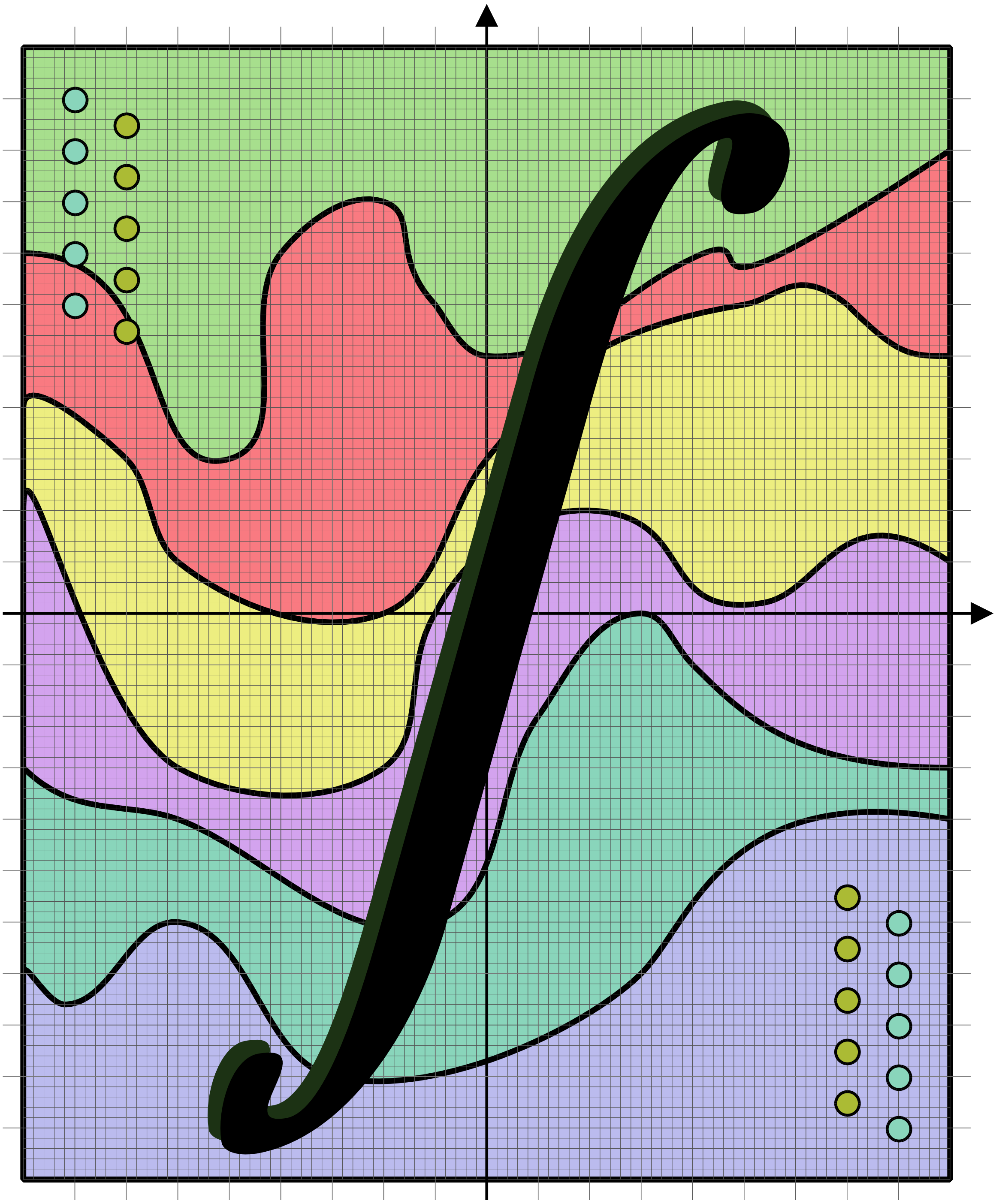
A-Level Pure Mathematics
Year 2

INTEGRATION

II



"Areas Between Curves"

Lesson 1

1.1 Calculus Compounded

Initial studies of Calculus focus upon the Differentiation and Integration of polynomials using The Power Rule.

The Calculus Power Rule

(i) If $y = x^n$ then $\dfrac{dy}{dx} = n\,x^{n-1}$

(ii) If $y = x^n$ then $\displaystyle\int y \; dx = \frac{x^{n+1}}{n+1} + c$ $\quad n \neq -1$

where n is a constant, and c is "the constant of integration"

In the Year 2 A-Level course, it was discovered that several other elementary functions, not just the polynomials, where amenable to being differentiated. For example, exponential, logarithmic and trigonometric functions all have derivatives. The chain, product and quotient rules widened further the variety of expressions that could be differentiated.

Here is a reminder of four key results that were obtained, where a is a constant;

Exponential

- If $y = e^{ax}$ then $\dfrac{dy}{dx} = a\,e^{ax}$

Natural Logarithm

- If $y = ln|ax|$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$

Sine

- If $y = sin\,(ax)$ then $\dfrac{dy}{dx} = a\,cos\,(ax)$

Cosine

- If $y = cos\,(ax)$ then $\dfrac{dy}{dx} = -\,a\,sin\,(ax)$

Our ability to differentiate has become much further advanced than our ability to integrate. The time has come to balance this state of affairs !

The key to progressing our knowledge of integration is a result already known;

The Fundamental Theorem of Calculus

The process of integration (finding areas under curves)
is the inverse of
the process of differentiation (finding gradients of curves)

In consequence, each of the four differentiation statements just made can be rephrased as an integration statement simply by "thinking backwards".
Again *a* is a constant and, in addition, as with the polynomials, there will be "a constant of integration", usually denoted *c*

Exponential

- If $y = e^{ax}$ then $\int y\,dx = \frac{1}{a}e^{ax} + c$

Natural Logarithm

- If $y = \frac{1}{x}$ then $\int y\,dx = ln\,|x| + c$

Sine

- If $y = sin\,(ax)$ then $\int y\,dx = -\frac{1}{a}cos\,(ax) + c$

Cosine

- *If* $y = cos\,(ax)$ then $\int y\,dx = \frac{1}{a}\,sin\,(ax) + c$

Note : **1)** RADIANS must be used whenever trigonometry and calculus mix.
2) With the natural logarithm function a step was missed out

More precisely • If $y = \frac{1}{x}$ then

$$\int y\,dx = ln|ax| + k$$
$$= ln|x| + ln|a| + k$$
$$= ln|x| + c$$

because one unknown number, $ln|a|$, plus a second unknown number, *k*, add together to give a third unknown number, *c*

1.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1
Write down the integral of each of the following.

(i) $\int 4x + \frac{1}{x}\, dx$

[2 marks]

(ii) $\int cos\,(4x)\;\, dx$

[2 marks]

(iii) $\int e^{5x}\, dx$

[2 marks]

(iv) $\int \sqrt{x}\; - \; sin\,(0.5x)\;\, dx$

[4 marks]

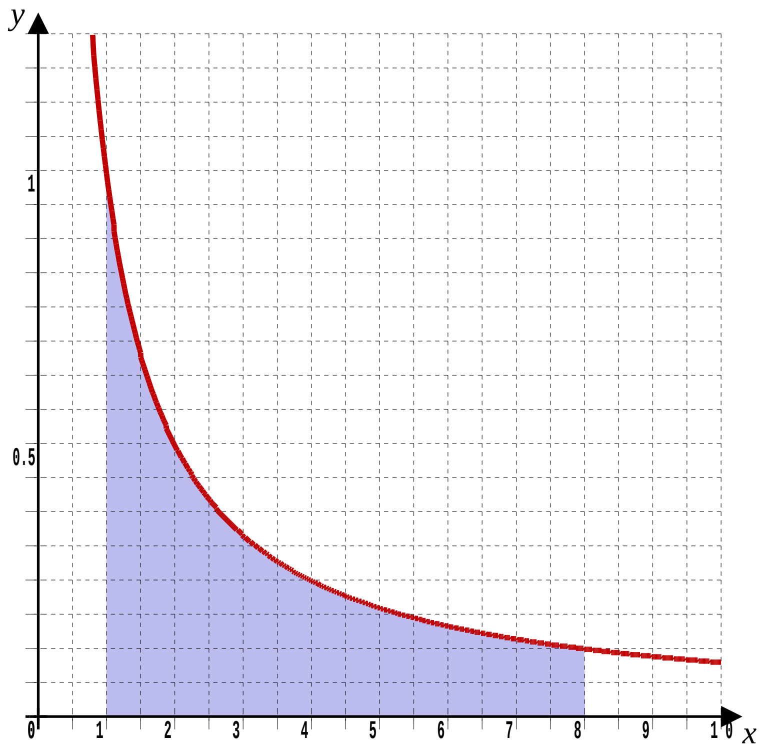
(v) $\int \left(1 \; + \; \frac{1}{x}\right)^2 \; dx$

[4 marks]

Top Tip : For each of the above, differentiating the answer should reverse the integration and get you back to the question. This is a handy way of checking any answer that you're not sure about.

