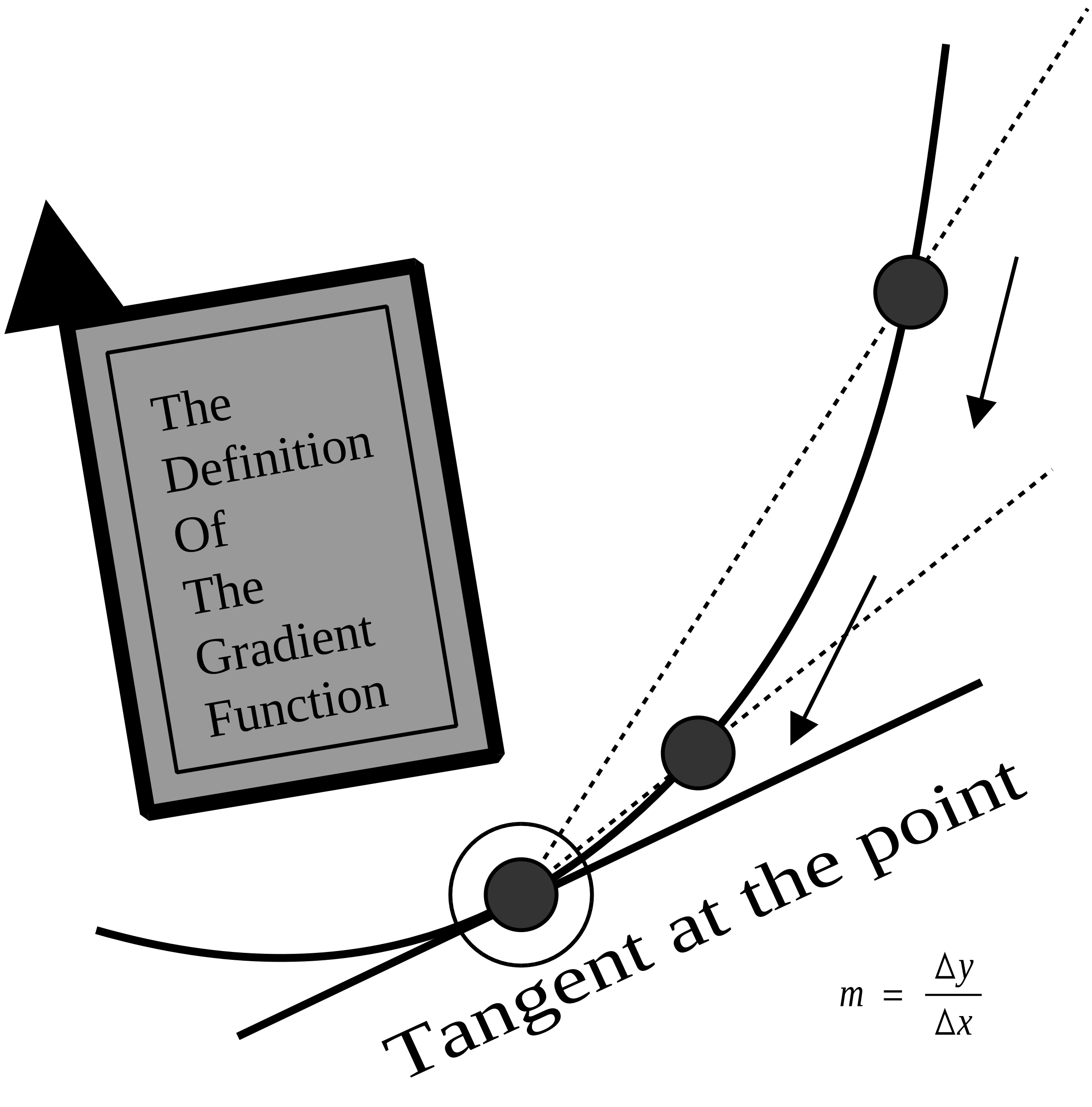
DIFFERENTIATION II

A-Level Pure Mathematics

~ Year 1 ~



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

D I F F E R E N T I A T I O N
II

Lesson 1

A-Level Pure Mathematics : Year 1
Differentiation II

1.1 Revision

In GCSE a technique is studied that allows the gradient equation of a simple curve to be immediately written down simply from looking at the equation of that curve. The key move is this:

If $y = x^n$ then $\frac{dy}{dx} = n\,x^{n-1}$ for any constant n

1.2 "Together" Examples

Each of the following is the equation of a curve.
For each curve, write down its gradient equation.

(i) $y = \frac{7}{2}x^4 + \frac{1}{3}x^2 + 5x + 3$ $\qquad \frac{dy}{dx} =$

(ii) $y = 8\sqrt{x} + \frac{3}{x^2}$ $\qquad \frac{dy}{dx} =$

(iii) $y = \frac{4x + 7}{x}$ $\qquad \frac{dy}{dx} =$

(iv) $y = (5 + \sqrt{x})^2$ $\qquad \frac{dy}{dx} =$

[8 marks]

1.3 Exercise

Marks Available : 40

Question 1

For each of these equations, write down the corresponding gradient equation.

(i) $y = \frac{9}{5}x + \frac{4}{3}$ $\frac{dy}{dx} =$

(ii) $y = \frac{7x^4}{6}$ $\frac{dy}{dx} =$

(iii) $y = (5x + 3)^2$ $\frac{dy}{dx} =$

(iv) $y = 3 - 17x$ $\frac{dy}{dx} =$

(v) $y = \frac{4}{x^{2.5}}$ $\frac{dy}{dx} =$

(vi) $y = 18\sqrt{x} + \frac{5}{x^3}$ $\frac{dy}{dx} =$

[12 marks]

Question 2

$$f(x) = \frac{5}{6}x^2 - \frac{7}{2}x^4$$

Find $f'(x)$

[2 marks]

Question 3

$$g(x) = \frac{4x^9}{3} + \frac{3x^6}{2}$$

Find $g'(x)$

[2 marks]

Question 4

$$h(x) = \frac{5}{4}x^2 + \frac{1}{3}x + 2$$

Find $h'(x)$

[2 marks]

Question 5

$$p(x) = \frac{4 + x^2}{x^3} \qquad \text{HINT : } p(x) = \frac{4}{x^3} + \frac{x^2}{x^3}$$

Find $p'(x)$

[3 marks]

Question 6

$$y = \frac{10x + 1}{x^2}$$

Find $\frac{dy}{dx}$

[3 marks]

Question 7

$$y = \frac{3x^2 + x}{\sqrt{x}}$$

Find $\frac{dy}{dx}$

[3 marks]

Question 8

$$y = (3 - 2x)^n$$

If you know the binomial theorem, find $\frac{dy}{dx}$ when $n = 4$ otherwise when $n = 2$

[4 marks]

Question 9

$$w(x) = 3x^3 + \frac{2}{x^3}$$

Find $w'(x)$

[3 marks]

Question 10

$$e(x) = \frac{2}{5x^4}$$

Find $e'(x)$

[3 marks]

Question 11

$$k(x) = 3x^3\left(2x^2 - x^5\right)$$

Find $k'(x)$

[3 marks]

1.5 Answers

1.5.1 Solutions to 1.2 Introductory Examples

(i) $y = \frac{7}{2}x^4 + \frac{1}{3}x^2 + 5x + 3$ $\quad \therefore \frac{dy}{dx} = 14x^3 + \frac{2}{3}x + 5$

(ii) $y = 8\sqrt{x} + \frac{3}{x^2}$

$= 8x^{0.5} + 3x^{-2}$ $\quad \therefore \frac{dy}{dx} = 4x^{-0.5} - 6x^{-3}$

$= \frac{4}{\sqrt{x}} - \frac{6}{x^3}$

(iii) $y = \frac{4x + 7}{x}$

$= \frac{4x}{x} + \frac{7}{x}$

$= 4 + 7x^{-1}$ $\quad \therefore \frac{dy}{dx} = -7x^{-2}$

$= -\frac{7}{x^2}$

(iv) $y = (5 + \sqrt{x})^2$

$= 25 + 10x^{0.5} + x$ $\quad \therefore \frac{dy}{dx} = 5x^{-0.5} + 1$

$= \frac{5}{\sqrt{x}} + 1$

[8 marks]

1.5.2 Solutions (1.3 Exercise)

Answer 1

(i) $y = \dfrac{9}{5}x + \dfrac{4}{3}$ $\therefore \dfrac{dy}{dx} = \dfrac{9}{5}$

(ii) $y = \dfrac{7x^4}{6}$ $\therefore \dfrac{dy}{dx} = \dfrac{14x^3}{3}$

(iii) $y = (5x + 3)^2$

$= 25x^2 + 30x + 9$ $\therefore \dfrac{dy}{dx} = 50x + 30$

(iv) $y = 3 - 17x$ $\therefore \dfrac{dy}{dx} = -17$

(v) $y = \dfrac{4}{x^{2.5}}$

$= 4x^{-2.5}$ $\therefore \dfrac{dy}{dx} = -10x^{-3.5}$

$= -\dfrac{10}{x^{3.5}}$

(vi) $y = 18\sqrt{x} + \dfrac{5}{x^3}$

$= 18x^{0.5} + 5x^{-3}$ $\therefore \dfrac{dy}{dx} = 9x^{-0.5} - 15x^{-4}$

$= \dfrac{9}{\sqrt{x}} - \dfrac{15}{x^4}$

[12 marks]

Answer 2

$f(x) = \dfrac{5}{6}x^2 - \dfrac{7}{2}x^4$ $\therefore$ $f'(x) = \dfrac{5}{3}x - 14x^3$

[2 marks]

Answer 3

$g(x) = \dfrac{4x^9}{3} + \dfrac{3x^6}{2}$ $\therefore$ $g'(x) = 12x^8 + 9x^5$

[2 marks]

Answer 4

$$h(x) = \frac{5}{4}x^2 + \frac{1}{3}x + 2 \qquad \therefore \qquad h'(x) = \frac{5}{2}x + \frac{1}{3}$$

[2 marks]

Answer 5

$$\begin{aligned} p(x) &= \frac{4 + x^2}{x^3} \\ &= \frac{4}{x^3} + \frac{x^2}{x^3} \\ &= 4x^{-3} + x^{-1} \end{aligned} \qquad \therefore \qquad \begin{aligned} p'(x) &= -12x^{-4} - x^{-2} \\ &= -\frac{12}{x^4} - \frac{1}{x^2} \end{aligned}$$

[3 marks]

Answer 6

$$\begin{aligned} y &= \frac{10x + 1}{x^2} \\ &= \frac{10x}{x^2} + \frac{1}{x^2} \\ &= 10x^{-1} + x^{-2} \end{aligned} \qquad \therefore \qquad \begin{aligned} \frac{dy}{dx} &= -10x^{-2} - 2x^{-3} \\ &= -\frac{10}{x^2} - \frac{2}{x^3} \end{aligned}$$

[3 marks]

Answer 7

$$\begin{aligned} y &= \frac{3x^2 + x}{\sqrt{x}} \\ &= \frac{3x^2}{\sqrt{x}} + \frac{x}{\sqrt{x}} \\ &= 3x^{1.5} + x^{0.5} \end{aligned} \qquad \therefore \qquad \begin{aligned} \frac{dy}{dx} &= 4.5x^{0.5} + 0.5x^{-0.5} \\ &= 4.5\sqrt{x} + \frac{0.5}{\sqrt{x}} \end{aligned}$$

[3 marks]

Answer 8

With $n = 4$: $\quad y = (3 - 2x)^4$

$$y = 81 - 216x + 216x^2 - 96x^3 + 16x^4$$

$$\therefore \quad \frac{dy}{dx} = -216 + 432x - 288x^2 + 64x^3$$

With $n = 2$: $\quad y = (3 - 2x)^2$

$$= 9 - 12x + 4x^2$$

$$\therefore \quad \frac{dy}{dx} = -12 + 8x$$

[4 marks]

