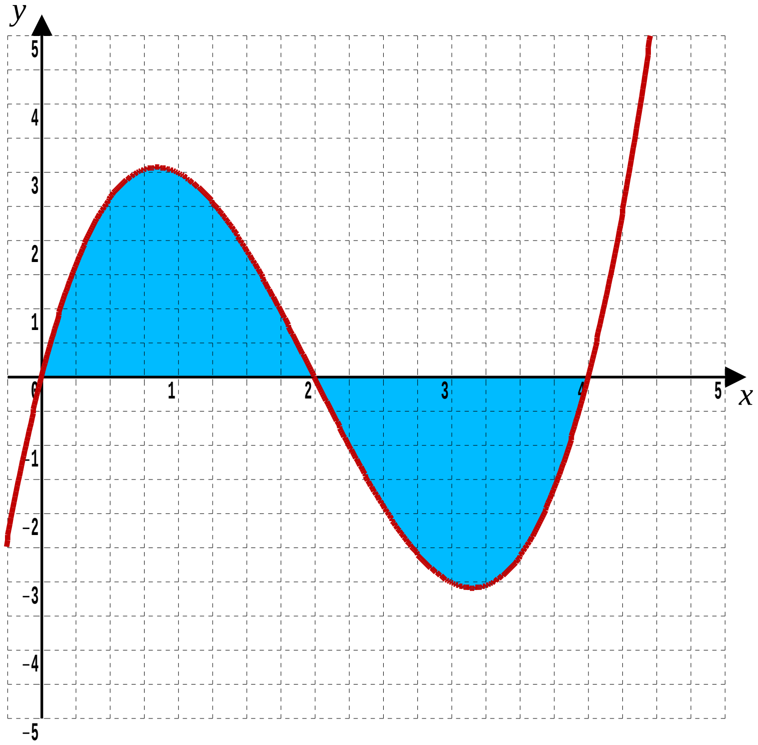
[2 marks]

(iii) Hence state the exact area that has been shaded on the graph

[1 mark]

Question 2

The graph is of the cubic curve $y = x(x - 2)(x - 4)$



(i) Without considering the graph, determine the value of $\int_0^4 y\,dx$

[4 marks]

(ii) Now consider the graph.
From your part (i) answer what can you deduce about the relationship between the area shown shaded above the *x*-axis and the area shown shaded below the *x*-axis ?

[1 mark]

Question 3

Determine the value of $\int_1^4 6x^2 - 5x^4 \; dx$

You should get a negative integer answer.

[4 marks]

Question 4

(i) Find the value of the upper limit that makes the following statement true;

$$\int_1^a (5 - 2x)\; dx = 0$$

[4 marks]

(ii) Give a geometric explanation of the part (i) result

[2 marks]

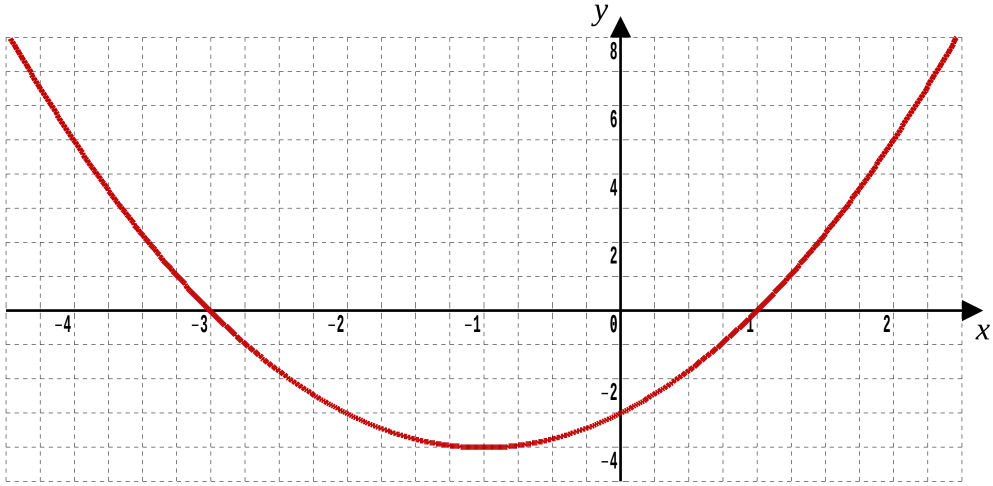
Question 5

Additional Mathematics Examination Question from June 2012, Q8 (OCR)

(i) Show that $\int_0^2 x^2 + 2x - 3 \; dx = \frac{2}{3}$

[3 marks]

The diagram shows part of the curve $y = x^2 + 2x - 3$



(ii) Marc claims that the total area between the curve, the x-axis and the lines $x = 0$ and $x = 2$ is $\frac{2}{3}$. Explain why he is wrong.

[1 mark]

(iii) Calculate total area between curve, x-axis and lines $x = 0$ and $x = 2$

[3 marks]

Question 6

This question is about using integration to find the area bounded by the curve $y = 3x - x^2$ and the x-axis and the vertical lines $x = 0$ and $x = 6$

(i) Sketch the graph of the curve and use your sketch to explain why

$$Area \neq \int_0^6 3x - x^2 \, dx$$

[4 marks]

(ii) Set up the correct integrations and evaluate them to find the area specified.

[6 marks]

2.4 Answers to 2.3 Exercise

Answer 1

(i)

$$\begin{aligned}
\text{LHS} &= \int_1^2 y\ dx \\
&= \int_1^2 x^4 - 5x^2 + 4\ dx \\
&= \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x \right]_1^2 \\
&= \left[\left(\frac{32}{5} - \frac{5 \times 8}{3} + 4 \times 2 \right) - \left(\frac{1}{5} - \frac{5}{3} + 4 \right) \right] \\
&= \left(\frac{16}{15} \right) - \left(\frac{38}{15} \right) \\
&= -\frac{22}{15} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[5 marks]

(ii)

$$\begin{aligned}
\int_0^1 y\ dx &= \int_0^1 x^4 - 5x^2 + 4\ dx \\
&= \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x \right]_0^1 \\
&= \left(\frac{1}{5} - \frac{5}{3} + 4 \right) - (0) \\
&= \frac{38}{15}
\end{aligned}$$

[2 marks]

(iii)

$$\begin{aligned}
\text{Area shaded} &= \frac{22}{15} + \frac{38}{15} \\
&= 4
\end{aligned}$$

[1 mark]

Answer 2

(i)

$$\int_0^4 y\ dx = \int_0^4 x(x-2)(x-4)\ dx$$

$$= \int_0^4 x\left(x^2 - 6x + 8\right)\ dx$$

$$= \int_0^4 x^3 - 6x^2 + 8x\ dx$$

$$= \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{8x^2}{2}\right]_0^4$$

$$= \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^4$$

$$= \left[\frac{256}{4} - 2 \times 64 + 4 \times 16\right] - [0]$$

$$= 0$$

[4 marks]

(ii) The two areas must be the same

[1 mark]

Answer 3

$$\int_1^4 6x^2 - 5x^4\ dx = \left[\frac{6x^3}{3} - \frac{5x^5}{5}\right]_1^4$$

$$= \left[2x^3 - x^5\right]_1^4$$

$$= \left[2 \times 4^3 - 4^5\right] - \left[2 \times 1^3 - 1^5\right]$$

$$= [128 - 1024] - [1]$$

$$= -897$$

[4 marks]

Answer 4

(i)

$$\int_1^a (5 - 2x)\ dx = 0$$

$$\Big[5x - x^2 \Big]_1^a = 0$$

$$\Big[5a - a^2 \Big] - \Big[5 - 1 \Big] = 0$$

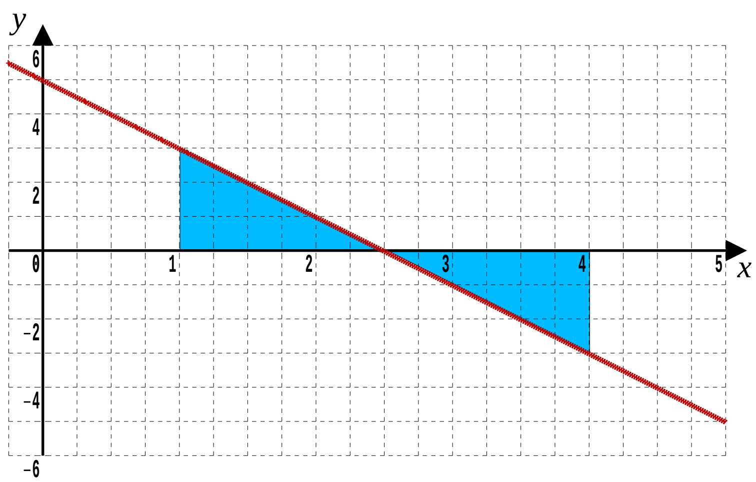
$$a^2 - 5a + 4 = 0$$

$$(a - 4)\ (a - 1) = 0$$

$a = 4$ is the only sensible solution

[4 marks]

(ii) The integration is finding the area under the 'curve' which, in this case, is the straight line $y = 5 - 2x$.
However, areas under the x-axis are seen as negative and so the upper limit is being moved to a place where the area above the x-axis is the same as the area below.
The graph below shows the situation when $a = 4$



[2 marks]

Answer 5

(i)

$$\text{LHS} = \int_0^2 x^2 + 2x - 3 \; dx$$

$$= \left[\frac{x^3}{3} + \frac{2\,x^2}{2} - 3x \right]_0^2$$

$$= \left[\frac{x^3}{3} + x^2 - 3x \right]_0^2$$

$$= \left[\frac{8}{3} + 4 - 6 \right] - [\,0\,]$$

$$= \frac{2}{3}$$

$$= \text{RHS} \qquad \square$$

[3 marks]

(ii) Marc has forgotten that integration will 'see' area under the x-axis as a negative quantity.
As the curve crosses the x-axis between $x = 0$ and $x = 1$ this will be an issue with some 'negative area' giving an incorrect result

[1 mark]

(iii) Area between $x = 0$ and $x = 1$;

$$\int_0^1 x^2 + 2x - 3 \; dx = \left[\frac{x^3}{3} + x^2 - 3x \right]_0^1$$

$$= \frac{1}{3} + 1 - 3$$

$$= -\frac{5}{3}$$

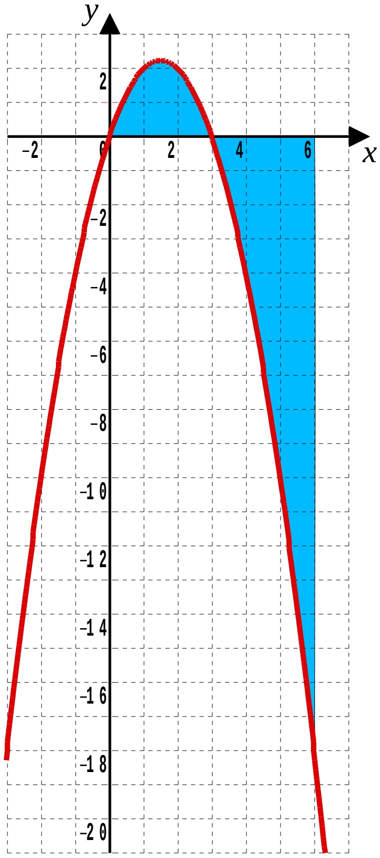
Total area requested will be,

$$\frac{2}{3} + 2 \times \frac{5}{3} = 4$$

[3 marks]

Answer 6

(i)



As the curve crosses the x-axis between $x = 0$ and $x = 6$ the integration over the whole interval in one go will not correspond to the area under the curve

(ii)

$$\int_0^3 3x - x^2 \; dx = \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3$$

$$= \left[\frac{27}{2} - 9 \right] - [\, 0 \,]$$

$$= 4.5$$

$$\int_3^6 3x - x^2 \; dx = \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_3^6$$

$$= [\, 54 - 72 \,] - [\, 4.5 \,]$$

$$= [\, -18 \,] - [\, 4.5 \,]$$

$$= -22.5$$

$\therefore$ Area specified is 27

[6 marks]

3.1 Integration with Fractional Powers

Fractional powers arise quite naturally in integration questions, not least because

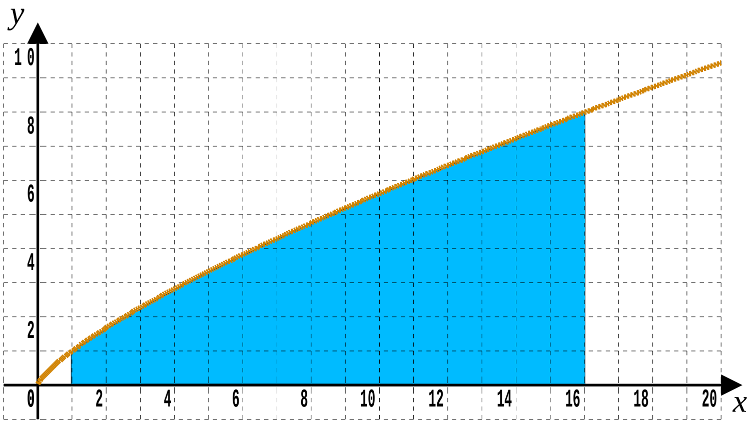
$$\sqrt{x} = x^{\frac{1}{2}}$$

Although the rule to integrate expressions containing such powers is no different to that for integrating integer powers, there is a skill in avoiding multi-decker fractions, as the next example will demonstrate.