Question 2

The graph is of the curve $y = \dfrac{1}{x}$

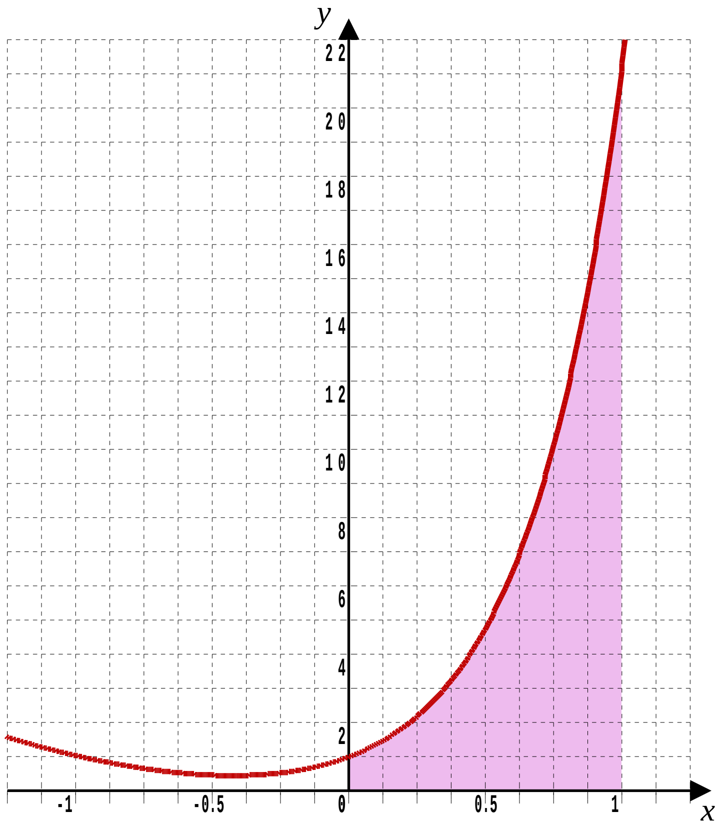


Use integration to show that the area shaded, bounded by the curve, the x-axis and the lines $x = 1$ and $x = 8$ is exactly $3 \ln 2$

[5 marks]

Question 3

The graph is of the curve $y = e^{3x} + x^2$



Use integration to show that the area shaded, bounded by the curve, the x-axis, the y-axis and $x = 1$ is exactly $\frac{1}{3}\left(e^{3}\right)$

[5 marks]

Question 4

Find the exact area under the following curve between the bounding lines given

$$y = 9\,cos\,(2x) \qquad x = 0, \qquad x = \frac{\pi}{4}$$

[5 marks]

Question 5

(i) Show that the area under the curve $y = 12\,e^{6x}$ between $x = 0$ and $x = 2$ is exactly $2\left(e^{12} - 1\right)$

[5 marks]

(ii) Give this area as a decimal correct to three decimal places.

[1 mark]

Question 6

Find the area under the curve between the bounding lines given

$$y = \frac{1}{9} sin \left(\frac{x}{2} \right) \qquad x = 0, \qquad x = \frac{\pi}{6}$$

Give your answer correct to 3 significant figures.

[5 marks]

1.3 Answers to 1.2 Exercise

Answer 1

(i) $\int 4x + \frac{1}{x}\, dx = 2x^2 + ln|x| + c$

[2 marks]

(ii) $\int cos(4x)\ dx = \frac{1}{4} sin(4x) + c$

[2 marks]

(iii) $\int e^{5x}\, dx = \frac{1}{5} e^{5x} + c$

[2 marks]

(iv) $\int \sqrt{x} - sin(0.5x)\ dx \quad = \int x^{\frac{1}{2}} - sin(0.5x)\ dx$

$$= \frac{2x^{\frac{3}{2}}}{3} + 2\, cos(0.5x) + c$$

[4 marks]

(v) $\int \left(1 + \frac{1}{x}\right)^2 dx = \int 1 + \frac{2}{x} + \frac{1}{x^2}\, dx$

$$= \int 1 + \frac{2}{x} + x^{-2}\, dx$$

$$= x + 2\, ln\, x - x^{-1} + c$$

[4 marks]

Answer 2

$$Area = \int_1^8 \frac{1}{x}\, dx$$

$$= [\ ln\ |x|\ ]_1^8$$

$$= ln\, 8 - ln\, 1$$

$$= ln\, 2^3 - 0$$

$$= 3\, ln\, 2 \qquad \square$$

[5 marks]

Answer 3

$$\int_0^1 e^{3x} + x^2\,dx = \left[\frac{1}{3} e^{3x} + \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{3} \left[e^{3x} + x^3 \right]_0^1$$

$$= \frac{1}{3} \left[e^3 + 1 - e^0 - 0 \right]$$

$$= \frac{1}{3} \left(e^3 \right)$$

[5 marks]

Answer 4

$$Area = \int_0^{\frac{\pi}{4}} 9\cos(2x)\,dx$$

$$= 9 \left[\frac{\sin(2x)}{2} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{9}{2} \sin\left(\frac{\pi}{2} \right) - 0$$

$$= \frac{9}{2}$$

[5 marks]

Answer 5

(i)

$$Area = \int_0^2 12\,e^{6x}\,dx$$

$$= 2 \left[e^{6x} \right]_0^2$$

$$= 2\left(e^{12} - e^0 \right)$$

$$= 2 (e^{12} - 1) \quad \square$$

[5 marks]

(ii)

325 507.583

[1 mark]

Answer 6

$$Area = \int_0^{\frac{\pi}{6}} \frac{1}{9} \sin\left(\frac{x}{2} \right) dx$$

$$= \frac{1}{9} \left[-2 \cos\left(\frac{x}{2} \right) \right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{9} \left\{ \left(-2 \cos\left(\frac{\pi}{12} \right) \right) - \left(-2 \cos 0 \right) \right\}$$

$$= 0.00757 \qquad (3 \text{ significant figures})$$

[5 marks]

2.1 Thinking Backwards

When $y = \dfrac{(4 + 5x)^6}{6}$ is differentiated the result is

$$\frac{dy}{dx} = \frac{6(4 + 5x)^5}{6} \times 5 \qquad \text{by the chain rule}$$

$$= 5(4 + 5x)^5$$

When $y = \dfrac{[f(x)]^{n+1}}{(n + 1)}$ is differentiated the result is

$$\frac{dy}{dx} = \frac{(n + 1)\ [f(x)]^n}{(n + 1)} \times f'(x) \quad \text{by the chain rule}$$

$$= f'(x)\ [f(x)]^n$$

By The Fundamental Theorem of Calculus,

The Chain Rule Backwards

$$\int f'(x)\ [f(x)]^n\ dx = \frac{[f(x)]^{n+1}}{(n + 1)} + c \qquad n \neq -1$$

The two key consequences of this result are;

- Every time a new integration question is tackled, a mental scan must be made to see if it is in the form of a function raised to a power with the derivative of that function sitting in front.

  In other words, watch out for $f'(x)\ [f(x)]^n$
- An alertness needs to be maintained for situations in which the desired set up of $f'(x)\ [f(x)]^n$ can be created by making use of a "fiddle factor". See example N° 2 and example N° 3.

Example N° 1

Determine: $\displaystyle\int 7(7x + 2)^3\ dx$

[3 marks]

Solution : It is spotted that with $f(x) = 7x + 2 \Rightarrow f'(x) = 7$ and that this situation is that of a chain rule backwards with $n = 3$

Thus, $\displaystyle\int 7(7x + 2)^3\ dx = \frac{(7x + 2)^4}{4} + c$

The teaching video will talk through Example N° 2 and Example N° 3

Teaching Video: http://www.NumberWonder.co.uk/v9045/2.mp4

Example N° 2

Determine: $\int (3x + 1)^4 \, dx$

[3 marks]

Example N° 3

Determine: $\int \frac{24}{(4x - 2)^4} \, dx$

[3 marks]

2.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

In each question use The Chain Rule Backwards to perform the integration given.
Most questions will require that a “fiddle factor” be introduced.