Answer 9

$$w(x) = 3x^3 + \frac{2}{x^3}$$

$$= 3x^3 + 2x^{-3} \qquad \therefore \quad w'(x) = 9x^2 - 6x^{-4}$$

$$= 9x^2 - \frac{6}{x^4}$$

[3 marks]

Answer 10

$$k(x) = \frac{2}{5x^4}$$

$$= \frac{2x^{-4}}{5} \qquad \therefore \quad k'(x) = -\frac{8x^{-5}}{5}$$

$$= -\frac{8}{5x^5}$$

[3 marks]

Answer 11

$$m(x) = 3x^3\left(2x^2 - x^5\right)$$

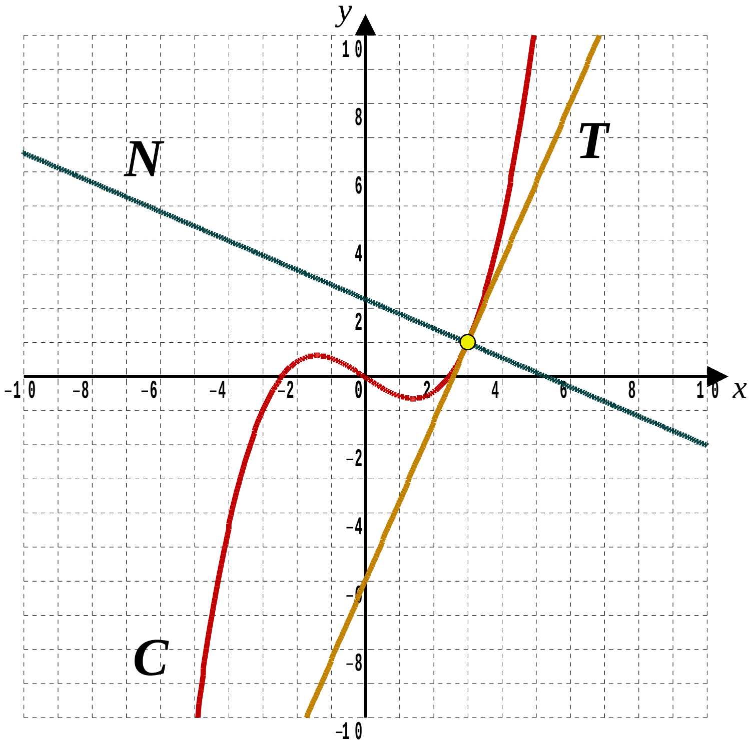
$$= 6x^5 - 3x^8 \qquad \therefore \quad m'(x) = 30x^4 - 24x^7$$

[3 marks]

2.1 The Tangent and The Normal to a Curve

Example

The graph shows a curve C with equation $y = \dfrac{x^3}{9} - \dfrac{2x}{3}$



Also shown is the tangent T and Normal N to the curve at the point (3, 1)
Use calculus to determine the equation of the tangent and the equation of the normal.

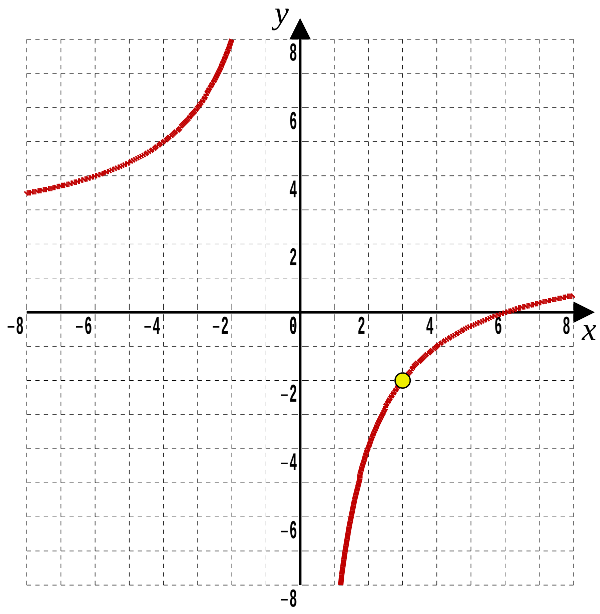
[8 marks]

2.2 Exercise

Marks Available : 52

Question 1

The graph is of the curve with equation $y = 2 - \dfrac{12}{x}$ with $x \neq 0$



(i) Determine the gradient equation of the curve, $\dfrac{dy}{dx}$

[2 marks]

(ii) Work out the equation of the tangent to the curve at the point $(3, -2)$

[3 marks]

(iii) Work out the equation on the normal to the curve at the point $(3, -2)$

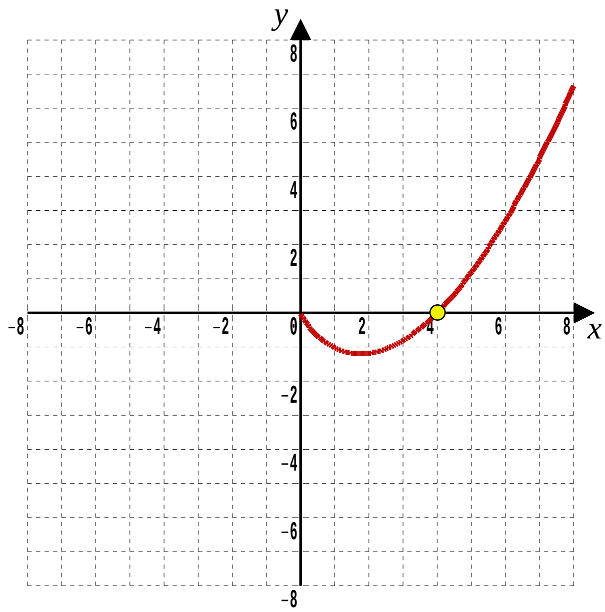
[3 marks]

(iv) Draw your part (ii) and (iii) straight lines onto the graph above.

[2 marks]

Question 2

The graph is of the curve with equation $y = x^{\frac{3}{2}} - 2x$ with $x \geqslant 0$



(i) Determine the gradient equation of the curve, $\dfrac{dy}{dx}$

[2 marks]

(ii) Work out the equation of the tangent to the curve at the point $(4, \; 0)$

[3 marks]

(iii) Work out the equation on the normal to the curve at the point $(4, \; 0)$

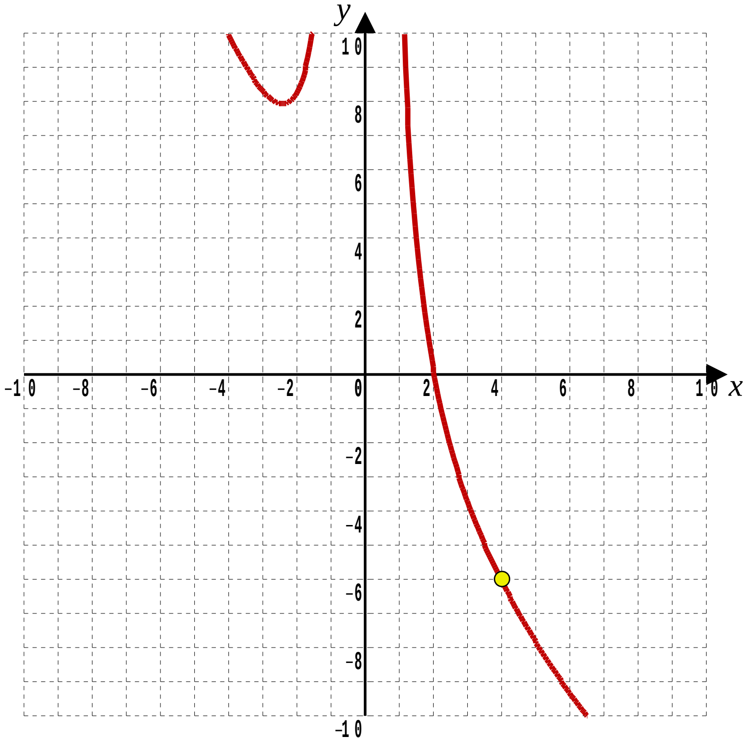
[3 marks]

(iv) Draw your part (ii) and (iii) straight lines onto the graph above.

[2 marks]

Question 3

The graph is of the curve with equation $y = \dfrac{x^2}{16} + \dfrac{16}{x^2} - 2x$ with $x \neq 0$



(i) Determine the gradient equation of the curve, $\dfrac{dy}{dx}$

[2 marks]

(ii) Work out the equation of the tangent to the curve at the point $(4, -6)$

[3 marks]

(iii) Work out the equation on the normal to the curve at the point $(4, -6)$

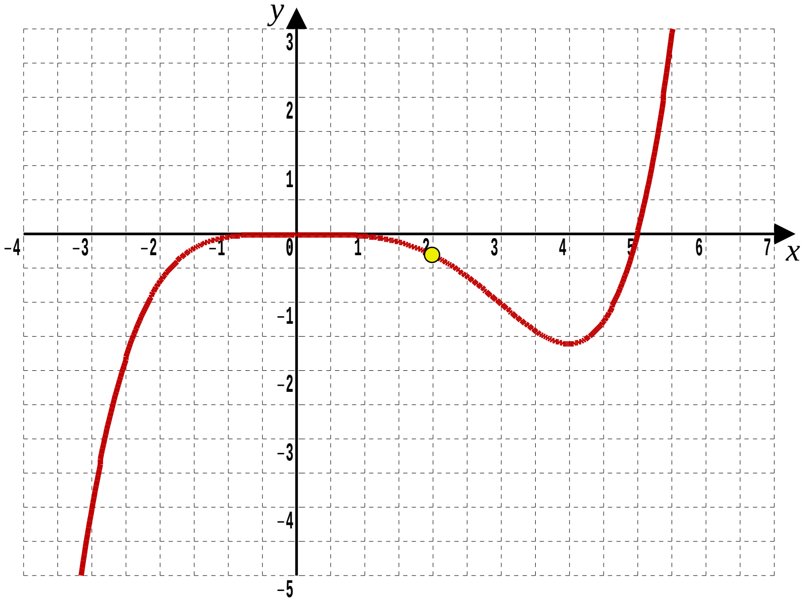
[3 marks]

(iv) Draw your part (ii) and (iii) straight lines onto the graph above.

[2 marks]

Question 4

The graph is of the curve with equation $y = \dfrac{x^4}{32}\left(\dfrac{x-5}{5}\right)$



(i) Given that the point $(2, a)$ is on the curve, find the value of a

[2 marks]

(ii) Determine the gradient equation of the curve, $\dfrac{dy}{dx}$

[2 marks]

(iii) Work out the equation of the tangent to the curve at the point $(2,\ a)$

[3 marks]

(iv) Work out the equation on the normal to the curve at the point $(2,\ a)$

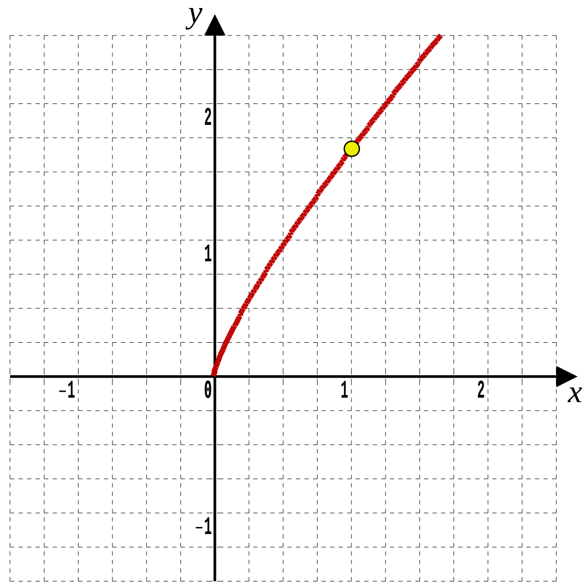
[3 marks]

(v) Draw your part (iii) and (iv) straight lines onto the graph above.

