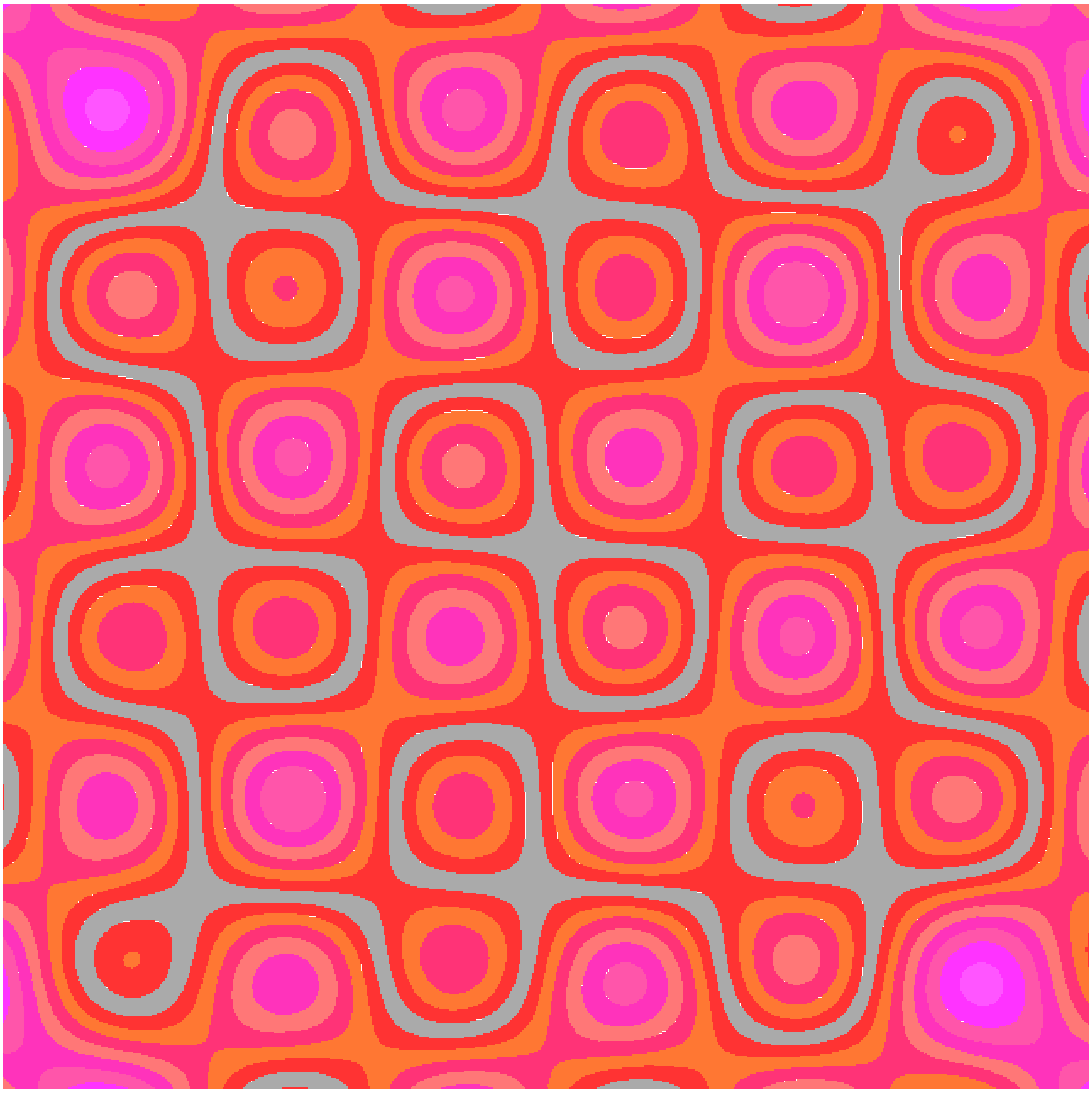
Trigonometry IV

Additional Mathematics
A-Level Pure Mathematics : Year 1

Equations and Identities



Art from the Trigonometric Equation

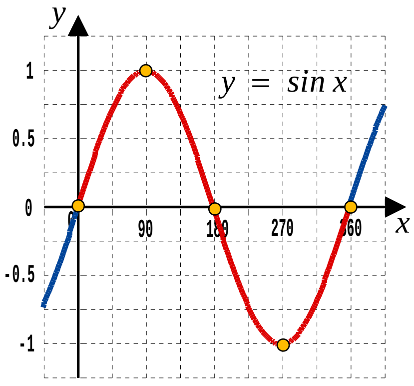
$$\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) + \left(\frac{y}{2}\right)\sin\left(\frac{y}{2}\right) + 8\sin x \sin y = 0$$

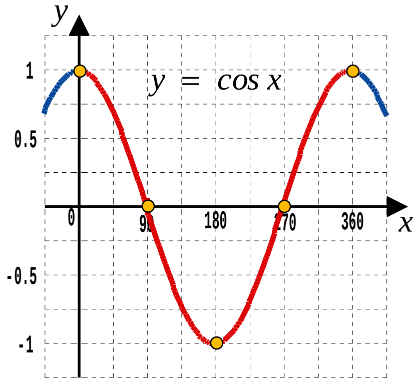
Lesson 1

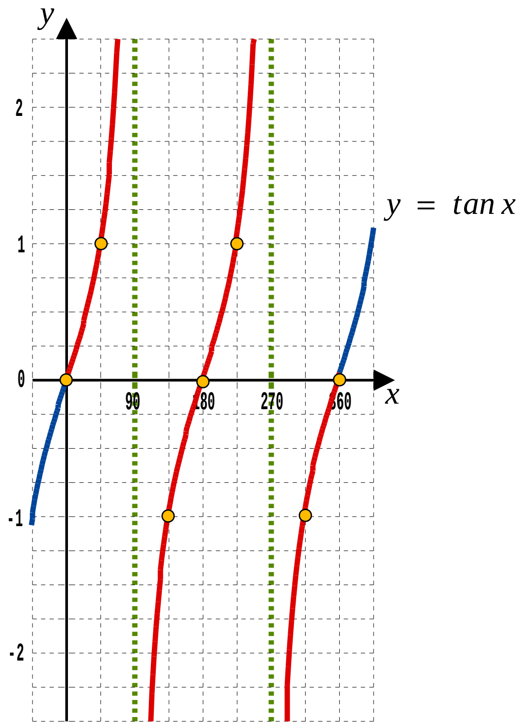
Additional Mathematics
A-Level Pure Mathematics : Year 1
Trigonometry IV

1.1 Graph Assisted Equation Solving

To solve equations that involve *sin*, *cos* and *tan* a calculator and, from memory, a sketch of the appropriate graph is required. Here are the graphs needed "in mind".







Teaching Video : http://www.NumberWonder.co.uk/v9044/1a.mp4

← Watch the video which talks through these essential graphs

1.2 Example

The Question : Solve the equation $\cos x \;=\; -0.283 \qquad 0° \leqslant x \leqslant 360°$

The Solution : It's tempting to start by using your calculator to get that

$$x = \arccos(-0.283)$$
$$= 106.4°$$

This is a solution, but it's not ALL of the solutions. Far better to begin by always getting the acute working angle by initially ignoring the minus sign.

Thus; $\cos x = 0.283$

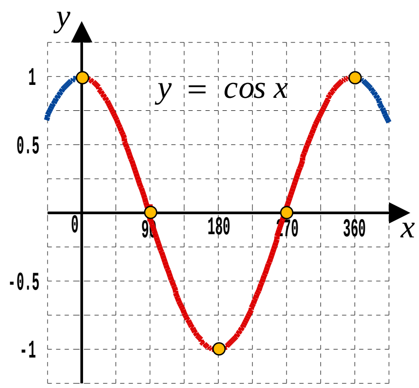
$x = \arccos 0.283$

$x = 73.6°$ ← The working (or principle) angle

The solutions sought in the range $0° \leqslant x \leqslant 360°$ will be two of

$x \qquad 180 - x \qquad 180 + x \qquad$ or $\qquad 360 - x$

Now sketch the appropriate graph, either for *sin*, *cos* or *tan*
In this case we want $y = \cos x$



From the sketch graph, solution sought are 180 – 73.6 and 180 + 73.6

$$\therefore\; x = 106.4°,\;\; 253.6°$$

[6 marks]

Teaching Video : http://www.NumberWonder.co.uk/v9044/1b.mp4

← Watch the Video which talks through the example's solution

1.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

(a) Find both solutions to the equation $\cos x = 0.417$ for $0° \leqslant x \leqslant 360°$
Your solution should include a sketch of the graph of $y = \cos x$

[4 marks]

(b) Find both solutions to the equation $\cos x = -0.728$ for $0° \leqslant x \leqslant 360°$

[4 marks]

Question 2

(i) Sketch the graph of $y = sin\, x$ for $0° \leqslant x \leqslant 360°$

[2 marks]

(ii) With the help of your part (i) sketch, find all solutions in the interval $0° \leqslant x \leqslant 360°$ of the following equations;

(a) $sin\, x = 0.622$

[2 marks]

(b) $sin\, x = -0.383$

[2 marks]

(c) $5\, sin\, x = 3.445$

[2 marks]

Question 3

(i) Sketch the graph of $y = tan\ x$ for $0° \leqslant x \leqslant 360°$

[2 marks]

(ii) With the help of your part (i) sketch, find all solutions in the interval $0° \leqslant x \leqslant 360°$ of the following equations;

(a) $tan\ x = 4.718$

[2 marks]

(b) $tan\ x = -1.383$

[2 marks]

(c) $11\ tan\ x = 8$

[2 marks]

Question 4

Additional Mathematics Examination Question from June 2007, Q4 (OCR)

Find all the values of x in the range $0° < x < 360°$ that satisfy

$$sin\,x = -4\,cos\,x$$

Hint : Divide both sides by $cos\,x$. It is OK to do this because $cos\,x = 0$ is not a solution so we're not dividing by zero. Then use the fact $\frac{sin\,x}{cos\,x} = tan\,x$.

[6 marks]

1.4 Answers to 1.3 Exercise

Additional Mathematics
A-Level Pure Mathematics : Year 1
Trigonometry IV

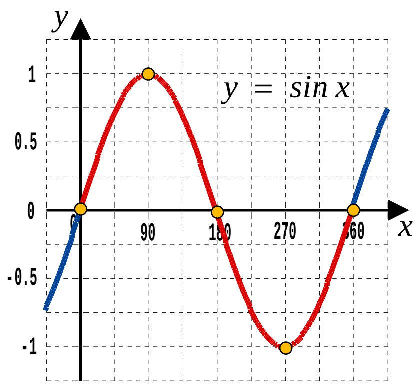
Answer 1

(a) 65.4°, 294.6° **(b)** 136.7°, 223.3°

[4, 4 marks]

Answer 2

(i)

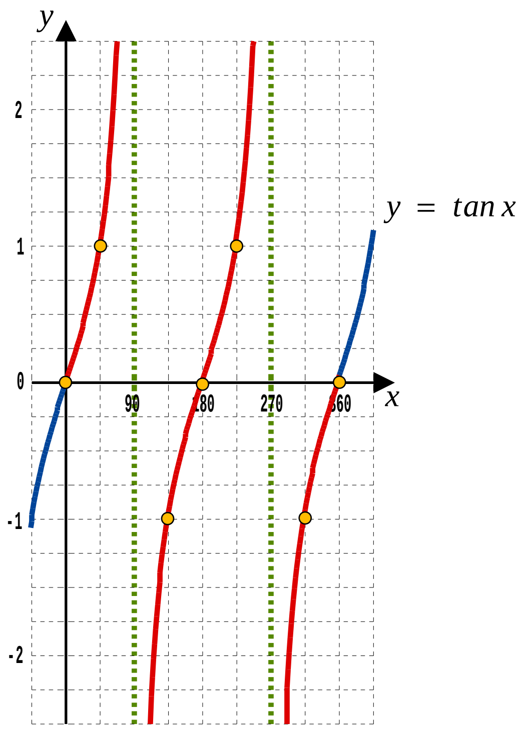


(ii) **(a)** 38.5°, 141.5° **(b)** 202.5°, 337.5° **(c)** 43.6°, 136.4°

[2, 2, 2, 2 marks]

Answer 3

(i)



(ii) **(a)** 78.0°, 258.0° **(b)** 125.9°, 305.9° **(c)** 36.0°, 216.0°

[2, 2, 2, 2 marks]

Answer 4

$$\sin x = -4\cos x$$

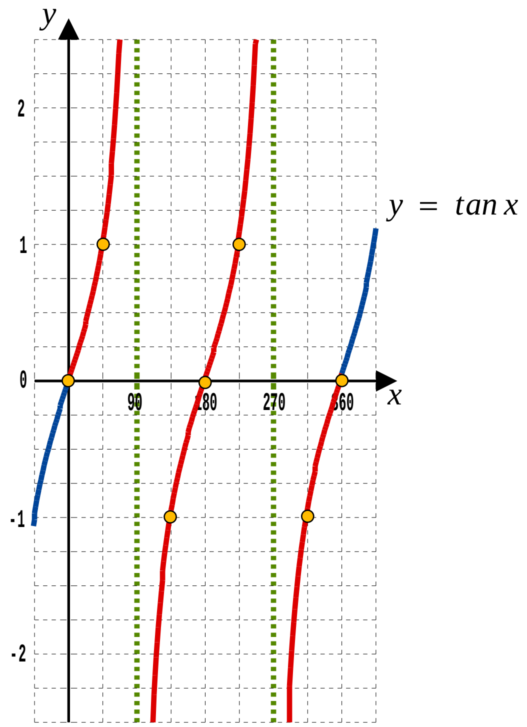
Divide both sides by $\cos x$

(which is OK as $\cos x = 0$ is not a solution)

$$\frac{\sin x}{\cos x} = -4$$

$$\tan x = -4$$

$$x = \arctan(-4)$$



$$x = 104°, \; 284°$$

[6 marks]

Lesson 2

2.1 In Disguise

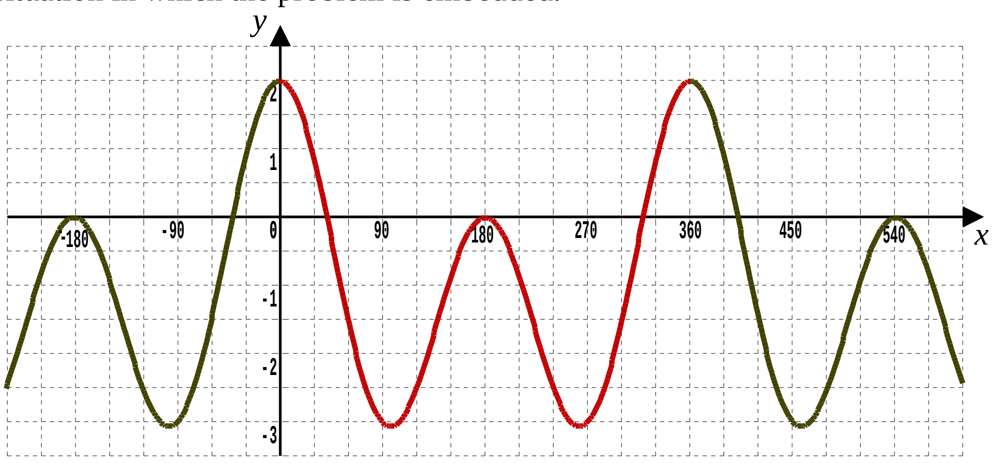
Faced with an unfamiliar mathematical situation, a technique frequently deployed is to seek a way to transform it into a more familiar situation for which a method of solution is already known.

