

5.3 Answers

A-Level Pure Mathematics : Year 2

Integration II

5.3.1 Solution to Example N° 2 (Teaching Video)

$$\frac{9x^2}{(6x+1)(3x+1)^2} = \frac{A}{(6x+1)} + \frac{B}{(3x+1)} + \frac{C}{(3x+1)^2}$$
$$\frac{9x^2 \text{ [... ...]}}{\text{[(6x+1)(3x+1)^2]}} = \frac{A \text{ [... ...]}}{(6x+1)} + \frac{B \text{ [... ...]}}{(3x+1)} + \frac{C \text{ [... ...]}}{(3x+1)^2}$$
$$9x^2 = A(3x+1)^2 + B(6x+1)(3x+1) + C(6x+1)$$

$$\text{Let } x = -\frac{1}{3} : 1 = C(-1) \Rightarrow C = -1$$

$$\text{Let } x = -\frac{1}{6} : \frac{1}{4} = A\left(\frac{1}{4}\right) \Rightarrow A = 1$$

$$\text{Let } x = 0 : 0 = A + B + C \Rightarrow B = 0$$

$$\begin{aligned} \int \frac{9x^2}{(6x+1)(3x+1)^2} dx &= \int \frac{1}{6x+1} dx - \int \frac{1}{(3x+1)^2} dx \\ &= \frac{1}{6} \int 6(6x+1)^{-1} dx - \frac{1}{3} \int 3(3x+1)^{-2} dx \\ &= \frac{1}{6} \ln|6x+1| - \frac{1}{3} \times \frac{(3x+1)^{-1}}{(-1)} + c \\ &= \frac{1}{6} \ln|6x+1| + \frac{1}{3(3x+1)} + c \end{aligned}$$

[8 marks]

5.3.2 Solutions to 5.2 Exercise

Answer 1

(i)

$$\frac{(10x + 19) [\dots \dots]}{[(2x + 1)(x + 4)]} = \frac{A [\dots \dots]}{(2x + 1)} + \frac{B [\dots \dots]}{(x + 4)}$$

$$10x + 19 = A(x + 4) + B(2x + 1)$$

$$\text{Let } x = -4 : -21 = B(-7) \Rightarrow B = 3$$

$$\text{Let } x = -\frac{1}{2} : 14 = A\left(\frac{7}{2}\right) \Rightarrow A = 4$$

$$\frac{10x + 19}{(2x + 1)(x + 4)} = \frac{4}{(2x + 1)} + \frac{3}{(x + 4)}$$

[4 marks]

(ii)

$$\begin{aligned} \int_0^1 \frac{10x + 19}{(2x + 1)(x + 4)} dx &= \int_0^1 \frac{4}{(2x + 1)} + \frac{3}{(x + 4)} dx \\ &= 4 \times \frac{1}{2} \int_0^1 2(2x + 1)^{-1} dx + 3 \int_0^1 (x + 4)^{-1} dx \\ &= \left[2 \ln|2x + 1| + 3 \ln(x + 4) \right]_0^1 \\ &= 2 \ln 3 + 3 \ln 5 - 2 \ln 1 - 3 \ln 4 \\ &= \ln 3^2 + \ln 5^3 - 0 - \ln 2^6 \\ &= \ln \left(\frac{3^2 \times 5^3}{2^6} \right) \quad \square \end{aligned}$$

[6 marks]

Answer 2

$$\frac{(x+2) [\dots]}{(x-1)(2x+1)} = \frac{A [\dots]}{(x-1)} + \frac{B [\dots]}{(2x+1)}$$

$$x+2 = A(2x+1) + B(x-1)$$

$$\text{Let } x = 1 : 3 = A(3) \Rightarrow A = 1$$

$$\text{Let } x = -\frac{1}{2} : \frac{3}{2} = B\left(-\frac{3}{2}\right) \Rightarrow B = -1$$

$$\frac{x+2}{(x-1)(2x+1)} = \frac{1}{(x-1)} - \frac{1}{(2x+1)}$$

$$\begin{aligned} \int_2^4 \frac{x+2}{(x-1)(2x+1)} dx &= \int_2^4 \frac{1}{(x-1)} - \frac{1}{(2x+1)} dx \\ &= \int_2^4 (x-1)^{-1} dx - \frac{1}{2} \int_2^4 2(2x+1)^{-1} dx \\ &= \left[\ln|x-1| - \frac{1}{2} \ln|2x+1| \right]_2^4 \\ &= \ln 3 - \frac{1}{2} \ln 9 - \ln 1 + \frac{1}{2} \ln 5 \\ &= \ln 3 - \ln 9^{\frac{1}{2}} - 0 + \frac{1}{2} \ln 5 \\ &= \ln 3 - \ln 3 + \frac{1}{2} \ln 5 \\ &= \frac{1}{2} \ln 5 \quad \square \end{aligned}$$