[2 marks]

Question 5

The graph is of the curve with equation $y = x^{\frac{2}{3}} + \dfrac{2x}{3}$ $\qquad x \geqslant 0$



(i) Determine the gradient equation of the curve, $\dfrac{dy}{dx}$

[2 marks]

(iii) Find the equation of the tangent to the curve at the point where $x = 1$

[3 marks]

(iii) Find the equation of the normal to the curve at the point where $x = 1$

[3 marks]

(iv) Draw your part **(ii)** and **(iii)** straight lines onto the graph above.

[2 marks]

2.3 Answers

2.3.1 Solution to Example 2.1

$$y = \frac{x^3}{9} - \frac{2x}{3}$$

$$\frac{dy}{dx} = \frac{x^2}{3} - \frac{2}{3}$$

$$\therefore \text{ When } x = 3, \qquad \frac{dy}{dx} = \frac{7}{3}$$

For the tangent, a line is required through $(3,\ 1)$ with a gradient of $\frac{7}{3}$

$$y = mx + c$$

$$1 = \frac{7}{3} \times 3 + c \quad \text{gives} \quad c = -6$$

$$\text{So the tangent is,} \quad y = \frac{7}{3}x - 6$$

Observe, on the graph, the tangent passes through (– 6, 0) and that from that point three across and seven up moves to (3, 1) so this result is visually verifiable.

For the normal, a line is required through $(3,\ 1)$ with a gradient of $-\frac{3}{7}$

$$y = mx + c$$

$$1 = -\frac{3}{7} \times 3 + c \quad \text{gives} \quad c = \frac{16}{7}$$

$$\text{So the normal is,} \quad y = -\frac{3}{7}x + \frac{16}{7}$$

Pleasingly, from an educational point of view, having been able to visually check the answer for the tangent, to get this exact answer the maths is needed, rather than simply looking at the graph of the Normal.

[8 marks]

2.3.2 Solutions to Exercise 2.2

Answer 1

(i) $\dfrac{dy}{dx} = \dfrac{12}{x^2}$

[2 marks]

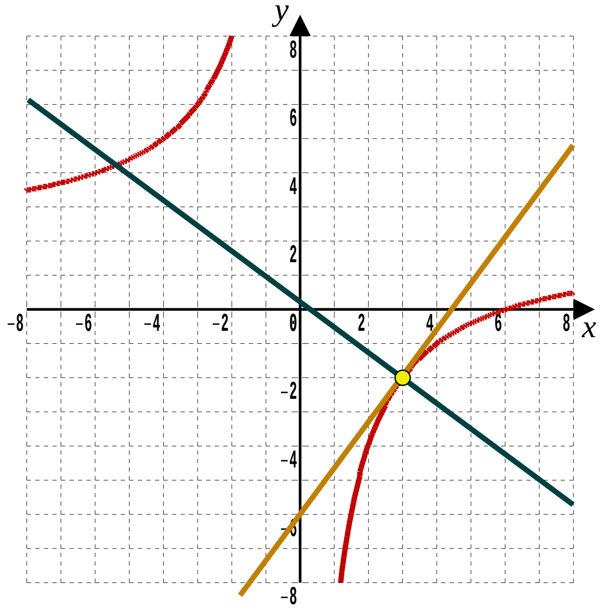
(ii) Tangent: $y = \dfrac{4}{3}x - 6$

[3 marks]

(iii) Normal: $y = -\dfrac{3}{4}x + \dfrac{1}{4}$

[3 marks]

(iv)



[2 marks]

Answer 2

(i) $\dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{1}{2}} - 2$ or $\dfrac{dy}{dx} = \dfrac{3}{2}\sqrt{x} - 2$

[2 marks]

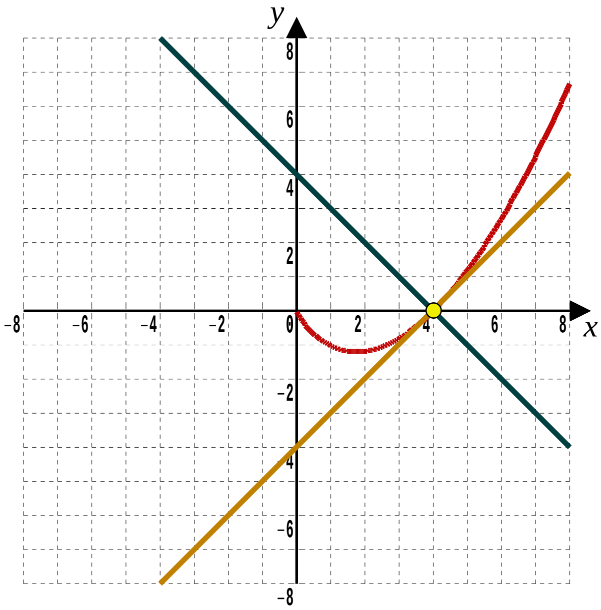
(ii) Tangent: $y = x - 4$

[3 marks]

(iii) Normal: $y = -x + 4$

[3 marks]

(iv)



[2 marks]

Answer 3

(i) $$\frac{dy}{dx} = \frac{x}{8} - \frac{32}{x^3} - 2$$

[2 marks]

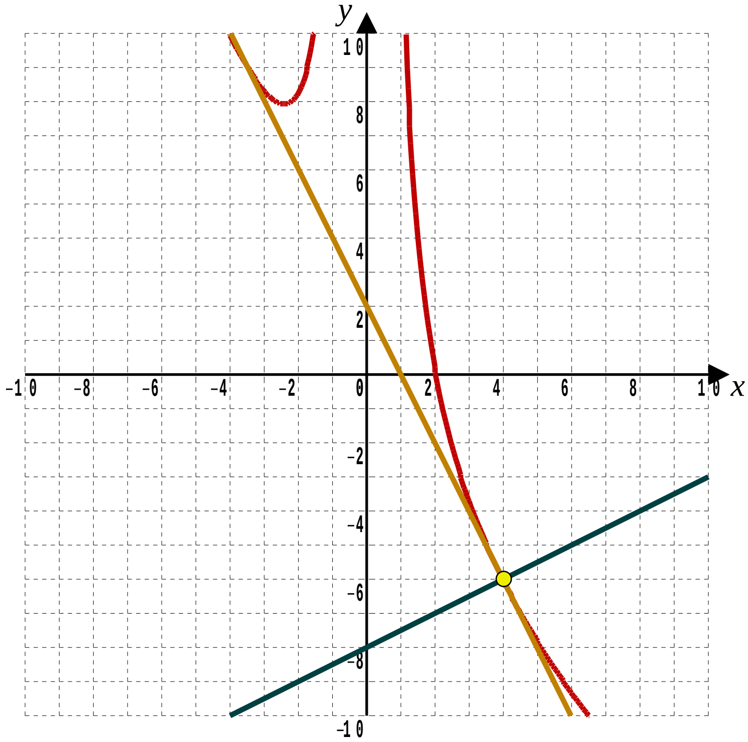
(ii) Tangent: $y = -2x + 2$

[3 marks]

(iii) Normal: $y = 0.5x - 8$

[3 marks]

(iv)



[3 marks]

Answer 4

(i) When $x = 2,\ y = \frac{2^4}{32}\left(\frac{2-5}{5}\right)$

$$= \frac{1}{2} \times \frac{-3}{5}$$

$$= -\frac{3}{10}$$

$$\therefore\ a = -0.3$$

[2 marks]

(ii) $y = \frac{x^4}{32}\left(\frac{x-5}{5}\right)$

$$= \frac{x^5}{160} - \frac{x^4}{32}$$

$$\frac{dy}{dx} = \frac{x^4}{32} - \frac{x^3}{8}$$

[2 marks]

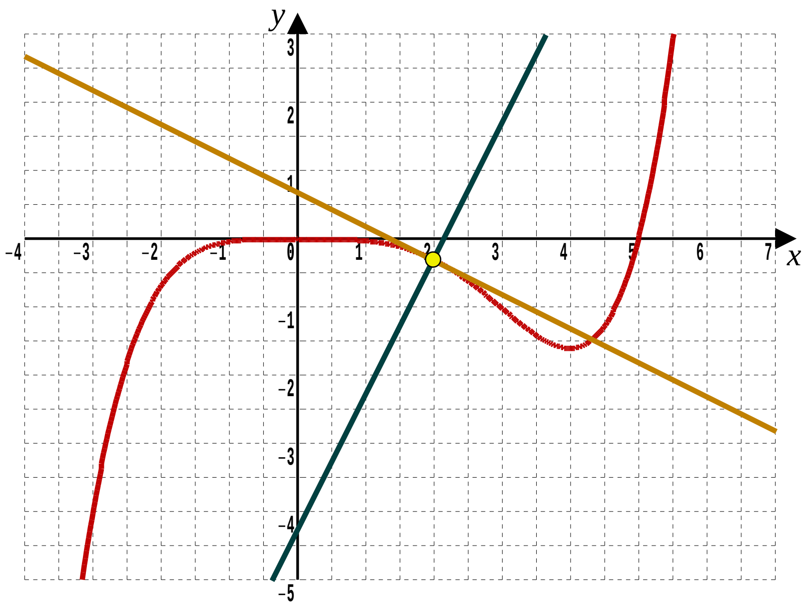
(iii) Tangent: $y = -0.5x + 0.7$

[3 marks]

(iv) Normal: $y = 2x - 4.3$

[3 marks]

(v)



[2 marks]

Answer 5

(i) $\dfrac{dy}{dx} = \dfrac{2}{3}x^{-\frac{1}{3}} + \dfrac{2}{3}$

[2 marks]

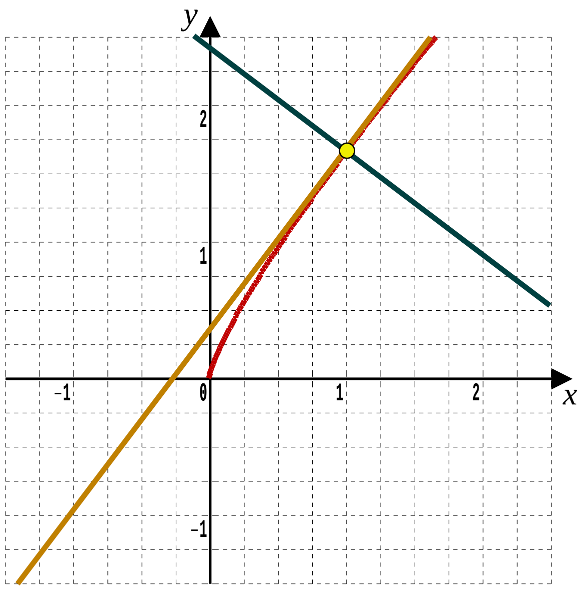
(ii) Tangent: $y = \dfrac{4}{3}x + \dfrac{1}{3}$

[3 marks]

(iii) Normal: $y = -\dfrac{3}{4}x + \dfrac{29}{12}$

[3 marks]

(iv)



[2 marks]

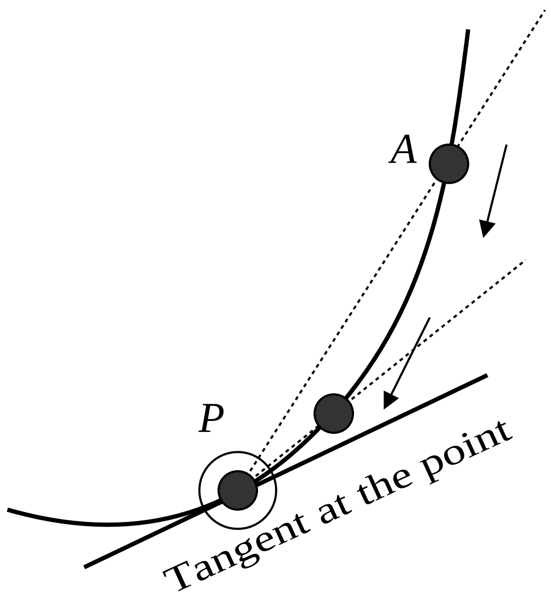
3.1 Differentiation from First Principles

In Lessons 1 and 2 good use was made of this rule:

If $y = x^n$ then $\dfrac{dy}{dx} = n\,x^{n-1}$ for any constant n

It is time to gain some understanding of where this rule comes from and why it works. A key idea stems from our work with the tangent to a curve.