Here is a seemingly more complicated trigonometric equation to solve;

$$4\cos^2 x + \cos x - 3 = 0 \qquad \text{for} \quad 0° \leqslant x \leqslant 360°$$

How might this be handled ?

Using a graphics calculator or a graph plotter would give some idea of the situation in which the problem is embedded.

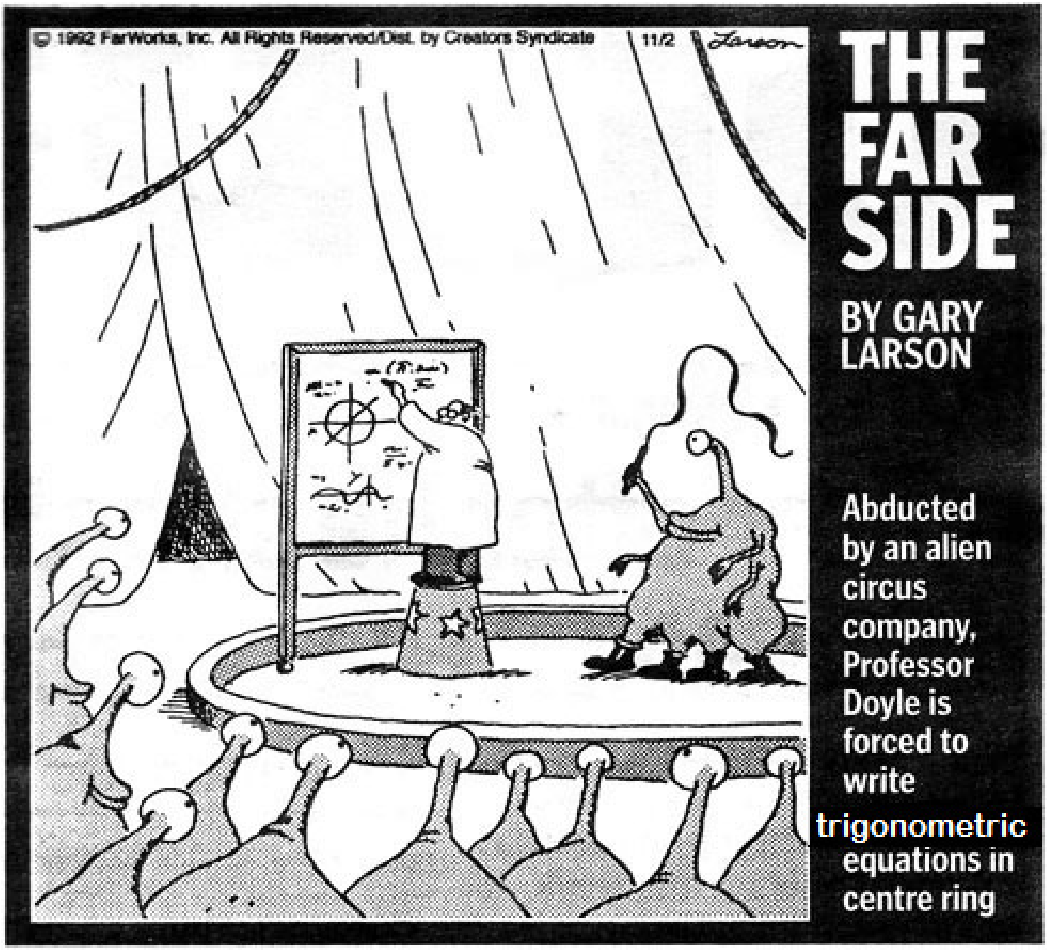


Where does this (red piece of) graph have zero height ?
It would seem that there is one solution around 45°, another around 315°, and possibly one or two more close to 180° although it's not clear if the curve gets to or crosses the *x*-axis near 180°.

Of course, what is sought is a mathematical method of solving the trigonometric equation which the teaching video will now reveal,

Teaching Video : http://www.NumberWonder.co.uk/v9044/2.mp4

[6 marks]



2.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

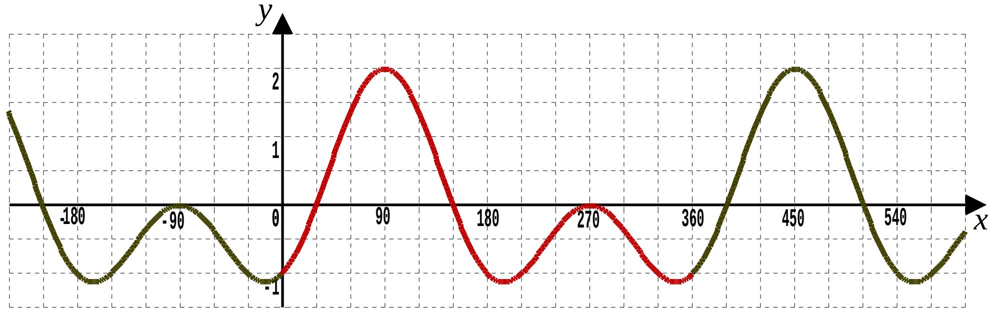
Marks Available : 30

Question 1

The graph is of the trigonometric equation,

$$f(x) = 2\sin^2 x + \sin x - 1$$

with a focus on the red piece on the interval $0° \leqslant x \leqslant 360°$



(i) From looking at the graph, write down the possible approximate values of x that would satisfy the equation

$$2\sin^2 x + \sin x - 1 = 0, \qquad 0° \leqslant x \leqslant 360°$$

[4 marks]

(ii) Use the mathematics of a *quadratic in disguise* to solve the equation

$$2\sin^2 x + \sin x - 1 = 0, \qquad 0° \leqslant x \leqslant 360°$$

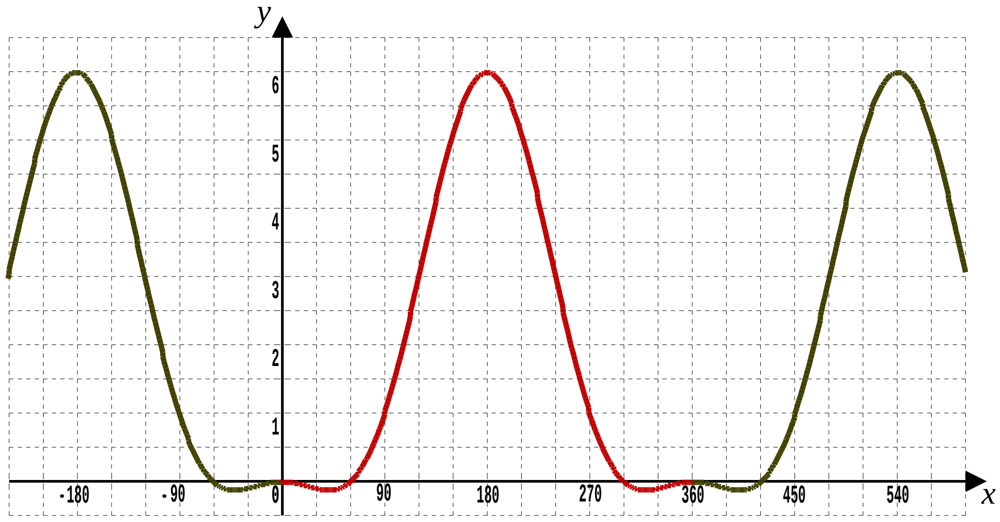
[6 marks]

Question 2

The graph is of the trigonometric equation,

$$f(x) = 2\cos^2 x - 3\cos x + 1$$

with a focus on the red piece on the interval $0° \leqslant x \leqslant 360°$



(i) From looking at the graph, write down the four approximate values of x that would satisfy the equation

$$2\cos^2 x - 3\cos x + 1 = 0, \qquad 0° \leqslant x \leqslant 360°$$

[4 marks]

(ii) Use the mathematics of a *quadratic in disguise* to solve the equation

$$2\cos^2 x - 3\cos x + 1 = 0, \qquad 0° \leqslant x \leqslant 360°$$

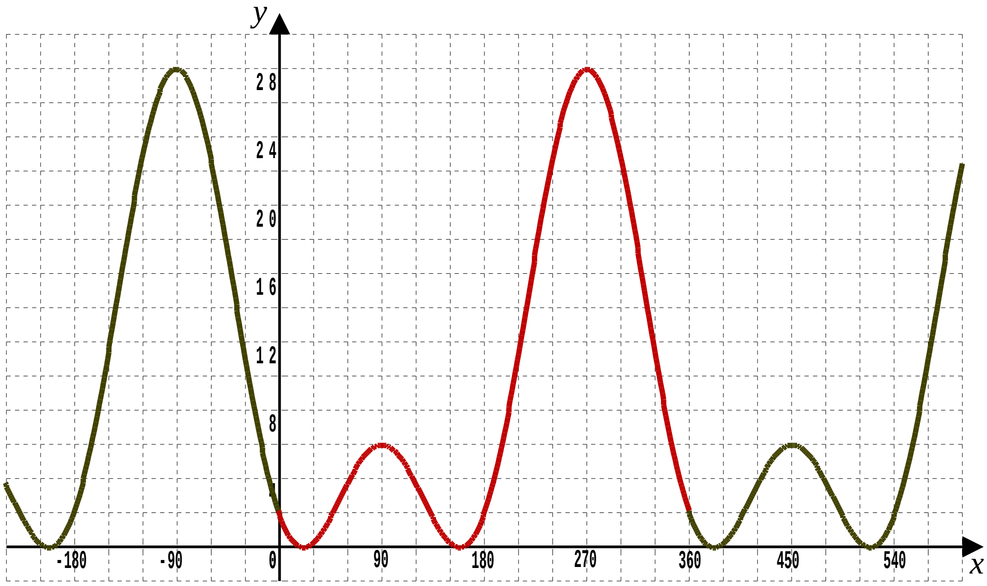
[6 marks]

Question 3

The graph is of the trigonometric equation,

$$f(x) = 15\,sin^2 x - 11\,sin\,x + 2$$

with a focus on the red piece on the interval $0° \leqslant x \leqslant 360°$

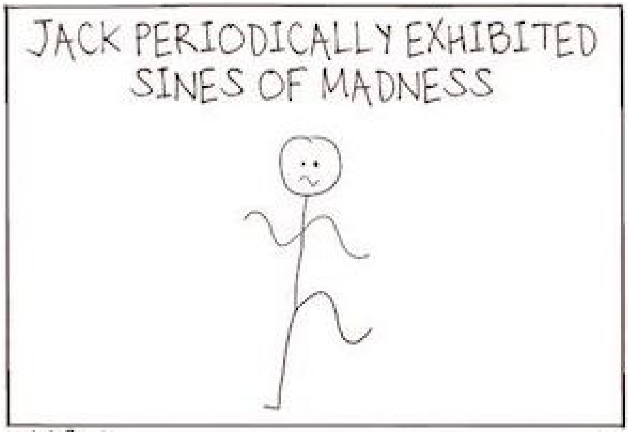


It's not clear from the graph how many, if any, solutions there are as the minimum points could be above or below the x-axis.

By viewing the following equation as a *quadratic in disguise* find the solutions, if there are any, to the equation

$$15\,sin^2 x - 11\,sin\,x + 2 = 0, \qquad 0° \leqslant x \leqslant 360°$$

[10 marks]



2.3 Answers

2.3.1 Solution to 2.1 Example (Teaching Video)

When a mathematician writes $cos^2 x$ they actually mean $(cos\,x)^2$
Unfortunately, this sloppy practice is widespread and you need to learn to live with it.
So,

$$4\,cos^2 x + cos\,x - 3 = 0 \qquad \text{for} \quad 0° \leqslant x \leqslant 360°$$

is better written as,

$$4\,(cos\,x)^2 + cos\,x - 3 = 0 \qquad \text{for} \quad 0° \leqslant x \leqslant 360°$$

which makes it more obvious that this is really a quadratic equation, but in disguise.

Let $z = cos\,x$ to get,

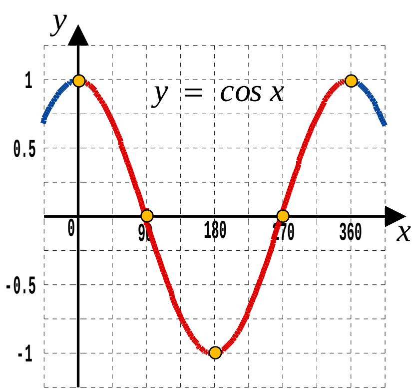
$$4z^2 + z - 3 = 0$$

$$(4z - 3)(z + 1) = 0$$

$$\text{Either } 4z - 3 = 0 \text{ or } z + 1 = 0$$

$$z = \frac{3}{4} \quad \text{or} \quad z = -1$$

$$cos\,x = \frac{3}{4} \quad \text{or} \quad cos\,x = -1$$



For $cos\,x = -1$ the answer is read off from the sketch graph as $x = 180°$

$$\text{For } cos\,x = \frac{3}{4} \Rightarrow x = arccos\left(\frac{3}{4}\right)$$

$$= 41.4,\ 360 - 41.4$$

$$= 41.4,\ 318.6$$

The three solutions to the original equation are $41.4°,\ 180°,\ 318.6°$

[6 marks]

2.3.2 Solutions to 2.2 Exercise

Answer 1

(i) From the graph it would seem that there is one solution around 30°, another around 150°, and possibly one or two more close to 270° although it's not clear if the curve gets to or crosses the *x*-axis near 270°.