3.2 "Fractional Power" Example

The curve graphed is of the equation $y = x^{\frac{3}{4}}$

Determine the shaded area.



Teaching Video : http://www.NumberWonder.co.uk/v9043/3.mp4

[4 marks]

3.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

Show that $\int_4^{25} \sqrt{x}\ dx = 78$

There is a bonus mark for not having any multi-decker fractions in your working !

[4 marks, plus 1 bonus mark]

Question 2

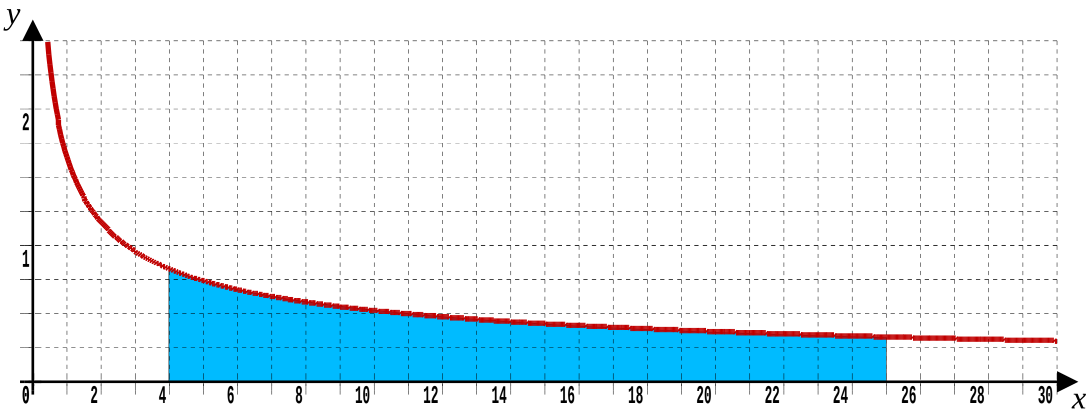
Determine the exact value of $\int_0^{3} \sqrt{3x}\ dx$

HINT **:** $\int_0^{3} \sqrt{3x}\ dx = \sqrt{3} \int_0^{3} \sqrt{x}\ dx$

[4 marks]

Question 3

The graph shows the curve with equation $y = \dfrac{5}{3\sqrt{x}}$



Determine the exact value of $\displaystyle\int_4^{25} \frac{5}{3\sqrt{x}}\,dx$

HINT : $\dfrac{5}{3x^{\frac{1}{2}}} = \dfrac{5x^{-\frac{1}{2}}}{3}$

[5 marks]

Question 4

Given that $y = \dfrac{\sqrt{x} + 1}{3}$ find in the simplest form $\int_1^4 y \; dx$

HINT : $\int_1^4 \dfrac{\sqrt{x} + 1}{3} \, dx = \dfrac{1}{3} \int_1^4 \sqrt{x} + 1 \, dx$

[5 marks]

Question 5

A-Level Examination Question from June 2009, Paper C2, Q1 (Edexcel)

Use calculus to find the exact value of

$$\int_1^4 \left(2x + 3\sqrt{x} \right) \, dx$$

[5 marks]

Question 6

A-Level Examination Question from January 2017, Paper C12, Q7(ii) (Edexcel)

Given that k is a constant and

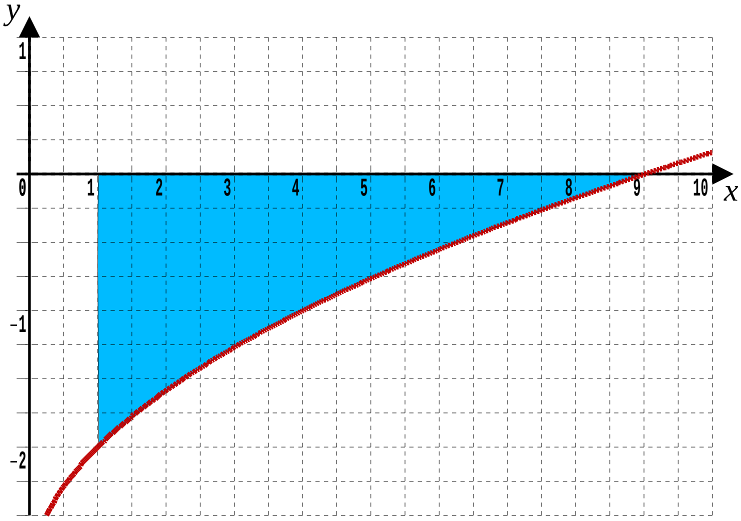
$$\int_2^4 \left(\frac{4}{\sqrt{x}} + k \right) dx = 30$$

find the exact value of k

[6 marks]

Question 7

The curve graphed below is of the equation $y = \sqrt{x} - 3$



Taking great care over the minus signs in the working, determine the shaded area.

[6 marks]

Question 8

A-Level Examination Question from May 2007, Paper C2, Q1 (Edexcel)

Evaluate

$$\int_1^8 \frac{1}{\sqrt{x}}\,dx$$

giving your answer in the form $a + b\sqrt{2}$, where a and b are integers

[4 marks]

3.4 Answers

3.4.1 Solution to 3.2 "Fractional Powers" Example (Teaching Video)

$$\int_{1}^{16} x^{\frac{3}{4}}\,dx = \left[\frac{4x^{\frac{7}{4}}}{7}\right]_{1}^{16}$$

$$= \frac{4}{7}\left[x^{\frac{7}{4}}\right]_{1}^{16}$$

$$= \frac{4}{7}\left[16^{\frac{7}{4}} - 1^{\frac{7}{4}}\right]$$

$$= \frac{4}{7}\left[128 - 1\right]$$

$$= \frac{508}{7} \qquad = 72\frac{4}{7}$$

3.4.2 Answers to 3.3 Exercise

Answer 1

$$\int_{4}^{25} x^{\frac{1}{2}}\,dx = \left[\frac{2x^{\frac{3}{2}}}{3}\right]_{4}^{25}$$

$$= \frac{2}{3}\left[x^{\frac{3}{2}}\right]_{4}^{25}$$

$$= \frac{2}{3}\left[25^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[125 - 8\right]$$

$$= 78$$

[3 marks plus 1 bonus mark]

Answer 2

$$\int_0^3 \sqrt{3x}\,dx = \sqrt{3}\int_0^3 x^{\frac{1}{2}}\,dx$$

$$= \sqrt{3}\left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^3$$

$$= \frac{2\sqrt{3}}{3}\left[x^{\frac{3}{2}}\right]_0^3$$

$$= \frac{2\sqrt{3}}{3}\left[3^{\frac{3}{2}} - 0\right]$$

$$= \frac{2\sqrt{3}}{3} \times 3\sqrt{3}$$

$$= 6$$

[4 marks]

Answer 3

$$\int_4^{25} \frac{5}{3\sqrt{x}}\,dx = \frac{5}{3}\int_4^{25} x^{-\frac{1}{2}}\,dx$$

$$= \frac{5}{3}\left[\frac{2x^{\frac{1}{2}}}{1}\right]_4^{25}$$

$$= \frac{10}{3}\left[\sqrt{x}\right]_4^{25}$$

$$= \frac{10}{3}\left\{[5] - [2]\right\}$$

$$= \frac{10}{3}\left\{3\right\}$$

$$= 10$$

[5 marks]

Answer 4

$$\int_1^4 y\ dx = \int_1^4 \frac{\sqrt{x} + 1}{3}\ dx$$

$$= \frac{1}{3} \int_1^4 \sqrt{x} + 1\ dx$$

$$= \frac{1}{3} \int_1^4 x^{\frac{1}{2}} + 1\ dx$$

$$= \frac{1}{3} \left[\frac{2x^{\frac{3}{2}}}{3} + x \right]_1^4$$

$$= \frac{1}{3}\left[\frac{2 \times 4^{\frac{3}{2}}}{3} + 4 - \left(\frac{2 \times 1^{\frac{3}{2}}}{3} + 1 \right) \right]$$

$$= \frac{1}{3}\left[\frac{16}{3} + 4 - \frac{2}{3} - 1 \right]$$

$$= \frac{1}{3}\left[\frac{23}{3} \right]$$

$$= \frac{23}{9} \qquad \text{or} \qquad 2\frac{5}{9}$$

[5 marks]

Answer 5

$$\int_1^4 \left(2x + 3\sqrt{x} \right) dx = \int_1^4 \left(2x + 3x^{\frac{1}{2}} \right) dx$$

$$= \left[x^2 + 2x^{\frac{3}{2}} \right]_1^4$$

$$= \{ [16 + 16] - [1 + 2] \}$$

$$= 29$$

[5 marks]

Answer 6

$$\int_2^4 \left(\frac{4}{\sqrt{x}} + k \right) dx = 30$$

$$\int_2^4 4x^{-\frac{1}{2}} + k\ dx = 30$$

$$\left[8x^{\frac{1}{2}} + kx \right]_2^4 = 30$$

$$16 + 4k - 8\sqrt{2} - 2k = 30$$

$$2k = 30 - 16 + 8\sqrt{2}$$

$$k = 7 + 4\sqrt{2}$$

[6 marks]

Answer 7

$$\int_1^9 x^{\frac{1}{2}} - 3 \; dx = \left[\frac{2x^{\frac{3}{2}}}{3} - 3x \right]_1^9$$

$$= \left\{ \left[\frac{2 \times 27}{3} - 3 \times 9 \right] - \left[\frac{2 \times 1}{3} - 3 \times 1 \right] \right\}$$

$$= \left\{ [\, 18 - 27 \,] - \left[\frac{2}{3} - 3 \right] \right\}$$

$$= \left\{ [-9\,] - \left[\; - \frac{7}{3} \right] \right\}$$

$$= \left[\; - 9 + \frac{7}{3} \right]$$

$$= - \frac{20}{3} \qquad \therefore \; Area = \frac{20}{3}$$

[6 marks]

Answer 8

$$\int_1^8 \frac{1}{\sqrt{x}} \; dx = \int_1^8 x^{-\frac{1}{2}} \; dx$$

$$= \left[\frac{2x^{\frac{1}{2}}}{1} \right]_1^8$$

$$= \left[2\sqrt{x} \; \right]_1^8$$

$$= \{ \, [2\sqrt{8}\,] - [\,2\,] \, \}$$

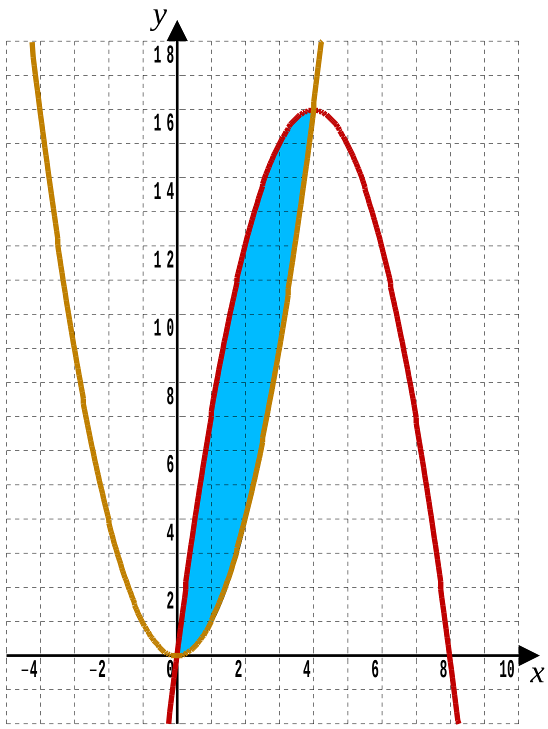
$$= \{ \, [2 \times 2\sqrt{2}\,] - [\,2\,] \, \}$$