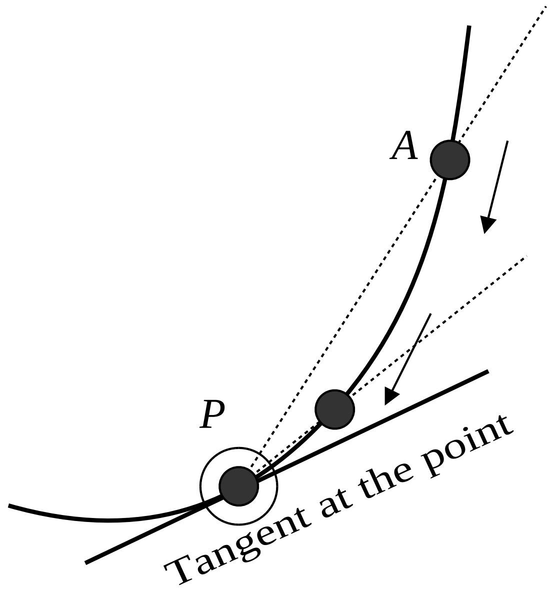
Definition: The Gradient of a Curve
The gradient of a curve at any given point on that curve is the same as the gradient of the tangent to the curve at that point



Suppose the gradient at the point P is sought.
Another point on the curve a short step away, A, is selected and the line between P and A is considered. The line between P and A is a poor approximation to what is wanted, the tangent at P.
However, as A slides along the curve towards P the approximating line becomes a better and better approximation of the desired line, the tangent at P.

Translating this argument into mathematics results in a 'first principles' mathematical definition of what a derivative is.

So, here is the same argument written as mathematics.



Let the curve be of the function, $f(x)$

Let the x coordinate of P also be x.

The point P then has coordinates $(x,\ f(x))$

Let the x coordinate at A be $x + h$

The point A then has coordinates $(x + h,\ f(x + h))$

The gradient of the line between P and A will be

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} \\ &= \frac{f(x + h) - f(x)}{x + h - x} \\ &= \frac{f(x + h) - f(x)}{h} \end{aligned}$$

To slide the point A towards P, h has to become smaller and smaller;

Definition of The Gradient Function (The Derivative)

$$f'(x) = \underset{h \to 0}{limit} \frac{f(x + h) - f(x)}{h}$$

Using this definition to find a derivative is “differentiating from first principles”.

3.2 Example

The point P with coordinates $(3,\ 36)$ lies on the curve with equation $y = 4x^2$

By differentiating from first principles, determine the gradient at P.

[5 marks]

3.3 Exercise

Marks Available : 40

Question 1

(i) Try to write down from memory, the first principles definition of a curve's gradient equation.

[2 marks]

(ii) The point P with coordinates $(5, \ 25)$ lies on the curve $y = x^2$
By differentiating from first principles, determine the gradient at P.

[4 marks]

Question 2

The point P with coordinates $(4, \ 48)$ lie on the curve $y = 3x^2$
By differentiating from first principles, determine the gradient at P.

[5 marks]

Question 3

The point *P* with coordinates $(2,\ 7)$ lie on the curve $y = x^2 + 3$
By differentiating from first principles, determine the gradient at *P*.

[6 marks]

Question 4

The point *P* with coordinates $(1,\ 5)$ lie on the curve $y = x^3 + 4x$
By differentiating from first principles, determine the gradient at *P*.

[8 marks]

Question 5

(i) By using the binomial theorem, or otherwise, expand $(3 + h)^4$

[2 marks]

(ii) The point P with coordinates $(3,\ 81)$ lie on the curve $y = x^4$
By differentiating from first principles, determine the gradient at P.

[5 marks]

Question 6
Consider the function

$$f(x) = \frac{1}{x}$$

The point P with coordinates (5, 0.2) lies on the curve of $f(x)$

By differentiating from first principles, determine the gradient at P.

[8 marks]

3.4 Answers

3.4.1 Solution to Example 3.2

The “grab the nettle” approach adopted here is to do the algebra first then insert the value of x at the very end.
This has the advantage of enabling students to do algebra only questions without having to learn anything new, but is a little harder to master initially.

$f(x) = 4x^2$

By definition

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{4(x+h)^2 - 4x^2}{h} \\
&= \lim_{h \to 0} \frac{4\left(x^2 + 2xh + h^2\right) - 4x^2}{h} \\
&= \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - 4x^2}{h} \\
&= \lim_{h \to 0} \frac{8xh + 4h^2}{h} \\
&= \lim_{h \to 0} \frac{h(8x + 4h)}{h} \\
&= \lim_{h \to 0} 8x + 4h \\
&= 8x \\
\therefore\ f'(3) &= 24
\end{aligned}
$$

[5 marks]

3.4.2 Solutions to Exercise 3.3

Answer 1

(i) By definition, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

[2 marks]

(ii) $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{(By definition)}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h)}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

$$\therefore f'(5) = 10$$

[4 marks]

Answer 2

$f(x) = 3x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{(By definition)}$$

$$= \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(6x + 3h)}{h}$$

$$= \lim_{h \to 0} 6x + 3h$$

$$= 6x$$

$$\therefore f'(4) = 24$$

[5 marks]

Answer 3

$f(x) = x^2 + 3$

By definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{(By definition)}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 3 - (x^2+3)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3 - x^2 - 3}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h)}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

$$\therefore \ f'(2) = 4$$

[6 marks]

Answer 4

$f(x) = x^3 + 4x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{(By definition)}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 + 4(x+h) - \left[x^3 + 4x \right]}{h}$$

METHOD

Using a Multiplication Grid,

$$(x+h)^3 = (x+h)\,(x+h)\,(x+h)$$

$$= (x+h)\left(x^2 + 2xh + h^2\right)$$

	x^2	$2xh$	h^2
x	x^3	$2x^2h$	xh^2
h	x^2h	$2xh^2$	h^3

$$= x^3 + 3x^2h + 3xh^2 + h^3$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 4x + 4h - x^3 - 4x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 + 4h}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 + 4)}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 + 4$$

$$= 3x^2 + 4$$

$$\therefore \; f'(1) = 7$$

[8 marks]

Answer 5

(i) METHOD 1

Using the Binomial Theorem,

$$(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

METHOD 2

Using a Multiplication Grid,

$$(x + h)^4 = \left((x + h)^2 \right)^2$$

$$= \left(x^2 + 2xh + h^2 \right)^2$$

	x^2	$2xh$	h^2
x^2	x^4	$2x^3h$	x^2h^2
$2xh$	$2x^3h$	$4x^2h^2$	$2xh^3$
h^2	x^2h^2	$2xh^3$	h^4

$$= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

[2 marks]

(ii) $f(x) = x^4$

$$f'(a) = \underset{h \to 0}{limit} \frac{f(x + h) - f(x)}{h} \qquad \text{(By definition)}$$

$$= \underset{h \to 0}{limit} \frac{(x + h)^4 - x^4}{h}$$

$$= \underset{h \to 0}{limit} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$= \underset{h \to 0}{limit} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

$$= \underset{h \to 0}{limit} \frac{h\left(4x^3 + 6x^2h + 4xh^2 + h^3\right)}{h}$$

$$= \underset{h \to 0}{limit}\ 4x^3 + 6x^2h + 4xh^2 + h^3$$

$$= 4x^3$$

$$\therefore\ f'(3) = 108$$

[5 marks]

Answer 6

$$f(x) = \frac{1}{x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{(By definition)}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{x(x+h)h}$$

$$= \lim_{h \to 0} \frac{-h}{x(x+h)h}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= \frac{-1}{x \times x}$$

$$= -\frac{1}{x^2}$$

$$\therefore \; f'(5) = -\frac{1}{25}$$

[8 marks]

Lesson 4

A-Level Pure Mathematics : Year 1
Differentiation II

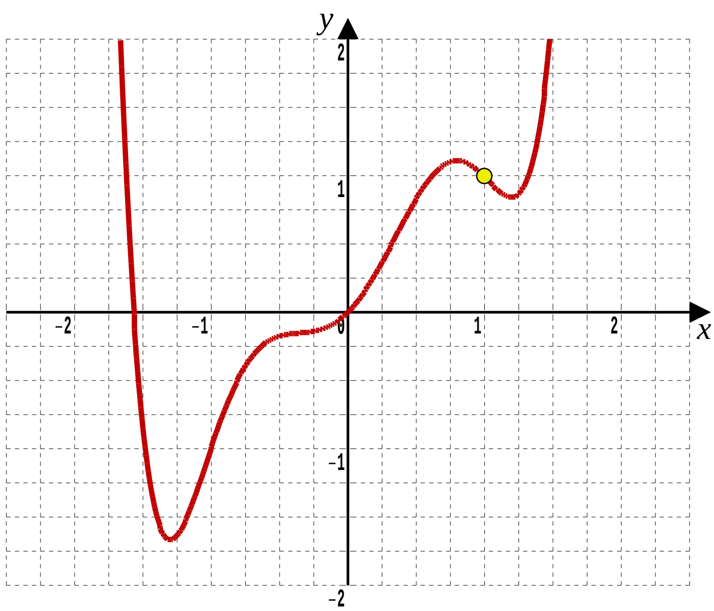
4.1 Point, Gradient, Bend

Consider the graph of the function, $f(x) = x^6 - 3x^4 + 2x^2 + x$

This is a **POINT** equation because, given a value of x, substituting that value into the function yields a **POINT** $(x, f(x))$ on the curve.

For example, if $x = 1$, then $f(1) = 1^6 - 3 \times 1^4 + 2 \times 1^2 + 1$

$= 1 \quad \therefore \ (1, 1)$ is a point on the curve.



By differentiation, $f'(x) = 6x^5 - 12x^3 + 4x + 1$

This is a **GRADIENT** equation because, given a value of x, substituting that value into $f'(x)$ yields the **GRADIENT** of the curve at $(x, f(x))$.

For example, if $x = 1$. then $f'(1) = 6 \times 1^5 - 12 \times 1^3 + 4 \times 1 + 1$

$= -1 \quad \therefore$ The gradient at $(1, 1)$ is -1

By differentiating a second time, $f''(x) = 30x^4 - 36x^2 + 4$

This is a **BEND** equation because, given a value of x, substituting that value into $f''(x)$ yields the **BEND** of the curve at $(x, f(x))$.

For example, if $x = 1$. then $f''(1) = 30 \times 1^4 - 36 \times 1^3 + 4$

$= -2$

The fact that this is negative reveals that the bend at (1, 1) is CLOCKWISE

4.2 Turning Points

The forgoing is of most use when trying to determine the nature of turning points.

Definition: Turning Point

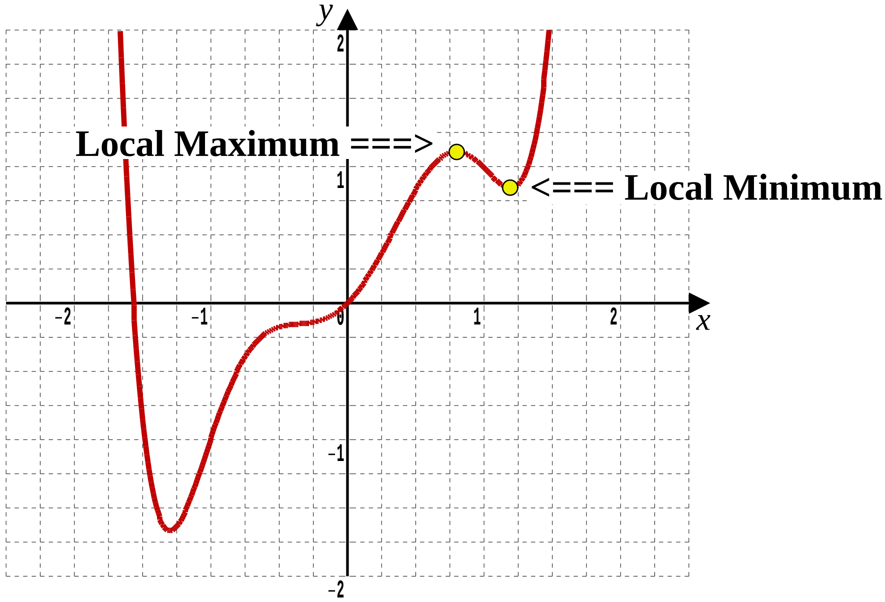
A turning point is a point where the gradient is zero and the sign of the gradient is different either side of the turning point.