[4 marks]

(ii) $2\,(\,sin\,x\,)^2 \;+\; sin\,x \;-\; 1 \;=\; 0$ for $0° \leqslant x \leqslant 360°$

Let $z \;=\; sin\,x$ to get,

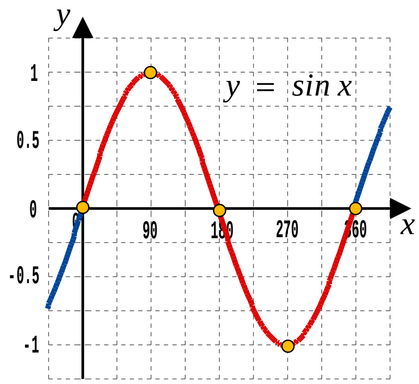
$$2\,z^2 + z - 1 = 0$$

$$(\,2z - 1\,)(\,z + 1\,) = 0$$

$$\text{Either } 2z - 1 = 0 \quad \text{or} \quad z + 1 = 0$$

$$z = \frac{1}{2} \quad \text{or} \quad z = -1$$

$$sin\,x = \frac{1}{2} \quad \text{or} \quad sin\,x = -1$$



For $sin\,x \;=\; -1$ the answer is read off from the sketch graph as $x \;=\; 270°$

$$\text{For } sin\,x = \frac{1}{2} \;\Rightarrow\; x = arcsin\left(\frac{1}{2}\right)$$

$$= 30,\; 180 - 30$$

$$= 30,\; 150$$

The three solutions to the original equation are $30°,\; 150°,\; 270°$

[6 marks]

Answer 2

(i) From the graph it would seem that there is one solution around 60°, another around 300°, and possibly one or two more close to 0° and likewise close to 360° although it's not clear if the curve gets to or crosses the *x*-axis near 0° and 360°

[4 marks]

(ii) $2\,(\cos x)^2 \;-\; 3\cos x \;+\; 1 \;=\; 0 \qquad \text{for} \quad 0° \leqslant x \leqslant 360°$

Let $z \;=\; \cos x$ to get,

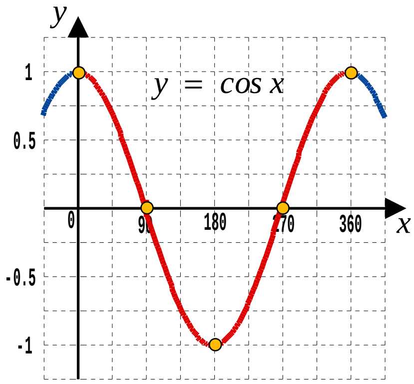
$$2\,z^2 \;-\; 3z \;+\; 1 \;=\; 0$$

$$(\,2z \;-\; 1\,)(\,z \;-\; 1\,) \;=\; 0$$

$$\text{Either } \; 2z \;-\; 1 \;=\; 0 \quad \text{or} \quad z \;-\; 1 \;=\; 0$$

$$z \;=\; \frac{1}{2} \quad \text{or} \quad z \;=\; 1$$

$$\cos x \;=\; \frac{1}{2} \quad \text{or} \quad \cos x \;=\; 1$$



For $\cos x \;=\; 1$ the answers read off from the sketch graph are $x \;=\; 0°,\; 360°$

$$\text{For } \cos x \;=\; \frac{1}{2} \;\Rightarrow\; x \;=\; \arccos\left(\frac{1}{2}\right)$$

$$=\; 60,\;\; 360 \;-\; 60$$

$$=\; 60,\;\; 300$$

The four solutions to the original equation are $0°,\; 60°,\; 300°,\; 360°$

[6 marks]

Answer 3

$$15\,(\sin x)^2 - 11 \sin x + 2 = 0$$

Let $z = \sin x$ to get,

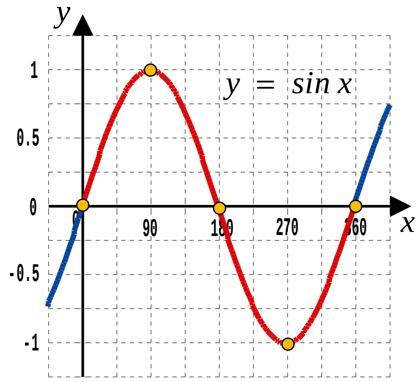
$$15 z^2 - 11z + 2 = 0$$

$$(3z - 1)(5z - 2) = 0$$

Either $3z - 1 = 0$ or $5z - 2 = 0$

$$z = \frac{1}{3} \quad \text{or} \quad z = \frac{2}{5}$$

$$\sin x = \frac{1}{3} \quad \text{or} \quad \sin x = \frac{2}{5}$$



$x = \arcsin\left(\frac{1}{3}\right)$ or $x = \arcsin\left(\frac{2}{5}\right)$

$x = 19.5°,\ 180 - 19.5$ or $x = 23.6,\ 180 - 23.6$

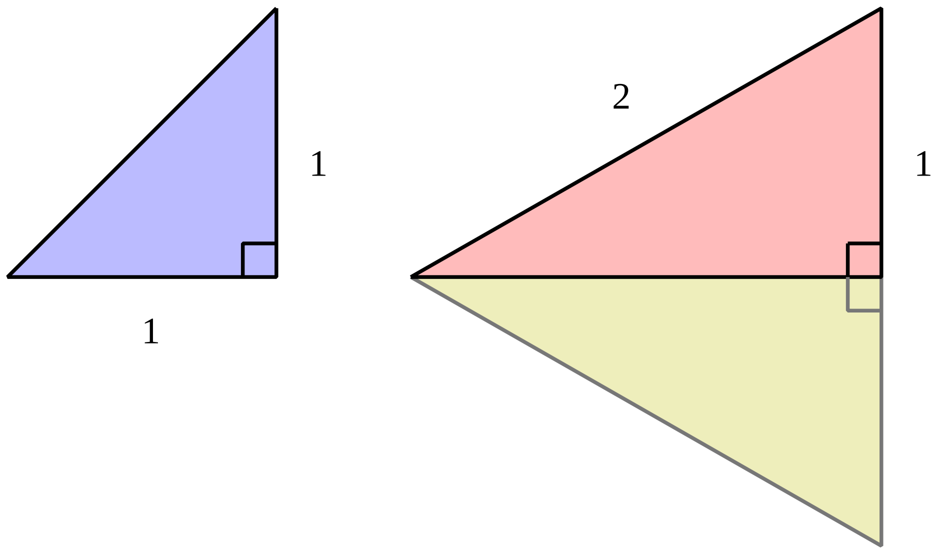
$$\therefore\ x = 19.5°,\ \ 23.6°,\ \ 156.4°,\ \ 160.5°$$

[10 marks]

Lesson 3

3.1 Exactitude

The following two simple right angled triangles are of great utility, for they can be used to determine exact values for the *sin*, *cos* and *tan* of certain angles.



The left-hand triangle has a base of length exactly 1 unit and a height of exactly 1 unit. On this triangle in the appropriate place write the exact length of the hypotenuse and the exact size of the two non right angled angles.

The right-hand triangle is equilateral with sides of length exactly 2 units, but has been cut in half. The focus is upon the upper half which has a hypotenuse of exact length 2 units and height of exactly 1 unit. On this triangle in the appropriate place write the exact length of the base and the exact size of the two non right angled angles.

Watch the teaching video and complete this “Exact Values” table without a calculator,

x	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin x$					
$\cos x$					
$\tan x$					

Teaching Video : http://www.NumberWonder.co.uk/v9044/3.mp4

[5 marks]

3.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

(i) Sketch the graph of $y = \sin x$ over the interval $0° \leqslant x \leqslant 360°$

[2 marks]

(ii) Use the symmetry of your part (i) sketch, and the “Exact Values” table from the Teaching Video to complete this “Extended Exact Values” table for *sin x*. You may like to use a calculator to check your answers.

x	0	30	45	60	90	120	135	150
$\sin x$								

x	180	210	225	240	270	300	315	330
$\sin x$								

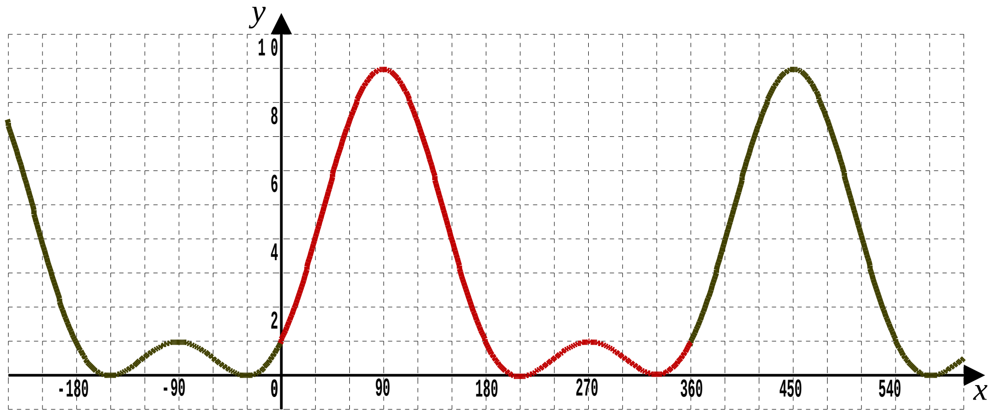
[4 marks]

Question 2

The graph is of the trigonometric equation,

$$f(x) = 4\sin^2 x + 4\sin x + 1$$

with a focus on the red piece on the interval $0° \leqslant x \leqslant 360°$



Use the mathematics of a *quadratic in disguise* to solve the equation

$$4\sin^2 x + 4\sin x + 1 = 0, \qquad 0° \leqslant x \leqslant 360°$$

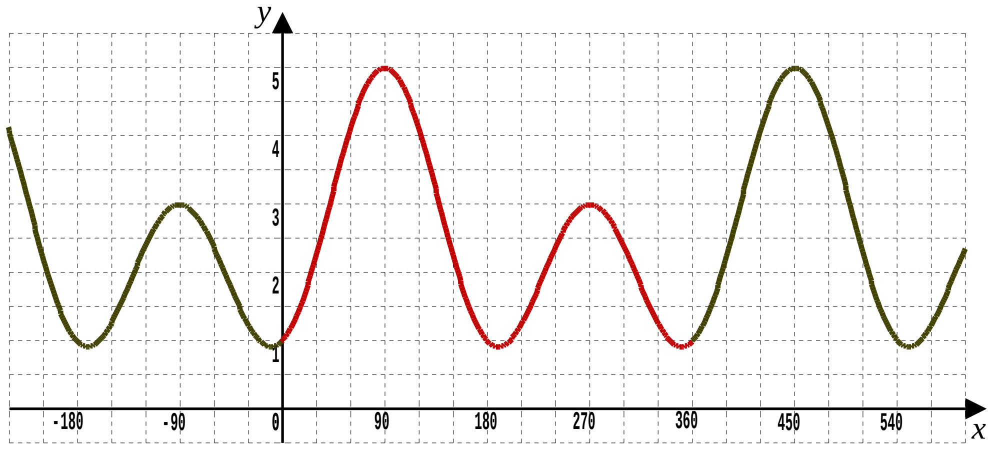
Check that your answers accord with what is suggested by the above graph.

[6 marks]

Question 3
The graph below is of the function

$$f(x) = 3\sin^2 x + \sin x + 1$$

with a focus, marked in red, on the interval $0° \leqslant x \leqslant 360°$



(i) Explain how, from looking at the graph, it can be seen that it would be futile to try and solve the equation,

$$3\sin^2 x + \sin x + 1 = 0 \qquad \text{for} \quad 0° \leqslant x \leqslant 360°$$

[2 marks]

(ii) Is trying to solve the equation

$$3\sin^2 x + \sin x + 1 = 0 \qquad \text{for} \quad -180° \leqslant x \leqslant 540°$$

a more worthwhile endeavour ?

Explain your answer.

[2 marks]

(iii) On the graph draw the horizontal straight line $y = 5$

By looking at the graph, now with your added line on, write down a probable solutions to the equation,

$$3\sin^2 x + \sin x + 1 = 5 \qquad \text{for} \quad 0° \leqslant x \leqslant 360°$$

[2 marks]

(iv) Use the mathematics of a *quadratic in disguise* to solve the equation

$$3\,sin^2 x + sin\,x + 1 = 5, \qquad 0° \leqslant x \leqslant 360°$$

Begin by forming a quadratic equation that equals zero before factorising that into two brackets.

You are expecting the mathematics to give you the solution identified in part (iii) but something must happen, mathematically, that prevents any other solutions from appearing.

Good luck on this voyage of discovery !

