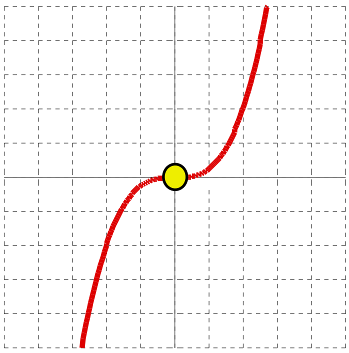
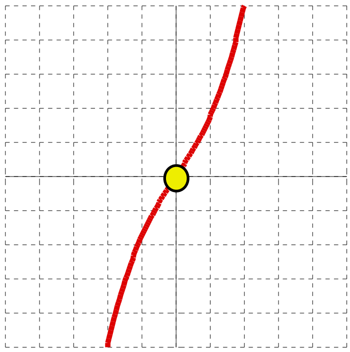
4.6 Answers

4.6.1 Solution to 4.3 Example





Left : An example of a stationary point of inflection
This function is increasing
(Note that increasing can include points with gradient zero)

Right : An example of a non-stationary point of inflection
This function is strictly increasing
(Note that the "strictly" means there are no points with gradient zero)

[2 marks]

4.6.1 Solution to Example 4.4

(i) $y = x^3 + 6x^2 + 9x \qquad \therefore \quad \frac{dy}{dx} = 3x^2 + 12x + 9$

[2 marks]

(ii) $\frac{d^2y}{dx^2} = 6x + 12$

[2 marks]

(iii) For a turning point, $3x^2 + 12x + 9 = 0$

$$x^2 + 4x + 3 = 0$$

$$(x + 3)(x + 1) = 0$$

Either $x = -3$ or $x = -1$

$y = (-3)^3 + 6(-3)^2 + 9(-3) = 0$

$y = (-1)^3 + 6(-1)^2 + 9(-1) = -4$

The turning points are $(-3, 0)$, $(-1, -4)$

For $x = -3$, $\frac{d^2y}{dx^2} = -6$ For $x = -1$, $\frac{d^2y}{dx^2} = +6$

$(-3, 0)$ is a local maximum $(-1, -4)$ is a local minimum

[4 marks]

4.6.2 Solutions to 4.5 Exercise

Answer 1

(i) $y = 5x^2 - 30x \quad \therefore \quad \frac{dy}{dx} = 10x - 30$

[1 mark]

(ii) $\frac{d^2y}{dx^2} = 10$

[1 mark]

(iii) For a turning point, $10x - 30 = 0 \quad \Leftrightarrow x = 3$

To get a ***point*** substitute this value of x back into the point equation of the curve. (NOT the gradient equation of the curve !)

$$\begin{aligned} y &= 5x^2 - 30x \\ &= 5 \times 3^2 - 30 \times 3 \\ &= -45 \Leftrightarrow (3, -45) \end{aligned}$$

When $x = 3$, $\frac{d^2y}{dx^2} = 10$

As this is positive $(3, -45)$ is a local minimum.

[3 marks]

Answer 2

(i) $y = 4x^2 + 16x + 21 \quad \therefore \quad \frac{dy}{dx} = 8x + 16$

[2 marks]

(ii) For a turning point, $8x + 16 = 0 \quad \Leftrightarrow \quad x = -2$

To get a ***point*** substitute this value of x back into the point equation of the curve. (NOT the gradient equation of the curve !)

$$\begin{aligned} y &= 4x^2 + 16x + 21 \\ &= 4 \times (-2)^2 + 16 \times (-2) + 21 \\ &= 16 - 32 + 21 \\ &= 5 \quad \Leftrightarrow \quad (-2, 5) \end{aligned}$$

[4 marks]

(iii) Regardless of the value of x, $\frac{d^2y}{dx^2} = 8$

As this is positive $(-2, 5)$ is a local minimum.

As this is positive is a local minimum.