They are so called because the gradient turns (BENDS) through zero.

If a function $f(x)$ has a turning point when $x = a$, then;

- if $f''(a) > 0$, the point is a local minimum i.e. if $f''(a)$ is positive
- if $f''(a) < 0$, the point is a local maximum i.e. if $f''(a)$ is negative

If $f''(a) = 0$, the bend has not been determined; the bend detector test has failed !



4.3 Example

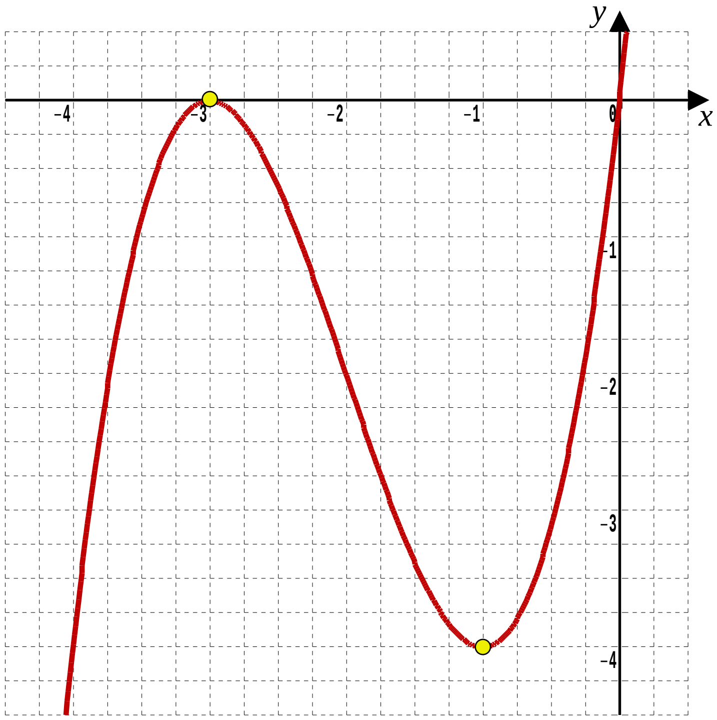
(i) Sketch an example of a curve which has a single point, P, with gradient zero, but for which P is not a turning point.
Such a point is described as a ***stationary point of inflection***.

(ii) Sketch an example of a curve with a point of inflection, but which has no point where the gradient is zero.
Such a point is described as a ***non-stationary point of inflection***.
Is your sketch of an ***increasing*** or ***decreasing*** function ?

[2 marks]

4.4 Example

Consider the curve, graphed below, with equation, $y = x^3 + 6x^2 + 9x$



(i) Find $\dfrac{dy}{dx}$

[2 marks]

(ii) Find $\dfrac{d^2y}{dx^2}$

[2 marks]

The curve has two turning points.

(iii) Work out the coordinates of both turning points and determine their nature.

[4 marks]

4.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Consider the curve with equation $y = 5x^2 - 30x$

(i) Find $\dfrac{dy}{dx}$

[1 mark]

(ii) Find $\dfrac{d^2y}{dx^2}$

[1 mark]

The curve with has one turning point.

(iii) Work out the coordinates of the turning point and determine its nature.

[3 marks]

Question 2

A curve has equation $y = 4x^2 + 16x + 21$

(i) Find $\dfrac{dy}{dx}$

[2 marks]

(ii) Find the coordinates of the turning point by solving the equation;

$$\frac{dy}{dx} = 0$$

Show your working clearly.

[4 marks]

(iii) Find $\dfrac{d^2y}{dx^2}$ and explain what this is telling you about the turning point.

[2 marks]

Question 3

A cubic curve has equation $y = x^3 + 9x^2 + 15x$

(i) Find $\dfrac{dy}{dx}$

[2 marks]

(ii) Find $\dfrac{d^2y}{dx^2}$

[2 marks]

The curve has two turning points.

(iii) Work out the coordinates of both turning points and determine their nature.

[4 marks]

Question 4

A function is described by $f(x) = x^3 + 6x^2 + 5$

(i) Find $f'(x)$

[2 marks]

(ii) Find $f''(x)$

[2 marks]

The curve has two turning points.

(iii) Work out the coordinates of both turning points and determine their nature.

[4 marks]

Question 5

A-Level Examination Question from January 2019, Paper C12, Q3 (Edexcel)

A curve has equation $y = \sqrt{2}\,x^2 - 6\sqrt{x} + 4\sqrt{2}, \quad x > 0$

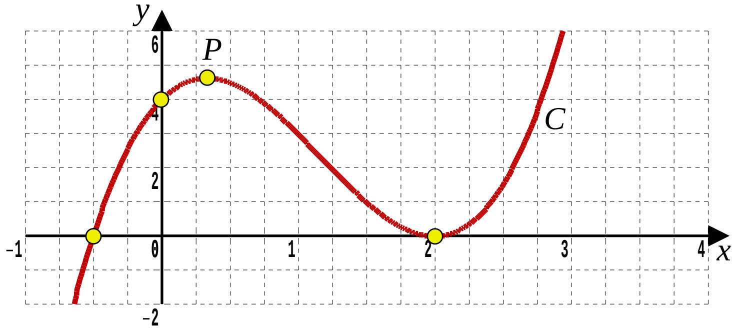
Find the gradient of the curve at the point $\left(2,\ 2\sqrt{2}\right)$

Write your answer in the form $a\sqrt{2}$, where a is a constant.

[4 marks]

Question 6

A-Level Examination Question from June 2018, Paper C12, Q14 (Edexcel)



The graph shows a curve, C, with equation $y = f(x)$ where

$$f(x) = (x - 2)^2(2x + 1) \qquad x \in \mathbb{R}$$

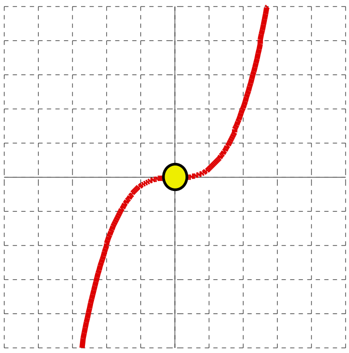
The curve crosses the x-axis at $\left(-\frac{1}{2},\ 0\right)$, touches it at $(2,\ 0)$ and crosses the y-axis at $(0,\ 4)$. There is a maximum turning point marked P.

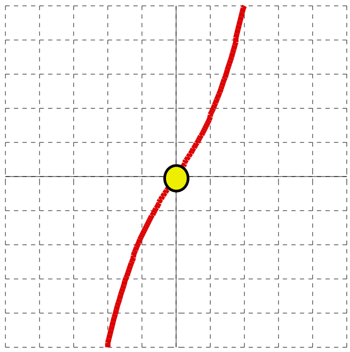
Use $f'(x)$ to find the exact coordinates of the turning point P.

[7 marks]

4.6 Answers

4.6.1 Solution to 4.3 Example





Left : An example of a stationary point of inflection
This function is increasing
(Note that increasing can include points with gradient zero)

Right : An example of a non-stationary point of inflection
This function is strictly increasing
(Note that the "strictly" means there are no points with gradient zero)

[2 marks]

4.6.1 Solution to Example 4.4

(i) $y = x^3 + 6x^2 + 9x \quad \therefore \quad \frac{dy}{dx} = 3x^2 + 12x + 9$

[2 marks]

(ii) $\frac{d^2y}{dx^2} = 6x + 12$

[2 marks]

(iii) For a turning point, $3x^2 + 12x + 9 = 0$

$$x^2 + 4x + 3 = 0$$

$$(x + 3)(x + 1) = 0$$

Either $x = -3$ or $x = -1$

$y = (-3)^3 + 6(-3)^2 + 9(-3)$
$= 0$

$y = (-1)^3 + 6(-1)^2 + 9(-1)$
$= -4$

The turning points are $(-3,\ 0)$, $(-1, -4)$

For $x = -3$, $\frac{d^2y}{dx^2} = -6$ For $x = -1$, $\frac{d^2y}{dx^2} = +6$

$(-3,\ 0)$ is a local maximum $(-1, -4)$ is a local minimum

[4 marks]

4.6.2 Solutions to 4.5 Exercise

Answer 1

(i) $y = 5x^2 - 30x \quad \therefore \quad \dfrac{dy}{dx} = 10x - 30$

[1 mark]

(ii) $\dfrac{d^2y}{dx^2} = 10$

[1 mark]

(iii) For a turning point, $10x - 30 = 0 \quad \Leftrightarrow x = 3$

To get a ***point*** substitute this value of *x* back into the point equation of the curve. (NOT the gradient equation of the curve !)

$$\begin{aligned} y &= 5x^2 - 30x \\ &= 5 \times 3^2 - 30 \times 3 \\ &= -45 \Leftrightarrow (3, -45) \end{aligned}$$

When $x = 3$, $\dfrac{d^2y}{dx^2} = 10$

As this is positive $(3, -45)$ is a local minimum.

[3 marks]

Answer 2

(i) $y = 4x^2 + 16x + 21 \quad \therefore \quad \dfrac{dy}{dx} = 8x + 16$

[2 marks]

(ii) For a turning point, $8x + 16 = 0 \quad \Leftrightarrow \quad x = -2$

To get a ***point*** substitute this value of *x* back into the point equation of the curve. (NOT the gradient equation of the curve !)

$$\begin{aligned} y &= 4x^2 + 16x + 21 \\ &= 4 \times (-2)^2 + 16 \times (-2) + 21 \\ &= 16 - 32 + 21 \\ &= 5 \quad \Leftrightarrow \quad (-2, 5) \end{aligned}$$

[4 marks]

(iii) Regardless of the value of *x*, $\dfrac{d^2y}{dx^2} = 8$

As this is positive $(-2, 5)$ is a local minimum.

As this is positive is a local minimum.