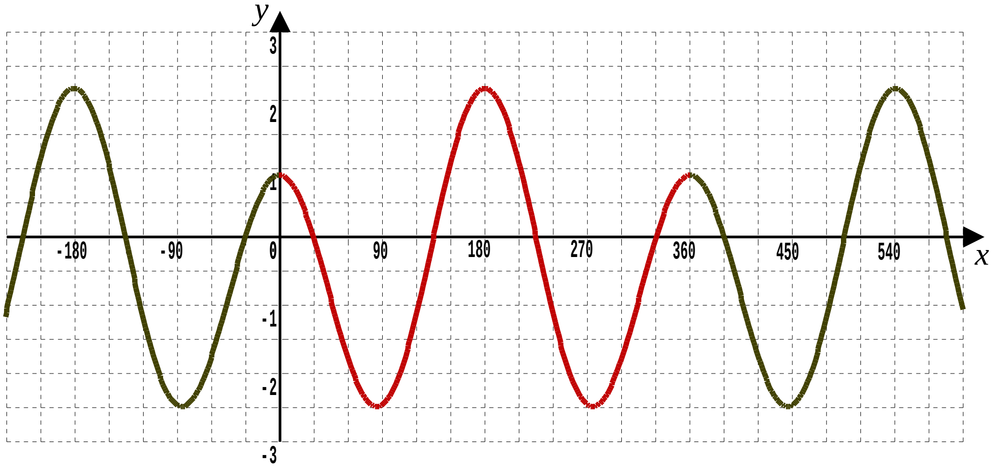
[6 marks]

Question 4

The graph is of the trigonometric equation,

$$f(x) = \left(2\cos x - \sqrt{3}\right)\left(2\cos x + \sqrt{2}\right)$$

with a focus on the red piece on the interval $0° \leqslant x \leqslant 360°$



Solve the equation $\left(2\cos x - \sqrt{3}\right)\left(2\cos x + \sqrt{2}\right) = 0$ for $0° \leqslant x \leqslant 360°$

Check that your answers accord with what is suggested by the above graph.

[6 marks]

3.3 Answers

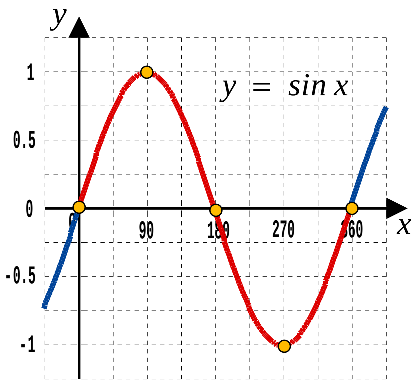
3.3.1 Solution to 3.1 Teaching Video

x	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	

3.3.2 Solutions to 3.2 Exercise

Answer 1

(i)



[2 marks]

(ii)

x	0	30	45	60	90	120	135	150
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$

x	180	210	225	240	270	300	315	330
$\sin x$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$

[6 marks]

Answer 2

$4(\sin x)^2 + 4\sin + 1 = 0 \qquad \text{for} \quad 0° \leqslant x \leqslant 360°$

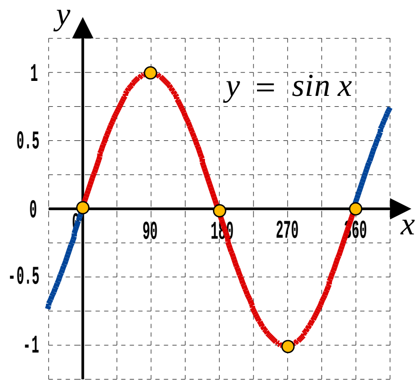
Let $z = \sin x$ to get,

$$4z^2 + 4z + 1 = 0$$
$$(2z + 1)(2z + 1) = 0$$
$$\therefore\ 2z + 1 = 0$$
$$z = -\frac{1}{2}$$
$$\sin x = -\frac{1}{2}$$

$$\text{For } \sin x = +\frac{1}{2} \Rightarrow x = \arcsin\left(\frac{1}{2}\right)$$
$$= 30 \leftarrow \text{ Working Angle}$$

The solutions sought in the range $0° \leqslant x \leqslant 360°$ will be two of

$x \qquad 180 - x \qquad 180 + x \qquad \text{or} \qquad 360 - x$



From the $y = \sin x$ sketch graph, solution sought are 180 + 30 and 360 – 30

$$\therefore\ x = 210°,\ \ 330°$$

Alternatively, these could have been grabbed from the table made in answer to question 1

[6 marks]

Answer 3

(i) The graph does not cross the x-axis over the domain of $0° \leqslant x \leqslant 360°$
That is, there is no value of x for which the function has no height.

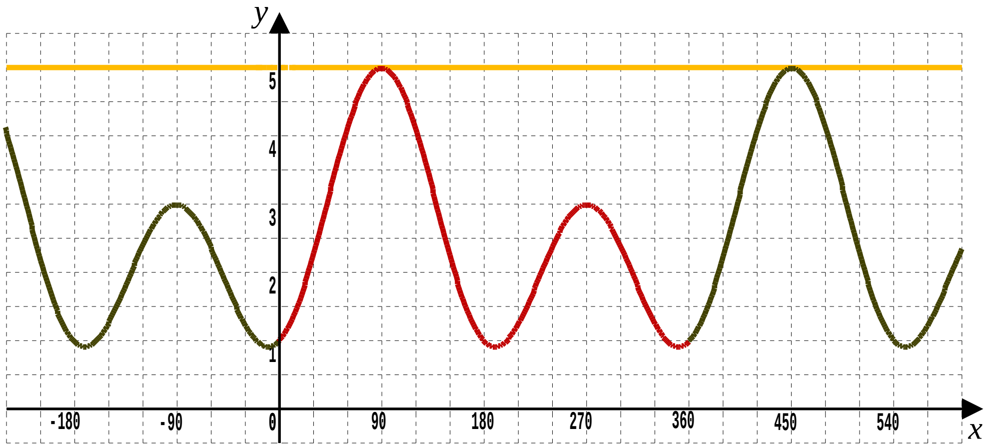
The equation $3\,sin^2\,x + sin\,x + 1 = 0$ will have no real solutions.
$\therefore$ Trying to find solutions is indeed futile.

[2 marks]

(ii) Over the expanded domain of $-180° \leqslant x \leqslant 540°$ the equation will still have no real solutions. This is because the function is made up of further copies of that already considered between $0° \leqslant x \leqslant 360°$. to both the left and the right of that interval.

[2 marks]

(iii) Here is the graph with the horizontal straight line $y = 5$ added,



The straight line intersects the curve at about 90°, and so that is the probable solution to

$$3\,sin^2\,x + sin\,x + 1 = 5 \qquad \text{for } 0° \leqslant x \leqslant 360°$$

[2 marks]

(iv)

$$3\,sin^2\,x + sin\,x + 1 = 5$$

$$3\,sin^2\,x + sin\,x - 4 = 0$$

Let $z = sin\,x$ to get,

$$3\,z^2 + z - 4 = 0$$

$$(3z + 4)(z - 1) = 0$$

$$\text{Either } 3z + 4 = 0 \text{ or } z - 1 = 0$$

$$z = -\frac{4}{3} \text{ or } z = 1$$

$sin\,x = -\dfrac{4}{3}$ has no solutions because that's less than it's minimum value

$sin\,x = 1$ gives $x = 90°$ as the only solution

[6 marks]

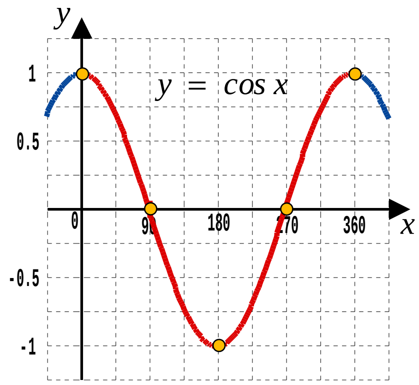
Answer 4

$$\left(2\cos x - \sqrt{3}\right)\left(2\cos x + \sqrt{2}\right) = 0$$

Either $\left(2\cos x - \sqrt{3}\right) = 0$ or $\left(2\cos x + \sqrt{2}\right) = 0$

$$\cos x = \frac{\sqrt{3}}{2} \quad \text{or} \quad \cos x = -\frac{\sqrt{2}}{2}$$

For $\cos x = \dfrac{\sqrt{3}}{2} \Rightarrow x = \arccos\left(\dfrac{\sqrt{3}}{2}\right) \Rightarrow x = 30° \leftarrow$ working angle

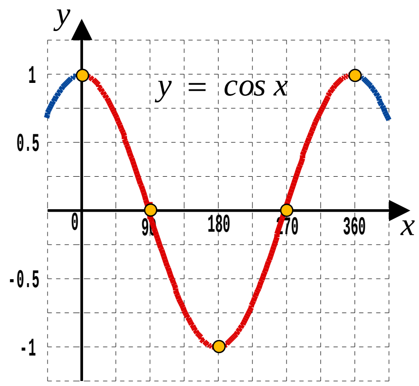


The solutions sought in the range $0° \leqslant x \leqslant 360°$ will be two of

$x \qquad 180 - x \qquad 180 + x \qquad$ or $\qquad 360 - x$

$$\therefore\ x = 30° \quad \text{or} \quad x = 330°$$

For $\cos x = +\dfrac{\sqrt{2}}{2} \Rightarrow x = \arccos\left(\dfrac{\sqrt{2}}{2}\right) \Rightarrow x = 45° \leftarrow$ working angle



The solutions sought in the range $0° \leqslant x \leqslant 360°$ will be two of

$x \qquad 180 - x \qquad 180 + x \qquad$ or $\qquad 360 - x$

$$\therefore\ x = 135° \quad \text{or} \quad x = 225°$$

Thus, in total there are four solutions; $30°,\ 135°,\ 225°,\ 330°$

[6 marks]

4.1 An Identity

When a trigonometric equation is solved, such as,

$$15\,sin^2 x \;-\; 11\,sin\,x \;+\; 2 \;=\; 0, \qquad 0° \;\leqslant\; x \;\leqslant\; 360°$$

the task is in finding the few values of *x* that make the equation true.

For this example, those values are, $x \;=\; 19.5°, \quad 23.6°, \quad 156.4°, \quad 160.5°$ †

A trigonometric identity is a very different beast : It is true for all values of *x*.
The perfect example of a trigonometric identity is the following:

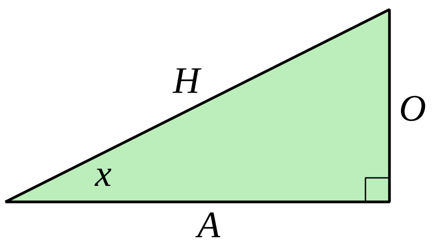
A Key Trigonometric Identity

$$cos^2 x \;+\; sin^2 x \;=\; 1$$

Proof

Consider a right angled triangle with sides labelled;

O for Opposite angle *x*
A for Adjacent to angle *x*
H for Hypotenuse



$$\begin{aligned}
\text{LHS} \;&=\; cos^2 x \;+\; sin^2 x \\
&=\; \left(\frac{A}{H}\right)^2 \;+\; \left(\frac{O}{H}\right)^2 \\
&=\; \frac{A^2}{H^2} \;+\; \frac{O^2}{H^2} \\
&=\; \frac{A^2 \;+\; O^2}{H^2} \qquad \text{By Pythagoras' Theorem, } A^2 \;+\; O^2 \;=\; H^2 \\
&=\; \frac{H^2}{H^2} \\
&=\; 1 \\
&=\; \text{RHS} \qquad \square
\end{aligned}$$

† This equation was solved in Lesson 2, Exercise 2.2, Question 3

4.2 Example

Solve the following trigonometric equation;

$$5\,sin\,x \;=\; 2\,cos^2\,x \;-\; 4 \qquad\qquad 0° \;\leqslant\; x \;\leqslant\; 360°$$

Teaching Video : http://www.NumberWonder.co.uk/v9044/4.mp4

[6 marks]

4.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

A-Level Examination Question from January 2010, Paper C2, Q2 (Edexcel)

(a) Show that the equation

$$5\,sin\,x \;=\; 1 \;+\; 2\,cos^2\,x$$

can be written in the form

$$2\,sin^2\,x \;+\; 5\,sin\,x \;-\; 3 \;=\; 0$$

[2 marks]

(b) Solve, for $0° \leqslant x \leqslant 360°$

$$2\,sin^2\,x \;+\; 5\,sin\,x \;-\; 3 \;=\; 0$$

[4 marks]

Question 2

Find all the solutions, in the interval $0 \leqslant x \leqslant 360°$, of the equation

$$cos^2 x - 3\,cos\,x = sin^2 x + 4$$

[6 marks]

Question 3

The next most useful trigonometric identity after $\cos^2 x + \sin^2 x = 1$ is;

Another Key Trigonometric Identity

$$\frac{\sin x}{\cos x} = \tan x \qquad (\text{ Provided } \cos x \neq 0 \text{ })$$

Your task now is to prove that this result is true.

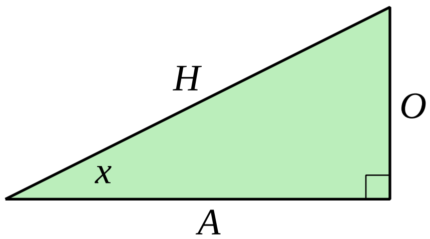
Your proof will be in a similar style to that for $\cos^2 x + \sin^2 x = 1$

To get you started here are the first few parts of the proof;

Proof

Consider a right angled triangle with sides labelled;

O for Opposite angle *x*
A for Adjacent to angle *x*
H for Hypotenuse



$$\text{LHS} = \frac{\sin x}{\cos x}$$

Complete the proof, started above.

[3 marks]

Question 4

Additional Mathematics Examination Question from June 2014, Q9 (OCR)

(i) Show that $\dfrac{1 - cos^2 x}{1 - sin^2 x} = tan^2 x$

[1 mark]

(ii) Hence solve the equation $\dfrac{1 - cos^2 x}{1 - sin^2 x} = 3 - 2 \, tan \, x$ for values of x in the range $0° \leqslant x \leqslant 180°$

[4 marks]

Question 5

A-Level Examination Question from October 2016, Paper C12, Q10 (edited) (Edexcel)

(a) Given that

$$8 \tan x = -3 \cos x$$

show that

$$3 \sin^2 x - 8 \sin x - 3 = 0$$

[3 marks]

(b) Hence solve, for $0 \leqslant x < 360°$,

$$8 \tan x = -3 \cos x$$

giving your answers to one decimal place, as appropriate.

[5 marks]

Question 6

A-Level Examination Question from January 2005, Paper C2, Q4 (Edexcel)

(a) Show that the equation

$$5\,cos^2 x = 3(1 + sin\,x)$$

can be written as

$$5\,sin^2 x + 3\,sin\,x - 2 = 0$$

[2 marks]

(b) Hence solve, for $0° \leqslant x \leqslant 360°$, the equation

$$5\,cos^2 x = 3(1 + sin\,x)$$

giving your answers to 1 decimal place where appropriate.