$$= -2 + 4\sqrt{2}$$

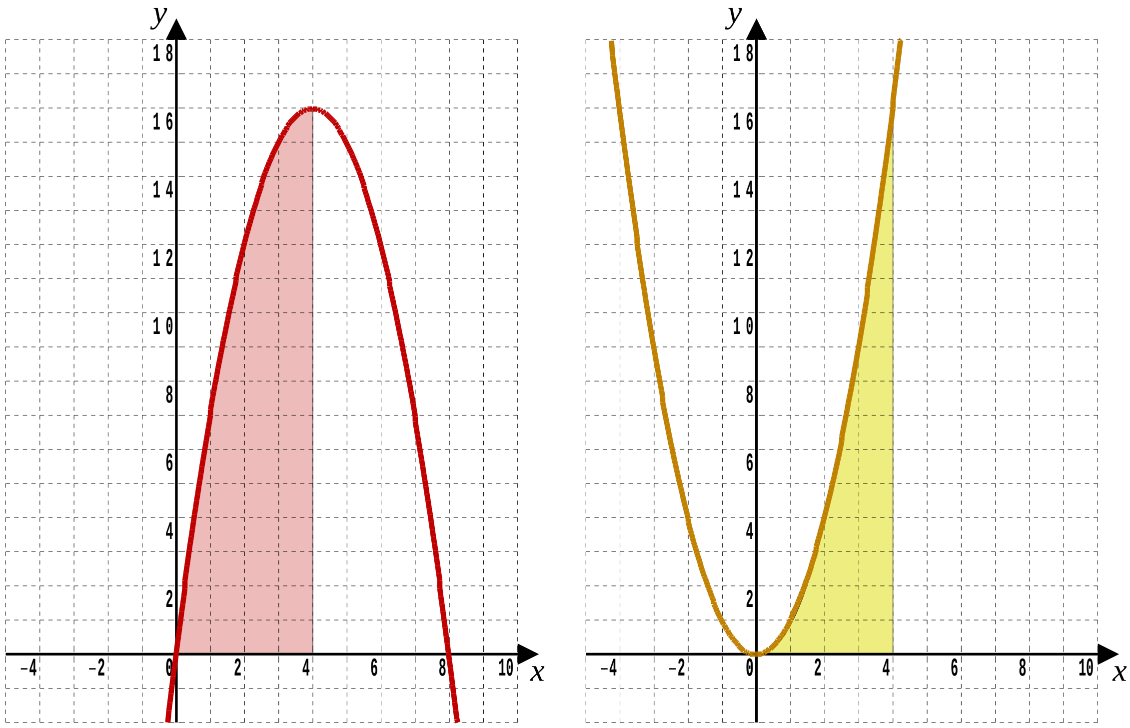
[4 marks]

4.1 Areas Between Curves

It is desired to find the area, shaded blue, between two curves.



The strategy is to use integration to find the pink area under the upper red curve and subtract the yellow area under the lower gold curve.



The two curves have the equations,

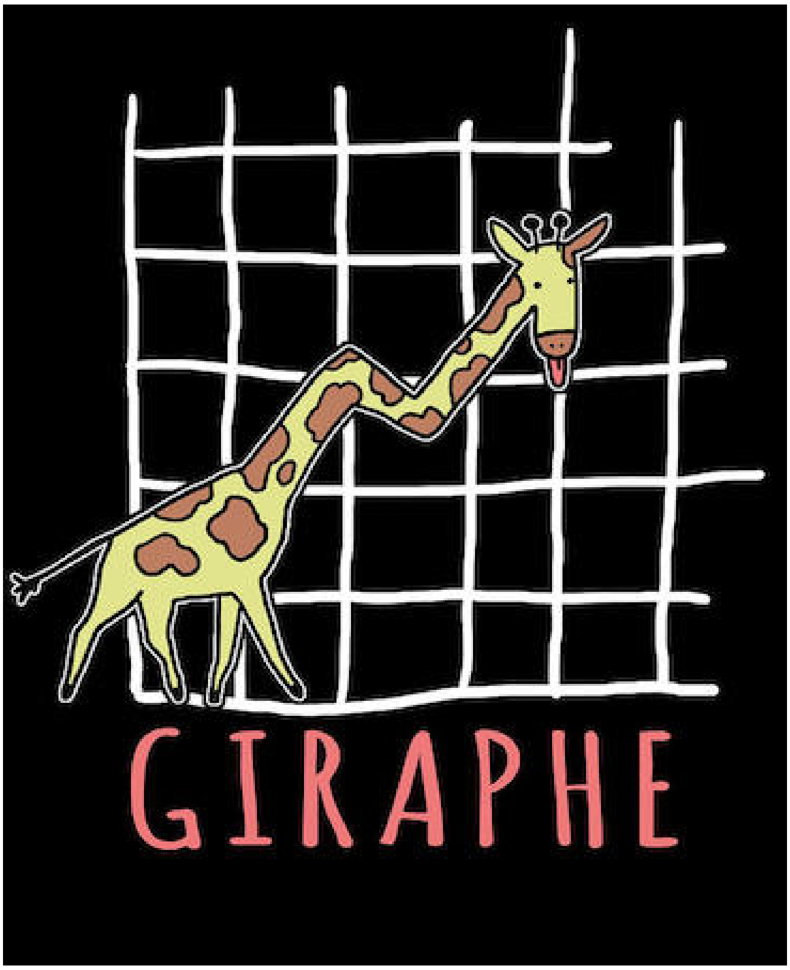
$$y = 8x - x^2 \qquad \text{The red 'upside down' parabola}$$

$$y = x^2 \qquad \text{The gold parabola}$$

Find area between the two curves.

Teaching Video : http://www.NumberWonder.co.uk/v9043/4.mp4

[6 marks]



4.2 Exercise

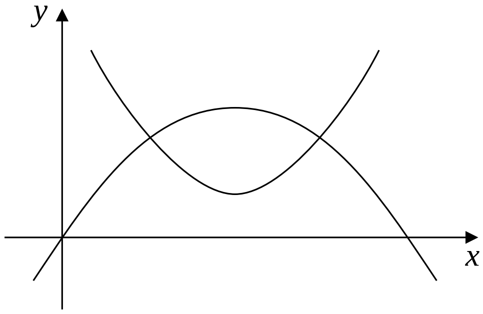
Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 52

Question 1

Additional Mathematics Examination Question from June 2007, Q8 (OCR) edited

The figure shows the graphs of $y = 4x - x^2$ and $y = x^2 - 4x + 6$



(i) Use an algebraic method to find the x-coordinates of the points where the curves intersect.

[3 marks]

(ii) Show that the area enclosed by the curves is $\int_1^3 8x - 2x^2 - 6 \, dx$

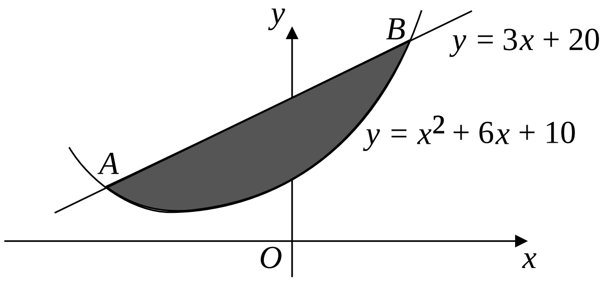
[2 marks]

(iii) Calculate the area enclosed by the two curves.

[4 marks]

Question 2

A-Level Examination Question from January 2005, Paper C2, Q8 (Edexcel)



The line with equation $y = 3x + 20$ cut the curve with equation $y = x^2 + 6x + 10$ at the points *A* and *B*, as shown.

(a) Use algebra to find the coordinates of *A* and the coordinates of *B*.

[5 marks]

The shaded region is bounded by the line and the curve, as shown.

(b) Use calculus to find the exact area shown shaded.

[7 marks]

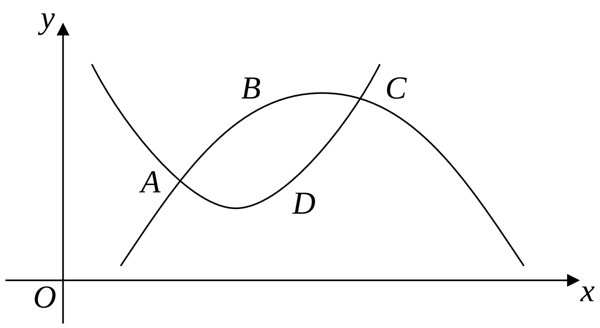
Question 3

Additional Mathematics Examination Question from June 2009, Q11 (OCR)

The shape *ABCD* below represents a leaf.

The curve *ABC* has equation $y = -x^2 + 8x - 9$

The curve *ADC* has equation $y = x^2 - 6x + 11$



(i) Find algebraically the coordinates of *A* and *C*, the points where the curves intersect.

[5 marks]

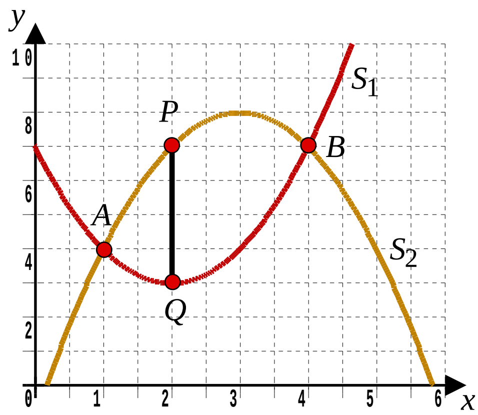
(ii) Find the area of the leaf.

[7 marks]

Question 4

Additional Mathematics Examination Question from June 2015, Q11 (OCR)

Two curves, S_1 and S_2 have equations $y = x^2 - 4x + 7$ and $y = 6x - x^2 - 1$ respectively. The curves meet at A and B.



(i) Show that the coordinates of A and B are $(1, \; 4)$ and $(4, \; 7)$ respectively.

[2 marks]

Points P and Q lie on S_2 and S_1 between A and B.

P and Q have the same x coordinate so that PQ is parallel to the y-axis, as shown.

(ii) Find an expression, in its simplest form, for the length PQ as a function of x.

[2 marks]

(iii) Use calculus to find the greatest length of PQ

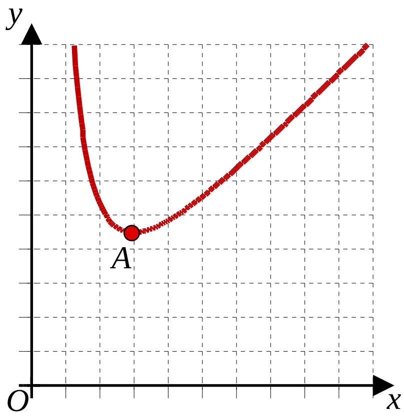
[4 marks]

(iv) Find the area between the two curves.

[4 marks]

Question 5

Additional Mathematics Examination Question, June 2018, Q11 (Cambridge IGCSE)



The diagram shows part of the graph of $y = 16x + \dfrac{27}{x^2}$ which has a minimum at A

(i) Find the coordinates of A

[4 marks]

The points P and Q lie of the curve $y = 16x + \dfrac{27}{x^2}$ and have x-coordinates 1 and 3 respectively.

(ii) Find the area enclosed by the curve and the line PQ.
You must show all your working.

[6 marks]

4.3 Answers

4.3.1 Solution to 4.1 Example (Teaching Video)

The two curves have the equations,

$y = 8x - x^2$ The red 'upside down' parabola

$y = x^2$ The gold parabola

Find area between the two curves.