Question 1

$$\int (3x + 2)^5 \, dx$$

[3 marks]

Question 2

$$\int (7x - 3)^3 \, dx$$

[3 marks]

Question 3

$$60 \int (4 + 5x)^3 \, dx$$

[3 marks]

Question 4

$$\int (3 - 2x)^5 \, dx$$

[3 marks]

Question 5

$$\int \left(7 - \frac{1}{2}x\right)^6 dx$$

[3 marks]

Question 6

$$18 \int \sqrt{1 + 4x}\, dx$$

[3 marks]

Question 7

$$\int \left(\frac{2x}{3} + 8\right)^{\frac{1}{2}} dx$$

[3 marks]

Question 8

$$\int \frac{1}{(2x - 1)^4}\, dx$$

[3 marks]

Question 9

$$\int 6\,(2x - 1)^{-2}\, dx$$

[3 marks]

Question 10

$$\int \frac{6}{\sqrt{3x - 1}}\, dx$$

[3 marks]

Question 11

$$\int \frac{1}{4\,(x+3)^2}\,dx$$

[3 marks]

Question 12

$$\int 14\,(3+2x)^{-2}\,dx$$

[3 marks]

Question 13

$$\int \frac{1}{(1-2x)^{\frac{3}{2}}}\,dx$$

[3 marks]

Question 14

$$\int_0^1 (4x + 1)^4 \, dx$$

[3 marks]

Question 15

$$\int_0^4 \sqrt{2x + 1} \, dx$$

[4 marks]

Question 16

$$\int_2^5 \frac{1}{(x - 1)^3} \, dx$$

[4 marks]

2.3 Answers

2.3.1 Solutions to 2.1 Examples (Teaching Video)

Example N° 2

Add in a fiddle factor of $\frac{1}{3} \times 3$

$$\int (3x + 1)^4 dx = \frac{1}{3} \int 3 (3x + 1)^4 dx$$
$$= \frac{1}{3} \times \frac{(3x + 1)^5}{5} + c$$
$$= \frac{(3x + 1)^5}{15} + c$$

[3 marks]

Example N° 3

Add in a fiddle factor of $\frac{1}{4} \times 4$

$$\int \frac{24}{(4x - 2)^4} dx = 24 \int (4x - 2)^{-4} dx$$
$$= 24 \times \frac{1}{4} \int 4 (4x - 2)^{-4} dx$$
$$= 6 \times \frac{(4x - 2)^{-3}}{(-3)} + c$$
$$= -2 (4x - 2)^{-3} + c$$
$$= -\frac{2}{(4x - 2)^3} + c$$

[3 marks]

2.3.2 Solutions to 3.2 Exercise

Answer 1

Add in a fiddle factor of $\frac{1}{3} \times 3$

$$\int (3x + 2)^5 dx = \frac{1}{3} \int 3 (3x + 2)^5 dx$$
$$= \frac{1}{3} \times \frac{(3x + 2)^6}{6} + c$$
$$= \frac{(3x + 2)^6}{18} + c$$

[3 marks]

Answer 2

Add in a fiddle factor of $\frac{1}{7} \times 7$

$$\int (7x - 3)^3 \, dx = \frac{1}{7} \int 7 (7x - 3)^3 \, dx$$
$$= \frac{1}{7} \times \frac{(7x - 3)^4}{4} + c$$
$$= \frac{(7x - 3)^4}{28} + c$$

[3 marks]

Answer 3

Add in a fiddle factor of $\frac{1}{5} \times 5$

$$60 \int (4 + 5x)^3 \, dx = 60 \times \frac{1}{5} \int 5 (4 + 5x)^3 \, dx$$
$$= 12 \times \frac{(4 + 5x)^4}{4} + c$$
$$= 3 (4 + 5x)^4 + c$$

[3 marks]

Answer 4

Add in a fiddle factor of $-\frac{1}{2} \times -2$

$$\int (3 - 2x)^5 \, dx = -\frac{1}{2} \int (-2) (3 - 2x)^5 \, dx$$
$$= -\frac{1}{2} \times \frac{(3 - 2x)^6}{6} + c$$
$$= -\frac{(3 - 2x)^6}{12} + c$$

[3 marks]

Answer 5

Add in a fiddle factor of $-2 \times -\frac{1}{2}$

$$\int \left(7 - \frac{1}{2}x\right)^6 dx = (-2) \int \left(-\frac{1}{2}\right)\left(7 - \frac{1}{2}x\right)^6 dx$$
$$= (-2) \frac{\left(7 - \frac{1}{2}x\right)^7}{7} + c$$
$$= -\frac{2}{7}\left(7 - \frac{1}{2}x\right)^7 + c$$

[3 marks]

Answer 6

Add in a fiddle factor of $\frac{1}{4} \times 4$

$$18 \int \sqrt{1 + 4x}\, dx = 18 \times \frac{1}{4} \int 4\,(1 + 4x)^{\frac{1}{2}}\, dx$$

$$= \frac{9}{2} \times \frac{2\,(1 + 4x)^{\frac{3}{2}}}{3} + c$$

$$= 3\,(1 + 4x)^{\frac{3}{2}} + c$$

[3 marks]

Answer 7

Add in a fiddle factor of $\frac{3}{2} \times \frac{2}{3}$

$$\int \left(\frac{2x}{3} + 8\right)^{\frac{1}{2}} dx = \frac{3}{2} \int \frac{2}{3}\left(\frac{2x}{3} + 8\right)^{\frac{1}{2}} dx$$

$$= \frac{3}{2} \times \frac{2\left(\frac{2x}{3} + 8\right)^{\frac{3}{2}}}{3} + c$$

$$= \left(\frac{2x}{3} + 8\right)^{\frac{3}{2}} + c$$

[3 marks]

Answer 8

Add in a fiddle factor of $\frac{1}{2} \times 2$

$$\int \frac{1}{(2x - 1)^4}\, dx = \int (2x - 1)^{-4}\, dx$$

$$= \frac{1}{2} \int 2\,(2x - 1)^{-4}\, dx$$

$$= \frac{1}{2} \times \frac{(2x - 1)^{-3}}{(-3)} + c$$

$$= -\frac{1}{6\,(2x - 1)^3} + c$$

[3 marks]

Answer 9

Add in a fiddle factor of $\frac{1}{2} \times 2$

$$\int 6\,(2x - 1)^{-2}\,dx = 6 \int (2x - 1)^{-2}\,dx$$

$$= 6 \times \frac{1}{2} \int 2\,(2x - 1)^{-2}\,dx$$

$$= 3 \int 2\,(2x - 1)^{-2}\,dx$$

$$= 3 \times \frac{(2x - 1)^{-1}}{(-1)} + c$$

$$= -\frac{3}{(2x - 1)} + c$$

[3 marks]

Answer 10

Add in a fiddle factor of $\frac{1}{3} \times 3$

$$\int \frac{6}{\sqrt{3x - 1}}\,dx = 6 \int (3x - 1)^{-\frac{1}{2}}\,dx$$

$$= 6 \times \frac{1}{3} \int 3\,(3x - 1)^{-\frac{1}{2}}\,dx$$

$$= 2 \int 3\,(3x - 1)^{-\frac{1}{2}}\,dx$$

$$= 2 \times 2\,(3x - 1)^{\frac{1}{2}} + c$$

$$= 4\sqrt{3x - 1} + c$$

[3 marks]

Answer 11

No fiddle factor needed !

$$\int \frac{1}{4\,(x + 3)^2}\,dx = \frac{1}{4} \int (x + 3)^{-2}\,dx$$

$$= \frac{1}{4} \times \frac{(x + 3)^{-1}}{(-1)} + c$$

$$= -\frac{1}{4\,(x + 3)} + c$$

[3 marks]

Answer 12

Add in a fiddle factor of $\frac{1}{2} \times 2$

$$\int 14\,(3 + 2x)^{-2}\,dx = 14 \int (3 + 2x)^{-2}\,dx$$

$$= 14 \times \frac{1}{2} \int 2\,(3 + 2x)^{-2}\,dx$$

$$= 7 \int (3 + 2x)^{-2}\,dx$$

$$= 7 \times \frac{(3 + 2x)^{-1}}{(-1)} + c$$

$$= -\frac{7}{(3 + 2x)} + c$$

[3 marks]

Answer 13

Add in a fiddle factor of $-\frac{1}{2} \times -2$

$$\int \frac{1}{(1 - 2x)^{\frac{3}{2}}}\,dx = -\frac{1}{2} \int (-2)\,(1 - 2x)^{-\frac{3}{2}}\,dx$$

$$= -\frac{1}{2} \times \frac{2\,(1 - 2x)^{-\frac{1}{2}}}{(-1)} + c$$

$$= \frac{1}{\sqrt{1 - 2x}} + c$$

[3 marks]