[5 marks]

Question 7

A-Level Examination Question from January 2011, Paper C2, Q7 (Edexcel)

(a) Show that the equation

$$3\,sin^2 x + 7\,sin\,x = cos^2 x - 4$$

can be written in the form

$$4\,sin^2 x + 7\,sin\,x + 3 = 0$$

[2 marks]

(b) Hence solve, for $0° \leqslant x \leqslant 360°$

$$3\,sin^2 x + 7\,sin\,x = cos^2 x - 4$$

giving your answers to 1 decimal place where appropriate.

[5 marks]

4.4 Answers

4.4.1 Solution to 4.2 Example (Teaching Video)

$$5\,sin\,x \;=\; 2\,cos^2 x \;-\; 4$$

$$5\,sin\,x \;=\; 2\,(\,1 \;-\; sin^2 x\,) \;-\; 4$$

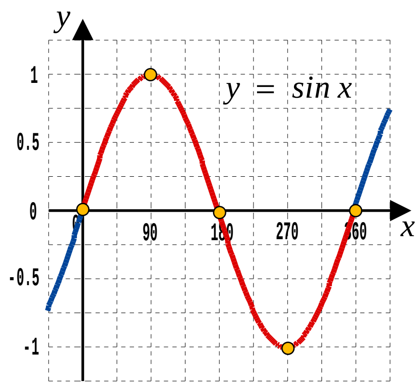
$$5\,sin\,x \;=\; 2 \;-\; 2\,sin^2 x \;-\; 4$$

$$2\,sin^2 x \;+\; 5\,sin\,x \;+\; 2 \;=\; 0$$

$$(\,2\,sin\,x \;+\; 1\,)\;(\,sin\,x \;+\; 2\,) \;=\; 0$$

Either $2\,sin\,x \;+\; 1 \;=\; 0$ or $sin\,x \;+\; 2 \;=\; 0$

$$sin\,x \;=\; -\frac{1}{2} \qquad\qquad sin\,x \;=\; -\,2$$



$$x \;=\; arcsin\left(\frac{1}{2}\right) \qquad\qquad x \;=\; arcsin\,(-\,2\,)$$

$x \;=\; 30°$ (working angle) $\qquad\qquad$ No solutions

Solutions are $\{\;180 \;+\; 30,\;\; 360 \;-\; 30\;\}$

i.e. $\{\;210°,\;\; 330°\;\}$

[6 marks]

4.4.2 Solutions to 4.3 Exercise

Answer 1

(a)

$$5\,sin\,x = 1 + 2\,cos^2 x$$

$$5\,sin\,x = 1 + 2\left(1 - sin^2 x\right)$$

$$5\,sin\,x = 1 + 2 - 2\,sin^2 x$$

$$2\,sin^2 x + 5\,sin\,x - 3 = 0 \qquad \square$$

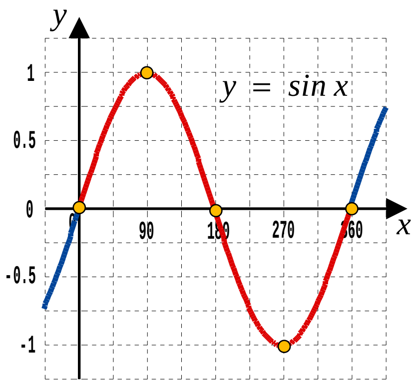
[2 marks]

(b)

$$(2\,sin\,x - 1)\,(sin\,x + 3) = 0$$

Either $2\,sin\,x - 1 = 0$ or $sin\,x + 3 = 0$

$$sin\,x = \frac{1}{2} \qquad\qquad sin\,x = -3$$



$$x = arcsin\left(\frac{1}{2}\right) \qquad\qquad x = arcsin(-3)$$

$x = 30°$ (working angle) No solutions

$$\therefore \quad x \in \{\, 30°,\ 150° \,\}$$

[4 marks]

Answer 2

$$cos^2 x - 3\,cos\,x = sin^2 x + 4$$

$$cos^2 x - 3\,cos\,x = \left(1 - cos^2 x\right) + 4$$

$$2\,cos^2 x - 3\,cos\,x - 5 = 0$$

$$(2\,cos\,x - 5)(cos\,x + 1) = 0$$

Either $2\,cos\,x - 5 = 0$ or $cos\,x = -1$

But $cos\,x = \dfrac{5}{2}$ has no solutions as the max value of the *cosine* function is 1

The only solution come from $cos\,x = -1 \Rightarrow x = 180°$

[6 marks]

Answer 3

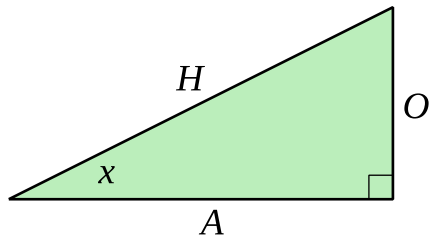
Another Key Trigonometric Identity

$$\frac{\sin x}{\cos x} = \tan x \qquad (\text{ Provided } \cos x \neq 0 \text{ })$$

Proof

Consider a right angled triangle with sides labelled;

O for Opposite angle x
A for Adjacent to angle x
H for Hypotenuse



$$\begin{aligned} \text{LHS} &= \frac{\sin x}{\cos x} \\ &= \sin x \div \cos x \\ &= \frac{O}{H} \div \frac{A}{H} \\ &= \frac{O}{H} \times \frac{H}{A} \\ &= \frac{O}{A} \\ &= \tan x \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

Answer 4

(i)

$$\text{LHS} = \frac{1 - cos^2 x}{1 - sin^2 x}$$

$$= \frac{sin^2 x}{cos^2 x}$$

$$= tan^2 x$$

$$= \text{RHS} \quad \square$$

[1 mark]

(ii)

$$\frac{1 - cos^2 x}{1 - sin^2 x} = 3 - 2\,tan\,x$$

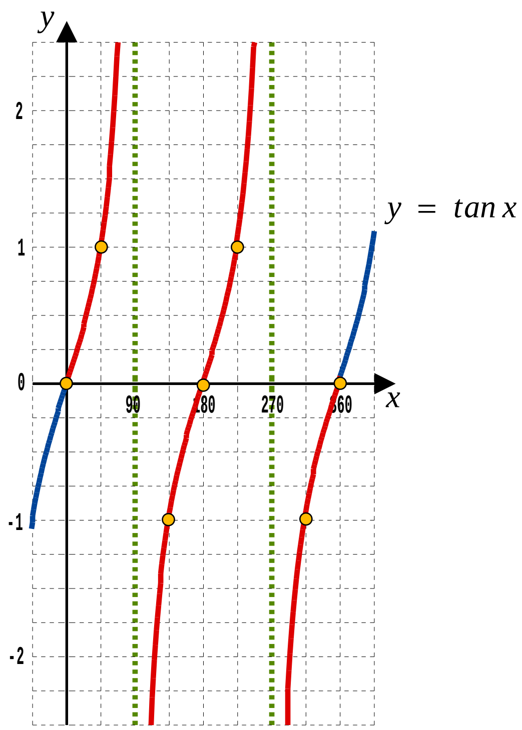
$$tan^2 x = 3 - 2\,tan\,x$$

$$tan^2 x + 2\,tan\,x - 3 = 0$$

$$(\,tan\,x + 3\,)(\,tanx - 1\,) = 0$$

$$\text{Either } tan\,x + 3 = 0 \text{ or } tanx - 1 = 0$$

$$tan\,x = -3 \text{ or } tan\,x = 1$$



Note that we're only asked for answers over the domain $0 \leqslant x \leqslant 180°$

$tan\,x = 3 \Rightarrow x = arctan(3) \Rightarrow x = 71.6°$ working angle

Thus for $tan\,x = -3$ we want $180 - 71.6 = 108.4°$

$tan\,x = 1 \Rightarrow x = arctan(1) \Rightarrow x = 45°$

So, two answers, $x \in \{\,45°,\ 108.4°\,\}$

[4 marks]

Answer 5

(a)

$$8\,tan\,x = -3\,cos\,x$$

$$8 \times \frac{sin\,x}{cos\,x} = -3\,cos\,x$$

Multiply both sides by $cos\,x$

$$8\,sin\,x = -3\,cos^2 x$$

$$8\,sin\,x = -3\left(1 - sin^2 x\right)$$

$$8\,sin\,x = -3 + 3\,sin^2 x$$

$$3\,sin^2 x - 8\,sin\,x - 3 = 0 \qquad \square$$

[3 marks]

(b)

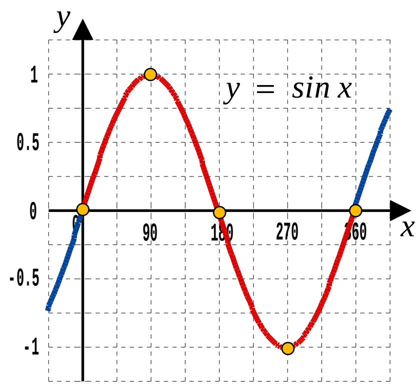
$$8\,tan\,x = -3\,cos\,x$$

$$3\,sin^2 x - 8\,sin\,x - 3 = 0 \qquad \text{from part (a)}$$

$$(3\,sin\,x + 1)(sin\,x - 3) = 0$$

Either $3\,sin\,x + 1 = 0$ or $sin\,x - 3 = 0$

$sin\,x = -\frac{1}{3}$ or $sin\,x = 3$ (No solutions)



Solving $sin\,x = +\frac{1}{3} \Rightarrow x = arcsin\left(\frac{1}{3}\right) \Rightarrow x = 19.5° \leftarrow$ working angle

$\therefore$ For $sin\,x = -\frac{1}{3}$ we'll have $x = 199.5°,\ 340.5°$

So, two answers, $x \in \{\ 199.5°,\ 340.5°\ \}$

[5 marks]

Answer 6

(a)

$$5\,cos^2 x = 3(1 + sin\,x)$$

$$5\,(1 - sin^2 x) = 3 + 3\,sin\,x$$

$$5 - 5\,sin^2 x = 3 + 3\,sin\,x$$

$$5\,sin^2 x + 3\,sin\,x - 2 = 0 \qquad \square$$

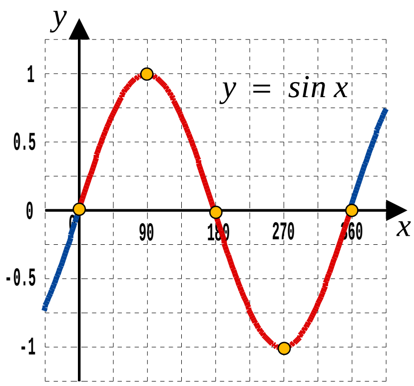
[2 marks]

(b)

$$(5\,sin\,x - 2)(sin\,x + 1) = 0$$

Either $5\,sin\,x - 2 = 0$ or $sin\,x + 1 = 0$

$$sin\,x = \frac{2}{5} \quad \text{or} \quad sin\,x = -1$$



$sin\,x = 0.4$ gives $23.6°,\ 156.4°$

$sin\,x = -1$ gives $270°$

So, three answers, $x \in \{ 23.6°,\ 156.4°,\ 270° \}$

[5 marks]

Answer 7

(a)

$$3\sin^2 x + 7\sin x = \cos^2 x - 4$$

$$3\sin^2 x + 7\sin x = \left(1 - \sin^2 x\right) - 4$$

$$4\sin^2 x + 7\sin x + 3 = 0 \qquad \square$$

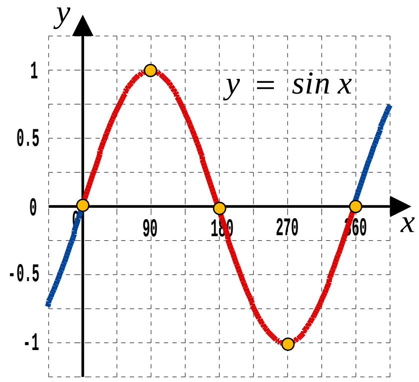
[2 marks]

(b)

$$(4\sin x + 3)\ (\sin x + 1) = 0$$

$$\text{Either } \ 4\sin x + 3 = 0 \ \text{ or } \ \sin x + 1 = 0$$

$$\sin x = -\frac{3}{4} \ \text{ or } \ \sin x = -1$$



$$\text{Solving } \sin x = +\frac{3}{4} \Rightarrow x = \arcsin\left(\frac{3}{4}\right) \Rightarrow x = 48.6° \leftarrow \text{ working angle}$$

$$\therefore \ \text{For } \sin x = -\frac{3}{4} \ \text{ we'll have } x = 228.6°, \ 311.4°$$

$$\text{The other solution } \sin x = -1 \text{ gives } x = 270°$$

$$\text{So, three answers, } \ x \in \{ 228.6°, \ 270°, \ 311.4° \}$$

[5 marks]

Lesson 5

Additional Mathematics
A-Level Pure Mathematics : Year 1
Trigonometry IV

5.1 Solving Quadratic Equations

The solution methods for trying to solve quadratic equations include,

- Factorising into two pairs of brackets
- Completing the square
- Using the Q Formula

The Q Formula

A quadratic equation that is written in the form

$$a x^2 + bx + c = 0 \qquad \text{where } a, b \text{ and } c \text{ are constants}$$

has real solutions, if any exist, given by the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the Q Formula, the expression under the square root sign, the $b^2 - 4ac$ piece, is called the discriminant, D, as it determines how many real solutions there are.

- If $b^2 - 4ac > 0$ then there are two distinct real solutions
- If $b^2 - 4ac = 0$ then there is one (repeated) real solution
- If $b^2 - 4ac < 0$ then there are no real solutions

Given that some trigonometric equations can be viewed as *quadratics in disguise* it should come as no surprise that the underlying quadratic may, in some questions, not factorise into two guessable brackets; instead, the Q formula may be needed.

Try the following example, then check your answer with mine, over the page;

5.2 Example For You To Try

For the equation $x^2 + x - 1 = 0$

(i) What is the value of the discriminant, D ?

[2 marks]

(ii) How many solutions will the equation have ?

[1 mark]

5.3 Answer to 5.2 Example

For the equation $x^2 + x - 1 = 0$

(i) What is the value of the discriminant, D ?