To find the x values of where the two curves intersect,

$$x^2 = 8x - x^2$$

$$2x^2 - 8x = 0$$

Divide through by 2

$$x^2 - 4x = 0$$

$$x(x - 4) = 0$$

Either $x = 0$ or $x = 4$

$$Area = \int_0^4 8x - x^2 \, dx - \int_0^4 x^2 \, dx$$

$$= \int_0^4 8x - x^2 - x^2 \, dx$$

$$= \int_0^4 8x - 2x^2 \, dx$$

$$= \left[4x^2 - \frac{2x^3}{3} \right]_0^4$$

$$= 4 \times 4^2 - \frac{2 \times 4^3}{3}$$

$$= 64 - \frac{128}{3}$$

$$= \frac{64}{3}$$

$$= 21\frac{1}{3}$$

[6 marks]

4.3.2 Solutions to 4.2 Exercise

Answer 1

(i)

$$x^2 - 4x + 6 = 4x - x^2$$
$$2x^2 - 8x + 6 = 0$$
$$x^2 - 4x + 3 = 0$$
$$(x - 1)(x - 3) = 0$$
$$\therefore \text{ Either } x = 1 \text{ or } x = 3$$

[3 marks]

(ii)

$$\begin{aligned} Area &= \int_1^3 4x - x^2\, dx - \int_1^3 x^2 - 4x + 6\, dx \\ &= \int_1^3 4x - x^2 - x^2 + 4x - 6\, dx \\ &= \int_1^3 8x - 2x^2 - 6\, dx \end{aligned}$$

[2 marks]

(iii)

$$\begin{aligned} &\int_1^3 8x - 2x^2 - 6\, dx \\ &= \left[4x^2 - \frac{2x^3}{3} - 6x \right]_1^3 \\ &= \left\{ \left[4 \times 9 - \frac{2 \times 27}{3} - 6 \times 3 \right] - \left[4 \times 1 - \frac{2}{3} - 6 \times 1 \right] \right\} \\ &= (36 - 18 - 18) - (4 - \frac{2}{3} - 6) \\ &= -\left(-\frac{8}{3}\right) \\ &= 2\frac{2}{3} \end{aligned}$$

[4 marks]

Answer 2

(a)

$$x^2 + 6x + 10 = 3x + 20$$

$$x^2 + 3x - 10 = 0$$

$$(x + 5)(x - 2) = 0$$

$$\therefore\ A(-5,\ 5) \text{ and } B(2,\ 26)$$

[5 marks]

(b)

$$Area = \int_{-5}^{2} 3x + 20 - \int_{-5}^{2} x^2 + 6x + 10\ dx$$

$$= \int_{-5}^{2} 3x + 20 - x^2 - 6x - 10\, dx$$

$$= \int_{-5}^{2} 10 - 3x - x^2\, dx$$

$$= \left[10x - \frac{3x^2}{2} - \frac{x^3}{3} \right]_{-5}^{2}$$

$$= \left\{ \left[20 - 6 - \frac{8}{3} \right] - \left[-50 - \frac{75}{2} + \frac{125}{3} \right] \right\}$$

$$= \left\{ \left[\frac{34}{3} \right] - \left[-\frac{275}{6} \right] \right\}$$

$$= \frac{343}{6} \qquad \text{or} \qquad 57\frac{1}{6}$$

[7 marks]

Answer 3

(i)

$$x^2 - 6x + 11 = -x^2 + 8x - 9$$

$$2x^2 - 14x + 20 = 0$$

$$x^2 - 7x + 10 = 0$$

$$(x - 2)(x - 5) = 0$$

$$\therefore A\ (2,\ 3) \text{ and } B\ (5,\ 6)$$

[5 marks]

(ii)

$$Area = \int_2^5 -x^2 + 8x - 9 \ dx - \int_2^5 x^2 - 6x + 11 \ dx$$

$$= \int_2^5 -2x^2 + 14x - 20 \ dx$$

$$= \left[-\frac{2x^3}{3} + 7x^2 - 20x \right]_2^5$$

$$= \left\{ \left[-\frac{250}{3} + 175 - 100 \right] - \left[-\frac{16}{3} + 28 - 40 \right] \right\}$$

$$= \left[-\frac{25}{3} \right] - \left[-\frac{52}{3} \right]$$

$$= 9$$

[7 marks]

Answer 4

(i)

$$x^2 - 4x + 7 = 6x - x^2 - 1$$

$$2x^2 - 10x + 8 = 0$$

$$x^2 - 5x + 4 = 0$$

$$(x - 1)(x - 4) = 0$$

Either $x = 1$ or $x = 4$

When $x = 1$: $x^2 - 4x + 7 = 1^2 - 4 \times 1 + 7$

$= 4 \quad \Rightarrow \quad A(1, 4)$ □

When $x = 4$: $x^2 - 4x + 7 = 4^2 - 4 \times 4 + 7$

$= 7 \quad \Rightarrow \quad B(4, 7)$ □

[2 marks]

(ii)

$$PQ = \left(6x - x^2 - 1\right) - \left(x^2 - 4x + 7\right)$$

$$= -2x^2 + 10x - 8 \quad \text{for } 1 \leqslant x \leqslant 4$$

[2 marks]

(iii) If $y = -2x^2 + 10x - 8$ then $\dfrac{dy}{dx} = -4x + 10$

For the greatest length, $\dfrac{dy}{dx} = 0 \quad \therefore \quad -4x + 10 = 0$

$$x = \frac{5}{2}$$

Thus, greatest length of $PQ = -2 \times \left(\dfrac{5}{2}\right)^2 + 10 \times \left(\dfrac{5}{2}\right) - 8$

$$= -\frac{25}{2} + 25 - 8$$

$$= \frac{9}{2}$$

[4 marks]

(iv)

$$Area = \int_1^4 -2x^2 + 10x - 8 \; dx$$

$$= \left[-\frac{2x^3}{3} + 5x^2 - 8x\right]_1^4$$

$$= \left[-\frac{128}{3} + 80 - 32\right] - \left[-\frac{2}{3} + 5 - 8\right]$$

$$= \left[\frac{16}{3}\right] - \left[-\frac{11}{3}\right]$$

$$= 9$$

[4 marks]

Answer 5

(i)

$$y = 16x + \frac{27}{x^2}$$

$$= 16x + 27x^{-2}$$

$$\frac{dy}{dx} = 16 - 54x^{-3}$$

$$= 16 - \frac{54}{x^3}$$

$$\frac{dy}{dx} = 0 \text{ when } 16 - \frac{54}{x^3} = 0$$

$$16 = \frac{54}{x^3}$$

$$16x^3 = 54$$

$$x^3 = \frac{27}{8}$$

$$x = \frac{3}{2}$$

$$\text{When } x = \frac{3}{2}, \quad y = 16 \times \left(\frac{3}{2} \right) + \frac{27}{\left(\frac{3}{2} \right)^2}$$

$$= 24 + 12$$

$$= 36 \qquad \Rightarrow \qquad A\left(\frac{3}{2},\ 36 \right)$$

[4 marks]

(ii)

$$P(1,\ 43) \text{ and } Q(3,\ 51)$$

$$Area = \frac{1}{2}(43 + 51) \times 2 - \int_1^3 16x + 27x^{-2}\, dx$$

$$= 94 - \left[8x^2 + \frac{27x^{-1}}{(-1)} \right]_1^3$$

$$= 94 - \left[8x^2 - \frac{27}{x} \right]_1^3$$

$$= 94 - [72 - 9] + [8 - 27]$$

$$= 12$$

[6 marks]

Lesson 5

A-Level Pure Mathematics, Year 1
Additional Mathematics
Integration I

5.1 Without Limits

In this lesson the focus switches from "Definite" to "Indefinite" integration.

◊ ***Definite*** Integration questions are those that involving limits.
A calculator can be used to check the numerical answer.

◊ ***Indefinite*** Integrations are the algebra part only, without any limits.

With "The Fundamental Theorem of Calculus" in mind (that integration is the reverse of differentiation) an issue that immediately arises is that illustrated by the following two statements;

The derivative of $f(x) = 4x^3 + 7x + \mathbf{3}$ is $f'(x) = 12x^2 + 7$

The derivative of $g(x) = 4x^3 + 7x + \mathbf{8}$ is $g'(x) = 12x^2 + 7$

When going the other way, in other words trying to find $\int 12x^2 + 7\,dx$, how can it be decided if the answer has an isolated **+ 3** or an isolated **+ 8** ?

The answer is that, without further information, there is no way of knowing what the isolated number should be. Traditionally, the letter c is used to represent an isolated unknown number.
Thus,

$$\int 12x^2 + 7\ dx = 4x^3 + 7x + c$$

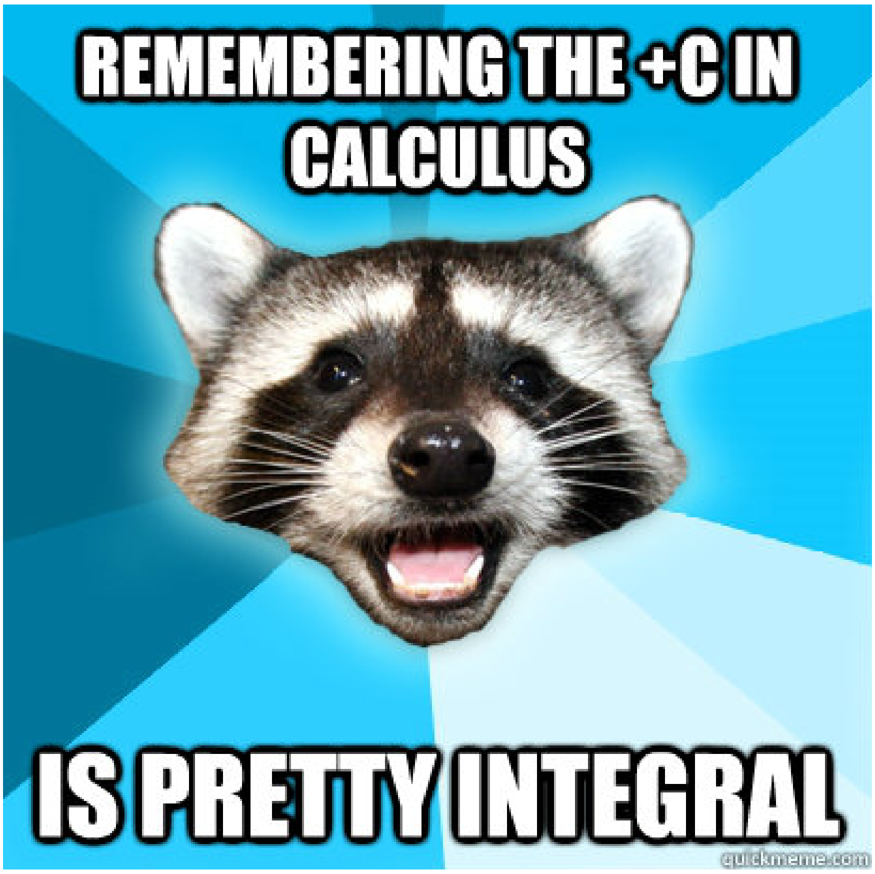
5.2 You Try

Work out the following indefinite integrations.
The answers are over the page to that you can self-check your answers.

(i) $\int 15x^4\,dx$

(ii) $14\int x^{\frac{3}{4}}\,dx$

(iii) $\frac{3}{2}\int \frac{1}{x^4}\,dx$



5.3 You Try Answers

(i) $\displaystyle \int 15x^4\,dx = \frac{15x^5}{5} + c$

$\displaystyle = 3x^5 + c$

(ii) $\displaystyle 14\int x^{\frac{3}{4}}\,dx = 14 \times \frac{4x^{\frac{7}{4}}}{7} + c$

$\displaystyle = 8x^{\frac{7}{4}} + c$

(iii) $\displaystyle \frac{3}{2}\int \frac{1}{x^4}\,dx = \frac{3}{2}\int x^{-4}\,dx$

$\displaystyle = \frac{3}{2} \times \frac{x^{-3}}{(-3)} + c$

$\displaystyle = -\frac{1}{2x^3} + c$

Tick which applies,

☐ MATHS = Mental Abuse To Homo Sapiens

☐ I forgot the **+ *c***

☐ I smashed it, 100%, give me more !

5.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 60

Question 1

Find the following integrals.
Simplify answers were possible.
Do not have any negative powers in your answers.

(i) $\int x^4 \, dx$

(ii) $\int 8x^3 \, dx$

(iii) $\int 6x^{-3} \, dx$

(iv) $\int \frac{5}{x^2} \, dx$

(v) $\int x^{\frac{3}{5}} \, dx$

(vi) $\int 12\sqrt{x} \, dx$

(vii) $\int x + 5 \, dx$

(viii) $\int \frac{x^3 + x^2}{x} \, dx$

[16 marks]

Question 2

A-Level Examination question from May 2012, Paper C1, Q1 (Edexcel)

Find

$$\int\left(6x^2 + \frac{2}{x^2} + 5\right)dx$$

giving each term in its simplest form.

[4 marks]

Question 3

A-Level Examination Question from January 2012, Q1 (Edexcel)

Given that

$$y = x^4 + 6x^{\frac{1}{2}}$$

find in their simplest form

(a) $\dfrac{dy}{dx}$

[3 marks]

(b) $\int y\ dx$

[3 marks]

Question 4

A-Level Examination Question from May 2011, Paper C1, Q2 (Edexcel)

Given that

$$y = 2x^5 + 7 + \frac{1}{x^3} \qquad x \neq 0$$

find in their simplest form

(a) $\dfrac{dy}{dx}$

[3 marks]

(b) $\int y \; dx$

[4 marks]

Question 5

A-Level Examination Question from January 2011, Q2 (Edexcel)

Find

$$\int \left(12x^5 - 3x^2 + 4x^{\frac{1}{3}} \right) dx$$

giving each term in its simplest form.

[5 marks]

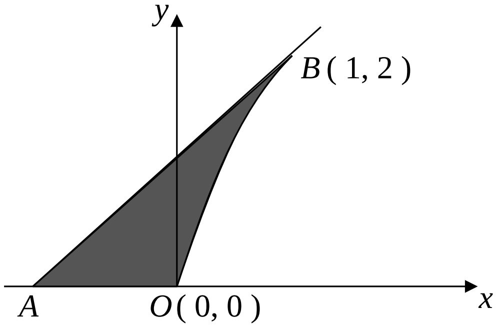
Question 6

Additional Mathematics Examination Question from June 2003, Q11 (OCR)

The shaded region on the diagram shows a boat's sail.
The units are metres and, referred to the axes shown, the coordinates of O and B are (0, 0) and (1, 2) respectively.