Answer 14

Add in a fiddle factor of $\frac{1}{4} \times 4$

$$\int_0^1 (4x + 1)^4\,dx = \frac{1}{4} \int_0^1 4\,(4x + 1)^4\,dx$$

$$= \frac{1}{4} \left[\frac{(4x + 1)^5}{5} \right]_0^1$$

$$= \frac{1}{4} \left\{ [\,625\,] - \left[\frac{1}{5} \right] \right\}$$

$$= 156.2$$

[3 marks]

Answer 15

Add in a fiddle factor of $\frac{1}{2} \times 2$

$$\int_0^4 \sqrt{2x + 1}\ dx = \frac{1}{2} \int_0^4 2\ (2x + 1)^{\frac{1}{2}}\ dx$$

$$= \frac{1}{2} \left[\frac{2\ (2x + 1)^{\frac{3}{2}}}{3} \right]_0^4$$

$$= \frac{1}{3} \left[(2x + 1)^{\frac{3}{2}} \right]_0^4$$

$$= \frac{1}{3} \{ [27] - [1] \}$$

$$= \frac{26}{3} \quad \text{or} \quad 8\frac{2}{3}$$

[4 marks]

Answer 16

No fiddle factor needed !

$$\int_2^5 \frac{1}{(x - 1)^3}\ dx = \int_2^5 (x - 1)^{-3}\ dx$$

$$= \left[\frac{(x - 1)^{-2}}{(-2)} \right]_2^5$$

$$= -\frac{1}{2} \left[\frac{1}{(x - 1)^2} \right]_2^5$$

$$= -\frac{1}{2} \left\{ \left[\frac{1}{16} \right] - \left[\frac{1}{1} \right] \right\}$$

$$= -\frac{1}{2} \times \left\{ -\frac{15}{16} \right\}$$

$$= \frac{15}{32}$$

[4 marks]

3.1 Unlocking The Backwards

The Chain Rule Backwards

$$\int f'(x)\ [\, f(x)\,]^n\ dx \ = \ \frac{[\, f(x)\,]^{n+1}}{(\, n + 1\,)} + c \qquad\qquad n \neq -1$$

Initial problems involving The Chain Rule Backwards kept $f(x)$ as a simple linear function. That is, $f(x) = mx + c$
However, this limitation will now be removed !

Example N° 1

Determine: $\int 120\, x^3 (\, 3\, x^4 + 7\,)^4\ dx$

Teaching Video: http://www.NumberWonder.co.uk/v9045/3a.mp4

[3 marks]

Example N° 1 is all that's needed for the first six questions in Exercise 3.2
So you may prefer to do those questions first then watch the Example N° 2 video ahead of Question 7 onwards.

Example N° 2

Determine: $\int 64\left(x^3 - 2\right)\left(x^4 - 8x\right)^3\, dx$

Teaching Video: http://www.NumberWonder.co.uk/v9045/3b.mp4

[4 marks]

3.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

In each question use The Chain Rule Backwards to perform the integration given.
Most questions will require that a "fiddle factor" be introduced.

Question 1

Determine: $\int 150x\left(3x^2 + 1\right)^4\, dx$

[3 marks]

Question 2

Determine: $\int 56x\left(x^2 - 5\right)^3\,dx$

[3 marks]

Question 3

Determine: $\int 42\,x^2\left(2 - x^3\right)^6\,dx$

[3 marks]

Question 4

Determine: $\int 12\,x\,\sqrt{x^2 + 6}\,dx$

[3 marks]

Question 5

Determine: $\int \frac{21\,x}{(1 + x^2)^2}\,dx$

[3 marks]

Question 6

Determine: $\int \frac{12\,x^3}{\sqrt{3 + x^4}}\,dx$

[3 marks]

Question 7

Determine: $\int 60\left(x^2 - 1\right)\left(x^3 - 3x + 7\right)^4\,dx$

[4 marks]

Question 8

Determine: $\int \frac{x + 1}{\left(x^2 + 2x + 3\right)^4}\, dx$

[4 marks]

Question 9

Determine: $\int \frac{x^2 + 2}{\sqrt{x^3 + 6x}}\, dx$

[4 marks]

Question 10

(i) Differentiate $y = sin\left(x^3 + 9x \right)$

[3 marks]

(ii) With part (i) in mind, find $\int \left(x^2 + 3 \right) cos\left(x^3 + 9x \right) dx$

[3 marks]

(iii) Determine: $\int x^2 \ cos\left(4 x^3 - 7 \right) dx$

[4 marks]

3.3 Answers

3.3.1 Solutions to 3.1 Examples

Example N° 1

Add in a fiddle factor of $\frac{1}{12} \times 12$

$$\int 120\,x^3\left(3\,x^4 + 7\right)^4\,dx = 120\int x^3\left(3\,x^4 + 7\right)^4\,dx$$

$$= 120 \times \frac{1}{12}\int 12\,x^3\left(3\,x^4 + 7\right)^4\,dx$$

$$= 10 \times \frac{\left(3\,x^4 + 7\right)^5}{5} + c$$

$$= 2\left(3\,x^4 + 7\right)^5 + c$$

[3 marks]

Example N° 2

Add in a fiddle factor of $\frac{1}{4} \times 4$

$$\int 64\left(x^3 - 2\right)\left(x^4 - 8x\right)^3\,dx = 64\int\left(x^3 - 2\right)\left(x^4 - 8x\right)^3\,dx$$

$$= 64 \times \frac{1}{4}\int 4\left(x^3 - 2\right)\left(x^4 - 8x\right)^3\,dx$$

$$= 16\int\left(4\,x^3 - 8\right)\left(x^4 - 8x\right)^3\,dx$$

$$= 16 \times \frac{\left(x^4 - 8x\right)^4}{4} + c$$

$$= 4\left(x^4 - 8x\right)^4 + c$$

[4 marks]

3.3.2 Solutions to 3.2 Exercise

Answer 1

Add in a fiddle factor of $\frac{1}{6} \times 6$

$$\int 150x\left(3x^2+1\right)^4 dx = 150 \int x\left(3x^2+1\right)^4 dx$$

$$= 150 \times \frac{1}{6} \int 6x\left(3x^2+1\right)^4 dx$$

$$= 25 \times \frac{\left(3x^2+1\right)^5}{5} + c$$

$$= 5\left(3x^2+1\right)^5 + c$$

[3 marks]

Answer 2

Add in a fiddle factor of $\frac{1}{2} \times 2$

$$\int 56x\left(x^2-5\right)^3 dx = 56 \int x\left(x^2-5\right)^3 dx$$

$$= 56 \times \frac{1}{2} \int 2x\left(x^2-5\right)^3 dx$$

$$= 28 \times \frac{\left(x^2-5\right)^4}{4} + c$$

$$= 7\left(x^2-5\right)^4 + c$$

[3 marks]

Answer 3

Add in a fiddle factor of $\left(-\frac{1}{3}\right) \times (-3)$

$$\int 42x^2\left(2-x^3\right)^6 dx = 42 \int x^2\left(2-x^3\right)^6 dx$$

$$= 42 \times \left(-\frac{1}{3}\right) \int (-3)x^2\left(2-x^3\right)^6 dx$$

$$= -14 \times \frac{\left(2-x^3\right)^7}{7} + c$$

$$= -2\left(2-x^3\right)^7 + c$$

[3 marks]

Answer 4

Add in a fiddle factor of $\frac{1}{2} \times 2$

$$\int 12x\sqrt{x^2+6}\,dx = 12\int x(x^2+6)^{\frac{1}{2}}\,dx$$
$$= 12 \times \frac{1}{2}\int 2x\,(x^2+6)^{\frac{1}{2}}\,dx$$
$$= 6 \times \frac{2(x^2+6)^{\frac{3}{2}}}{3} + c$$
$$= 4(x^2+6)^{\frac{3}{2}} + c$$

[3 marks]

Answer 5

Add in a fiddle factor of $\frac{1}{2} \times 2$

$$\int \frac{21x}{(1+x^2)^2}\,dx = 21\int x(1+x^2)^{-2}\,dx$$
$$= 21 \times \frac{1}{2}\int 2x\,(1+x^2)^{-2}\,dx$$
$$= \frac{21}{2} \times \frac{(1+x^2)^{-1}}{(-1)} + c$$
$$= -\frac{21}{2(1+x^2)} + c$$

[3 marks]

Answer 6

Add in a fiddle factor of $\frac{1}{4} \times 4$

$$\int \frac{12x^3}{\sqrt{3+x^4}}\,dx = \int 12x^3(3+x^4)^{-\frac{1}{2}}\,dx$$
$$= 12\int x^3(3+x^4)^{-\frac{1}{2}}\,dx$$
$$= 12 \times \frac{1}{4}\int 4x^3\,(3+x^4)^{-\frac{1}{2}}\,dx$$
$$= 3 \times \frac{2\,(3+x^4)^{\frac{1}{2}}}{1} + c$$
$$= 6\sqrt{3+x^4} + c$$