$a = 1$, $b = 1$, $c = -1$ so the discriminant, D, will be;

$$\begin{aligned} D &= b^2 - 4ac \\ &= 1^2 - 4 \times 1 \times (-1) \\ &= 1 + 4 \\ &= 5 \end{aligned}$$

[2 marks]

(ii) How many solutions will the equation have ?

As $D > 0$, the equation $x^2 + x - 1 = 0$ will have 2 distinct solutions

Notice that you were not asked to solve the equation !

[1 mark]

5.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

For the equation $3x^2 + 4x + 2 = 0$

(i) What is the value of the discriminant, D ?

[2 marks]

(ii) How many solutions will the equation have ?

[1 mark]

Question 2

By considering the discriminant, D, of the underlying quadratic equation, prove that the following trigonometric equation has no solutions;

$$2\,tan^2 x - 5\,tan\, x + 4 = 0$$

[3 marks]

Question 3

(i) Use a trigonometric identity to turn the following equation into one suitable for analysing as a *quadratic in disguise*;

$$sin^2 x + 3\,cos\, x - 8 = 0$$

[3 marks]

(ii) By considering the discriminant, D, of your part (i) equation show that $sin^2 x + 3\,cos\, x - 8 = 0$ has no solutions.

[3 marks]

Question 4

A quadratic equation of the form $ax^2 + bx + c = 0$

can be solved by using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(i) Show how to use this formula to find the ***exact*** solutions to the equation

$$3x^2 - 4\sqrt{3}x + 3 = 0$$

giving your answers in the form $k\sqrt{3}$ for rational values of k

[5 marks]

(ii) Hence, or otherwise, solve over the interval $0° \leqslant x \leqslant 360°$ the equation

$$3\tan^2 x - 4\sqrt{3}\tan x + 3 = 0$$

[4 marks]

Question 5

(i) Show how to use the Q Formula to find the solutions to the equation

$$7x^2 + 5\sqrt{7}x + 6 = 0$$

giving your answers in the form $k\sqrt{7}$ for rational values of k.

[5 marks]

(ii) Hence, or otherwise, solve over the interval $0° \leqslant x \leqslant 360°$ the equation

$$7\sin^2 x + 5\sqrt{7}\sin x + 6 = 0$$

[4 marks]

5.5 Answers to 5.4 Exercise

Answer 1

(i)

$$3x^2 + 4x + 2 = 0$$

$$a = 3,\ b = 4 \text{ and } c = 2$$

$$\begin{aligned} D &= b^2 - 4ac \\ &= 16 - 4 \times 3 \times 2 \\ &= 16 - 24 \\ &= -8 \end{aligned}$$

[2 marks]

(ii) As $D < 0$, the equation will not have any real solutions

[1 mark]

Answer 2

$$2\tan^2 x - 5\tan x + 4 = 0$$

Let $z = \tan x$ to reveal the underlying quadratic equation as,

$$2z^2 - 5z + 4 = 0$$

$$a = 2,\ b = -5 \text{ and } c = 4$$

$$\begin{aligned} D &= b^2 - 4ac \\ &= 25 - 4 \times 2 \times 4 \\ &= 25 - 32 \\ &= -7 \end{aligned}$$

As $D < 0$, the equation will not have any real solutions ☐

[3 marks]

Answer 3

(i) The trigonometric identity referred to must be, $cos^2 x + sin^2 x = 1$

$$sin^2 x + 3\cos x - 8 = 0$$

$$\left(1 - cos^2 x\right) + 3\cos x - 8 = 0$$

$$1 - cos^2 x + 3\cos x - 8 = 0$$

$$cos^2 x - 3\cos x + 7 = 0$$

[3 marks]

(ii) $a = 1,\ b = -3$ and $c = 7$

$$D = b^2 - 4ac$$

$$= 9 - 4 \times 1 \times 7$$

$$= 9 - 28$$

$$= -19$$

As $D < 0$, the equation will not have any real solutions □

[3 marks]

Answer 4

(i)

$$3x^2 - 4\sqrt{3}x + 3 = 0$$

$$a = 3,\ b = -4\sqrt{3} \text{ and } c = 3$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4\sqrt{3} \pm \sqrt{48 - 4 \times 3 \times 3}}{2 \times 3}$$

$$= \frac{4\sqrt{3} \pm \sqrt{48 - 36}}{6}$$

$$= \frac{4\sqrt{3} \pm \sqrt{12}}{6}$$

$$= \frac{4\sqrt{3} \pm \sqrt{4 \times 3}}{6}$$

$$= \frac{4\sqrt{3} \pm 2\sqrt{3}}{6}$$

$$= \frac{6\sqrt{3}}{6} \quad \text{or} \quad \frac{2\sqrt{3}}{6}$$

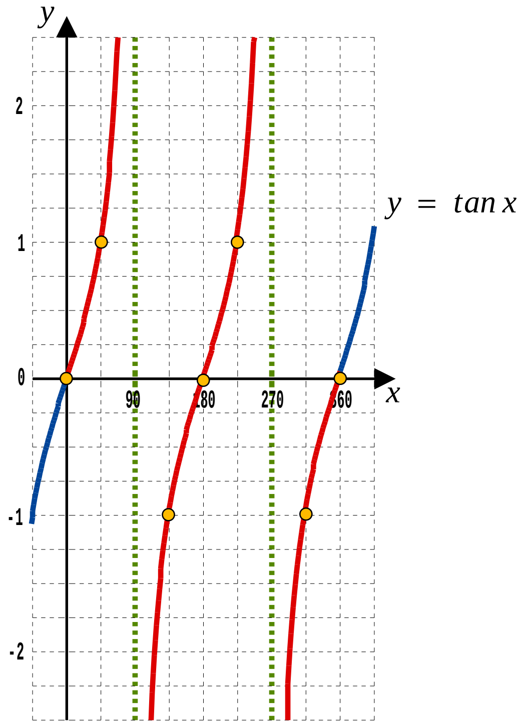
$$= \sqrt{3} \quad \text{or} \quad \frac{\sqrt{3}}{3}$$

[5 marks]

(ii)

$$3\,tan^2 x \;-\; 4\sqrt{3}\,tan\,x \;+\; 3 \;=\; 0$$

$$\text{Either} \quad tan\,x \;=\; \sqrt{3} \quad \text{or} \quad tan\,x \;=\; \frac{\sqrt{3}}{3}$$



$$\text{For} \;\; tan\,x \;=\; \sqrt{3} \;\;\Rightarrow\;\; x \;=\; 60° \text{ as working angle}$$

$$\text{Then, from the graph, } x \;=\; 60°,\;\; 240°$$

$$\text{For} \;\; tan\,x \;=\; \frac{\sqrt{3}}{3} \;\;\Rightarrow\;\; x \;=\; 30° \text{ as working angle}$$

$$\text{Then, from the graph, } x \;=\; 30°,\;\; 210°$$

In total, four solutions $x \in \{\; 30°, 60°, 210°, 240° \;\}$

[4 marks]

Answer 5

(i)

$$7x^2 + 5\sqrt{7}x + 6 = 0$$

$$a = 7, \; b = 5\sqrt{7} \text{ and } c = 6$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5\sqrt{7} \pm \sqrt{175 - 4 \times 7 \times 6}}{2 \times 7}$$

$$= \frac{-5\sqrt{7} \pm \sqrt{175 - 168}}{14}$$

$$= \frac{-5\sqrt{7} \pm \sqrt{7}}{14}$$

$$= -\frac{4\sqrt{7}}{14} \quad \text{or} \quad -\frac{6\sqrt{7}}{14}$$

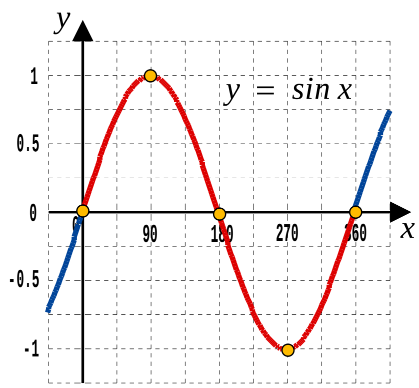
$$= -\frac{2\sqrt{7}}{7} \quad \text{or} \quad -\frac{3\sqrt{7}}{7}$$

[5 marks]

(ii)

$$7\sin^2 x + 5\sqrt{7}\sin x + 6 = 0$$

$$\text{Either } \sin x = -\frac{2\sqrt{7}}{7} \quad \text{or} \quad \sin x = -\frac{3\sqrt{7}}{7}$$



For $\sin x = -\frac{3\sqrt{7}}{7}$: No solutions as the $\sin x$ curve is never less than – 1

For $\sin x = +\frac{2\sqrt{7}}{7}$: $x = 49.1°$ is the working angle

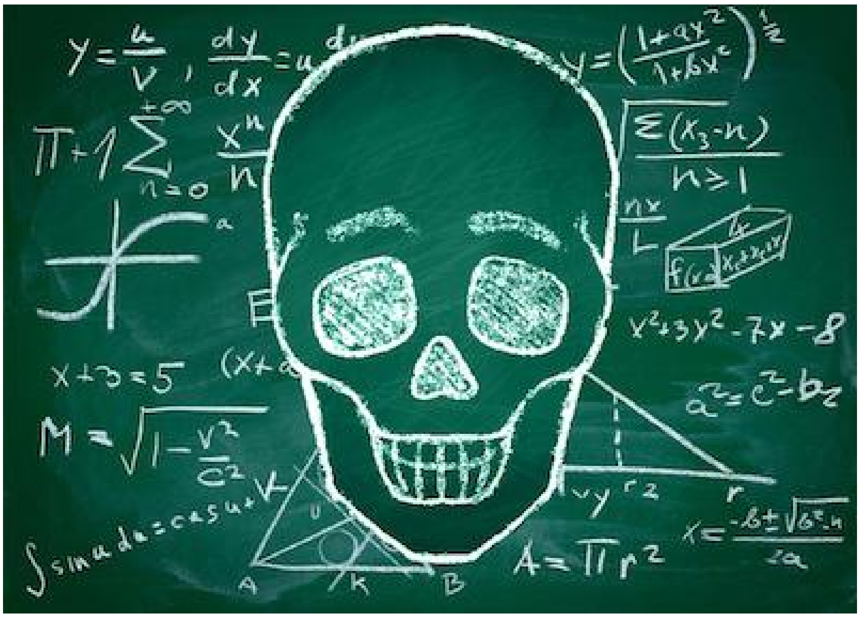
From the graph this gives two answers: $x \in \{ 229.1°, \; 310.9° \}$

[4 marks]

Lesson 6

Additional Mathematics
A-Level Pure Mathematics : Year 1
Trigonometry IV

6.1 Beware The Multi–Angle



Suppose that the following trigonometric equation is to be solved;

$$2\,sin\left(2x\right) \;=\; -\sqrt{3}\,, \qquad 0 \;\leqslant\; x \;\leqslant\; 360°$$

This equation contains a Multi-Angle, the $2x$, buried inside the *sin* function. The Teaching Video shows how to solve this equation and the importance of ignoring the Multi-Angle until the very end of the solution.

Teaching Video : http://www.NumberWonder.co.uk/v9044/6.mp4

[6 marks]

6.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Find all solutions to the following equation for $0 \leqslant x \leqslant 360°$

$$4\ cos(2x) = -3$$

Give your answers to one decimal place.

[6 marks]

Question 2

A-Level Examination Question from June 2005, Paper C2, Q5b edited (Edexcel)

Solve $cos(2x) = -0.9$ for $0° \leqslant x \leqslant 360°$

Give your answers to one decimal place.

[4 marks]

Question 3

A-Level Examination Question from June 2008, Paper C2, Q9b edited (Edexcel)

Solve

$$cos(3x) = -\frac{1}{2} \qquad 0° \leqslant x \leqslant 360°$$

[6 marks]

Question 4

Solve the equation $sin^3(2x) = \frac{1}{8}$ for $0° \leqslant x \leqslant 360°$

[6 marks]

Question 5

$$\tan^2(3x) = 2\tan(3x) - 1 \qquad 0° \leqslant x \leqslant 360°$$

[8 marks]

6.3 Answers

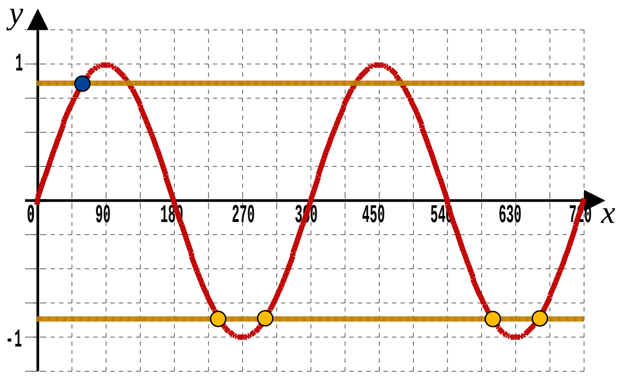
Additional Mathematics
A-Level Pure Mathematics : Year 1
Trigonometry IV

6.3.1 Solution to 6.1 Example (Teaching Video)

$$2\,sin\,(2x) \;=\; -\sqrt{3} \qquad\qquad 0 \;\leqslant\; x \;\leqslant\; 360°$$

$$sin\,(2x) \;=\; -\frac{\sqrt{3}}{2} \;\Rightarrow\; \text{Solve}\;\; sin\,(2x) \;=\; +\frac{\sqrt{3}}{2} \;\; \text{for the working angle}$$

$$(2x) \;=\; 60°$$



$$(2x) \;=\; 240,\;\; 300,\;\; 600,\;\; 660 \qquad \text{from the sketch graph of}\;\; y \;=\; sin\,x$$

$$x \;=\; 120°,\;\; 150°,\;\; 300°,\;\; 330°$$

[6 marks]