OB is part of the curve $y = 3x - x^2$

The tangent to the curve at B meets the x-axis at A.



Find

(i) $\dfrac{dy}{dx}$ for the curve $y = 3x - x^2$

[2 marks]

(ii) the equation of the tangent at B

[3 marks]

(iii) the coordinates of the point A

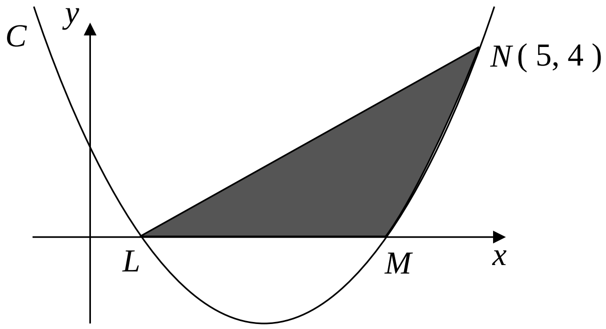
[1 mark]

(iv) the area of the sail

[6 marks]

Question 7

A-Level Examination Question from January 2010, Paper C2, Q7 (Edexcel)



The curve C has equation $y = x^2 - 5x + 4$

It cuts the x-axis at the points L and M as shown.

(a) Find the coordinates of the point L and the point M.

[2 marks]

(b) Show that the point $N\ (5, 4)$ lies on C.

[1 mark]

(c) Find $\int x^2 - 5x + 4 \ dx$

[2 marks]

The region bounded by LN, LM and the curve C is shown shaded.

(d) Use your answer to part (c) to find the exact area of the shaded region.

[5 marks]

5.5 Answers to 5.4 Exercise

Answer 1

(i) $$\int x^4 \, dx = \frac{x^5}{5} + c$$

(ii) $$\int 8x^3 \, dx = \frac{8x^4}{4} + c = 2x^4 + c$$

(iii) $$\int 6x^{-3} dx = \frac{6x^{-2}}{(-2)} + c = -\frac{3}{x^2} + c$$

(iv) $$\int \frac{5}{x^2} dx = \int 5x^{-2} \, dx = \frac{5x^{-1}}{(-1)} + c = -\frac{5}{x} + c$$

(v) $$\int x^{\frac{3}{5}} \, dx = \frac{5x^{\frac{8}{5}}}{8} + c$$

(vi) $$\int 12\sqrt{x} \, dx = \int 12x^{\frac{1}{2}} \, dx = \frac{12 \times 2x^{\frac{3}{2}}}{3} + c = 8x^{\frac{3}{2}} + c$$

(vii) $$\int x + 5 \, dx = \frac{x^2}{2} + 5x + c$$

(viii) $$\int \frac{x^3 + x^2}{x} dx = \int \frac{x^3}{x} + \frac{x^2}{x} dx = \int x^2 + x \, dx = \frac{x^3}{3} + \frac{x^2}{2} + c$$

[16 marks]

Answer 2

$$\int \left(6x^2 + \frac{2}{x^2} + 5 \right) dx = \int \left(6x^2 + 2x^{-2} + 5 \right) dx$$

$$= \frac{6x^3}{3} + \frac{2x^{-1}}{(-1)} + 5x + c$$

$$= 2x^3 - \frac{2}{x} + 5x + c$$

[4 marks]

Answer 3

(a)

$$y = x^4 + 6x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 4x^3 + 3x^{-\frac{1}{2}}$$

$$= 4x^3 + \frac{3}{\sqrt{x}}$$

[3 marks]

(b)

$$y = x^4 + 6x^{\frac{1}{2}}$$

$$\int y\,dx = \int x^4 + 6x^{\frac{1}{2}}\,dx$$

$$= \frac{x^5}{5} + \frac{6 \times 2x^{\frac{3}{2}}}{3} + c$$

$$= \frac{x^5}{5} + 4x^{\frac{3}{2}} + c$$

[3 marks]

Answer 4

(a)

$$y = 2x^5 + 7 + \frac{1}{x^3}$$

$$y = 2x^5 + 7 + x^{-3}$$

$$\frac{dy}{dx} = 10x^4 - 3x^{-4}$$

$$\frac{dy}{dx} = 10x^4 - \frac{3}{x^4}$$

[3 marks]

(b)

$$y = 2x^5 + 7 + \frac{1}{x^3}$$

$$\int y\,dx = \int 2x^5 + 7 + x^{-3}\ dx$$

$$= \frac{2x^6}{6} + 7x + \frac{x^{-2}}{(-2)} + c$$

$$= \frac{x^6}{3} + 7x - \frac{1}{2x^2} + c$$

[4 marks]

Answer 5

$$\int \left(12\,x^5 - 3\,x^2 + 4\,x^{\frac{1}{3}} \right) dx = \frac{12\,x^6}{6} - \frac{3\,x^3}{3} + \frac{4 \times 3\,x^{\frac{4}{3}}}{4} + c$$

$$= 2\,x^6 - x^3 + 3\,x^{\frac{4}{3}} + c$$

[5 marks]

Answer 6

(i)

$$y = 3x - x^2$$

$$\frac{dy}{dx} = 3 - 2x$$

[2 marks]

(ii)

Gradient at B, where $x = 1$ is thus 1

$$y = mx + c$$

$$2 = 1 \times 1 + c \Rightarrow c = 1$$

$$\therefore\ y = x + 1$$

[3 marks]

(iii)

$$A\,(-1,\ 0)$$

[1 mark]

(iv)

Area sail $=$ area Δ $-$ area under curve

$$= \frac{1}{2} \times 2 \times 2 - \int_0^1 3x - x^2\ dx$$

$$= 2 - \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= 2 - \left\{ \left[\frac{3}{2} - \frac{1}{3} \right] - [\,0\,] \right\}$$

$$= 2 - \frac{7}{6}$$

$$= \frac{5}{6}\ \ \mathrm{m}^2$$

[6 marks]

Answer 7

(a)

$$x^2 - 5x + 4 = 0$$

$$(x - 1)(x - 4) = 0$$

$$\text{Either} \quad x = 1 \quad \text{or} \quad x = 4$$

$$\text{Thus,} \; L(1, 0) \; \text{and} \; M(4, 0)$$

[2 marks]

(b)

$$\begin{aligned} y &= x^2 - 5x + 4 \\ &= 5^2 - 5 \times 5 + 4 \\ &= 4 \qquad \text{so } N(5, 4) \text{ lies on } C \end{aligned}$$

[1 mark]

(c)

$$\int (x^2 - 5x + 4)\, dx = \frac{x^3}{3} - \frac{5x^2}{2} + 4x + c$$

[2 marks]

(d)

$$\begin{aligned} Area &= \frac{1}{2} \times 4 \times 4 - \int_4^5 x^2 - 5x + 4 \, dx \\ &= 8 - \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x \right]_4^5 \\ &= 8 - \left\{ \left[\frac{5^3}{3} - \frac{5 \times 5^2}{2} + 4 \times 5 \right] - \left[\frac{4^3}{3} - \frac{5 \times 4^2}{2} + 4 \times 4 \right] \right\} \\ &= 8 - \left\{ \left[\frac{2 \times 125}{6} - \frac{3 \times 125}{6} + 20 \right] - \left[\frac{64}{3} - 40 + 16 \right] \right\} \\ &= 8 - \left\{ \left[-\frac{125}{6} + 20 \right] - \left[21\frac{1}{3} - 24 \right] \right\} \\ &= 8 - \left\{ \left[-20\frac{5}{6} + 20 \right] - \left[-2\frac{2}{3} \right] \right\} \\ &= 8 - \left\{ -\frac{5}{6} + \frac{16}{6} \right\} \\ &= 8 - \frac{11}{6} \\ &= 6\,\frac{1}{6} \end{aligned}$$

[5 marks]

Lesson 6

6.1 The Constant Of Integration

Last lesson made use of the following rule,

Indefinite Integration Power Rule

$$\text{If} \quad y = x^n \quad \text{then} \quad \int y \; dx = \frac{x^{n+1}}{n+1} + c$$

Proof

Let $f(x) = \dfrac{x^{n+1}}{n+1} + c$ where c is a constant

in which case the derivative is given by,

$$f'(x) = \frac{(n+1)\,x^{n+1-1}}{n+1}$$
$$= x^n$$

As the derivative of $\dfrac{x^{n+1}}{n+1} + c$ is x^n then, by the fundamental theorem of calculus, the integral of x^n must be $\dfrac{x^{n+1}}{n+1} + c$ $\square$

A logical question at this point would be to ask why c, often called "the constant of integration" was ignored when considering integrations with limits;

Definite Integration Power Rule

$$\text{If} \quad y = x^n \quad \text{then} \quad \int_a^b y \; dx = \left[\frac{x^{n+1}}{n+1} \right]_a^b$$

Here is the answer :

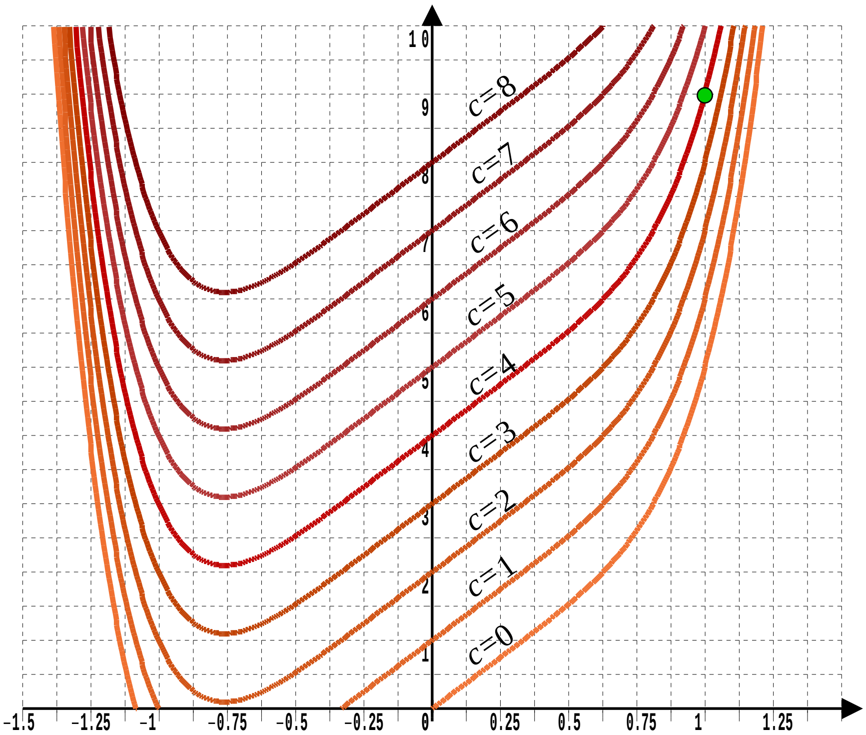
$$\left[\frac{x^{n+1}}{n+1} + c \right]_a^b = \left[\frac{b^{n+1}}{n+1} + c \right] - \left[\frac{a^{n+1}}{n+1} + c \right]$$
$$= \left[\frac{b^{n+1}}{n+1} \right] + c - \left[\frac{a^{n+1}}{n+1} \right] - c$$
$$= \left[\frac{x^{n+1}}{n+1} \right]_a^b$$

6.2 Example

The gradient function of a curve is given by $\frac{dy}{dx} = 12x^5 + 3$

Find the equation of the curve given that it passes through the point $(1, 9)$

Teaching Video : http://www.NumberWonder.co.uk/v9043/6.mp4



All these curves have gradient equation $\frac{dy}{dx} = 12x^5 + 3$

but only one passes through the point $(1, 9)$

6.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Additional Mathematics Examination Question from June 2011, Q9 (OCR)

The gradient function of a curve is given by;

$$\frac{dy}{dx} = 3x^2 - 2x + 4$$

Find the equation of the curve given that it passes through the point (2, 2)

[4 marks]

Question 2

Additional Mathematics Examination Question from June 2018, Q2 (OCR)

The gradient function of a curve is given by;

$$\frac{dy}{dx} = 2 + 2x - 3x^2$$

Find the equation of the curve given that it passes through the point (2, 3)

[4 marks]

Question 3

A-Level Examination Question from January 2012, Paper C1, Q7 (Edexcel)

A curve with equation $y = f(x)$ passes through the point $(2, 10)$

Given that

$$f'(x) = 3x^2 - 3x + 5$$

find the value of $f(1)$

[5 marks]

Question 4

A-Level Examination Question from January 2018, Paper C12, Q1 (Edexcel)

Given that $y = \dfrac{2x^{\frac{2}{3}} + 3}{6}$, $x > 0$ find, in the simplest form,

(a) $\dfrac{dy}{dx}$

[2 marks]

(b) $\int y \ dx$

[3 marks]

Question 5

A-Level Examination Question from January 2017, Paper C12, Q7(i) (Edexcel)

Find $\int \frac{2 + 4x^3}{x^2}\, dx$ giving each term in its simplest form

[4 marks]

Question 6

A-Level Examination Question from January 2013, Paper C1, Q8 (Edexcel)

$$\frac{dy}{dx} = -x^3 + \frac{4x - 5}{2x^3}, \qquad x \neq 0$$

Given that $y = 7$ at $x = 1$, find y in terms of x, giving each term in its simplest form

[6 marks]

Question 7

A-Level Examination Question from January 2014, Paper C1, Q9 (Edexcel)

A curve with equation $y = f(x)$ passes through the point $(3, 6)$

Given that $f'(x) = (x - 2)(3x + 4)$

(a) use integration to find $f(x)$

Give your answer as a polynomial in its simplest form

[5 marks]

(b) Show that $f(x) = (x - 2)^2(x + p)$, where p is a positive constant.

