A-Level

~ Year 2 ~

Pure Mathematics

ARITHMETIC PROGRESSIONS

Carl Friedrich Gauss [1777-1855] and the problem of
summing the first one hundred positive integers.
This calculation is an example of summing an Arithmetic Progression.

Arithmetic Progressions

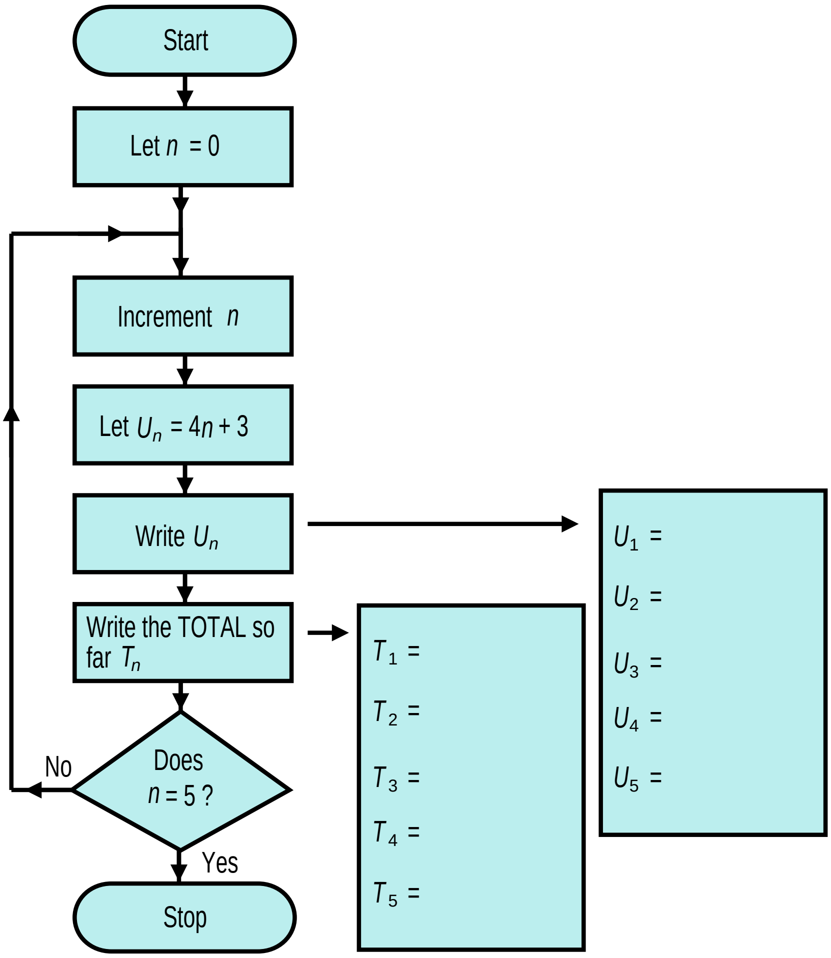
Lesson 1

A-Level Pure Mathematics
Sequences & Series : Year 2

1.1 Sigma Notation

Teaching Video : http://www.NumberWonder.co.uk/Video/v9049(1).mp4

(i) Study and complete the flow chart below;