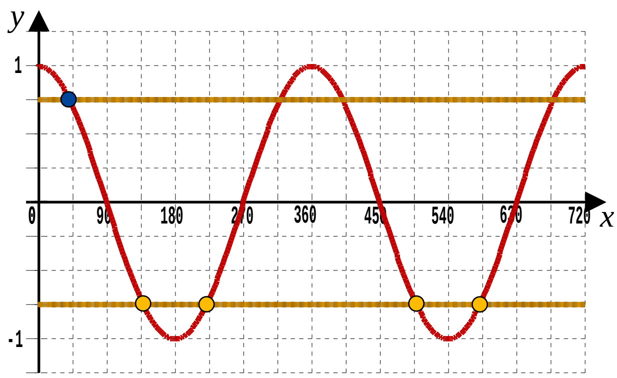
6.3.2 Solutions to 6.2 Exercise

Answer 1

$$4\,cos\,(2x) \;=\; -3 \qquad\qquad 0 \;\leqslant\; x \;\leqslant\; 360°$$

$$cos\,(2x) \;=\; -\frac{3}{4} \;\Rightarrow\; \text{Solve}\;\; cos\,(2x) \;=\; +\frac{3}{4} \;\; \text{for the working angle}$$

$$(2x) \;=\; 41.4°$$



$$(2x) \;=\; 138.6,\;\; 221.4,\;\; 498.6,\;\; 581.4 \qquad \text{from the sketch graph of}\;\; y \;=\; cos\,x$$

$$x \;=\; 69.3°,\;\; 110.7°,\;\; 249.3°,\;\; 290.7°$$

[6 marks]

Answer 2

$$\cos(2x) = -0.9 \Rightarrow \text{Solve } \cos(2x) = +0.9 \text{ for the working angle}$$

$$(2x) = 25.8°$$

$$(2x) = 154.2,\ 205.8,\ 514.2,\ 565.8 \quad \text{from the sketch graph of } y = \cos x$$

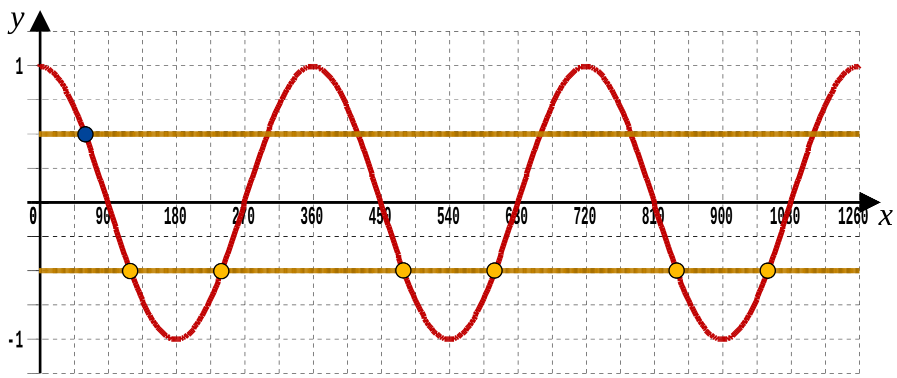
$$x = 77.1°,\ 102.9°,\ 257.1°,\ 282.9°$$

[4 marks]

Answer 3

$$\cos(3x) = -\frac{1}{2} \Rightarrow \text{Solve } \cos(3x) = +\frac{1}{2} \text{ for the working angle}$$

$$(3x) = 60°$$



$$(3x) = 120,\ 240,\ 480,\ 600,\ 840,\ 960 \quad \text{from the sketch graph of } y = \cos x$$

$$x = 40°,\ 80°,\ 160°,\ 200°,\ 280°,\ 320°$$

[6 marks]

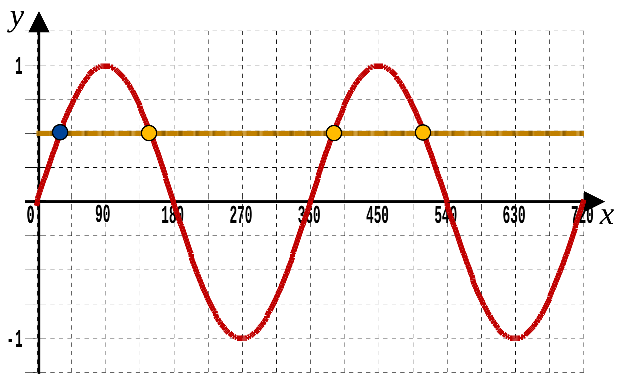
Answer 4

$$sin^3(2x) = \frac{1}{8}$$

Cube root both sides

$$sin(2x) = \frac{1}{2}$$

$$(2x) = 30° \text{ for the working angle}$$



$(2x) = 30, 150, 390, 510$ from the sketch graph of $y = sin\, x$

$x = 15°, 75°, 195°, 255°$

[6 marks]

Answer 5

$$\tan^2(3x) = 2\tan(3x) - 1$$

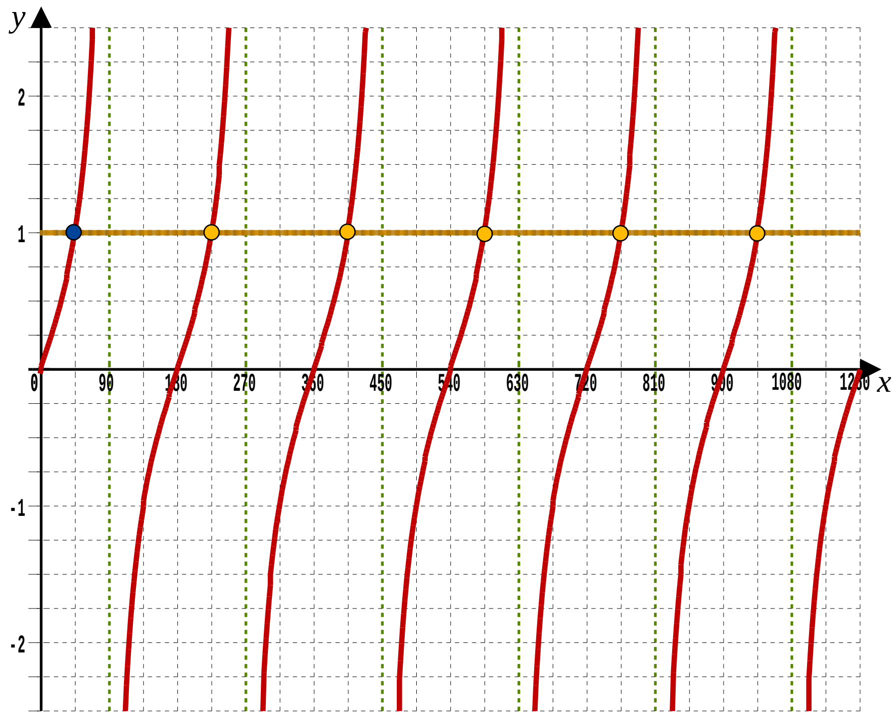
$$\tan^2(3x) - 2\tan(3x) + 1 = 0$$

$$(\tan(3x) - 1)(\tan(3x) - 1) = 0$$

$$\therefore\ (\tan(3x) - 1) = 0$$

$$\tan(3x) = 1$$

$$(3x) = 45° \text{ for the working angle}$$



$(3x) = 45,\ 225,\ 405,\ 585,\ 765,\ 945$ from the sketch graph of $y = \tan x$

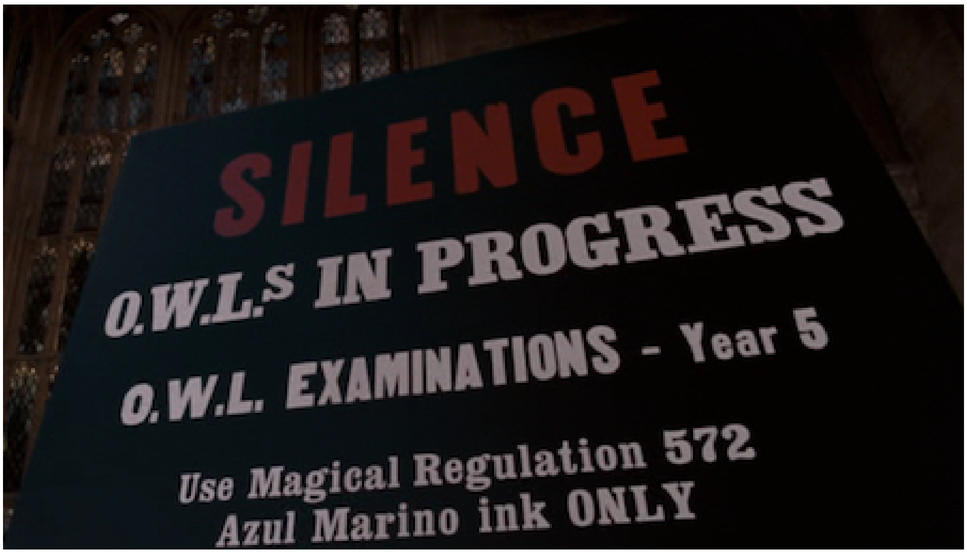
$x = 15°,\ 75°,\ 135°,\ 195°,\ 255°,\ 315°$

[8 marks]

Lesson 7

Additional Mathematics
A-Level Pure Mathematics : Year 1
Trigonometry IV

7.1 From The Examination Hall



Here's some wand waving wizard mathemagic that mysteriously proves 2 = 1

The Classic Proof that 2 = 1

Given any number, x, let y be the same number.

0) In other words $\quad x = y$

1) Multiply both sides by x $\quad x^2 = xy$

2) Subtract y^2 from both sides $\quad x^2 - y^2 = xy - y^2$

3) Factorise both sides $\quad (x+y)(x-y) = y(x-y)$

4) Divide both sides by $(x-y)$ $\quad \dfrac{(x+y)(x-y)}{(x-y)} = \dfrac{y(x-y)}{(x-y)}$

5) Simplify $\quad (x+y) = y$

6) Recall that $x = y$ $\quad (y+y) = y$

7) Simplify $\quad 2y = y$

8) Divide both sides by y $\quad \dfrac{2y}{y} = \dfrac{y}{y}$

9) Simplify $\quad 2 = 1 \quad \square$

Clearly, this proof cannot be correct. On which line is there a mistake ?
The lines are numbered from 1 to 9.
Write down the line number containing a mistake :

7.2 An Examination Temptation

An old examination question that tried to tempt candidates into making the same mistake as in the false proof is this;

Solve the equation $\tan^2 x = \tan x$ for $0° \leqslant x < 360°$

Teaching Video: http://www.NumberWonder.co.uk/v9044/7.mp4

[5 marks]

7.3 The Equation $a \sin x = b \cos x$

In spite of the worry over dividing by zero, in practice it's often not the primary focus in examination questions.
In particular look out for the following popular question, easily tackled by dividing both sides by $\cos \theta$

Example : Solve $\sin x = \sqrt{3} \cos x$ for $0 \leqslant x \leqslant 360°$

[4 marks]

Solution :

$$\sin x = \sqrt{3} \cos x$$

Divide both sides by $\cos x$

(OK as $\cos x = 0$ is not a solution, i.e. $x = 90°$ is not a solution)

$$\frac{\sin x}{\cos x} = \sqrt{3}$$

$$\tan x = \sqrt{3}$$

$$x = 60°, \ 240°$$ (From a sketch graph of $y = \tan x$)

[4 marks]

7.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 43

Question 1

Additional Mathematics Examination Question from June 2016, Q3 (OCR)

Find all the values of x in the domain $0 < x < 360°$ that satisfy,

$$3\, sin\, x = 4\, cos\, x$$

[4 marks]

Question 2

Additional Mathematics Examination Question from June 2007, Q4 (OCR)

Find all the values of x in the range $0° < x < 360°$ that satisfy

$$sin\, x = -4\, cos\, x$$

[5 marks]

Question 3

Additional Mathematics Examination Question from June 2011, Q4 (OCR)

Solve the equation

$$5\,sin\,2x \;=\; 2\,cos\,2x$$

in the interval $0° < x < 360°$

Give your answers correct to 1 decimal place.

[5 marks]

Question 4

A-Level Examination Question from January 2018, Paper C12, Q5(i) edited (Edexcel)

Solve, for $0 \;<\; x \;<\; 90°$

$$5\,sin\,(3x) \;-\; 7\,cos\left(3x\right) \;=\; 0$$

Give each solution to 3 significant figures

[5 marks]

Question 5

A-Level Examination Question from January 2009, Paper C2, Q8 (Edexcel)

(a) Show that the equation

$$4\,sin^2 x + 9\,cos\,x - 6 = 0$$

can be written as

$$4\,cos^2 x - 9\,cos\,x + 2 = 0$$

[2 marks]

(b) Hence solve, for $0° < x < 720°$

$$4\,sin^2 x + 9\,cos\,x - 6 = 0$$

giving your answers to 1 decimal place.

[6 marks]

Question 6

Solve $\cos^2 x = \cos x$ for $0° \leqslant x < 360°$

[5 marks]

Question 7

A-Level Examination Question from January 2006, Paper C2, Q8b edited (Edexcel)

Find all the values of x, to 1 decimal place, in the interval $0° < x < 360°$ for which

$$\tan^2 x = 4$$

[5 marks]

Question 8

$$3\,tan^4 x \;-\; 10\,tan^2 x \;+\; 3 \;=\; 0 \qquad\qquad 0^\circ \;\leqslant\; x \;\leqslant\; 360^\circ$$

[6 marks]

7.5 Answers

7.5.1 Solution to 7.2 An Examination Temptation (Teaching Video)

Th mistake to avoid is dividing both sides by *tan x* because this would be dividing by zero.

In other words, $tan\, x = 0$ is a solution to the equation for then $tan^2\, x = 0$ and the LHS = RHS.