State the value of p

[3 marks]

(c) Sketch the graph of $y = f(x)$, showing the coordinates of any points where the curve touches or crosses the coordinate axes.

[4 marks]

6.4 Answers

6.4.1 Solution to 6.2 Example (Teaching Video)

$$y = \int 12x^5 + 3\,dx$$

$$y = \frac{12x^6}{6} + 3x + c$$

$$y = 2x^6 + 3x + c$$

Told that this curve passes through (1, 9)

$$9 = 2 \times 1^6 + 3 \times 1 + c$$

$$9 = 5 + c$$

$$c = 4$$

Thus, the equation of the curve is

$$y = 2x^6 + 3x + 4$$

6.4.2 Solutions to 6.3 Exercise

Answer 1

$$y = \int \frac{dy}{dx}\,dx$$

$$= \int 3x^2 - 2x + 4\,dx$$

$$= \frac{3x^3}{3} - \frac{2x^2}{2} + 4x + c$$

$$= x^3 - x^2 + 4x + c$$

Told that this curve passes through (2, 2)

$$2 = 2^3 - 2^2 + 4 \times 2 + c$$

$$2 = 8 - 4 + 8 + c$$

$$c = -10$$

Thus, the equation of the curve is

$$y = x^3 - x^2 + 4x - 10$$

[4 marks]

Answer 2

$$y = \int \frac{dy}{dx}\, dx$$

$$= \int 2 + 2x - 3x^2 \; dx$$

$$= 2x + \frac{2x^2}{2} - \frac{3x^3}{3} + c$$

$$= 2x + x^2 - x^3 + c$$

Told that this curve passes through (2, 3)

$$3 = 2 \times 2 + 2^2 - 2^3 + c$$

$$3 = 4 + 4 - 8 + c$$

$$c = 3$$

Thus, the equation of the curve is

$$y = 2x + x^2 - x^3 + 3$$

[4 marks]

Answer 3

$$\int f'(x) \; dx = \int 3x^2 - 3x + 5\, dx$$

$$f(x) = \frac{3x^3}{3} - \frac{3x^2}{2} + 5x + c$$

$$y = x^3 - \frac{3x^2}{2} + 5x + c$$

Told that this curve passes through (2, 10)

$$10 = 2^3 - \frac{3 \times 2^2}{2} + 5 \times 2 + c$$

$$10 = 8 - 6 + 10 + c$$

$$c = -2$$

Thus, the equation of the curve is

$$y = x^3 - \frac{3x^2}{2} + 5x - 2$$

Finally;

$$f(1) = 1^3 - \frac{3 \times 1^2}{2} + 5 \times 1 - 2$$

$$= 2.5$$

[5 marks]

Answer 4

$$y = \frac{2x^{\frac{2}{3}} + 3}{6}$$

$$= \frac{2x^{\frac{2}{3}}}{6} + \frac{3}{6}$$

$$= \frac{1}{3}x^{\frac{2}{3}} + \frac{1}{2}$$

(a)

$$\frac{dy}{dx} = \frac{1}{3} \times \frac{2}{3}x^{-\frac{1}{3}}$$

$$= \frac{2}{9x^{\frac{1}{3}}}$$

[2 marks]

(b)

$$\int y\ dx = \int \frac{x^{\frac{2}{3}}}{3} + \frac{1}{2}\ dx$$

$$= \frac{3x^{\frac{5}{3}}}{3 \times 5} + \frac{x}{2} + c$$

$$= \frac{1}{5}x^{\frac{5}{3}} + \frac{1}{2}x + c$$

[3 marks]

Answer 5

$$\int \frac{2 + 4x^3}{x^2}\ dx = \int \frac{2}{x^2} + \frac{4x^3}{x^2}\ dx$$

$$= \int 2x^{-2} + 4x\ dx$$

$$= 2 \times \frac{x^{-1}}{(-1)} + 4 \times \frac{x^2}{2} + c$$

$$= -\frac{2}{x} + 2x^2 + c$$

[4 marks]

Answer 6

$$y = \int \frac{dy}{dx}\, dx$$

$$= \int -x^3 + \frac{4x - 5}{2x^3}\, dx$$

$$= \int -x^3 + \frac{4x}{2x^3} - \frac{5}{2x^3}\, dx$$

$$= \int -x^3 + 2x^{-2} - \frac{5}{2}x^{-3}\, dx$$

$$= -\frac{x^4}{4} + \frac{2x^{-1}}{(-1)} - \frac{5}{2} \times \frac{x^{-2}}{(-2)} + c$$

$$= -\frac{x^4}{4} - \frac{2}{x} + \frac{5}{4x^2} + c$$

Using the fact that $y = 7$ when $x = 1$ gives,

$$7 = -\frac{1^4}{4} - \frac{2}{1} + \frac{5}{4 \times 1^2} + c$$

$$7 = -\frac{1}{4} - 2 + \frac{5}{4} + c$$

$$7 = 1 - 2 + c$$

$$c = 8$$

$$\therefore\ y = -\frac{x^4}{4} - \frac{2}{x} + \frac{5}{4x^2} + 8$$

[6 marks]

Answer 7

(a)

$$f(x) = \int f'(x)\, dx$$

$$= \int (x - 2)(3x + 4)\, dx$$

$$= \int 3x^2 - 2x - 8\, dx$$

$$= x^3 - x^2 - 8x + c$$

Making use of the point (3, 6) to get that,

$$6 = 3^3 - 3^2 - 8 \times 3 + c$$

$$6 = 27 - 9 - 24 + c$$

$$c = 12$$

$$\therefore\ f(x) = x^3 - x^2 - 8x + 12$$

[5 marks]

(b) Observe that, $f(2) = 2^3 - 2^2 - 8 \times 2 + 12$

$$= 8 - 4 - 16 + 12$$

$$= 0$$

$\therefore$ $(x - 2)$ is a factor of $f(x)$

$$\begin{array}{r|rrrr} & x^2 & +\ x & -\ 6 & \\ \hline x - 2 & x^3 & -\ x^2 & -\ 8x & +\ 12 \\ & x^3 & -\ 2x^2 & & \\ \hline & & x^2 & -\ 8x & \\ & & x^2 & -\ 2x & \\ \hline & & & -\ 6x & +\ 12 \\ & & & -\ 6x & +\ 12 \\ \hline & & & & 0 \end{array}$$

$0 \leftarrow$ Remainder, as expected

$$\therefore\ f(x) = (x - 2)\left(x^2 + x - 6\right)$$

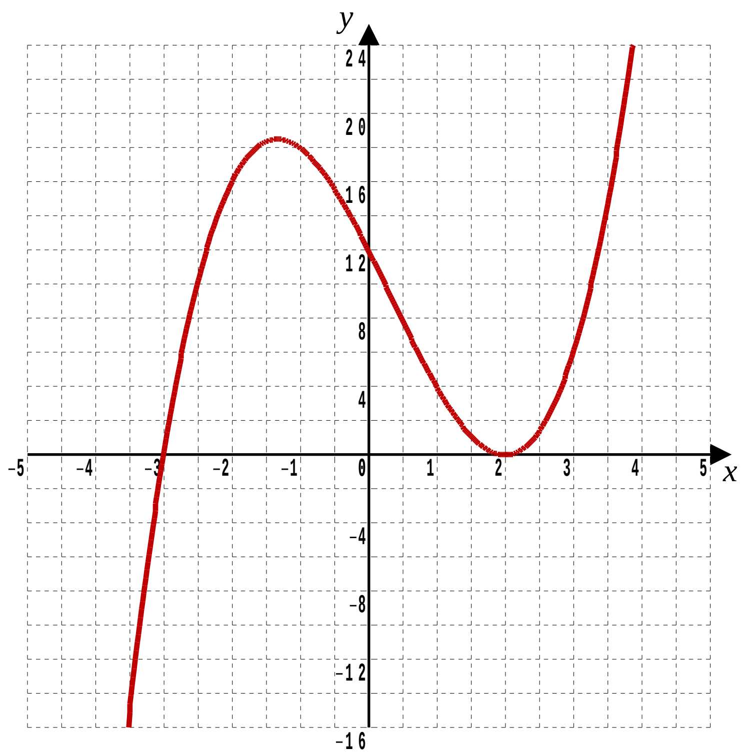
$$= (x - 2)(x + 3)(x - 2)$$

$$= (x - 2)^2 (x + 3)$$

$$\therefore\ p = 3$$

[3 marks]

(c)



[4 marks]

Lesson 7

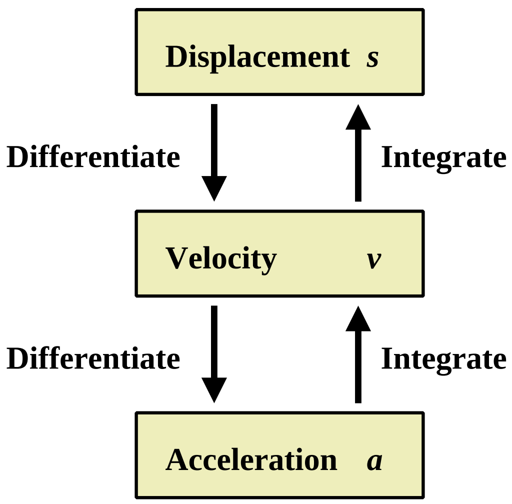
A-Level Pure Mathematics, Year 1
Additional Mathematics
Integration I

7.1 Calculus Kinematics

The calculus of Differentiation and Integration has a straightforward application in Kinematics, the study of motion. It allows the skilled user to hop between the quantities of displacement, *s*, velocity, *v*, and acceleration, *a*.

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt}$$

$$s = \int v\,dt \qquad v = \int a\,dt$$



7.2 Example

A particle moves in a straight line.
Its velocity, v m s^{-1}, t seconds after passing a point O is given by the equation

$$v = 8 + \frac{3t^2}{4}$$

(i) Show that the particle's acceleration when $t = 5$ seconds is 7.5 m s^{-2}

[3 marks]

(ii) Find the distance travelled between the times $t = 1$ and $t = 4$

[4 marks]

7.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 80

Question 1

Additional Mathematics Examination Question from June 2007, Q2 (OCR)

A particle moves in a straight line.

Its velocity, v m s^{-1}, t seconds after passing a point O is given by the equation

$$v = 6 + 3t^2$$

Find the distance travelled between the times $t = 1$ and $t = 3$

[4 marks]

Question 2

Additional Mathematics Examination Question from June 2008, Q6 (OCR)

A speedboat accelerates from rest so that t seconds after starting its velocity, in m s^{-1}, is given by the formula

$$v = 0.36\,t^2 - 0.024\,t^3$$

(i) Find the acceleration at time t

[3 marks]

(ii) Find the distance travelled in the first 10 seconds.

[4 marks]

Question 3

Additional Mathematics Examination Question from June 2010, Q8 (OCR)

A train moves between two stations, taking 5 minutes for the journey.
The velocity of the train may be modelled by the equation

$$v = 60\left(t^4 - 10t^3 + 25t^2\right)$$

where v is measured in metres per minute and t is measured in minutes.

Calculate the distance between the two stations.

[5 marks]

Question 4

Additional Mathematics Examination Question from June 2005, Q8 (OCR)

A car moves in a straight line.
Its velocity in metres per second, t seconds after passing a point A, is given by

$$v = 27 - \frac{1}{8}t^3$$

It comes to rest at a point B.

(i) Show that the car is at B when $t = 6$

[1 mark]

(ii) Find the distance AB

[5 marks]

Question 5

Additional Mathematics Examination Question from June 2009, Q9 (OCR)

A car accelerates from rest.

At time t seconds, its acceleration is given (in metres per second squared) by

$$a = 4 - 0.2t$$

until $t = 20$

(i) Find the velocity after 5 seconds

[3 marks]

(ii) What is happening to the velocity at $t = 20$?