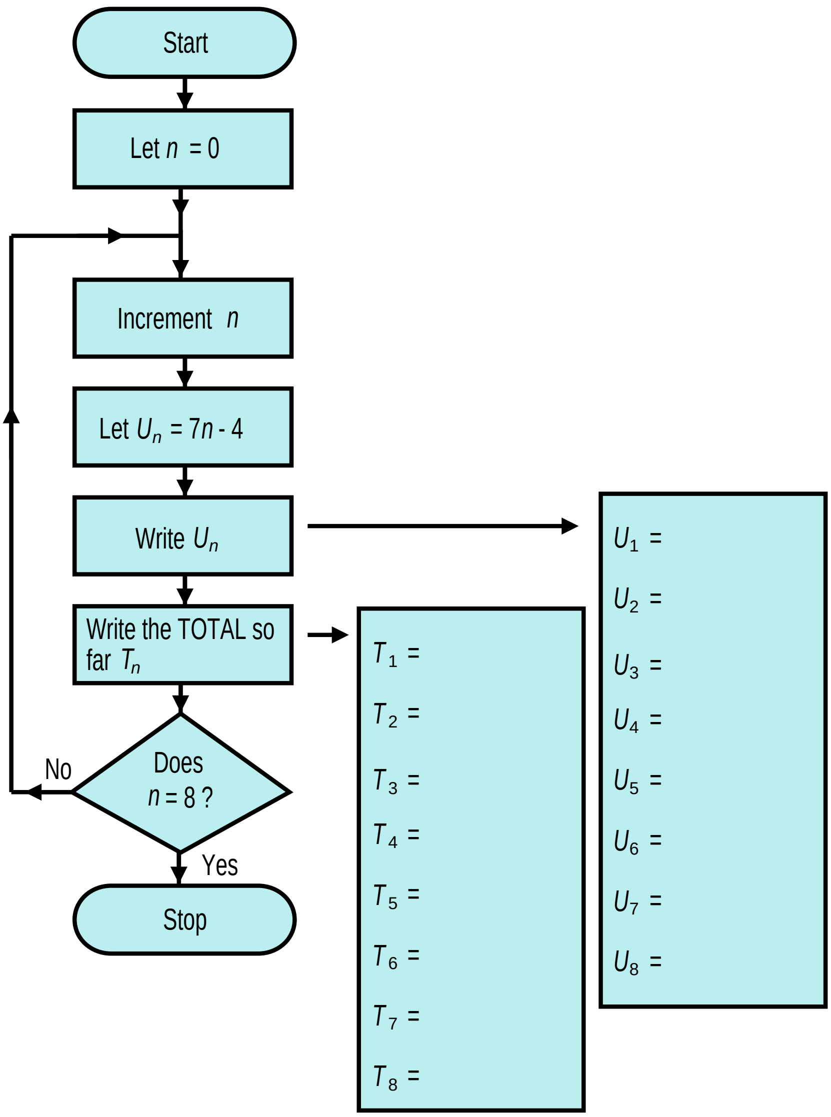
(ii) Determine the value of;

(a) $\sum_{1}^{5} (4n + 3)$ **(b)** $\sum_{2}^{5} (4n + 3)$

1.2 Exercise

Question 1

(i) Study and complete the flow chart below;



(ii) Determine the value of;

(a)

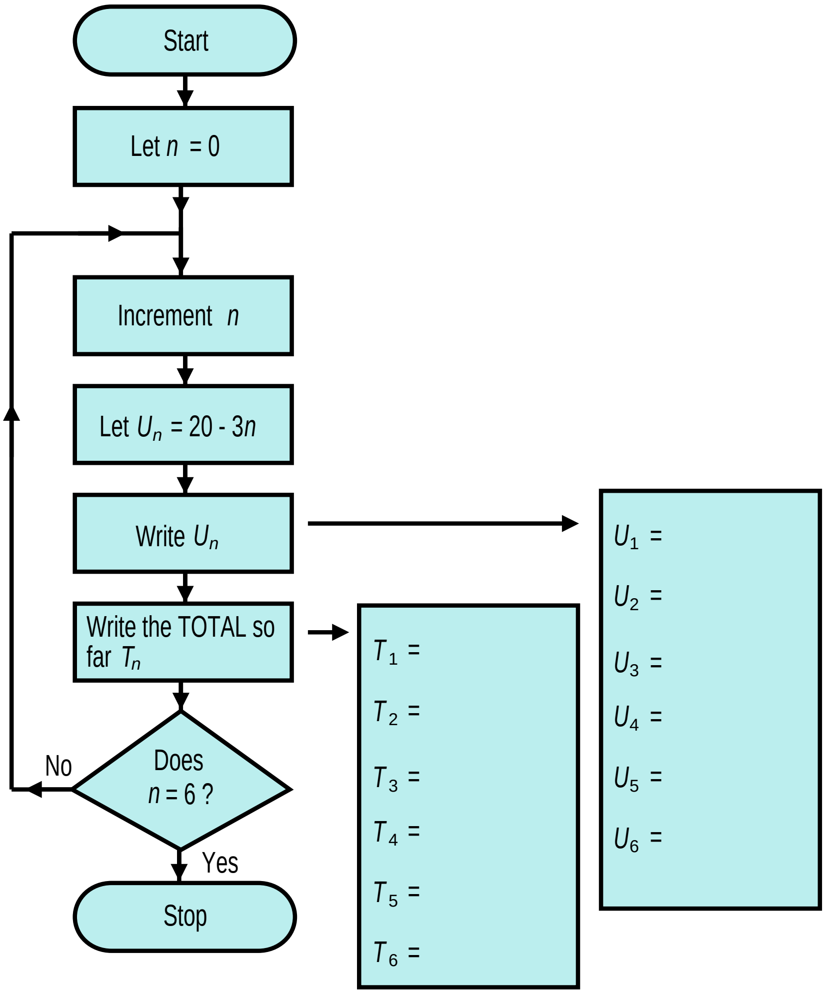
$$\sum_{1}^{8} (7n - 4)$$

(b)

$$\sum_{4}^{8} (7n - 4)$$

Question 2

(i) Study and complete the flow chart below;