[3 marks]

Answer 7

Add in a fiddle factor of $\frac{1}{3} \times 3$

$$\int 60\left(x^2 - 1\right)\left(x^3 - 3x + 7\right)^4 dx = 60 \int \left(x^2 - 1\right)\left(x^3 - 3x + 7\right)^4 dx$$

$$= 60 \times \frac{1}{3} \int 3\left(x^2 - 1\right)\left(x^3 - 3x + 7\right)^4 dx$$

$$= 20 \int \left(3x^2 - 3\right)\left(x^3 - 3x + 7\right)^4 dx$$

$$= 20 \times \frac{\left(x^3 - 3x + 7\right)^5}{5} + c$$

$$= 4\left(x^3 - 3x + 7\right)^5 + c$$

[4 marks]

Answer 8

Add in a fiddle factor of $\frac{1}{2} \times 2$

$$\int \frac{x + 1}{\left(x^2 + 2x + 3\right)^4} dx = \int \left(x + 1\right)\left(x^2 + 2x + 3\right)^{-4} dx$$

$$= \frac{1}{2} \int 2\left(x + 1\right)\left(x^2 + 2x + 3\right)^{-4} dx$$

$$= \frac{1}{2} \int \left(2x + 2\right)\left(x^2 + 2x + 3\right)^{-4} dx$$

$$= \frac{1}{2} \frac{\left(x^2 + 2x + 3\right)^{-3}}{(-3)} + c$$

$$= -\frac{1}{6\left(x^2 + 2x + 3\right)^3} + c$$

[4 marks]

Answer 9

Add in a fiddle factor of $\frac{1}{3} \times 3$

$$\int \frac{x^2 + 2}{\sqrt{x^3 + 6x}}\, dx = \int (x^2 + 2)(x^3 + 6x)^{-\frac{1}{2}}\, dx$$

$$= \frac{1}{3} \int 3(x^2 + 2)(x^3 + 6x)^{-\frac{1}{2}}\, dx$$

$$= \frac{1}{3} \int (3x^2 + 6)(x^3 + 6x)^{-\frac{1}{2}}\, dx$$

$$= \frac{1}{3} \times \frac{2(x^3 + 6x)^{\frac{1}{2}}}{1} + c$$

$$= \frac{2}{3}\sqrt{x^3 + 6x} + c$$

[4 marks]

Answer 10

(i) $\frac{dy}{dx} = (3x^2 + 9)\cos(x^3 + 9x)$

[3 marks]

(ii) $\int (x^2 + 3)\cos(x^3 + 9x)\, dx = \frac{1}{3} \int 3(x^2 + 3)\cos(x^3 + 9x)\, dx$

$$= \frac{1}{3} \int (3x^2 + 9)\cos(x^3 + 9x)\, dx$$

$$= \frac{1}{3} \sin(x^3 + 9x) + c \qquad \text{From part (i)}$$

[3 marks]

(iii) $\int x^2 \cos(4x^3 - 7)\, dx = \frac{1}{12} \int 12x^2 \cos(4x^3 - 7)\, dx$

$$= \frac{1}{12} \sin(4x^3 - 7) + c$$

[4 marks]

4.1 The Hole In The Sequence

Mathematician's like integer sequences, and they absolutely love it when there is a mysterious hole in a sequence; a term that should be there but isn't !

Here is a Calculus integer sequence of derivatives;

Function $f(x)$	Derivative $f'(x)$
...	...
$\frac{x^3}{3}$	x^2
$\frac{x^2}{2}$	x^1
$\frac{x^1}{1}$	x^0
The Hole	x^{-1}
$\frac{x^{-1}}{-1}$	x^{-2}
$\frac{x^{-2}}{-2}$	x^{-3}
...	...

The reason for the hole is that the function sequence is hitting a division by zero;

$$\ldots,\ \frac{x^3}{3},\ \frac{x^2}{2},\ \frac{x^1}{1},\ \frac{x^0}{0},\ \frac{x^{-1}}{-1},\ \frac{x^{-2}}{-2},\ \ldots$$

Fortunately, from our work on differentiation†, it is known what is in the hole.

In other words, we know what function differentiates to $\frac{1}{x}$; it's $ln(x)$.

Consequently, the statement of The Chain Rule Backwards can be extended to include this special case;

The Chain Rule Backwards

$$\int f'(x)\ [\,f(x)\,]^n\ dx \ = \ \frac{[\,f(x)\,]^{n+1}}{(\,n+1\,)} + c \qquad n \neq -1$$

$$\int f'(x)\ [\,f(x)\,]^{-1}\ dx \ = \ ln\left|\,f(x)\,\right| + c \qquad \text{i.e. with } n = -1$$

---

† Differentiation III, Lesson 7

Example

Determine: $\int \dfrac{36\,x^2}{4\,x^3 - 9}\ dx$

Teaching Video: http://www.NumberWonder.co.uk/v9045/4.mp4

[3 marks]

4.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 30

Question 1

Determine: $\int \dfrac{40\,x^3}{1 + 2\,x^4}\ dx$

[3 marks]

Question 2

Determine: $\int \frac{35\,x^4}{3 - x^5}\ dx$

[3 marks]

Question 3

Determine: $\int \frac{x^3 + 1}{x^4 + 4x}\, dx$

[4 marks]

Question 4

Determine: $\int \frac{8\,x^3}{(\,x^2 + 1\,)(\,x^2 - 1\,)}\, dx$

[4 marks]

Question 5

(i) Explain why finding $\int \frac{x + 3}{x^2 + x} \, dx$ can not be done by a straight forward application of The Chain Rule Backwards.

[2 marks]

(ii) Prove that $\frac{x + 3}{x^2 + x} = \frac{3}{x} - \frac{2}{x + 1}$

Begin your proof "RHS ="

[2 marks]

(iii) Use the part (ii) result to show $\int \frac{x + 3}{x^2 + x} \, dx = ln \left| \frac{x^3}{(x + 1)^2} \right| + c$

[4 marks]

Question 6

Show that,

$$\int_1^3 \frac{6x^3 + 5x}{3x^4 + 5x^2 + 1}\, dx = ln\left(\frac{17}{3}\right)$$

[8 marks]

4.3 Answers

4.3.1 Solution to 4.1 Example (Teaching Video)

Add in a fiddle factor of $\frac{1}{12} \times 12$

$$\int \frac{36x^2}{4x^3 - 9}\, dx = 36 \int x^2 (4x^3 - 9)^{-1}\, dx$$

$$= 36 \times \frac{1}{12} \int 12x^2 (4x^3 - 9)^{-1}\, dx$$

$$= 3 \int 12x^2 (4x^3 - 9)^{-1}\, dx$$

$$= 3\, ln\, |4x^3 - 9| + c$$

[3 marks]

4.3.2 Solutions to 4.2 Exercise

Answer 1

Add in a fiddle factor of $\frac{1}{8} \times 8$

$$\int \frac{40x^3}{1 + 2x^4}\, dx = 40 \int x^3 (1 + 2x^4)^{-1}\, dx$$

$$= 40 \times \frac{1}{8} \int 8x^3 (1 + 2x^4)^{-1}\, dx$$

$$= 5\, ln\, |1 + 2x^4| + c$$

[3 marks]

Answer 2

Add in a fiddle factor of $\left(-\frac{1}{5}\right) \times (-5)$

$$\int \frac{35x^4}{3 - x^5}\, dx = 35 \int x^4 (3 - x^5)^{-1}\, dx$$

$$= 35 \times \left(-\frac{1}{5}\right) \int (-5)\, x^4 (3 - x^5)^{-1}\, dx$$

$$= -7\, ln\, |3 - x^5| + c$$

[3 marks]

Answer 3

Add in a fiddle factor of $\frac{1}{4} \times 4$

$$\int \frac{x^3 + 1}{x^4 + 4x}\, dx = \int (x^3 + 1)(x^4 + 4x)^{-1}\, dx$$

$$= \frac{1}{4} \int 4\,(x^3 + 1)(x^4 + 4x)^{-1}\, dx$$

$$= \frac{1}{4} \int (4x^3 + 4)(x^4 + 4x)^{-1}\, dx$$

$$= \frac{1}{4}\, ln\left| x^4 + 4x \right| + c$$

[4 marks]

Answer 4

Add in a fiddle factor of $\frac{1}{4} \times 4$

$$\int \frac{8x^3}{(x^2 + 1)(x^2 - 1)}\, dx = 8 \int \frac{x^3}{x^4 - 1}\, dx$$

$$= 8 \times \frac{1}{4} \int 4x^3 (x^4 - 1)^{-1}\, dx$$

$$= 2 \int 4x^3 (x^4 - 1)^{-1}\, dx$$

$$= 2\, ln\left| x^4 - 1 \right| + c$$

[4 marks]