[1 mark]

(iii) Find the distance travelled in the first 20 seconds

[3 marks]

Question 6

GCSE Examination Question from May 2004, Q13 (OCR)

A bod is moving in a straight line which passes through a fixed point *O*.

The displacement, s metres, of the body from *O* at time t seconds is given by

$$s = t^3 + 4t^2 - 5t$$

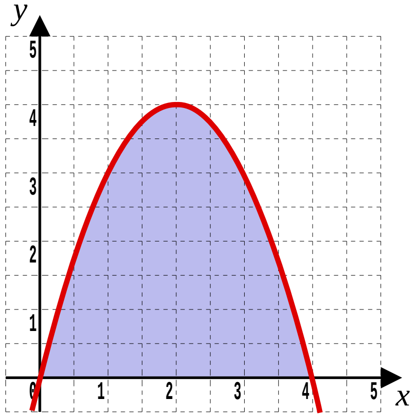
(a) Find an expression for the velocity, v m s^{-1}, at time t seconds

[2 marks]

(b) Find the acceleration after 2 seconds

[2 marks]

Question 7



A cyclist is waiting at a red traffic light.
When the light turns green, he travels along a straight road before stopping at a second red traffic light.
The velocity-time graph shows the motion of the cyclist between the two sets of lights.
The units of velocity are metres per second, those of time are seconds, and those of distance are metres.
The curve has equation

$$v = t(4 - t)$$

(i) Use calculus to determine the shaded area.

[3 marks]

(ii) What does the shaded area tell you about the cyclist's journey ?

[1 mark]

(iii) Determine the gradient of the curve at the point (1, 3)

[2 marks]

(iv) What does the gradient at (1, 3) tell you about the cyclist's journey ?

[1 mark]

Question 8

Additional Mathematics Specimen Examination Question from 2002, Q12 (OCR)

A body is falling through a liquid, and the distance fallen is modelled by the formula

$$s = 48t - t^3$$

until it comes to rest.

In the model; s centimetres is the distance fallen
t seconds is the time

(a) Find

(i) the velocity when $t = 1$

[2 marks]

(ii) the initial velocity

[1 mark]

(iii) the acceleration when $t = 1$

[3 marks]

(iv) the time when the body comes to rest

[2 marks]

(v) the distance fallen when the body comes to rest

[2 marks]

(b) Sketch the velocity-time graph for the period of time until the body comes to rest.

[2 marks]

Question 9

Additional Mathematics Examination Question from June 2013, Q12 (OCR)

An object sinks through a thick liquid such that at time t seconds after being released on the surface the depth, s metres is given by

$$s = 4t^2 - \frac{2t^3}{3} \qquad for\ 0 \leqslant t \leqslant 4$$

(a) Find the formula for the velocity, v metres per second, t seconds after being released. Hence show that the object stops sinking when $t = 4$.

[4 marks]

(b) Find

(i) the acceleration of the object when it's released on the surface of the liquid

[4 marks]

(ii) the greatest depth of the object

[2 marks]

(c) Sketch the velocity-time & the acceleration-time graphs

[2 marks]

Question 10

Additional Mathematics Examination Question from June 2004, Q13 (OCR)

I regularly travel a journey of 200 kilometres.

When I travel by day I average v kilometres per hour. When I travel at night the traffic is not so bad, so I can average 20 kilometres per hour faster. This means that I am able to complete the journey in 50 minutes less time.

(i) Write down expressions for the journey times during day and at night.

[2 marks]

(ii) Hence form an equation in v and show that it simplifies to

$$v^2 + 20v - 4800 = 0$$

[5 marks]

(iii) Hence find the times it takes me to complete the journey during the day and at night.

[5 marks]

Question 11

GCSE Examination Question from November 2009, Q23 (Edexcel)

In a race, Paula runs 25 laps of a track.

Each lap of the track is 400 m, correct to the nearest metre.

Paula's average speed is 5.0 m s^{-1}, correct to one decimal place.

Calculate the upper bound for the time that Paula takes to run the race.

Give your answer in minutes and seconds, correct to the nearest second.

[4 marks]

7.4 Answers

7.4.1 Solution to 7.2 Example

(i)

$$a = \frac{dv}{dt}$$

$$= \frac{d}{dt}\left(8 + \frac{3t^2}{4}\right)$$

$$= \frac{3 \times 2t^1}{4}$$

$$= \frac{3t}{2}$$

When $t = 5$, $a = 7.5 \text{ m s}^{-2}$

[3 marks]

(ii)

$$s = \int v \, dt$$

$$= \int_1^4 8 + \frac{3t^2}{4} \, dt$$

$$= \left[8t + \frac{3t^3}{3 \times 4} \right]_1^4$$

$$= \left[8t + \frac{t^3}{4} \right]_1^4$$

$$= \{ [32 + 16] - [8 + 0.25] \}$$

$$= 48 - 8.25$$

$$= 39.75 \text{ m}$$

[4 marks]

7.4.2 Solution to 7.3 Exercise

Answer 1

$$s = \int v \, dt$$

$$= \int_1^3 6 + 3t^2 \, dt$$

$$= \left[6t + t^3 \right]_1^3$$

$$= [18 + 27] - [6 + 1]$$

$$= 38 \text{ metres}$$

[4 marks]

Answer 2

(i)

$$a = \frac{dv}{dt}$$

$$= 0.72\,t - 0.072\,t^2$$

[3 marks]

(ii)

$$s = \int v \;dt$$

$$= \int_0^{10} 0.36\,t^2 - 0.024\,t^3 \;dt$$

$$= \Big[\, 0.12\,t^3 - 0.006\,t^4 \,\Big]_0^{10}$$

$$= [\,120 - 60\,] - [\,0 - 0\,]$$

$$= 60 \text{ metres}$$

[4 marks]

A flaw in this examination question is that the correct answer can be obtained from the incorrect method of

$$s = \left(\frac{u + v}{2}\right)t = \left(\frac{0 + 12}{2}\right) \times 10$$

Answer 3

$$s = \int v \;dt$$

$$= 60 \int_0^{5} t^4 - 10\,t^3 + 25\,t^2 \;dt$$

$$= 60 \times \left[\frac{t^5}{5} - \frac{5\,t^4}{2} + \frac{25\,t^3}{3}\right]_0^5$$

$$= 6250 \text{ metres}$$

[5 marks]

Answer 4

(i)

$$v = 0$$

$$27 - \frac{1}{8}t^3 = 0$$

$$\frac{1}{8}t^3 = 27$$

$$t^3 = 27 \times 8$$

$$t = 3 \times 2$$

$$= 6 \qquad \text{as expected}$$

[1 mark]

(ii)

$$s = \int v \ dt$$

$$= \int_0^6 27 - \frac{1}{8} t^3 \, dt$$

$$= \left[27t - \frac{t^4}{32} \right]_0^6$$

$$= [\ 162 - 40.5\ ] - [0 - 0\ ]$$

$$= 121.5 \text{ metres}$$

[5 marks]

Answer 5

(i)

$$v = \int a \ dt$$

$$= \int 4 - 0.2\, t \ dt$$

$$= 4t - 0.1\, t^2 + c$$

As $v = 0$ when $t = 0$, $c = 0$

$$\therefore \ v = 4t - 0.1\, t^2$$

When $t = 5$, $v = 17.5 \text{ m s}^{-1}$

[3 marks]

(ii) When $t = 20$, $a = 0$

Also, the velocity-time graph is of an inverted parabola, ∩, because of the minus t^2 in the velocity equation.

Thus at $t = 20$, the car is at its maximum velocity.

[1 mark]

(iii)

$$s = \int v \ dt$$

$$= \int_0^{20} 4t - 0.1\, t^2 \, dt$$

$$= \left[2\, t^2 - 0.1 \times \frac{t^3}{3} \right]_0^{20}$$

$$= \left[800 - \frac{800}{3} \right] - [0 - 0\ ]$$

$$= 533 \text{ metres}$$

[3 marks]

Answer 6

(a)

$$v = \frac{ds}{dt}$$

$$= 3t^2 + 8t - 5$$

[2 marks]

(b)

$$a = \frac{dv}{dt}$$

$$= 6t + 8$$

Thus, when $t = 2$, $a = 20 \text{ m s}^{-2}$

[2 marks]

Answer 7

(i)

$$s = \int v \; dt$$

$$= \int_0^4 t\,(4 - t)\; dt$$

$$= \int_0^4 4t - t^2 \; dt$$

$$= \left[2t^2 - \frac{t^3}{3} \right]_0^4$$

$$= 32 - \frac{64}{3}$$

$$= 32 - 21\frac{1}{3}$$

$$= 10\frac{2}{3}$$

[3 marks]

(ii) The distance between the lights is $10\frac{2}{3}$ metres.

[1 mark]

(iii)

$$a = \frac{dv}{dt}$$

$$= \frac{d}{dt}\,(4t - t^2)$$

$$= 4 - 2t$$

When $t = 1$, the gradient is 2

[2 marks]

(iv) The acceleration when $t = 1$ second was 2 m s^{-2}

[1 mark]

Answer 8

$$v = \frac{ds}{dt}$$

$$= 48 - 3t^2$$

(i) When $t = 1$, $v = 45$ cm s^{-1}

[2 marks]

(ii) When $t = 0$, $v = 48$ cm s^{-1}

[1 mark]

$$a = \frac{dv}{dt}$$

$$= -6t$$

(iii) When $t = 1$, $a = -6$ cm s^{-2}

[3 marks]

(iv) Come to rest when,

$$v = 0$$

$$48 - 3t^2 = 0$$

$$16 - t^2 = 0$$

$$t = 4 \text{ seconds} \qquad (\text{reject } t = -4)$$

[2 marks]

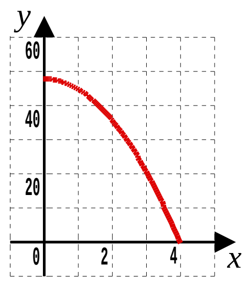
(v)

$$s = 48t - t^3$$

When $t = 4$, $s = 128$ cm

[2 marks]

(b)



[2 marks]

Answer 9

(a)

$$v = \frac{ds}{dt} = 8t - 2t^2$$

Object is not moving when, $v = 0$

$$8t - 2t^2 = 0$$

$$2t(4 - t) = 0$$

$$\therefore\ v = 0 \text{ when } t = 0 \text{ or } t = 4$$

So, the object started sinking at $t = 0$ and stopped sinking when $t = 4$

[4 marks]

(b) **(i)**

$$a = \frac{dv}{dt} = 8 - 4t$$

When $t = 0$, $a = 8$ m s^{-2} is the initial acceleration

[4 marks]

(ii) Greatest depth will be when $t = 4$ seconds

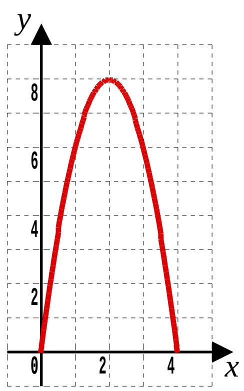
$$s = 4t^2 - \frac{2t^3}{3}$$

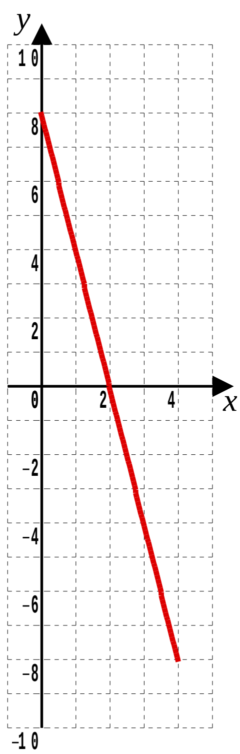
$$= 64 - \frac{128}{3}$$

$$= \frac{64}{3} = 21.3 \text{ metres}$$

[2 marks]

(c)





Velocity-time (left) and acceleration-time (right) graphs

[2 marks]

Answer 10

(i)

$$time = \frac{distance}{average\ speed}$$

$$T_{DAY} = \frac{200}{v} \qquad \& \qquad T_{NIGHT} = \frac{200}{v + 20}$$

[2 marks]

(ii)

$$T_{NIGHT} = T_{DAY} - \frac{50}{60}$$

$$\frac{200}{v + 20} = \frac{200}{v} - \frac{5}{6}$$

$$200\,v = 200\,(v + 20) - \frac{5}{6} \times v \times (v + 20)$$

$$1200\,v = 1200\,(v + 20) - 5\,v\,(v + 20)$$

$$1200\,v = 1200\,v + 24000 - 5\,v^2 - 100\,v$$

$$5\,v^2 + 100\,v + 24000 = 0$$

$$v^2 + 20\,v + 4800 = 0 \qquad \text{as expected}$$

[5 marks]

(iii)

$$(v + 80)\,(v - 60) = 0$$

Either $v = -80$ (reject) or $v = 60$ km h^{-1}

Thus, $$T_{DAY} = \frac{200}{60}$$

$$= \frac{10}{3} = 3 \text{ hours and 20 minutes}$$

$$T_{NIGHT} = \frac{200}{60 + 20}$$

$$= \frac{5}{2} = 2 \text{ hours and 30 minutes}$$

[5 marks]

Answer 11

$$distance = average\ speed \times time$$

$$time = \frac{25 \times 400.5}{4.95}$$

$$= 2022.7272... \qquad \text{33 minutes and 43 seconds}$$

[4 marks]

8.1 TEST

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 66

Question 1

Show that the following integral has an exact value of 100

$$\int_1^3 12x^2 - 2 \; dx$$

[4 marks]

Question 2

Find this indefinite integral, giving your answer in a simplified form.

$$\int \frac{9x^5}{2} \, dx$$

[3 marks]

Question 3

A-Level Examination Question from May 2010, Paper C1, Q2 (Edexcel)

Find

$$\int \left(8x^3 + 6x^{\frac{1}{2}} - 5\right)\, dx$$

giving each term in its simplest form

[4 marks]

Question 4

A-Level Examination Question from June 2009, Paper C1, Q3 (Edexcel)

Given that

$$y = 2x^3 + \frac{3}{x^2}, \qquad x \neq 0$$

Find **(a)** $\dfrac{dy}{dx}$

[3 marks]

(b) $\int y\, dx$

simplifying each term

[3 marks]

Question 5

Additional Mathematics Examination Question from June 2006, Q6 (OCR)

A curve has gradient given by

$$\frac{dy}{dx} = 2x + 2$$

The curve passes through the point (3, 0)

Find the equation of the curve.