(ii) Determine the value of;

(a)

$$\sum_{1}^{6} (20 - 3n)$$

(b)

$$\sum_{2}^{5} (20 - 3n)$$

Question 3

Determine the value of

$$\sum_{1}^{5} (10n + 3)$$

Question 4

(i) Write down the formula for the n^{th} term of the following sequence;

$$3, \quad 5, \quad 7, \quad 9, \quad 11, \ \ldots$$

(ii) Express the following series in sigma notation;

$$3 + 5 + 7 + 9 + 11$$

Question 5

Showing some working, determine the value of

$$\sum_{10}^{15} (100 - 5n)$$

Question 6

(i) Write down the formula for the n^{th} term of the following sequence;

$$5, \ 8, \ 11, \ 14, \ 17, \ 20, \ \ldots$$

(ii) Write the following series in sigma notation;

$$5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29$$

Question 7

Write the following series in sigma notation;

$$10 + 21 + 32 + 43 + 54 + 65 + 76 + 87 + 98 + 109 + 120$$

Question 8

Write the following series in sigma notation;

$$1 - 2 - 5 - 8 - 11 - 14 - 17$$

Question 9

The sequences and series considered have all been in in Arithmetic Progression.
Two numbers determine an Arithmetic Progression.

- The first term, a
- The common difference, d

Example

When $a = 6$ and $d = 4$ the sequence specified is;

$$6,\ (6 + 4),\ (6 + 4 + 4),\ (6 + 4 + 4 + 4 + 4),\ \ldots$$

$$6,\ 10,\ 14,\ 22,\ \ldots$$

Write down the first five terms of the Arithmetic Progression specified by,

(i) $a = 5$, $d = 6$

(ii) $a = 16$, $d = -3$

(iii) $a = 11$, $d = -7$

Question 10

Write down the formula for the n^{th} term of the following Arithmetic Progressions;

(i) $a = 3,\ d = 8$

(ii) $a = 7,\ d = 5$

(iii) $a = 9,\ d = -4$

Question 11

(i) Explain why the following series has 4 terms;

$$\sum_{3}^{6} (6n + 1)$$

(ii) How many terms has the following series ?

$$\sum_{5}^{11} (6n - 1)$$

(iii) How many terms has the following series ?

$$\sum_{15}^{47} (17n + 13)$$

1.3 Answers

1.3.1 Solution to the Introductory Example

(i)

$T_1 = 7$ $U_1 = 7$

$T_2 = 18$ $U_2 = 11$

$T_3 = 33$ $U_3 = 15$

$T_4 = 52$ $U_4 = 19$

$T_5 = 75$ $U_5 = 23$

(ii) **(a)** 75 **(b)** 68

1.3.2 Solutions to the 1.2 Exercise

Answer 1

(i)

$T_1 = 3$ $U_1 = 3$

$T_2 = 13$ $U_2 = 10$

$T_3 = 30$ $U_3 = 17$

$T_4 = 54$ $U_4 = 24$

$T_5 = 85$ $U_5 = 31$

$T_6 = 123$ $U_6 = 38$

$T_7 = 168$ $U_7 = 45$

$T_8 = 220$ $U_8 = 52$

(ii) **(a)** 220 **(b)** 190

Answer 2

(i)

$T_1 = 17$ $U_1 = 17$

$T_2 = 31$ $U_2 = 14$

$T_3 = 42$ $U_3 = 11$

$T_4 = 50$ $U_4 = 8$

$T_5 = 55$ $U_5 = 5$

$T_6 = 57$ $U_6 = 2$

(ii) **(a)** 57 **(b)** 38

Answer 3

165

Answer 4

(i)

$U_n = 2n + 1$

(ii)

$$\sum_{1}^{5} (2n + 1)$$

Answer 5

225

Answer 6

(i)

$U_n = 3n + 2$

(ii)

$$\sum_{1}^{9} (3n + 2)$$

Answer 7

$$\sum_{1}^{11} (11n - 1)$$

Answer 8

$$\sum_{1}^{7} (4 - 3n)$$

Answer 9

(i) 5, 11, 17, 23, 29, ...