Instead, subtract $tan\, x$ from both sides and factorise.

$$tan^2\, x = tan\, x$$

$$tan^2\, x - tan\, x = 0$$

$$tanx\ (\ tanx - 1\ ) = 0$$

Either $tan\, x = 0 \Rightarrow x = 0,\ 180$: Reject 360 as it's not in the domain

Or $tan\, x - 1 = 0 \Rightarrow tan\, x = 1 \Rightarrow x = 45,\ 225$

$$\text{Answers } x \in \{\, 0°,\ 45°,\ 180°,\ 225° \,\}$$

[5 marks]

7.5.2 Solutions to 7.4 Exercise

Answer 1

$$3\, sin\, x = 4\, cos\, x$$

$$\frac{sin\, x}{cos\, x} = \frac{4}{3}$$

$$tan\, x = \frac{4}{3}$$

$$x = arctan\left(\frac{4}{3}\right)$$

$x = 53.1°,\ 233.1°$ (From a sketch graph of $y = tan\, x$)

[4 marks]

Answer 2

$$\sin x = -4\cos x$$

$$\frac{\sin x}{\cos x} = -4$$

$$\tan x = -4 \qquad \text{Solve } \tan x = +4$$

$$x = \arctan(+4)$$

$$= 76° \text{ working angle}$$

$$x = 104°,\ 284° \qquad \text{(From a sketch graph of } y = \tan x\text{)}$$

[5 marks]

Answer 3

$$5\sin 2x = 2\cos 2x$$

$$\frac{\sin 2x}{\cos 2x} = \frac{2}{5}$$

$$\tan 2x = \frac{2}{5}$$

$$2x = \arctan\left(\frac{2}{5}\right)$$

$$2x = 21.8°,\ 201.8°,\ 381.8°,\ 561.8°$$

(From a sketch graph of $y = \tan x$)

$$x = 10.9°,\ 100.9°,\ 190.9°,\ 280.9°$$

[5 marks]

Answer 4

$$5\sin(3x) - 7\cos(3x) = 0$$

$$5\sin(3x) = 7\cos(3x)$$

$$\frac{\sin(3x)}{\cos(3x)} = \frac{7}{5}$$

$$\tan(3x) = \frac{7}{5}$$

$$3x = \arctan\left(\frac{7}{5}\right)$$

$$3x = 54.5°,\ 234.5°$$

(From a sketch graph of $y = \tan x$)

$$x = 18.2°,\ 78.2°$$

[5 marks]

Answer 5

(a)

$$4\,sin^2 x + 9\,cos\,x - 6 = 0$$

$$4\,(1 - cos^2 x) + 9\,cos\,x - 6 = 0$$

$$4 - 4\,cos^2 x + 9\,cos\,x - 6 = 0$$

$$4\,cos^2 x - 9\,cos\,x + 2 = 0 \qquad \square$$

[2 marks]

(b)

$$4\,sin^2 x + 9\,cos\,x - 6 = 0$$

$$4\,cos^2 x - 9\,cos\,x + 2 = 0 \qquad \text{from part (a)}$$

$$(4\,cos\,x - 1)\,(cos\,x - 2) = 0$$

Either $cos\,x = \dfrac{1}{4}$ or $cos\,x = 2$

$$x = arccos\left(\frac{1}{4}\right) \qquad cos\,x = 2 \quad \text{(No solutions)}$$

$$x = 75.5°,\ 284.5°,\ 435.5°,\ 644.5° \qquad \text{(From sketch graph)}$$

[6 marks]

Answer 6

$$cos^2 x = cos\,x$$

$$cos^2 x - cos\,x = 0$$

$$cos\,x\,(cos\,x - 1) = 0$$

Either $cos\,x = 0 \Rightarrow x = 90,\ 270$

Or $cos\,x = 1 \Rightarrow x = 0$

(360 is not in the domain for the question)

$\therefore$ Answer : $x \in \{0,\ 90,\ 270\}$

[5 marks]

Answer 7

$$tan^2 x = 4$$

$$tan\,x = \pm 2$$

$$x = arctan(\pm 2)$$

$$x = 63.4°,\ 116.6°,\ 243.4°,\ 296.6°$$

[5 marks]

Answer 8

Let $z = tan^2 x$ in which case the given equation becomes,

$$3z^2 - 10z + 3 = 0$$

$$(3z - 1)(z - 3) = 0$$

$$\text{Either } z = \frac{1}{3} \qquad \text{or} \qquad z = 3$$

$$tan^2 x = \frac{1}{3} \qquad \text{or} \qquad tan^2 x = 3$$

$$tan\, x = \pm \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \qquad \text{or} \qquad tan\, x = \pm\sqrt{3}$$

$$= \pm \frac{\sqrt{3}}{3}$$

$$x = 30°,\ 150°,\ 210°,\ 330° \qquad x = 60,\ 120,\ 240,\ 300$$

$$x \in \{ 30°,\ 60°,\ 120°,\ 150°,\ 210°,\ 240°,\ 300°,\ 330° \}$$

[6 marks]

Lesson 8

Additional Mathematics
A-Level Pure Mathematics : Year 1
Trigonometry IV

8.1 Revision

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

(a) **(i)** Sketch the graph of $y = \sin x$ over the interval $0° \leqslant x \leqslant 360°$

[2 marks]

(ii) Correctly place $0°$, $90°$, $180°$, $270°$, and $360°$ on the x-axis.

[1 mark]

(iii) Correctly place -1, 0, and $+1$ on the y-axis

[1 mark]

(b) Calculate the values of x in the domain $0° \leqslant x \leqslant 360°$ for which

$$\sin x = -0.416$$

[4 marks]

Question 2

You are given that

$$\cos x = \frac{2}{3}$$

with $0° < x < 90°$

(a) Use the identity $\cos^2 x + \sin^2 x = 1$ to find an **exact** value for $\sin x$ and, if appropriate, simplify your answer.

[3 marks]

(b) Use the identity $\tan x = \dfrac{\sin x}{\cos x}$ and your part (a) answer to find an **exact** value for $\tan x$ and, if appropriate, simplify your answer.

[3 marks]

Question 3

Find the four values of x in the domain $0° < x < 360°$ that satisfy the equation

$$cos\left(2x\right) = \frac{4}{5}$$

Give your answers to 1 decimal place.

[5 marks]

Question 4

Solve, for $0° \leqslant x < 360°$

$$sin^2 x = \frac{3}{4}$$

[5 marks]

Question 5

(a) Show that the equation

$$5\,cos^2\,x\;+\;16\,sin\,x\;-\;8\;=\;0$$

can be written as

$$5\,sin^2\,x\;-\;16\,sin\,x\;+\;3\;=\;0$$

[3 marks]

(b) Hence solve, for $0° \leqslant x \leqslant 720°$

$$5\,cos^2\,x\;+\;16\,sin\,x\;-\;8\;=\;0$$

giving your answers to 1 decimal place.

[6 marks]

Question 6

Solve, for $0° < x < 360°$

$$\tan x\ (\tan x - 2) = 3$$

[6 marks]

Question 7

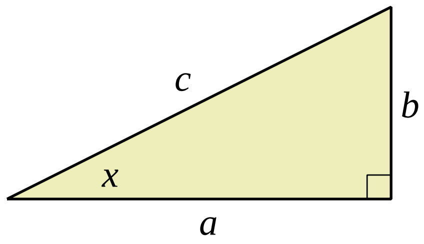
A-Level Examination Question from June 2009, Paper C2, Q7b (Edexcel)

Solve for $0° \leqslant x \leqslant 360°$

$$4 \sin x = 3 \tan x$$

[6 marks]

Question 8



Using the triangle, or otherwise, prove that, for $0° \leqslant x \leqslant 90°$

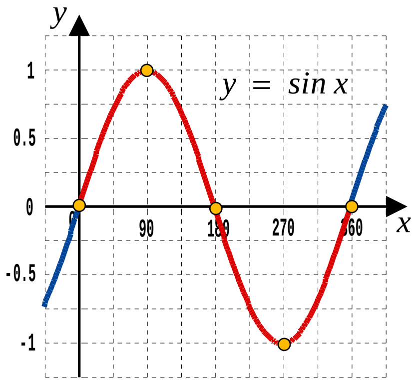
$$tan^2\, x \times cos^2\, x + cos^2\, x - 1 = 0$$

[5 marks]

8.2 Answers

Answer 1

(a)



[2, 1, 1 marks]

(b) 204.6°, 335.4°

[4 marks]

Answer 2

(a) $\sin x = \dfrac{\sqrt{5}}{3}$

(b) $\tan x = \dfrac{\sqrt{5}}{2}$

[3, 3 marks]

Answer 3

18.4°, 161.6°, 198.4°, 341.6°

Answer 4

60°, 120°, 240°, 300°

[5, 5 marks]

Answer 5

(a)

$$5\cos^2 x + 16\sin x - 8 = 0$$
$$5(1 - \sin^2 x) + 16\sin x - 8 = 0$$
$$5 - 5\sin^2 x + 16\sin x - 8 = 0$$
$$5\sin^2 x - 16\sin x + 3 = 0$$

[3 marks]

(b) 11.5°, 168.5°, 371.5°, 528.5°

[6 marks]

Answer 6

71.6°, 135°, 251.6°, 315°

Answer 7

0°, 41.4°, 180°, 318.6°, 360°

[6, 6 marks]

Answer 8

$$
\begin{aligned}
\text{LHS} &= tan^2\, x \times cos^2\, x + cos^2\, x - 1 \\
&= \left(\frac{b}{a}\right)^2 \left(\frac{a}{c}\right)^2 + \left(\frac{a}{c}\right)^2 - 1 \\
&= \frac{b^2 \times a^2}{a^2 \times c^2} + \frac{a^2}{c^2} - 1 \\
&= \frac{b^2}{c^2} + \frac{a^2}{c^2} - 1 \\
&= \frac{b^2 + a^2}{c^2} - 1 \\
&= \frac{c^2}{c^2} - 1 \\
&= 1 - 1 \\
&= 0 \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]

Lesson 9

Additional Mathematics
A-Level Pure Mathematics : Year 1
Trigonometry IV

9.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

(a) **(i)** Sketch the graph of $y = \cos x$ over the interval $0° \leqslant x \leqslant 360°$

[2 marks]

(ii) Correctly place $0°$, $90°$, $180°$, $270°$, and $360°$ on the x-axis.

[1 mark]

(iii) Correctly place -1, 0, and $+1$ on the y-axis

[1 mark]

(b) Calculate the values of x in the domain $0° \leqslant x \leqslant 360°$ for which

$$\cos x = -0.728$$

[4 marks]

Question 2

You are given that

$$\sin x = \frac{5}{6}$$

with $0° < x < 90°$

(a) Use the identity $\cos^2 x + \sin^2 x = 1$ to find an **exact** value for $\cos x$ and, if appropriate, simplify your answer.

[3 marks]

(b) Use the identity $\tan x = \dfrac{\sin x}{\cos x}$ and your part (a) answer to find an **exact** value for *tan x* and, if appropriate, simplify your answer.

[3 marks]

Question 3

Find the four values of x in the domain $0° < x < 360°$ that satisfy the equation

$$sin\left(2x\right) = \frac{3}{4}$$

Give your answers to 1 decimal place.

[5 marks]

Question 4

Solve, for $0° \leqslant x < 360°$

$$cos^2 x = \frac{2}{3}$$

[5 marks]

Question 5

(a) Show that the equation

$$7\,sin^2 x - 19\,cos\,x - 1 = 0$$

can be written as

$$7\,cos^2 x + 19\,cos\,x - 6 = 0$$

[3 marks]

(b) Hence solve, for $0° \leqslant x \leqslant 720°$

$$7\,sin^2 x - 19\,cos\,x - 1 = 0$$

giving your answers to 1 decimal place.

[6 marks]

Question 6

Solve, for $0° < x < 360°$

$$tan\,x\ (\,tan\,x + 4\,) = 5$$

[6 marks]

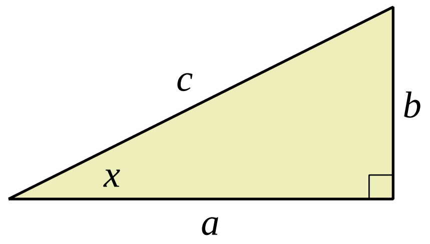
Question 7

Solve for $0^\circ \leqslant x \leqslant 360^\circ$

$$4\,cos\,x \;=\; 15\,tan\,x$$

[6 marks]

Question 8



Using the triangle, or otherwise, prove that, for $0° \leqslant x \leqslant 90°$

$$tan^2 x - sin^2 x - tan^2 x \times sin^2 x = 0$$

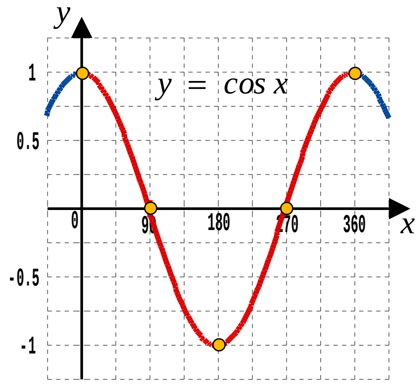
[5 marks]

9.2 Answers

Additional Mathematics
A-Level Pure Mathematics : Year 1
Trigonometry IV

Answer 1

(a)



[2, 1, 1 marks]

(b) 136.7°, 223.3°

[4 marks]

Answer 2

(a) $\cos x = \dfrac{\sqrt{11}}{6}$

(b) $\tan x = \dfrac{5\sqrt{11}}{11}$

[3, 3 marks]

Answer 3

24.3°, 65.7°, 204.3°, 245.7°

Answer 4

35.3°, 144.7°, 215.3°, 324.7°

[5, 5 marks]

Answer 5

(a)

$$7\sin^2 x - 19\cos x - 1 = 0$$

$$7\left(1 - \cos^2 x\right) - 19\cos x - 1 = 0$$

$$7 - 7\cos^2 x - 19\cos x - 1 = 0$$

$$7\cos^2 x + 19\cos x - 6 = 0$$

[3 marks]

(b) 73.4°, 286.6°, 433.4°, 646.6°

[6 marks]

Answer 6

45°, 101.3°, 225°, 281.3°

Answer 7

14.5°, 165.5°

[6, 6 marks]

Answer 8

$$
\begin{aligned}
\text{LHS} &= tan^2 x - sin^2 x - tan^2 x \times sin^2 x \\
&= \left(\frac{b}{a}\right)^2 - \left(\frac{b}{c}\right)^2 - \left(\frac{b}{a}\right)^2 \times \left(\frac{b}{c}\right)^2 \\
&= \frac{c^2 \times b^2}{c^2 \times a^2} + \frac{a^2 \times b^2}{a^2 \times c^2} - \frac{b^2 \times b^2}{a^2 \times c^2} \\
&= \frac{b^2\left(c^2 - a^2\right) - b^2 \times b^2}{c^2 \times a^2} \\
&= \frac{b^2 \times b^2 - b^2 \times b^2}{c^2} \\
&= 0 \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]