[5 marks]

Question 6

A-Level Examination Question from May 2006, Paper C2, Q2 (Edexcel)

Use calculus to find the exact value of

$$\int_1^2 \left(3x^2 + 5 + \frac{4}{x^2} \right) dx$$

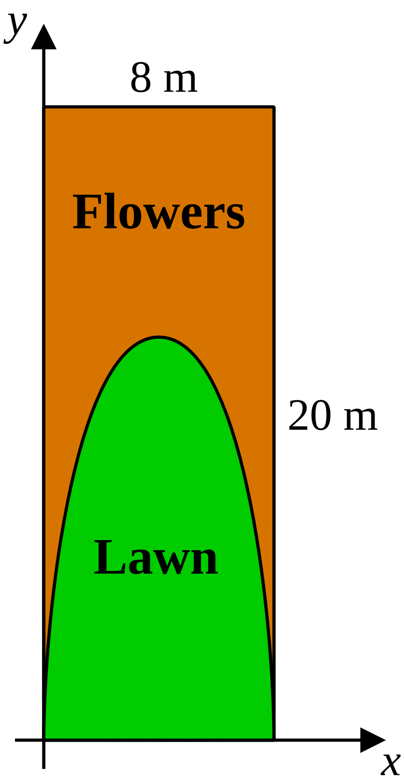
[5 marks]

Question 7

Additional Mathematics Examination Question from June 2013, Q8 (OCR)

A mathematical gardener has a garden which is rectangular in shape measuring 20 metres by 8 metres. He wishes to arrange the garden so that approximately half of it is lawn and the rest flower bed. He sets up a coordinate system as shown in the diagram below and maps out the graph of the curve,

$$y = 8x - x^2$$

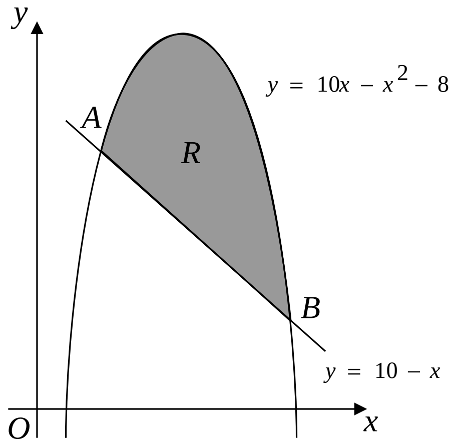


Show that the area of the lawn is approximately 53% of the total area.

[6 marks]

Question 8

A-Level Examination Question from May 2012, Paper C2, Q5 (Edexcel)



The diagram shows the line with equation $y = 10 - x$ and the curve with equation $y = 10x - x^2 - 8$

The line and the curve intersect at the points *A* and *B*, and *O* is the origin.

(a) Calculate the coordinates of *A* and the coordinates of *B*.

[5 marks]

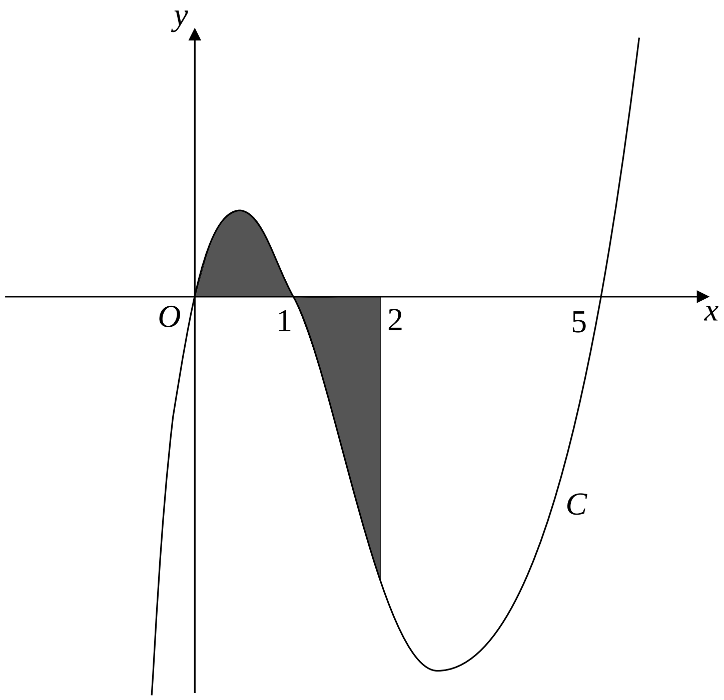
The shaded area *R* is bounded by the line and the curve as shown.

(b) Calculate the exact area of *R*.

[7 marks]

Question 9

A-Level Examination Question from January 2007, Paper C2, Q7 (Edexcel)



The sketch shows part of the curve C with equation

$$y = x(x-1)(x-5)$$

Use calculus to find the total area of the finite region, shown shaded, that is between $x = 0$ and $x = 2$ and is bounded by C, the x-axis and the line $x = 2$.

[9 marks]

Question 10

Additional Mathematics Examination Question from June 2008, Q10 (OCR)

Simon and Gavin drive a distance of 140 km along a motorway, both at constant speed. Simon drives at 5 km per hour faster than Gavin.

Let Gavin's speed be v km per hour.

(i) Write down expressions in terms of v for the times, in hours, taken by Gavin and Simon.

[2 marks]

Simon completes the journey in 15 minutes less than Gavin.

(ii) Explain why

$$\frac{140}{v} - \frac{140}{v + 5} = \frac{1}{4}$$

and show that this equation reduces to the equation

$$v^2 + 5v - 2800 = 0$$

[5 marks]

(iii) Solve this equation to find v and hence find the times taken by Simon and Gavin. Give your answers correct to the nearest minute.

[5 marks]

8.2 Answers

Answer 1

$$\int_1^3 12x^2 - 2 \ dx = \left[4x^3 - 2x \right]_1^3$$

$$= (4 \times 27 - 2 \times 3) - (4 \times 1 - 2 \times 1)$$

$$= 102 - 2$$

$$= 100 \qquad \square$$

[4 marks]

Answer 2

$$\int \frac{9x^5}{2} dx = \frac{9x^6}{2 \times 6} + c$$

$$= \frac{3x^6}{4} + c$$

[3 marks]

Answer 3

$$\int (8x^3 + 6x^{\frac{1}{2}} - 5) \ dx = 2x^4 + 4x^{\frac{3}{2}} - 5x + c$$

[4 marks]

Answer 4

$$y = 2x^3 + \frac{3}{x^2}$$

$$= 2x^3 + 3x^{-2}$$

(a)

$$\frac{dy}{dx} = 6x^2 - 6x^{-3}$$

[3 marks]

(b)

$$\int y \, dx = \frac{x^4}{2} - 3x^{-1} + c$$

$$= \frac{x^4}{2} - \frac{3}{x} + c$$

[3 marks]

Answer 5

$$y = \int \frac{dy}{dx}\, dx$$

$$= \int 2x + 2 \ dx$$

$$= x^2 + 2x + c$$

Told that this curve passes through (3, 0)

$$0 = 3^2 + 2 \times 3 + c$$

$$0 = 15 + c$$

$$c = -15$$

Thus, the equation of the curve is

$$y = x^2 + 2x - 15$$

[5 marks]

Answer 6

$$\int_1^2 \left(3x^2 + 5 + \frac{4}{x^2} \right) dx = \int_1^2 \left(3x^2 + 5 + 4x^{-2} \right) dx$$

$$= \left[x^3 + 5x - 4x^{-1} \right]_1^2$$

$$= \left[x^3 + 5x - \frac{4}{x} \right]_1^2$$

$$= (8 + 10 - 2) - (1 + 5 - 4)$$

$$= 16 - 2$$

$$= 14$$

[5 marks]

Answer 7

Area of the lawn is given by,

$$\int_0^8 8x - x^2 \ dx = \left[4x^2 - \frac{x^3}{3} \right]_0^8$$

$$= \left[4 \times 8^2 - \frac{8^3}{3} \right]$$

$$= \frac{256}{3} \text{ m}^2$$

Area whole garden is $20 \times 8 = 160$ m^2

Percentage that's lawn is $\frac{256}{3 \times 160} \times 100 = 53\frac{1}{3}$ □

[6 marks]

Answer 8

(a) To find the points of intersection of line and curve,

$$10 - x = 10x - x^2 - 8$$

$$x^2 - 11x + 18 = 0$$

$$(x - 2)(x - 9) = 0$$

$$\text{Either } x = 2 \text{ or } x = 9$$

$$\text{That is, } A(2,\ 8),\ B(9,\ 1)$$

[5 marks]

(b)

$$\begin{aligned}
Area\ R &= \int_2^9 10x - x^2 - 8\ dx - \int_2^9 10 - x\ dx \\
&= \int_2^9 11x - x^2 - 18\ dx \\
&= \left[\frac{11x^2}{2} - \frac{x^3}{3} - 18x \right]_2^9 \\
&= \left[\frac{891}{2} - 243 - 162 \right] - \left[22 - \frac{8}{3} - 36 \right] \\
&= \left[\frac{81}{2} \right] - \left[-\frac{50}{3} \right] \\
&= \frac{343}{6} = 57\frac{1}{6}
\end{aligned}$$

[7 marks]

Answer 9

$$\int_0^1 x(x-1)(x-5)\,dx = \int_0^1 x^3 - 6x^2 + 5x\,dx$$

$$= \left[\frac{x^4}{4} - 2x^3 + \frac{5x^2}{2}\right]_0^1$$

$$= \left[\frac{1}{4} - 2 + \frac{5}{2}\right]$$

$$= \frac{3}{4}$$

$$\int_1^2 x(x-1)(x-5)\,dx = \int_1^2 x^3 - 6x^2 + 5x\,dx$$

$$= \left[\frac{x^4}{4} - 2x^3 + \frac{5x^2}{2}\right]_1^2$$

$$= [4 - 16 + 10] - \left[\frac{1}{4} - 2 + \frac{5}{2}\right]$$

$$= -2 - \frac{3}{4}$$

$$= -\frac{11}{4}$$

$$= -2\frac{3}{4}$$

Thus, the area shaded will be,

$$\frac{3}{4} + \frac{11}{4} = \frac{7}{2}$$

[9 marks]

Answer 10

(i)

$$time = \frac{distance}{speed}$$

$$time_{GAVIN} = \frac{140}{v} \qquad time_{SIMON} = \frac{140}{v + 5}$$

[2 marks]

(ii) As 15 minutes is quarter of an hour,

Gavin's time minus Simon's time will be one quarter

$$\frac{140}{v} - \frac{140}{v + 5} = \frac{1}{4}$$

Multiply through by $4v(v + 5)$

$$4(v + 5)\,140 - 4v\,140 = v(v + 5)$$

$$560v + 2800 - 560v = v^2 + 5v$$

$$v^2 + 5v - 2800 = 0$$

[5 marks]

(iii)

$$v = \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times (-2800)}}{2 \times 1}$$

$$= \frac{-5 \pm \sqrt{25 + 11200}}{2}$$

$$= 50.474, \quad \text{(reject } -55.474\text{)}$$

$$\therefore\ time_{GAVIN} = \frac{140}{50.474} = \text{2 hours 46 minutes}$$

$$time_{SIMON} = \frac{140}{55.474} = \text{2 hours 31 minutes}$$

[5 marks]