(ii) 16, 13, 10, 7, 4, ...

(iii) 11, 4, –3, –10, –17, ...

Answer 10

(i) $U_n = 8n - 5$

(ii) $U_n = 5n + 2$

(iii) $U_n = 13 - 4n$

Answer 11

(i) When $n = 3$, $6n - 1 = 19$
When $n = 4$, $6n - 1 = 25$
When $n = 5$, $6n - 1 = 31$
When $n = 6$, $6n - 1 = 37$ There are four terms ☐

(ii) 11 – 5 + 1 is 7 terms

(iii) 47 – 15 + 1 = 33 terms

2.1 The Sum of a famous Arithmetic Progression

One of Germany's finest mathematicians, Carl Friedrich Gauss, [1777-1855] when young, was asked to sum the first 100 integers.
i.e. What is 1 + 2 + 3 = 4 + ... + 98 + 99 + 100 ?

Instantly, Gauss had the answer.
More than that, he had an interesting mental method.

	S =	1	+ 2	+ 3	+ 4	+ ...	+ 98	+ 99	+ 100
ADD	S =	100	+ 99	+ 98	+ 97	+ ...	+ 3	+ 2	+ 1
	2S =	101	+ 101	+ 101	+ 101	+ ...	+ 101	+ 101	+ 101

$\therefore$ 2S = 101 × 100
2S = 10100
S = 5050

2.2 The Sum of any Arithmetic Progression

Algebraically, an Arithmetic Progression is a number sequence of the form;

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad \ldots, \quad a + (n - 1) d$$

where a is the initial term
d is the common difference
n is number of terms

Notice that the last term, L, is given by

$$L = a + (n - 1) d \qquad n \geqslant 1$$

$$\begin{array}{ccccccc}
S = a & + a + d & + a + 2d & + \ldots & + L - 2d & + L - d & + L \\
S = L & + L - d & + L - 2d & + \ldots & + a + 2d & + a + d & + a \\
\hline
2S = a + L & + a + L & + a + L & + \ldots & + a + L & + a + L & + a + L
\end{array}$$

$$\therefore \quad 2S = n \{ a + L \}$$

$$\therefore \quad S = \frac{n}{2} \{ a + L \} \qquad n \geqslant 1$$

n times the average of the first and last terms

The formula book result is now obtained by substituting into this the formula for L:

$$S = \frac{n}{2} \{ a + a + (n - 1) d \}$$

$$\therefore S = \frac{n}{2} \{ 2a + (n - 1) d \} \qquad n \geqslant 1$$

2.3 Exercise

Question 1

(i) Write out the first six terms of the arithmetic progression
with first term, *a*, of 11
and common difference, *d*, of 7

(ii) Use the formula,

$$L = a + (n - 1) d \qquad n \geqslant 1$$

to calculate the 400^{th} term.

(iii) Use the formula,

$$S = \frac{n}{2} \{ a + L \} \qquad n \geqslant 1$$

to find the sum of the first 400 terms of the Arithmetic progression.

Question 2

C1 Examination question from May 2007, Q4.

A girl saves money over a period of 200 weeks. she saves 5 p in Week 1,
7 p in Week 2, 9 p in Week 3, and so on until Week 200.
Her weekly savings form an arithmetic sequence.

(a) Find the amount she saves in Week 200.

[3 marks]

(b) Calculate her total savings over the complete 200 week period.

[3 marks]

Need help ?
Exam Solutions Video : http://www.NumberWonder.co.uk/Video/v9049(2a).mp4

Question 3

C1 Examination question from May 2013, Q7

A company, which is making 200 mobile phones each week, plans to increase its production.

The number of mobile phones produced is to be increased by 20 each week from 200 in week 1 to 220 in week 2, to 240 in week 3 and so on, until it is producing 600 in week N.

(a) Find the value of N

[2 marks]

The company then plans to continue to make 600 mobile phones each week.

(b) Find the total number of mobile phones that will be made in the first 52 weeks starting from and including week 1.

[5 marks]

Need help ?