[10 marks]

Answer 3**(i)**

$$\frac{1}{(2x-1)x^2} = \frac{A}{(2x-1)} + \frac{B}{x} + \frac{C}{x^2}$$

$$\frac{1 \text{ [...]}}{(2x-1)x^2} = \frac{A \text{ [...]}}{(2x-1)} + \frac{B \text{ [...]}}{x} + \frac{C \text{ [...]}}{x^2}$$

$$1 = Ax^2 + B(2x-1)x + C(2x-1)$$

$$\text{Let } x = 0 : 1 = C(-1) \Rightarrow C = -1$$

$$\text{Let } x = \frac{1}{2} : 1 = A\left(\frac{1}{4}\right) \Rightarrow A = 4$$

$$\text{Let } x = 1 : 1 = A + B + C \Rightarrow B = -2$$

That is,

$$\frac{1}{(2x-1)x^2} = \frac{4}{(2x-1)} - \frac{2}{x} - \frac{1}{x^2}$$

[4 marks]**(ii)** Leaving out the limits for now (to reduce clutter),

$$\begin{aligned} \int \frac{1}{(2x-1)x^2} dx &= 4 \int \frac{1}{2x-1} dx - 2 \int \frac{1}{x} dx - \int \frac{1}{x^2} dx \\ &= 4 \times \frac{1}{2} \int 2(2x-1)^{-1} dx - 2 \int x^{-1} dx - \int x^{-2} dx \\ &= 2 \ln |2x-1| - 2 \ln |x| + \frac{1}{x} + c \end{aligned}$$

With an upper limit of 2 and a lower limit of 1,

$$\begin{aligned} &= 2 \ln 3 - 2 \ln 2 + \frac{1}{2} - 2 \ln 1 + 2 \ln 1 - \frac{1}{1} \\ &= \ln 3^2 - \ln 2^2 - \frac{1}{2} \\ &= \ln 9 - \ln 4 - \frac{1}{2} \\ &= \ln\left(\frac{9}{4}\right) - \frac{1}{2} \quad \square \end{aligned}$$

[6 marks]

Answer 4**(a)**

$$\frac{1}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{(3x-1)} + \frac{C}{(3x-1)^2}$$

$$\frac{1 \text{ [... ...]}}{[x(3x-1)^2]} = \frac{A \text{ [... ...]}}{x} + \frac{B \text{ [... ...]}}{(3x-1)} + \frac{C \text{ [... ...]}}{(3x-1)^2}$$

$$1 = A(3x-1)^2 + Bx(3x-1) + Cx$$

$$\text{Let } x = 0 : 1 = A(-1)^2 \Rightarrow A = 1$$

$$\text{Let } x = \frac{1}{3} : 1 = C\left(\frac{1}{3}\right) \Rightarrow C = 3$$

$$\text{Let } x = 1 : 1 = 4A + 2B + C \Rightarrow B = -3$$

That is,

$$\frac{1}{x(3x-1)^2} = \frac{1}{x} - \frac{3}{(3x-1)} + \frac{3}{(3x-1)^2}$$

[4 marks]**(b) (i)**

$$\begin{aligned} \int \frac{1}{x(3x-1)^2} dx &= \int \frac{1}{x} dx - \int \frac{3}{(3x-1)} dx + \int \frac{3}{(3x-1)^2} dx \\ &= \int x^{-1} dx - \int 3(3x-1)^{-1} dx + \int 3(3x-1)^{-2} dx \\ &= \ln|x| - \ln|3x-1| + \frac{(3x-1)^{-1}}{(-1)} + c \\ &= \ln|x| - \ln|3x-1| - \frac{1}{3x-1} + c \end{aligned}$$

[3 marks]**(ii)** With an upper limit of 2 and a lower limit of 1,

$$\begin{aligned} \int_1^2 \frac{1}{x(3x-1)^2} dx &= \left[\ln|x| - \ln|3x-1| - \frac{1}{3x-1} \right]_1^2 \\ &= \ln 2 - \ln 5 - \frac{1}{5} - \ln 1 + \ln 2 + \frac{1}{2} \\ &= \frac{1}{2} - \frac{1}{5} + 2 \ln 2 - \ln 5 \\ &= \frac{3}{10} + \ln\left(\frac{4}{5}\right) \end{aligned}$$

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