[2 marks]

Answer 3

(i) $y = x^3 + 9x^2 + 15x$ $\therefore$ $\dfrac{dy}{dx} = 3x^2 + 18x + 15$

[2 marks]

(ii) $\dfrac{d^2y}{dx^2} = 6x + 18$

[2 marks]

(iii) For a turning point, $3x^2 + 18x + 15 = 0$

$$x^2 + 6x + 5 = 0$$

$$(x + 5)(x + 1) = 0$$

Either $x = -5$ or $x = -1$

$y = x^3 + 9x^2 + 15x$
$= (-5)^3 + 9(-5)^2 + 15(-5)$
$= -125 + 225 - 75$
$= 25 \Leftrightarrow (-5,\ 25)$

$y = x^3 + 9x^2 + 15x$
$= (-1)^3 + 9(-1)^2 + 15(-1)$
$= -1 + 9 - 15$
$= -7 \Leftrightarrow (-1, -7)$

When $x = -5$, $\dfrac{d^2y}{dx^2} = -12$

$(-5,\ 25)$ is a local maximum

When $x = -1$, $\dfrac{d^2y}{dx^2} = +12$

$(-1, -7)$ is a local minimum

[4 marks]

Answer 4

(i) $f(x) = x^3 + 6x^2 + 5$ $\therefore$ $f'(x) = 3x^2 + 12x$

[2 marks]

(ii) $f''(x) = 6x + 12$

[2 marks]

(iii) For a turning point, $3x^2 + 12x = 0$

$$x^2 + 4x = 0$$

$$x(x + 4) = 0$$

Either $x = 0$ or $x = -4$

$y = x^3 + 6x^2 + 5$
$= (0)^3 + 6(0)^2 + 5$
$= 5 \Leftrightarrow (0,\ 5)$

$y = x^3 + 6x^2 + 5$
$= (-4)^3 + 6(-4)^2 + 5$
$= 37 \Leftrightarrow (-4,\ 37)$

When $x = 0$, $\dfrac{d^2y}{dx^2} = 12$

$(0,\ 5)$ is a local minimum

When $x = -4$, $\dfrac{d^2y}{dx^2} = -12$

$(-4,\ 37)$ is a local maximum

[4 marks]

Answer 5

$$y = \sqrt{2}\,x^2 - 6\sqrt{x} + 4\sqrt{2}$$
$$= \sqrt{2}\,x^2 - 6x^{\frac{1}{2}} + 4\sqrt{2}$$
$$\frac{dy}{dx} = 2\sqrt{2}\,x - 3x^{-\frac{1}{2}}$$
$$= 2\sqrt{2}\,x - \frac{3}{\sqrt{x}}$$
$$\text{When } x = 2,\ \frac{dy}{dx} = 4\sqrt{2} - \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= 4\sqrt{2} - \frac{3}{2}\sqrt{2}$$
$$= \frac{5}{2}\sqrt{2} \qquad \therefore\ a = \frac{5}{2}$$

[4 marks]

Answer 6

Given that students do not yet know of "The Product Rule"...

$$f(x) = (x - 2)^2(2x + 1)$$
$$= \left(x^2 - 4x + 4\right)\left(2x + 1\right)$$
$$= 2x^3 - 8x^2 + 8x + x^2 - 4x + 4$$
$$= 2x^3 - 7x^2 + 4x + 4$$
$$f'(x) = 6x^2 - 14x + 4$$

For turning points, $6x^2 - 14x + 4 = 0$

$$3x^2 - 7x + 2 = 0$$
$$(3x - 1)(x - 2) = 0$$

Either $x = \frac{1}{3}$ or $x = 2$

The minimum at $(2,\ 0)$ is already known about.

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{3} - 2\right)^2\left(\frac{2}{3} + 1\right)$$
$$= \left(\frac{5}{3}\right)^2 \times \frac{5}{3}$$
$$= \frac{125}{27} \qquad \therefore\ P \text{ is } \left(\frac{1}{3},\ \frac{125}{27}\right)$$

[7 marks]

Lesson 5

5.1 Local Minimum & Local Maximum

Differentiation is used to find the optimal solutions to problems.
On a graph, such 'best' solutions are often found where there is either a **local maximum** or a **local minimum**.

Mathematically, to find all turning points on a curve:

STEP 1 : Differentiate the **POINTS** equation to get the **GRADIENT** equation.

STEP 2 : Set the **GRADIENT** equation equal to zero and solve.

STEP 3 : Put the solution(s) from STEP 2 into the **POINTS** equation to get a list of possible turning points.
Be aware that there may be points of inflection in this list.

STEP 4 : Differentiate the **GRADIENT** equation to get the **BEND** equation.

STEP 5 : Put the solution(s) from STEP 2 back into the **BEND** equation.

- A turning point with positive bend is a local MINimum.
- A turning point with negative bend is a local MAXimum.

Alas, a bend of zero does not determine the nature of the turning point; it could be a minimum, a maximum or a point of inflection.

5.2 Example

(i) The curve $y = f(x)$ where $f(x) = x^3 + 12x$ has no turning points.
Show that this is the case by trying to find them via the mathematical method.

[3 marks]

(ii) Circle which one of the following best describes $f(x)$;

$f(x)$ is a decreasing function — $f(x)$ is a strictly decreasing function

$f(x)$ is an increasing function — $f(x)$ is a strictly increasing function

[1 mark]

5.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 52

Question 1

A curve has equation $y = x^3 + 3x^2 - 24x$

Using calculus, find the coordinates of the turning points of the curve and also determine their nature.

[5 marks]

Question 2

The curve with equation $y = 8x^2 + \frac{2}{x}$ has one turning point.

Find the coordinates of this turning point and determine its nature.
Use calculus, and show your working clearly.

[5 marks]

Question 3

Consider the polynomial curve, $y = 5x^5 - 7x^3$

(i) Write down the **gradient equation** of the polynomial curve.

$$\frac{dy}{dx} =$$

[2 marks]

(ii) Write down the **bend detector equation** of the polynomial curve.

$$\frac{d^2y}{dx^2} =$$

[1 mark]

(iii) Use the appropriate equation to find the **point** on the curve when $x = 1$

[1 mark]

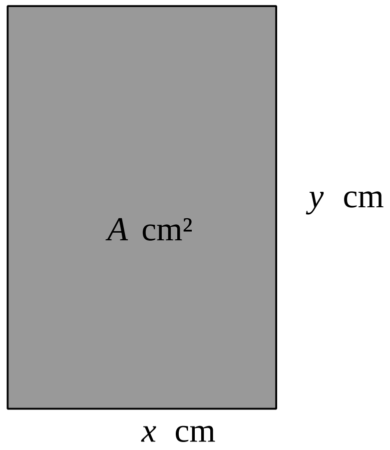
(iv) Use the appropriate equation to find the **gradient** of the curve when $x = 1$

[1 mark]

(v) Use the appropriate equation to determine, when $x = 1$, if the curve is **bending** clockwise or anticlockwise.

[2 marks]

Question 4



The diagram shows a rectangular photo frame of area A cm^2.
The width of the photo frame is x cm.
The height of the photo frame is y cm.
The perimeter of the photo frame is 72 cm.

(i) Show that $A = 36x - x^2$

[3 marks]

(ii) Find $\dfrac{dA}{dx}$

[2 marks]

(iii) Find the maximum value of A.

[3 marks]

(iv) Show that $\dfrac{d^2A}{dx^2}$ detects correctly that your part (iii) answer is a maximum.

[2 marks]

Question 5

(a) For the equation $y = 5000x - 625x^2$, find $\dfrac{dy}{dx}$

[2 marks]

(b) Find the coordinates of the turning point of $y = 5000x - 625x^2$

[3 marks]

(c) **(i)** State whether this turning point is a maximum or a minimum.

(ii) Give a reason for your answer.

[2 marks]

(d) A publisher has to set the price for a new book.
The profit, £y, depends on the price of the book, £x, where

$$y = 5000x - 625x^2$$

(i) What price would you advise the publisher to set for the book ?

(ii) Give a reason for your answer.

[2 marks]

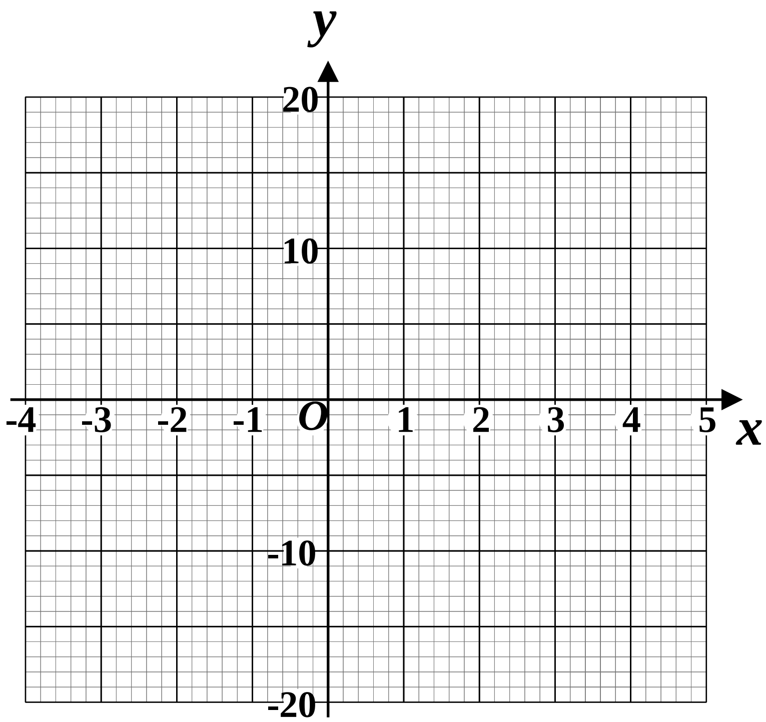
Question 6

(a) Complete the table of values for $y = x^3 - 12x + 2$

x	-3	-2	-1	0	1	2	3	4
y	11						-7	18

[2 marks]

(b) On the grid, draw the graph of $y = x^3 - 12x + 2$



[2 marks]

(c) For the curve with equation $y = x^3 - 12x + 2$

(i) Find $\dfrac{dy}{dx}$

(ii) find the gradient of the curve at the point where $x = 5$

[4 marks]

Question 7

GCSE Examination question from June 2011, 3H, Q21.

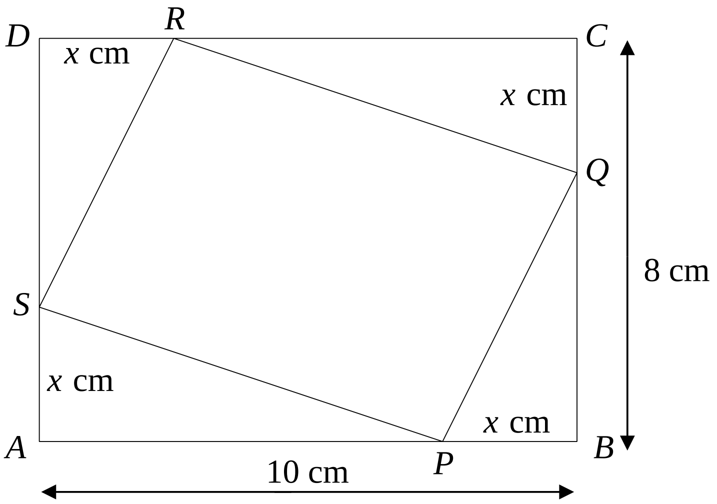


Diagram NOT accurately drawn

ABCD is a rectangle

$AB = 10$ cm

$BC = 8$ cm

P, *Q*, *R* and *S* are points on the sides of the rectangle.

$BP = CQ = DR = AS = x$ cm

(a) Show that the area, A cm^2, of the quadrilateral *PQRS* is given by the formula

$$A = 2x^2 - 18x + 80$$

[3 marks]

(b) For $A = 2x^2 - 18x + 80$

(i) find $\dfrac{dA}{dx}$

(ii) find the value of x for which A is a minimum.

(iii) Explain how you know that A is a minimum for this value of x.

[5 marks]

5.4 Answers

5.4.1 Solution to Example 5.2

(i)

$$y = x^3 + 12x$$

$$\frac{dy}{dx} = 3x^2 + 12$$

To find turning points,

$$\frac{dy}{dx} = 0 \quad \text{when} \quad 3x^2 + 12 = 0$$

$$x^2 + 4 = 0$$

$$x^2 = -4$$

This has no real solutions; the curve has no turning points. ☐

[3 marks]

This is an example of a strictly increasing function,.
Its gradient is always positive.