Exam Solutions Video : http://www.NumberWonder.co.uk/Video/v9049(2b)

Question 4

C1 Examination question from June 2009, Q5

A 40-year building programme for new houses began in Oldtown in the year 1951 (Year 1) and finished in 1990 (Year 40).

The numbers of houses built each year form an arithmetic sequence with first term a and common difference d.

Given that 2400 new houses were built in 1960 and 600 new houses were built in 1990, find

(a) the value of d

[3 marks]

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(b) the value of a

[2 marks]

Need help ?

Exam Solutions Video : http://www.NumberWonder.co.uk/Video/v9049(2d).mp4

(c) the total number of houses built in Oldtown over the 40-year period

[3 marks]

Need help ?
Exam Solutions Video : http://www.NumberWonder.co.uk/Video/v9049(2e).mp4

Question 5

Without using a calculator, show how to determine the value of,

$$\sum_{1}^{21}(7n + 3)$$

Question 6

C1 Examination question from January 2009, Q9

The first term of an arithmetic series is a and the common difference is d.

The 18th term of the series is 25 and the 21st term of the series is 32½.

(a) Use this information to write down two equations for a and d.

[2 marks]

(b) Show that $a = -17.5$ and find the value of d.

[2 marks]

The sum of the first n terms of the series is 2750.

(c) Show that n is given by

$$n^2 - 15n = 55 \times 40 \qquad n \geqslant 1$$

[4 marks]

(d) Hence find the value of n.

[3 marks]

Question 7

C1 Examination question from January 2005, Q5

The r^{th} term of an arithmetic series is $(2r - 5)$.

(a) Write down the first three terms of this series.

[2 marks]

(b) State the value of the common difference.

[1 mark]

(c) Show that

$$\sum_{r=1}^{n} (2r - 5) = n(n - 4)$$

[3 marks]

2.4 Answers

2.4.1 Solution (Exercise 2.3)

Answer 1

(i) 11, 18, 25, 32, 39, 46, ...

(ii) $a = 11$, $d = 7$ and $n = 400$

$$\begin{aligned} L_n &= a + (n - 1)\, d \qquad\qquad n \geqslant 1 \\ &= 11 + (400 - 1)\, 7 \\ &= 11 + 2800 - 7 \\ &= 2804 \end{aligned}$$

(iii)

$$\begin{aligned} S_n &= \frac{n}{2}\,\{ a + L \} \qquad\qquad n \geqslant 1 \\ &= \frac{400}{2}\,\{ 11 + 2804 \} \\ &= 200 \times 2815 \\ &= 563000 \end{aligned}$$

Answer 2

(a) $a = 5$, $d = 2$ and $n = 200$

$$\begin{aligned} L_n &= a + (n - 1)\, d \qquad\qquad n \geqslant 1 \\ &= 5 + (200 - 1)\, 2 \\ &= 5 + 400 - 2 \\ &= 403 \text{ pence} \end{aligned}$$

(b)

$$\begin{aligned} S_n &= \frac{n}{2}\,\{ a + L \} \qquad\qquad n \geqslant 1 \\ &= \frac{200}{2}\,\{ 5 + 403 \} \\ &= 100 \times 408 \\ &= 40800 \text{ pence} \quad (\text{£}408) \end{aligned}$$

Answer 3

(a) $a = 200$, $d = 20$, $L_n = 600$, and $n = N$

$$L_n = a + (n - 1)d \qquad n \geqslant 1$$
$$600 = 200 + (N - 1)\,20$$
$$400 = 20N - 20$$
$$420 = 20N$$
$$N = 21$$

(b)

$$\begin{aligned} S_n &= \text{N° of phones to week 21 + N° of phones after week 21} \\ &= \frac{n}{2}\{a + L\} + 600 \times (52 - 21) \\ &= \frac{21}{2}\{200 + 600\} + 600 \times 31 \\ &= \frac{21 \times 800}{2} + 18600 \\ &= 8400 + 18600 \\ &= 27000 \text{ phones} \end{aligned}$$

Answer 4

(a) 1960 is Year 10

$$L_{10} = 2400 \qquad L_{40} = 600$$
$$2400 = a + 9d \qquad 600 = a + 39d$$

Combine these two equations by subtraction,

$$1800 = -30d$$
$$d = -60$$

(b) Substituting this answer back into the first equation,

$$2400 = a + 9 \times (-60)$$
$$a = 2940$$

(c)

$$\begin{aligned} S_n &= \frac{n}{2}\{a + L\} \qquad n \geqslant 1 \\ &= \frac{40}{2}\{2940 + 600\} \\ &= 20 \times 3540 \\ &= 70800 \text{ houses} \end{aligned}$$