Answer 5

(i) $\displaystyle \int \frac{x + 3}{x^2 + x}\, dx = \int (x + 3)(x^2 + x)^{-1}\, dx$

- If $f(x) = x^2 + x$ then $f'(x) = 2x + 1$
- But we have $x + 3$ where we want $2x + 1$
- The $x + 3$ isn't adjustable into $2x + 1$ by a fiddle factor

[2 marks]

(ii)

$$\text{RHS} = \frac{3}{x} - \frac{2}{x + 1}$$

$$= \frac{3}{x} \times \frac{x + 1}{x + 1} - \frac{2}{x + 1} \times \frac{x}{x}$$

$$= \frac{3x + 3 - 2x}{x(x + 1)}$$

$$= \frac{x + 3}{x(x + 1)}$$

$$= \text{LHS} \qquad \square$$

[2 marks]

(iii)

$$\int \frac{x + 3}{x^2 + x}\,dx = \int \frac{3}{x}\,dx - \int \frac{2}{x + 1}\,dx$$

$$= 3\int \frac{1}{x}\,dx - 2\int \frac{1}{x + 1}\,dx$$

$$= 3\,ln\,|\,x\,| - 2\,ln\,|\,x + 1\,| + c$$

$$= ln\,|\,x\,|^3 - ln\,|\,x + 1\,|^2 + c$$

$$= ln\left|\frac{x^3}{(\,x + 1\,)^2}\right| + c$$

[4 marks]

Answer 6

Add in a fiddle factor of $\frac{1}{2} \times 2$

$$\int_1^3 \frac{6\,x^3 + 5x}{3\,x^4 + 5\,x^2 + 1}\,dx = \int_1^3 (\,6\,x^3 + 5x\,)(\,3\,x^4 + 5\,x^2 + 1\,)^{-1}\,dx$$

$$= \frac{1}{2}\int_1^3 2\,(\,6\,x^3 + 5x\,)(\,3\,x^4 + 5\,x^2 + 1\,)^{-1}\,dx$$

$$= \frac{1}{2}\int_1^3 (\,12\,x^3 + 10x\,)(\,3\,x^4 + 5\,x^2 + 1\,)^{-1}\,dx$$

$$= \frac{1}{2}\Big[\,ln\,|\,3\,x^4 + 5\,x^2 + 1\,|\,\Big]_1^3$$

$$= \frac{1}{2}\Big[\,ln\,|\,289\,| - ln\,|\,9\,|\,\Big]$$

$$= \frac{1}{2}\,ln\left(\frac{289}{9}\right)$$

$$= ln\left(\frac{289}{9}\right)^{\frac{1}{2}}$$

$$= ln\left(\frac{17}{3}\right) \qquad \square$$

[8 marks]

5.1 Partial Fractions : The Comeback

The “special case” of The Chain Rule Backwards, when $n = -1$, occurs more often than might initially be expected. This is because complicated algebraic fraction integrations are handled by first rewriting them using partial fractions with the resulting integrations often being the $n = -1$ case.

Here is an example to give an overall idea of how partial fractions, used at the start, gives rise to “special case” integrations. Don't worry about the details as these will be revised shortly. This first example is just to give a feel for the direction of travel.

Example N° 1

$$\begin{aligned}\int \frac{4x+1}{(x+1)(x-2)}\,dx &= \int \frac{1}{x+1}\,dx + \int \frac{3}{x-2}\,dx \\ &= \int (x+1)^{-1}\,dx + 3\int (x-2)^{-1}\,dx \\ &= \ln|x+1| + 3\ln|x-2| + c\end{aligned}$$

~ Only a fraction of you will find this funny ~

The Chain Rule Backwards

$$\int f'(x)\ [\,f(x)\,]^{n}\ dx \;=\; \frac{[\,f(x)\,]^{n+1}}{(\,n\,+\,1\,)} \;+\; c \qquad n \neq -1$$

$$\int f'(x)\ [\,f(x)\,]^{-1}\ dx \;=\; ln\left|\ f(x)\ \right| + c \qquad \text{i.e. with } n = -1$$

Example N° 2

Determine $\displaystyle\int \frac{9\,x^{2}}{(\,6x\,+\,1\,)(\,3x\,+\,1\,)^{2}}\,dx$

Teaching Video: http://www.NumberWonder.co.uk/v9045/5a.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9045/5b.mp4 (Part 2)

<= Part 1 Part 2 =>

[8 marks]

5.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

(i) Find the value of A and the value of B, given that,

$$\frac{10x + 19}{(2x + 1)(x + 4)} = \frac{A}{(2x + 1)} + \frac{B}{(x + 4)}$$

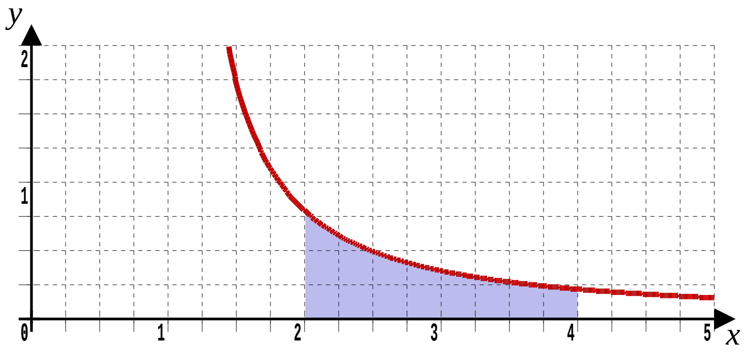
[4 marks]

(ii) Hence show that $\int_0^1 \frac{10x + 19}{(2x + 1)(x + 4)}\, dx = ln\left(\frac{3^2 \times 5^3}{2^6}\right)$

[6 marks]

Question 2

The graph is on the function, $f(x) = \dfrac{(x+2)}{(x-1)(2x+1)}$



Show that $\int_2^4 f(x)\,dx = \dfrac{1}{2}\,ln\,5$

[10 marks]

Question 3

(i) Find the values of *A*, *B* and *C* given that,

$$\frac{1}{(2x - 1)x^2} = \frac{A}{(2x - 1)} + \frac{B}{x} + \frac{C}{x^2}$$

[4 marks]

(ii) Hence show that $\int_1^2 \frac{1}{(2x - 1)x^2}\,dx = ln\left(\frac{9}{4}\right) - \frac{1}{2}$

[6 marks]

Question 4

A-Level Examination Question from June 2012, Paper C4, Q1

$$f(x) = \frac{1}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{(3x-1)} + \frac{C}{(3x-1)^2}$$

(a) Find the values of the constants A, B and C

[4 marks]

(b) **(i)** Hence find

$$\int f(x)\, dx$$

[3 marks]

(ii) Find $\int_1^2 f(x)\ dx$ leaving your answer in the form $a + ln\, b$ where a and b are constants

[3 marks]

5.3 Answers

5.3.1 Solution to Example N° 2 (Teaching Video)

$$\frac{9x^2}{(6x+1)(3x+1)^2} = \frac{A}{(6x+1)} + \frac{B}{(3x+1)} + \frac{C}{(3x+1)^2}$$

$$\frac{9x^2\,[\,\dots\dots\,]}{[\,(6x+1)(3x+1)^2\,]} = \frac{A\,[\,\dots\dots\,]}{(6x+1)} + \frac{B\,[\,\dots\dots\,]}{(3x+1)} + \frac{C\,[\,\dots\dots\,]}{(3x+1)^2}$$

$$9x^2 = A(3x+1)^2 + B(6x+1)(3x+1) + C(6x+1)$$

$$\text{Let } x = -\frac{1}{3} \;:\; 1 = C(-1) \;\Rightarrow\; C = -1$$

$$\text{Let } x = -\frac{1}{6} \;:\; \frac{1}{4} = A\left(\frac{1}{4}\right) \;\Rightarrow\; A = 1$$

$$\text{Let } x = 0 \;:\; 0 = A + B + C \;\Rightarrow\; B = 0$$

$$\int \frac{9x^2}{(6x+1)(3x+1)^2}\,dx = \int \frac{1}{6x+1}\,dx - \int \frac{1}{(3x+1)^2}\,dx$$

$$= \frac{1}{6}\int 6(6x+1)^{-1}\,dx - \frac{1}{3}\int 3(3x+1)^{-2}\,dx$$

$$= \frac{1}{6}\ln|6x+1| - \frac{1}{3} \times \frac{(3x+1)^{-1}}{(-1)} + c$$

$$= \frac{1}{6}\ln|6x+1| + \frac{1}{3(3x+1)} + c$$

[8 marks]