[1 mark]

5.4.2 Solution to Exercise 5.3

Answer 1

$$y = x^3 + 3x^2 - 24x$$

$$\frac{dy}{dx} = 3x^2 + 6x - 24$$

$$\frac{dy}{dx} = 0 \quad \text{when} \quad 3x^2 + 6x - 24 = 0$$

$$x^2 + 2x - 8 = 0$$

$$(x + 4)(x - 2) = 0$$

Either $x = -4$ or $x = 2$

The turning points are $(-4,\ 80),\ (2, -28)$

$$\frac{d^2y}{dx^2} = 6x + 6$$

For $x = -4$, $\dfrac{d^2y}{dx^2} = -18$, so $(-4,\ 80)$ is a MAXimum

For $x = 2$, $\dfrac{d^2y}{dy^2} = 18$, so $(2, -28)$ is a MINimum

[5 marks]

Answer 2

$$y = 8x^2 + \frac{2}{x}$$

$$= 8x^2 + 2x^{-1}$$

$$\frac{dy}{dx} = 16x - 2x^{-2}$$

$$= 16x - \frac{2}{x^2}$$

$$\frac{dy}{dx} = 0 \quad \text{when} \quad 16x - \frac{2}{x^2} = 0$$

$$16x^3 - 2 = 0$$

$$8x^3 - 1 = 0$$

$$x^3 = \frac{1}{8}$$

$$x = \frac{1}{2}$$

The turning point is (0.5, 6)

$$\frac{d^2y}{dx^2} = 16 + 4x^{-3}$$

$$= 16 + \frac{4}{x^3}$$

For $x = 0.5$, $\frac{d^2y}{dx^2} = 48$, so $(0.5, 6)$ is a MINimum

[5 marks]

Answer 3

(i) $\frac{dy}{dx} = 25x^4 - 21x^2$

[2 marks]

(ii) $\frac{d^2y}{dx^2} = 100x^3 - 42x$

[1 mark]

(iii) $(1, -2)$

[1 mark]

(iv) 4

[1 mark]

(v) 58

That is, positive bend $\therefore$ Anti-Clockwise

[2 marks]

Answer 4

(i) Consider perimeter:

$$2x + 2y = 72$$
$$x + y = 36$$
$$\therefore\ y = 36 - x$$

Consider area:

$$A = x \times y$$
$$= x\,(36 - x)$$
$$= 36x - x^2$$

[3 marks]

(ii) $36 - 2x$

[2 marks]

(iii) $x = 18 \text{ cm} \Rightarrow A = 324 \text{ cm}^2$

[3 marks]

(iv) $\dfrac{d^2A}{dx^2} = -2$ As this is negative, the turning point is a MAXimum

[2 marks]

Answer 5

(a) $\dfrac{dy}{dx} = 5000 - 1250x$

[2 marks]

(b) (4, 10 000)

[3 marks]

(c) **(i)** Maximum

(ii) The curve is an upside down quadratic equation, $\cap$,
because of the negative in front of the x^2
and so only has a local maximum

[2 marks]

(d) **(i)** £4

(ii) It maximises his profit

[2 marks]

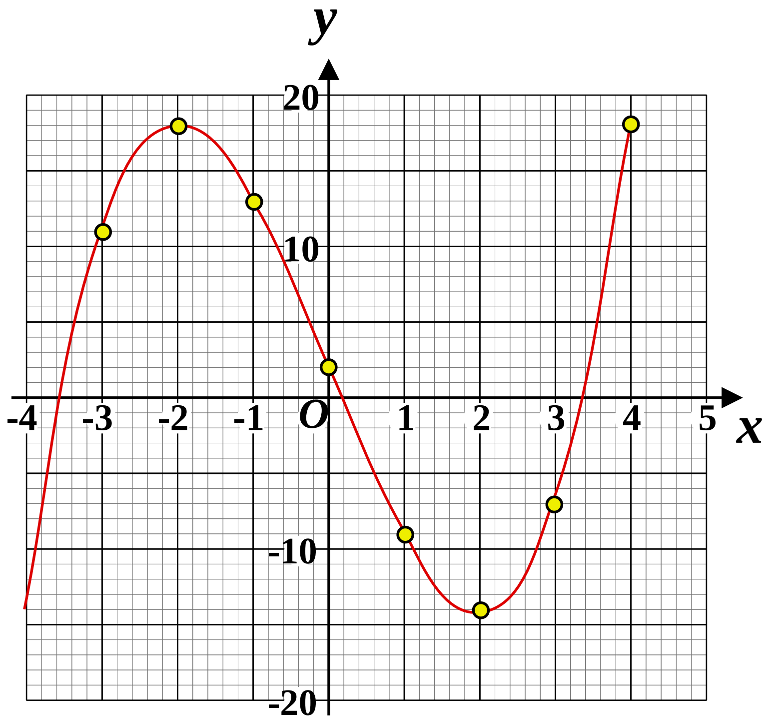
Answer 6

(a)

x	– 3	– 2	– 1	0	1	2	3	4
y	11	**18**	**13**	**2**	**– 9**	**– 14**	– 7	18

[2 marks]

(b)



[2 marks]

(c) **(i)** $\frac{dy}{dx} = 3t^2 - 12$

(ii) 63

[4 marks]

Answer 7

(a)

$$A = \textit{area rectangle} - \textit{area 4 triangles}$$

$$= 10 \times 8 - \frac{1}{2}(10 - x) \times 2 - \frac{1}{2}(8 - x)(x) \times 2$$

$$= 80 - x(10 - x) - x(8 - x)$$

$$= 80 - 10x + x^2 - 8x + x^2$$

$$= 2x^2 - 18x + 80$$

[3 marks]

(b) **(i)** $4x - 18$

(ii) 4.5

(iii) Second derivative is 4 which is positive

$\therefore$ Anticlockwise bend $\therefore$ MINimum

[5 marks]

6.1 Stationary Points

Previously, the definition of a turning point was considered;

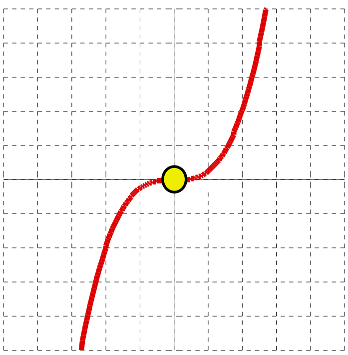
Definition: Turning Point
A turning point is a point where the gradient is zero and the sign of the gradient is different either side of the turning point.
They are so called because the gradient turns (BENDS) through zero.

Mathematicians also talk of stationary points; there is a subtle difference !

Definition: Stationary Point
A stationary point is a point where the gradient is zero.

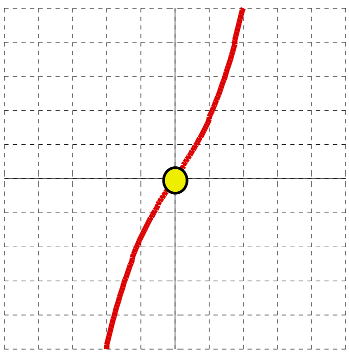
So, a local maximum or a local minimum are both turning and stationary points.
A point of inflection, which has gradient zero, is not a turning point but it is a stationary point. Technically, it's a ***stationary point of inflection***.

Note that it is possible to have a point of inflection which does not have gradient zero and so is neither a turning point nor a stationary point.
Technically, this is a ***non-stationary point of inflection***.



$y = x^3$

(0, 0) is not a turning point
but is a stationary point.
It's a stationary point of inflection.
This is an increasing function were
a gradient of zero is allowed.



$y = x^3 + x$

(0, 0) is not a turning point
and is not a stationary point.
It's a non-stationary point of inflection
This is a strictly increasing function.
The strictly emphasises that there
was no gradient of zero.

6.3 Example

$$f(x) = \frac{x^5}{5} - 2x^3 + 13x$$

Prove that $f(x)$ is a strictly increasing function.

[3 marks]

6.4 Exercise (Revision)

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available: 34

Question 1

$$g(x) = x^4 + 32x + 50$$

Use calculus to find the stationary point on the graph of g and determine if it is a local maximum, a local minimum or a point of inflection.

[3 marks]

Question 2

A function has equation,

$$y = 4x^5 + 8\sqrt{x} \qquad x \geqslant 0$$

Find the gradient equation of this curve.

[3 marks]

Question 3

A curve has equation

$$y = \frac{8}{x} - x + 3x^2 \qquad x > 0$$

Find the equation of the tangent to the curve at the point where $x = 2$

[4 marks]

Question 4

(i) Expand the brackets;

$$(x + h)^2$$

[1 mark]

(ii) A derivative of a smooth and continuous function $f(x)$ is defined to be;

$$f'(x) = \underset{h \to 0}{limit} \frac{f(x + h) - f(x)}{h}$$

Prove, from first principles, that the derivative of $5x^2$ is $10x$

[4 marks]

Question 5

A farmer has 80 m of fencing, and wishes to make a rectangular enclosure using a long straight wall as one of the four sides.
Let x be the length of wall used as a side, and y be the other dimension.

(i) Show that the area enclosed, A, is given by the formula

$$A = y(80 - 2y)$$

[2 marks]

(ii) Given that the farmer would like the area enclosed by this length of fence to be as large as possible, find the dimensions of the enclosure and its area.

[3 marks]

Question 6

Differentiate with respect to x

$$y = \frac{2x^3 + 5x}{x^3}$$

[3 marks]

Question 7

The curve C has equation

$$y = 4x + 3x^{\frac{3}{2}} - 2x^2 \qquad x > 0$$

(i) Find an expression for $\dfrac{dy}{dx}$

[3 marks]

(ii) Show that the point $P(4, 8)$ lies on C

[2 marks]

(iii) Show than an equation of the normal to C at point P is $3y = x + 20$

[3 marks]

The normal to C at P cuts the x-axis at point Q

(iv) Find the length PQ, giving your answer in simplified surd form

[3 marks]

6.5 Answers

6.5.1 Solution to Example 6.3

$$f(x) = \frac{x^5}{5} - 2x^3 + 13x$$

$$f'(x) = x^4 - 6x^2 + 13$$

This is a quadratic as can be made clear by letting $z = x^2$

$$\begin{aligned} f'(z) &= \left[z^2 - 6z \right] + 13 \\ &= \left[(z-3)^2 - 9 \right] + 13 \\ &= (z-3)^2 + 4 \end{aligned}$$

As the square has least value 0 and, even then 4 is added on 4
this expression is always positive

$\therefore\ f(x)$ is a strictly increasing function $\quad \square$

[3 marks]

6.5.2 Solutions to Exercise 6.4

Answer 1

$$g(x) = x^4 + 32x + 50$$

$$g'(x) = 4x^3 + 32$$

$$\begin{aligned} g'(x) = 0 \quad \text{when} \quad 4x^3 + 32 &= 0 \\ x^3 &= -8 \\ x &= -2 \end{aligned}$$

$$\begin{aligned} g(-2) &= 16 - 64 + 50 \\ &= 2 \end{aligned}$$

$$g''(x) = 12x^2$$

$g''(-2) = 48$ As this is positive, $(-2,\ 2)$ is a MINimum

[3 marks]

Answer 2

$$y = 4x^5 + 8\sqrt{x} \qquad x \geqslant 0$$

$$= 4x^5 + 8x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 20x^4 + 4x^{-\frac{1}{2}}$$

$$= 20x^4 + \frac{4}{\sqrt{x}}$$

[3 marks]

Answer 3

$$y = \frac{8}{x} - x + 3x^2 \qquad x > 0$$

$$= 8x^{-1} - x + 3x^2$$

When $\quad x = 2 \quad y = 4 - 2 + 12$

$$= 14$$

$$\frac{dy}{dx} = -8x^{-2} - 1 + 6x$$

$$= -\frac{8}{x^2} - 1 + 6x$$

When $x = 2, \quad \frac{dy}{dx} = -2 - 1 + 12$

$$= 9 \quad \text{at} \quad (2, \; 14)$$

Tangent: $y = mx + c$

$$14 = 9 \times 2 + c$$

$$c = -4$$

The equation of the tangent is $y = 9x - 4$

[4 marks]

Answer 4

(i) $\quad (x + h)^2 = x^2 + 2xh + h^2$

[1 mark]

(ii) $\quad f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{5(x + h)^2 - 5x^2}{h}$$

$$= \lim_{h \to 0} \frac{5\left(x^2 + 2xh + h^2\right) - 5x^2}{h}$$

$$= \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - 5x^2}{h}$$

$$= \lim_{h \to 0} \frac{10xh + 5h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(10x + 5h)}{h}$$

$$= \lim_{h \to 0} 10x + 5h$$

$$= 10x \qquad \square$$

[4 marks]

Answer 5

(i) The 80 m of fencing has to run along two sides of length y and one side of length x.

