5.3 Answers

5.3.1 Solution to Example N° 2 (Teaching Video)

$$\frac{9x^2}{(6x+1)(3x+1)^2} = \frac{A}{(6x+1)} + \frac{B}{(3x+1)} + \frac{C}{(3x+1)^2}$$

$$\frac{9x^2\,[\ \dots\dots\ ]}{[\ (6x+1)(3x+1)^2\ ]} = \frac{A\,[\ \dots\dots\ ]}{(6x+1)} + \frac{B\,[\ \dots\dots\ ]}{(3x+1)} + \frac{C\,[\ \dots\dots\ ]}{(3x+1)^2}$$

$$9x^2 = A(3x+1)^2 + B(6x+1)(3x+1) + C(6x+1)$$

$$\text{Let } x = -\frac{1}{3} \ : \ 1 = C(-1) \Rightarrow C = -1$$

$$\text{Let } x = -\frac{1}{6} \ : \ \frac{1}{4} = A\left(\frac{1}{4}\right) \Rightarrow A = 1$$

$$\text{Let } x = 0 \ : \ 0 = A + B + C \Rightarrow B = 0$$

$$\int \frac{9x^2}{(6x+1)(3x+1)^2}\,dx = \int \frac{1}{6x+1}\,dx - \int \frac{1}{(3x+1)^2}\,dx$$

$$= \frac{1}{6}\int 6(6x+1)^{-1}\,dx - \frac{1}{3}\int 3(3x+1)^{-2}\,dx$$

$$= \frac{1}{6}\,ln|6x+1| - \frac{1}{3} \times \frac{(3x+1)^{-1}}{(-1)} + c$$

$$= \frac{1}{6}\,ln|6x+1| + \frac{1}{3(3x+1)} + c$$

[8 marks]

5.3.2 Solutions to 5.2 Exercise

Answer 1

(i)

$$\frac{(10x + 19)[\,\dots\dots\,]}{[\,(2x + 1)(x + 4)\,]} = \frac{A[\,\dots\dots\,]}{(2x + 1)} + \frac{B\,[\,\dots\dots\,]}{(x + 4)}$$

$$10x + 19 = A\,(x + 4) + B\,(2x + 1)$$

$$\text{Let } x = -4 \; : \; -21 = B(-7) \;\Rightarrow\; B = 3$$

$$\text{Let } x = -\frac{1}{2} \; : \quad 14 = A\left(\frac{7}{2}\right) \;\Rightarrow\; A = 4$$

$$\frac{10x + 19}{(2x + 1)(x + 4)} = \frac{4}{(2x + 1)} + \frac{3}{(x + 4)}$$

[4 marks]

(ii)

$$\int_0^1 \frac{10x + 19}{(2x + 1)\,(x + 4)}\,dx = \int_0^1 \frac{4}{(2x + 1)} + \frac{3}{(x + 4)}\,dx$$

$$= 4 \times \frac{1}{2}\int_0^1 2\,(2x + 1)^{-1}\,dx + 3\int_0^1 (x + 4)^{-1}\,dx$$

$$= \Big[\,2\,ln|2x + 1| + 3\,ln(x + 4)\Big]_0^1$$

$$= 2\,ln\,3 + 3\,ln\,5 - 2\,ln\,1 - 3\,ln\,4$$

$$= ln\,3^2 + ln\,5^3 - 0 - ln\,2^6$$

$$= ln\left(\frac{3^2 \times 5^3}{2^6}\right) \qquad \square$$

[6 marks]

Answer 2

$$\frac{(x+2)[\ \dots\dots\ ]}{[\ (x-1)(2x+1)\ ]} = \frac{A[\ \dots\dots\ ]}{(x-1)} + \frac{B[\ \dots\dots\ ]}{(2x+1)}$$

$$x+2 = A(2x+1) + B(x-1)$$

$$\text{Let } x = 1 : \quad 3 = A(3) \qquad \Rightarrow \quad A = 1$$

$$\text{Let } x = -\frac{1}{2} : \quad \frac{3}{2} = B\left(-\frac{3}{2}\right) \quad \Rightarrow \quad B = -1$$

$$\frac{x+2}{(x-1)(2x+1)} = \frac{1}{(x-1)} - \frac{1}{(2x+1)}$$

$$\int_2^4 \frac{x+2}{(x-1)(2x+1)}\,dx = \int_2^4 \frac{1}{(x-1)} - \frac{1}{(2x+1)}\,dx$$

$$= \int_2^4 (x-1)^{-1}\,dx - \frac{1}{2}\int_2^4 2(2x+1)^{-1}\,dx$$

$$= \Big[\ ln|x-1| - \frac{1}{2}\,ln|2x+1|\ \Big]_2^4$$

$$= ln\,3 - \frac{1}{2}\,ln\,9 - ln\,1 + \frac{1}{2}\,ln\,5$$

$$= ln\,3 - ln\,9^{\frac{1}{2}} - 0 + \frac{1}{2}\,ln\,5$$

$$= ln\,3 - ln\,3 + \frac{1}{2}\,ln\,5$$

$$= \frac{1}{2}\,ln\,5 \qquad \square$$

[10 marks]