[2 marks]

Answer 3

(i) $y = x^3 + 9x^2 + 15x \qquad \therefore \qquad \frac{dy}{dx} = 3x^2 + 18x + 15$

[2 marks]

(ii) $\frac{d^2y}{dx^2} = 6x + 18$

[2 marks]

(iii) For a turning point, $3x^2 + 18x + 15 = 0$

$$x^2 + 6x + 5 = 0$$

$$(x + 5)(x + 1) = 0$$

Either $x = -5$ or $x = -1$

$y = x^3 + 9x^2 + 15x$

$= (-5)^3 + 9(-5)^2 + 15(-5)$

$= -125 + 225 - 75$

$= 25 \Leftrightarrow (-5,\ 25)$

$y = x^3 + 9x^2 + 15x$

$= (-1)^3 + 9(-1)^2 + 15(-1)$

$= -1 + 9 - 15$

$= -7 \Leftrightarrow (-1, -7)$

When $x = -5$, $\frac{d^2y}{dx^2} = -12$

$(-5,\ 25)$ is a local maximum

When $x = -1$, $\frac{d^2y}{dx^2} = +12$

$(-1, -7)$ is a local minimum

[4 marks]

Answer 4

(i) $f(x) = x^3 + 6x^2 + 5 \qquad \therefore\ f'(x) = 3x^2 + 12x$

[2 marks]

(ii) $f''(x) = 6x + 12$

[2 marks]

(iii) For a turning point, $3x^2 + 12x = 0$

$$x^2 + 4x = 0$$

$$x(x + 4) = 0$$

Either $x = 0$ or $x = -4$

$y = x^3 + 6x^2 + 5$

$= (0)^3 + 6(0)^2 + 5$

$= 5 \Leftrightarrow (0,\ 5)$

$y = x^3 + 6x^2 + 5$

$= (-4)^3 + 6(-4)^2 + 5$

$= 37 \Leftrightarrow (-4,\ 37)$

When $x = 0$, $\frac{d^2y}{dx^2} = 12$

$(0,\ 5)$ is a local minimum

When $x = -4$, $\frac{d^2y}{dx^2} = -12$

$(-4,\ 37)$ is a local maximum

[4 marks]

Answer 5

$$y = \sqrt{2}\,x^2 - 6\sqrt{x} + 4\sqrt{2}$$
$$= \sqrt{2}\,x^2 - 6x^{\frac{1}{2}} + 4\sqrt{2}$$
$$\frac{dy}{dx} = 2\sqrt{2}\,x - 3x^{-\frac{1}{2}}$$
$$= 2\sqrt{2}\,x - \frac{3}{\sqrt{x}}$$
$$\text{When } x = 2,\ \frac{dy}{dx} = 4\sqrt{2} - \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= 4\sqrt{2} - \frac{3}{2}\sqrt{2}$$
$$= \frac{5}{2}\sqrt{2} \qquad \therefore\ a = \frac{5}{2}$$

[4 marks]

Answer 6

Given that students do not yet know of “The Product Rule”...

$$f(x) = (x - 2)^2(2x + 1)$$
$$= \left(x^2 - 4x + 4\right)\left(2x + 1\right)$$
$$= 2x^3 - 8x^2 + 8x + x^2 - 4x + 4$$
$$= 2x^3 - 7x^2 + 4x + 4$$
$$f'(x) = 6x^2 - 14x + 4$$

For turning points, $6x^2 - 14x + 4 = 0$

$$3x^2 - 7x + 2 = 0$$
$$(3x - 1)(x - 2) = 0$$

Either $x = \frac{1}{3}$ or $x = 2$

The minimum at $(2,\ 0)$ is already known about.

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{3} - 2\right)^2\left(\frac{2}{3} + 1\right)$$
$$= \left(\frac{5}{3}\right)^2 \times \frac{5}{3}$$
$$= \frac{125}{27} \qquad \therefore P \text{ is } \left(\frac{1}{3},\ \frac{125}{27}\right)$$

[7 marks]