5.3.2 Solutions to 5.2 Exercise

Answer 1

(i)

$$\frac{(10x + 19)[\ldots\ldots]}{[(2x+1)(x+4)]} = \frac{A[\ldots\ldots]}{(2x+1)} + \frac{B[\ldots\ldots]}{(x+4)}$$

$$10x + 19 = A(x+4) + B(2x+1)$$

$$\text{Let } x = -4 \;:\; -21 = B(-7) \Rightarrow B = 3$$

$$\text{Let } x = -\frac{1}{2} \;:\quad 14 = A\left(\frac{7}{2}\right) \Rightarrow A = 4$$

$$\frac{10x + 19}{(2x+1)(x+4)} = \frac{4}{(2x+1)} + \frac{3}{(x+4)}$$

[4 marks]

(ii)

$$\int_0^1 \frac{10x + 19}{(2x+1)(x+4)}\,dx = \int_0^1 \frac{4}{(2x+1)} + \frac{3}{(x+4)}\,dx$$

$$= 4 \times \frac{1}{2}\int_0^1 2(2x+1)^{-1}\,dx + 3\int_0^1 (x+4)^{-1}\,dx$$

$$= \Big[\,2\ln|2x+1| + 3\ln(x+4)\Big]_0^1$$

$$= 2\ln 3 + 3\ln 5 - 2\ln 1 - 3\ln 4$$

$$= \ln 3^2 + \ln 5^3 - 0 - \ln 2^6$$

$$= \ln\left(\frac{3^2 \times 5^3}{2^6}\right) \qquad \square$$

[6 marks]

Answer 2

$$\frac{(x+2)[\ldots\ldots]}{[(x-1)(2x+1)]} = \frac{A[\ldots\ldots]}{(x-1)} + \frac{B[\ldots\ldots]}{(2x+1)}$$

$$x+2 = A(2x+1) + B(x-1)$$

$$\text{Let } x = 1 : \quad 3 = A(3) \quad \Rightarrow A = 1$$

$$\text{Let } x = -\frac{1}{2} : \frac{3}{2} = B\left(-\frac{3}{2}\right) \quad \Rightarrow B = -1$$

$$\frac{x+2}{(x-1)(2x+1)} = \frac{1}{(x-1)} - \frac{1}{(2x+1)}$$

$$\int_2^4 \frac{x+2}{(x-1)(2x+1)}\,dx = \int_2^4 \frac{1}{(x-1)} - \frac{1}{(2x+1)}\,dx$$

$$= \int_2^4 (x-1)^{-1}\,dx - \frac{1}{2}\int_2^4 2(2x+1)^{-1}\,dx$$

$$= \left[\ln|x-1| - \frac{1}{2}\ln|2x+1| \right]_2^4$$

$$= \ln 3 - \frac{1}{2}\ln 9 - \ln 1 + \frac{1}{2}\ln 5$$

$$= \ln 3 - \ln 9^{\frac{1}{2}} - 0 + \frac{1}{2}\ln 5$$

$$= \ln 3 - \ln 3 + \frac{1}{2}\ln 5$$

$$= \frac{1}{2}\ln 5 \qquad \square$$

[10 marks]

Answer 3

(i)

$$\frac{1}{(2x-1)x^2} = \frac{A}{(2x-1)} + \frac{B}{x} + \frac{C}{x^2}$$

$$\frac{1\,[\,\ldots\ldots\,]}{[\,(2x-1)x^2\,]} = \frac{A\,[\,\ldots\ldots\,]}{(2x-1)} + \frac{B\,[\,\ldots\ldots\,]}{x} + \frac{C\,[\,\ldots\ldots\,]}{x^2}$$

$$1 = Ax^2 + B(2x-1)x + C(2x-1)$$

$$\text{Let } x = 0 : 1 = C(-1) \quad \Rightarrow C = -1$$

$$\text{Let } x = \frac{1}{2} : 1 = A\left(\frac{1}{4}\right) \quad \Rightarrow A = 4$$

$$\text{Let } x = 1 : 1 = A + B + C \Rightarrow B = -2$$

That is,

$$\frac{1}{(2x-1)x^2} = \frac{4}{(2x-1)} - \frac{2}{x} - \frac{1}{x^2}$$

[4 marks]

(ii) Leaving out the limits for now (to reduce clutter),

$$\int \frac{1}{(2x-1)x^2}\,dx = 4\int \frac{1}{2x-1}\,dx - 2\int \frac{1}{x}\,dx - \int \frac{1}{x^2}\,dx$$

$$= 4 \times \frac{1}{2} \int 2(2x-1)^{-1}\,dx - 2\int x^{-1}\,dx - \int x^{-2}\,dx$$

$$= 2\,ln\,|2x-1| - 2\,ln|x| + \frac{1}{x} + c$$

With an upper limit of 2 and a lower limit of 1,

$$= 2\,ln\,3 - 2\,ln\,2 + \frac{1}{2} - 2\,ln\,1 + 2\,ln\,1 - \frac{1}{1}$$

$$= ln\,3^2 - ln\,2^2 - \frac{1}{2}$$

$$= ln\,9 - ln\,4 - \frac{1}{2}$$

$$= ln\left(\frac{9}{4}\right) - \frac{1}{2} \qquad \square$$

[6 marks]

Answer 4

(a)

$$\frac{1}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{(3x-1)} + \frac{C}{(3x-1)^2}$$

$$\frac{1\,[\dots\dots]}{[\,x(3x-1)^2\,]} = \frac{A\,[\dots\dots]}{x} + \frac{B\,[\dots\dots]}{(3x-1)} + \frac{C\,[\dots\dots]}{(3x-1)^2}$$

$$1 = A(3x-1)^2 + Bx(3x-1) + Cx$$

$$\text{Let } x = 0 : 1 = A(-1)^2 \qquad \Rightarrow A = 1$$

$$\text{Let } x = \frac{1}{3} : 1 = C\left(\frac{1}{3}\right) \qquad \Rightarrow C = 3$$

$$\text{Let } x = 1 : 1 = 4A + 2B + C \Rightarrow B = -3$$

That is,

$$\frac{1}{x(3x-1)^2} = \frac{1}{x} - \frac{3}{(3x-1)} + \frac{3}{(3x-1)^2}$$

[4 marks]

(b) **(i)**

$$\int \frac{1}{x(3x-1)^2}\,dx = \int \frac{1}{x}\,dx - \int \frac{3}{(3x-1)}\,dx + \int \frac{3}{(3x-1)^2}\,dx$$

$$= \int x^{-1}\,dx - \int 3(3x-1)^{-1}\,dx + \int 3(3x-1)^{-2}\,dx$$

$$= ln|x| - ln|3x-1| + \frac{(3x-1)^{-1}}{(-1)} + c$$

$$= ln|x| - ln|3x-1| - \frac{1}{3x-1} + c$$

[3 marks]

(ii) With an upper limit of 2 and a lower limit of 1,

$$\int_1^2 \frac{1}{x(3x-1)^2}\,dx = \left[\, ln|x| - ln|3x-1| - \frac{1}{3x-1} \,\right]_1^2$$

$$= ln\,2 - ln\,5 - \frac{1}{5} - ln\,1 + ln\,2 + \frac{1}{2}$$

$$= \frac{1}{2} - \frac{1}{5} + 2\,ln\,2 - ln\,5$$

$$= \frac{3}{10} + ln\left(\frac{4}{5}\right)$$

[3 marks]

Lesson 6

A-Level Pure Mathematics : Year 2
Integration II

6.1 Beyond The Power Function

Suitable integrals, *I*, that can be determined by applying The Chain Rule Backwards have been spotted by identifying the key situation;

$$I = a \int f'(x) \; [\, f(x) \,]^n \; dx \qquad \text{where } n \text{ and } a \text{ are constants}$$

That power of *n* can be thought of as a function in its own right.

With $g(x) = x^n$ another way of expressing what we've been looking for is;

$$I = a \int f'(x) \; g[\, f(n) \,]$$

There is no reason why function *g* has to be a power function; The Chain Rule Backwards can be wielded when *g* is other functions such as *cos* or *sin*

The Chain Rule Backwards

$$\int f'(x) \; [\, f(x) \,]^n \; dx = \frac{[\, f(x) \,]^{n+1}}{(\, n + 1 \,)} + c \qquad n \neq -1$$

$$\int f'(x) \; [\, f(x) \,]^{-1} \; dx = ln|\, f(x) \,| + c \qquad \text{i.e. with } n = -1$$

$$\int f'(x) \; cos[\, f(x) \,] \; dx = sin\,[\, f(x) \,] + c$$

$$\int f'(x) \; sin\,[\, f(x) \,] \; dx = -cos\,[\, f(x) \,] + c$$

$$\int f\,'(x) \; e^{[\, f(x) \,]} \, dx = e^{[\, f(x) \,]} + c$$

Example

Determine: $\int 63\, x^2 \, e^{7x^3} \; dx$

Teaching Video: http://www.NumberWonder.co.uk/v9045/6.mp4

[3 marks]

6.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Determine $\int 30x\, e^{5x^2}\, dx$

[3 marks]

Question 2

Evaluate: $\int_0^2 x^2\, e^{x^3}\, dx$

Give the answer to the nearest integer.