Answer 3

(i)

$$\frac{1}{(2x-1)x^2} = \frac{A}{(2x-1)} + \frac{B}{x} + \frac{C}{x^2}$$

$$\frac{1\,[\ \ldots\ldots\ ]}{[\ (2x-1)x^2\ ]} = \frac{A\,[\ \ldots\ldots\ ]}{(2x-1)} + \frac{B\,[\ \ldots\ldots\ ]}{x} + \frac{C\,[\ \ldots\ldots\ ]}{x^2}$$

$$1 = Ax^2 + B(2x-1)x + C(2x-1)$$

$$\text{Let } x = 0 : 1 = C(-1) \quad \Rightarrow C = -1$$

$$\text{Let } x = \frac{1}{2} : 1 = A\left(\frac{1}{4}\right) \quad \Rightarrow A = 4$$

$$\text{Let } x = 1 : 1 = A + B + C \Rightarrow B = -2$$

That is,

$$\frac{1}{(2x-1)x^2} = \frac{4}{(2x-1)} - \frac{2}{x} - \frac{1}{x^2}$$

[4 marks]

(ii) Leaving out the limits for now (to reduce clutter),

$$\int \frac{1}{(2x-1)x^2}\,dx = 4\int \frac{1}{2x-1}\,dx - 2\int \frac{1}{x}\,dx - \int \frac{1}{x^2}\,dx$$

$$= 4 \times \frac{1}{2}\int 2(2x-1)^{-1}\,dx - 2\int x^{-1}\,dx - \int x^{-2}\,dx$$

$$= 2\,ln\,|2x-1| - 2\,ln|x| + \frac{1}{x} + c$$

With an upper limit of 2 and a lower limit of 1,

$$= 2\,ln\,3 - 2\,ln\,2 + \frac{1}{2} - 2\,ln\,1 + 2\,ln\,1 - \frac{1}{1}$$

$$= ln\,3^2 - ln\,2^2 - \frac{1}{2}$$

$$= ln\,9 - ln\,4 - \frac{1}{2}$$

$$= ln\left(\frac{9}{4}\right) - \frac{1}{2} \qquad \square$$

[6 marks]

Answer 4

(a)

$$\frac{1}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{(3x-1)} + \frac{C}{(3x-1)^2}$$

$$\frac{1\,[\,\ldots\ldots\,]}{[\,x(3x-1)^2\,]} = \frac{A\,[\,\ldots\ldots\,]}{x} + \frac{B\,[\,\ldots\ldots\,]}{(3x-1)} + \frac{C\,[\,\ldots\ldots\,]}{(3x-1)^2}$$

$$1 = A(3x-1)^2 + Bx(3x-1) + Cx$$

$$\text{Let } x = 0 : 1 = A(-1)^2 \qquad \Rightarrow A = 1$$

$$\text{Let } x = \frac{1}{3} : 1 = C\left(\frac{1}{3}\right) \qquad \Rightarrow C = 3$$

$$\text{Let } x = 1 : 1 = 4A + 2B + C \Rightarrow B = -3$$

That is,

$$\frac{1}{x(3x-1)^2} = \frac{1}{x} - \frac{3}{(3x-1)} + \frac{3}{(3x-1)^2}$$

[4 marks]

(b) **(i)**

$$\int \frac{1}{x(3x-1)^2}\,dx = \int \frac{1}{x}\,dx - \int \frac{3}{(3x-1)}\,dx + \int \frac{3}{(3x-1)^2}\,dx$$

$$= \int x^{-1}\,dx - \int 3(3x-1)^{-1}\,dx + \int 3(3x-1)^{-2}\,dx$$

$$= ln|x| - ln|3x-1| + \frac{(3x-1)^{-1}}{(-1)} + c$$

$$= ln|x| - ln|3x-1| - \frac{1}{3x-1} + c$$

[3 marks]

(ii) With an upper limit of 2 and a lower limit of 1,

$$\int_1^2 \frac{1}{x(3x-1)^2}\,dx = \left[\, ln|x| - ln|3x-1| - \frac{1}{3x-1} \,\right]_1^2$$

$$= ln\,2 - ln\,5 - \frac{1}{5} - ln\,1 + ln\,2 + \frac{1}{2}$$

$$= \frac{1}{2} - \frac{1}{5} + 2\,ln\,2 - ln\,5$$

$$= \frac{3}{10} + ln\left(\frac{4}{5}\right)$$

[3 marks]