In other words, $80 = 2y + x \quad \Leftrightarrow \quad x = 80 - 2y$

The area enclosed is;

$$A = yx$$
$$= y\,(80 - 2y) \qquad \square$$

[2 marks]

(ii)

$$A = 80y - 2y^2$$
$$\frac{dA}{dy} = 80 - 4y$$

For a turning point $\frac{dA}{dy} = 0$

$$\therefore\ 80 - 4y = 0$$
$$4y = 80$$
$$y = 20$$

The enclosure has dimensions 20 m by 40 m and area 800 m^2

[3 marks]

Answer 6

$$y = \frac{2x^3 + 5x}{x^3}$$
$$= \frac{2x^3}{x^3} + \frac{5x}{x^3}$$
$$= 2 + 5x^{-2}$$
$$\frac{dy}{dx} = -10x^{-3}$$
$$= -\frac{10}{x^3}$$

[3 marks]

Answer 7

$$y = 4x + 3x^{\frac{3}{2}} - 2x^2 \qquad x > 0$$

(i)
$$\frac{dy}{dx} = 4 + \frac{9}{2}x^{\frac{1}{2}} - 4x$$

[3 marks]

(ii) When $x = 4$,

$$y = 4 \times 4 + 3 \times 4^{\frac{3}{2}} - 2 \times 4^2$$
$$= 16 + 24 - 32$$
$$= 8$$

$\therefore$ (4, 8) is on the curve C ☐

[2 marks]

(iii) When $x = 4$,
$$\frac{dy}{dx} = 4 + \frac{9}{2}\sqrt{4} - 4 \times 4$$
$$= 4 + 9 - 16$$
$$= -3$$

Thus $m_N = \dfrac{1}{3}$

Normal: $y = mx + c$

$$8 = \frac{1}{3} \times 4 + c$$
$$c = \frac{24 - 4}{3}$$
$$= \frac{20}{3}$$

The equation of the normal is thus $y = \dfrac{1}{3}x + \dfrac{20}{3}$

That is $3y = x + 20$ ☐

[3 marks]

(iv) The normal will cut the x-axis when $y = 0$, when $x = -20$
Required, is the distance, d, between $(-20, 0)$ and $(4, 8)$

$$d = \sqrt{(\Delta y)^2 + (\Delta x)^2}$$
$$= \sqrt{8^2 + 24^2}$$
$$= 8\sqrt{10}$$

[3 marks]

Lesson 7

A-Level Pure Mathematics : Year 1
Differentiation II

7.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Total Marks Available : 53

Question 1

$$g(x) = x^4 - 256x + 800$$

Use calculus to find the stationary point on the graph of g and determine if it is a local maximum, a local minimum or a point of inflection.

[5 marks]

Question 2

A function has equation,

$$y = 4x^{1.5} + \frac{9}{x^3} \qquad x \neq 0$$

Find the gradient equation of this curve.

[3 marks]

Question 3

A curve has equation

$$y = \sqrt{x} \qquad\qquad x > 0$$

Find the equation of the tangent to the curve at the point where $x = 4$

[5 marks]

Question 4

$$f(x) = \frac{x^3}{3} - 3x^2 + 11x$$

Prove that $f(x)$ is a strictly increasing function.

[4 marks]

Question 5

(i) Expand the brackets;

$$(x + h)^3$$

[2 marks]

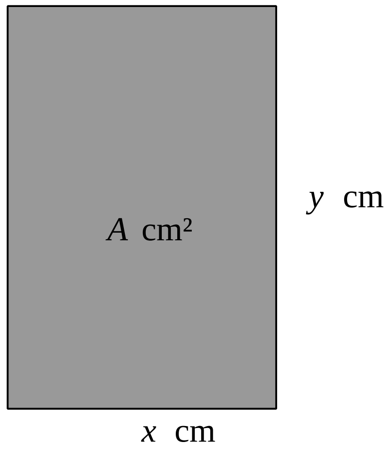
(ii) A derivative of a smooth and continuous function $f(x)$ is defined to be;

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Prove, from first principles, that the derivative of $4x^3$ is $12x^2$

[4 marks]

Question 6



The diagram shows a rectangular photo frame of area A cm^2.
The width of the photo frame is x cm.
The height of the photo frame is y cm.
The perimeter of the photo frame is 84 cm.

(i) Show that $A = 42x - x^2$

[3 marks]

(ii) Find $\dfrac{dA}{dx}$

[2 marks]

(iii) Find the maximum value of A.

[1 mark]

(iv) Show that $\dfrac{d^2A}{dx^2}$ detects that your part (iii) answer is a maximum.

[2 marks]

Question 7

The curve C has equation, $y = 17x + 3x^{\frac{2}{3}} - x^2 - 80 \qquad x > 0$

(i) Find an expression for $\dfrac{dy}{dx}$

[3 marks]

(ii) Show that the point $P(8, 4)$ lies on C

[2 marks]

(iii) Show that an equation of the normal to C at point P is $2y + x = 16$

[4 marks]

The normal to C at P cuts the x-axis at point Q

(iv) Find the length PQ, giving your answer in simplified surd form

[3 marks]

Question 8

Differentiate with respect to x

$$y = \frac{5x^4 + 7x}{x^3}$$

[3 marks]

Question 9

(i) Expand the brackets;

$$y = \left(x^{\frac{3}{2}} - 1\right)\left(x^{-\frac{1}{2}} + 1\right)$$

[2 marks]

(ii) For the curve described in part (i), find $\dfrac{dy}{dx}$

[3 marks]

(iii) Calculate the gradient of the part (i) curve at the point where $x = 4$

[2 marks]

Answer 1

$$g(x) = x^4 - 256x + 800$$

$$g'(x) = 4x^3 - 256$$

$$g'(x) = 0 \quad \text{when} \quad 4x^3 - 256 = 0$$

$$x^3 - 64 = 0$$

$$x^3 = 64$$

$$x = 4$$

$$g(4) = 4^4 - 256 \times 4 + 800$$

$$= 256 - 1024 + 800$$

$$= 32$$

$\therefore$ Point is $(4, 32)$

$$g''(x) = 12x^2$$

$$g''(4) = 192$$

As this is positive $(4, 32)$ is a local minimum

[5 marks]

Answer 2

$$y = 4x^{1.5} + \frac{9}{x^3} \qquad x \neq 0$$

$$= 4x^{1.5} + 9x^{-3}$$

$$\frac{dy}{dx} = 6x^{0.5} - 27x^{-4}$$

$$= 6\sqrt{x} - \frac{27}{x^4}$$

[3 marks]

Question 3

$$y = \sqrt{x} \qquad x > 0$$

When $x = 4$ $y = 2$ $\therefore$ tangent passes through $(4, 2)$

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$When \;\; x = 4, \qquad \frac{dy}{dx} = \frac{1}{4}$$

Tangent: $y = mx + c$

$$2 = \frac{1}{4} \times 4 + c$$

$$c = 1$$

The equation of the tangent is $y = \frac{1}{4}x + 1$

[5 marks]

Answer 4

$$f(x) = \frac{x^3}{3} - 3x^2 + 11x$$

$$f'(x) = x^2 - 6x + 11$$

$$= (x - 3)^2 - 9 + 11$$

$$= (x - 3)^2 + 2$$

The least value of $(x - 3)^2$ is zero.

Even then, 2 is added on.

That is,

$$f'(x) > 0$$

$\therefore$ $f(x)$ is a strictly increasing function.

[4 marks]

Answer 5

(i) $(x + h)^3 = x^3 + 3x^2 h + 3x h^2 + h^3$

[2 marks]

$$f(x) = 4x^3$$

(ii) $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{4(x + h)^3 - 4x^3}{h}$$

$$= \lim_{h \to 0} \frac{4\left(x^3 + 3x^2 h + 3x h^2 + h^3\right) - 4x^3}{h}$$

$$= \lim_{h \to 0} \frac{4x^3 + 12x^2 h + 12x h^2 + 4h^3 - 4x^3}{h}$$

$$= \lim_{h \to 0} \frac{12x^2 h + 12x h^2 + 4h^3}{h}$$

$$= \lim_{h \to 0} \frac{h\left(12x^2 + 12xh + 4h^2\right)}{h}$$

$$= \lim_{h \to 0} 12x^2 + 12xh + 4h^2$$

$$= 12x^2 \qquad \square$$

[4 marks]

Answer 6

(i) The 84 cm of frame stretches over two sides of length *y* and two of length *x*.

In other words, $84 = 2y + 2x \Rightarrow y = 42 - x$

The area enclosed is, $A = yx$

$$= (42 - x)x$$

$$= 42x - x^2 \qquad \square$$

[3 marks]

(ii) $\dfrac{dA}{dx} = 42 - 2x$

For a turning point $\dfrac{dA}{dy} = 0 \ \therefore \ 42 - 2x = 0 \Rightarrow x = 21$ cm

[2 marks]

(iii) $\therefore A_{MAX} = 21 \times (42 - 21) = 441 \text{ cm}^2$

[1 mark]

(iv) $\dfrac{d^2A}{dx^2} = -2$ That this is negative indicates a maximum $\square$

[2 marks]

Answer 7

$$y = 17x + 3x^{\frac{2}{3}} - x^2 - 80 \qquad x > 0$$

(i)
$$\frac{dy}{dx} = 17 + 2x^{-\frac{1}{3}} - 2x$$

$$= 17 + \frac{2}{\sqrt[3]{x}} - 2x$$

[3 marks]

(ii) When $x = 8$,

$$y = 17 \times 8 + 3 \times 8^{\frac{2}{3}} - 8^2 - 80$$

$$= 136 + 12 - 64 - 80$$

$$= 4$$

$\therefore$ (8, 4) is on the curve C ☐

[2 marks]

(iii) When $x = 8$,

$$\frac{dy}{dx} = 17 + \frac{2}{\sqrt[3]{8}} - 2 \times 8$$

$$= 17 + 1 - 16$$

$$= 2$$

Thus $m_N = -\frac{1}{2}$

Normal: $y = mx + c$

$$4 = -\frac{1}{2} \times 8 + c$$

$$c = 8$$

The equation of the normal is thus $y = -\frac{1}{2}x + 8$

In other words, $2y + x = 16$ ☐

[4 marks]

(iv) The normal will cut the x-axis when $y = 0$, when $x = 16$
Require the distance, d, between $Q(16, 0)$ and $P(8, 4)$

$$d = \sqrt{(\Delta y)^2 + (\Delta x)^2}$$

$$= \sqrt{4^2 + 8^2}$$

$$= \sqrt{80}$$

$$= 4\sqrt{5}$$

[3 marks]

Answer 8

$$y = \frac{5x^4 + 7x}{x^3}$$

$$= \frac{5x^4}{x^3} + \frac{7x}{x^3}$$

$$= 5x + 7x^{-2}$$

$$\frac{dy}{dx} = 5 - 14x^{-3}$$

$$= 5 - \frac{14}{x^3}$$

[3 marks]

Answer 9

(i)

$$y = \left(x^{\frac{3}{2}} - 1\right)\left(x^{-\frac{1}{2}} + 1\right)$$

$$= x - 1 - x^{-\frac{1}{2}} + x^{\frac{3}{2}}$$

[2 marks]

(ii)

$$\frac{dy}{dx} = 1 + \frac{1}{2}x^{-\frac{3}{2}} + \frac{3}{2}x^{\frac{1}{2}}$$

[3 marks]

(iii)

$$\frac{dy}{dx} = 1 + \frac{1}{2} \times 4^{-\frac{3}{2}} + \frac{3}{2} \times 4^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2} \times \frac{1}{8} + \frac{3}{2} \times 2$$

$$= 1 + \frac{1}{16} + 3$$

$$= 4\frac{1}{16}$$

[2 marks]