[4 marks]

Question 3

Determine: $\int 100\,x^3 \cos\left(6 + 5x^4\right)\,dx$

[4 marks]

Question 4

Determine: $\int \frac{3}{2}\,x \sin\left(x^2\right)\,dx$

[4 marks]

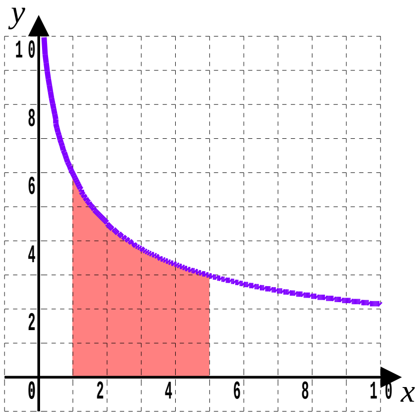
Question 5

Determine : $\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

[4 marks]

Question 6

The graph is of the function $f(x) = \dfrac{12}{\sqrt{3x + 1}}$

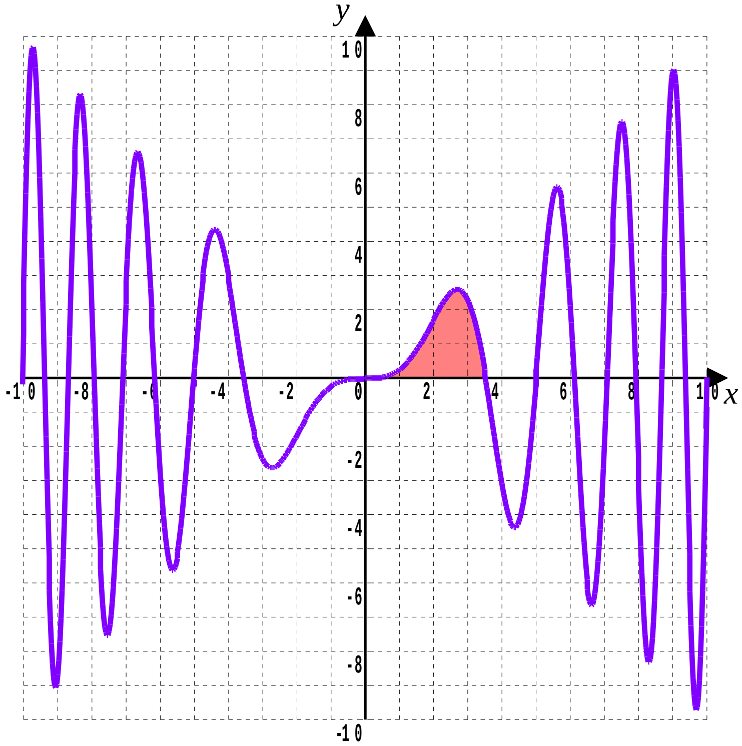


Find the area shown shaded in the graph which is bound by the curve, the *x*-axis and the vertical lines $x = 1$ and $x = 5$

[7 marks]

Question 7

The graph is of the function, $g(x) = x\,sin\left(\dfrac{x^2}{4}\right)$



Find the exact area of the 'bump' shown shaded in the graph which is bounded by $g(x)$, the x-axis and the lines $x = 0$ and $x = 2\sqrt{\pi}$

[7 marks]

Question 8

$$f(x) = e^{\sqrt{x}}$$

(i) Find $f'(x)$

[2 marks]

(ii) Evaluate to 3 decimal places;

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$$

[5 marks]

6.3 Answers

6.3.1 Solution to 6.1 Example

Apply a fiddle factor of $\frac{1}{21} \times 21$

$$\begin{aligned} \int 63\, x^2\, e^{7x^3}\, dx &= 63 \int x^2\, e^{7x^3}\, dx \\ &= 63 \times \frac{1}{21} \int 21\, x^2\, e^{7x^3}\, dx \\ &= 3 \int 21\, x^2\, e^{7x^3}\, dx \\ &= 3\, e^{7x^3} + c \end{aligned}$$

[3 marks]

6.3.2 Solution to 6.2 Exercise

Answer 1

Apply a fiddle factor of $\frac{1}{10} \times 10$

$$\begin{aligned} \int 30x\, e^{5x^2}\, dx &= 30 \int x\, e^{5x^2}\, dx \\ &= 30 \times \frac{1}{10} \int 10x\, e^{5x^2}\, dx \\ &= 3\, e^{5x^2} + c \end{aligned}$$

[3 marks]

Answer 2

Apply a fiddle factor of $\frac{1}{3} \times 3$

$$\begin{aligned} \int_0^2 x^2\, e^{x^3}\, dx &= \frac{1}{3} \int_0^2 3\, x^2\, e^{x^3}\, dx \\ &= \frac{1}{3} \left[e^{x^3} \right]_0^2 \\ &= \frac{1}{3} \left\{ e^8 - 1 \right\} \\ &= 993 \text{ to the nearest integer} \end{aligned}$$

[4 marks]

Answer 3

Apply a fiddle factor of $\frac{1}{20} \times 20$

$$\int 100x^3 \cos(6 + 5x^4)\, dx = 100 \int x^3 \cos(6 + 5x^4)\, dx$$

$$= 100 \times \frac{1}{20} \int 20x^3 \cos(6 + 5x^4)\, dx$$

$$= 5 \int 20x^3 \cos(6 + 5x^4)\, dx$$

$$= 5 \sin(6 + 5x^4) + c$$

[4 marks]

Answer 4

Apply a fiddle factor of $\frac{1}{2} \times 2$

$$\int \frac{3}{2} x \sin(x^2)\, dx = \frac{3}{2} \times \frac{1}{2} \int 2x \sin(x^2)\, dx$$

$$= \frac{3}{4} \int 2x \sin(x^2)\, dx$$

$$= \frac{3}{4}(-\cos(x^2)) + c$$

$$= -\frac{3}{4} \cos(x^2) + c$$

[4 marks]

Answer 5

Apply a fiddle factor of $2 \times \frac{1}{2}$

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\, dx = \int x^{-\frac{1}{2}} \cos\left(x^{\frac{1}{2}}\right) dx$$

$$= 2 \int \frac{1}{2} x^{-\frac{1}{2}} \cos\left(x^{\frac{1}{2}}\right) dx$$

$$= 2 \sin\left(x^{\frac{1}{2}}\right) + c$$

[4 marks]

Answer 6

Apply a fiddle factor of $\frac{1}{3} \times 3$

$$\int_1^5 \frac{12}{\sqrt{3x+1}}\,dx = 12\int_1^5 (3x+1)^{-\frac{1}{2}}\,dx$$

$$= 12 \times \frac{1}{3}\int_1^5 3(3x+1)^{-\frac{1}{2}}\,dx$$

$$= 4\left[\frac{2(3x+1)^{\frac{1}{2}}}{1}\right]_1^5$$

$$= 8\left[(3x+1)^{\frac{1}{2}}\right]_1^5$$

$$= 8\{4-2\}$$

$$= 16$$

[7 marks]

Answer 7

Apply a fiddle factor of $2 \times \frac{1}{2}$

$$\int_0^{2\sqrt{\pi}} x\sin\left(\frac{x^2}{4}\right)dx = 2\int_0^{2\sqrt{\pi}} \frac{1}{2}x\sin\left(\frac{x^2}{4}\right)dx$$

$$= 2\left[-\cos\left(\frac{x^2}{4}\right)\right]_0^{2\sqrt{\pi}}$$

$$= (-2)\{[\cos\pi] - [\cos 0]\}$$

$$= (-2)\{[(-1)] - [1]\}$$

$$= (-2)\{-2\}$$

$$= 4$$

[7 marks]

Answer 8

(i)

$$y = e^{\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} e^{\sqrt{x}}$$

$$= \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

[2 marks]

(ii)

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx = 2\int_1^4 \frac{e^{\sqrt{x}}}{2\sqrt{x}}\ dx$$

$$= 2\left[e^{\sqrt{x}} \right]_1^4$$

$$= 2\left\{ e^2 - e^1 \right\}$$

$$= 9.342 \qquad \text{to 3 decimal places}$$

[5 marks]