3.4 Answers

3.4.1 Solution to Example 3.2

The "grab the nettle" approach adopted here is to do the algebra first then insert the value of x at the very end.
This has the advantage of enabling students to do algebra only questions without having to learn anything new, but is a little harder to master initially.

$f(x) = 4x^2$

By definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{4(x+h)^2 - 4x^2}{h}$$

$$= \lim_{h \to 0} \frac{4\left(x^2 + 2xh + h^2\right) - 4x^2}{h}$$

$$= \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - 4x^2}{h}$$

$$= \lim_{h \to 0} \frac{8xh + 4h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(8x + 4h)}{h}$$

$$= \lim_{h \to 0} 8x + 4h$$

$$= 8x$$

$$\therefore\ f'(3) = 24$$

[5 marks]

3.4.2 Solutions to Exercise 3.3

Answer 1

(i) By definition, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

[2 marks]

(ii) $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(By definition)}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h)}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

$$\therefore f'(5) = 10$$

[4 marks]

Answer 2

$f(x) = 3x^2$

$$f'(x) = \underset{h \to 0}{\text{limit}} \frac{f(x+h) - f(x)}{h} \qquad \text{(By definition)}$$

$$= \underset{h \to 0}{\text{limit}} \frac{3(x+h)^2 - 3x^2}{h}$$

$$= \underset{h \to 0}{\text{limit}} \frac{3(x^2 + 2xh + h^2) - 3x^2}{h}$$

$$= \underset{h \to 0}{\text{limit}} \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h}$$

$$= \underset{h \to 0}{\text{limit}} \frac{6xh + 3h^2}{h}$$

$$= \underset{h \to 0}{\text{limit}} \frac{h(6x + 3h)}{h}$$

$$= \underset{h \to 0}{\text{limit}} 6x + 3h$$

$$= 6x$$

$$\therefore f'(4) = 24$$

[5 marks]

Answer 3

$f(x) = x^2 + 3$

By definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{(By definition)}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 3 - (x^2 + 3)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3 - x^2 - 3}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h)}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

$$\therefore \; f'(2) = 4$$

[6 marks]

Answer 4

$f(x) = x^3 + 4x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{(By definition)}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 + 4(x+h) - \left[x^3 + 4x\right]}{h}$$

METHOD

Using a Multiplication Grid,

$$(x+h)^3 = (x+h)(x+h)(x+h)$$

$$= (x+h)\left(x^2 + 2xh + h^2\right)$$

	x^2	$2xh$	h^2
x	x^3	$2x^2h$	xh^2
h	x^2h	$2xh^2$	h^3

$$= x^3 + 3x^2h + 3xh^2 + h^3$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 4x + 4h - x^3 - 4x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 + 4h}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 + 4)}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 + 4$$

$$= 3x^2 + 4$$

$$\therefore \; f'(1) = 7$$

[8 marks]

Answer 5

(i) METHOD 1

Using the Binomial Theorem,

$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

METHOD 2

Using a Multiplication Grid,

$$(x+h)^4 = \left((x+h)^2 \right)^2$$

$$= \left(x^2 + 2xh + h^2 \right)^2$$

	x^2	$2xh$	h^2
x^2	x^4	$2x^3h$	x^2h^2
$2xh$	$2x^3h$	$4x^2h^2$	$2xh^3$
h^2	x^2h^2	$2xh^3$	h^4

$$= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

[2 marks]

(ii) $f(x) = x^4$

$$f'(a) = \underset{h \to 0}{limit} \frac{f(x+h) - f(x)}{h} \qquad \text{(By definition)}$$

$$= \underset{h \to 0}{limit} \frac{(x+h)^4 - x^4}{h}$$

$$= \underset{h \to 0}{limit} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$= \underset{h \to 0}{limit} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

$$= \underset{h \to 0}{limit} \frac{h\left(4x^3 + 6x^2h + 4xh^2 + h^3\right)}{h}$$

$$= \underset{h \to 0}{limit} \; 4x^3 + 6x^2h + 4xh^2 + h^3$$

$$= 4x^3$$

$$\therefore \; f'(3) = 108$$

[5 marks]

Answer 6

$$f(x) = \frac{1}{x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{(By definition)}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{x(x+h)h}$$

$$= \lim_{h \to 0} \frac{-h}{x(x+h)h}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= \frac{-1}{x \times x}$$

$$= -\frac{1}{x^2}$$

$$\therefore \; f'(5) = -\frac{1}{25}$$

[8 marks]