Answer 5

$$\sum_{1}^{21} (7n + 3) = 10 + 17 + 24 + \ldots + 150$$

The series described is in Arithmetic Progression with $a = 10$, $d = 7$, $L = 150$, $n = 21$
Thus,

$$\sum_{1}^{21} (7n + 3) = \frac{21}{2} \{ 10 + 150 \}$$

$$= 1680$$

Answer 6

(a)

$$25 = a + 17d \qquad\qquad 32.5 = a + 20d$$

(b) Combine these two equations by subtraction,

$$7.5 = 3d$$

$$d = 2.5$$

Substitute back into the first equation,

$$25 = a + 17 \times 2.5$$

$$a = 25 - 42.5$$

$$= -17.5 \qquad \text{as expected}$$

Doing the question this way uses the check on a to also check d

(c)

$$S_n = \frac{n}{2} \{ 2a + (n - 1) d \} \qquad\qquad n \geqslant 1$$

$$2750 = \frac{n}{2} \{-35 + 2.5n - 2.5\}$$

$$5500 = n \{-37.5 + 2.5n \}$$

$$2.5n^2 - 37.5n - 5500 = 0$$

$$n^2 - 15n - 2200 = 0$$

$$\therefore n^2 - 15n = 55 \times 40 \qquad\qquad \square$$

(d)

$$n^2 - 15n - 55 \times 40 = 0$$

$$(n - 55) (n + 40) = 0$$

$$\therefore n = 55 \qquad \text{reject } n = -40 \text{ as } n \text{ must be positive}$$

Answer 7

(a) $-3, -1, 1, \ldots$

(b) 2

(c)

$$\sum_{r=1}^{n} (2r - 5) = \frac{n}{2}\{-3 \times 2 + (n - 1) \times 2\}$$

$$= n\{-3 + n - 1\}$$

$$= n(n - 4) \qquad \square$$

3.1 APs : Formulae Summary

Algebraically, an Arithmetic Progression is a number sequence of the form;

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad \ldots, \quad a + (n - 1)\, d$$

where a is the initial term,
d is the common difference,
and n is number of terms.

The n^{th} term, L_n, is given by,

$$L_n = a + (n - 1)\, d \qquad n \geqslant 1$$

The sum of an AP is given by,

$$S = \frac{n}{2}\{ a + L \} \qquad n \geqslant 1$$

In words this can be remembered as :

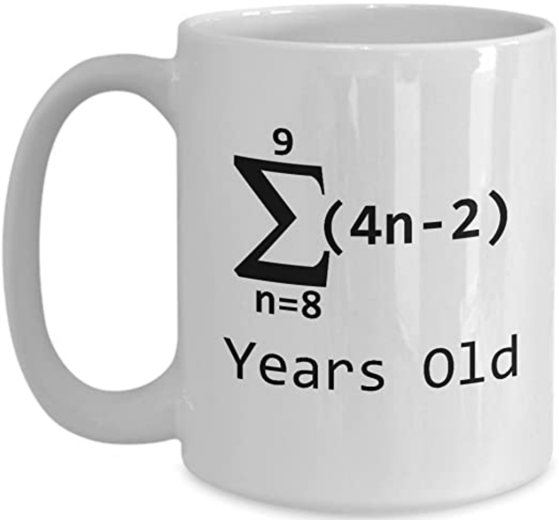
"n times the average of the first and last terms"

From substituting the first formula into the second, another formula is obtained for the sum of an AP. It is,

$$S = \frac{n}{2}\{ 2a + (n - 1)\, d \}, \qquad n \geqslant 1$$

Sigma notation is often used to compact the additions.

For example, which birthday does the following mug* celebrate ?
The answer is on the next page but try to do the question yourself before looking at it.



* Available from Amazon.co.uk

3.2 Coffee Cup Sigma Notation Answer

$$\sum_{n=8}^{9} (4n - 2) = (4 \times 8 - 2) + (4 \times 9 - 2)$$
$$= 30 + 34$$
$$= 64$$

3.3 Subscript AP Questions

In this topic, the letter n gets used in a variety of subtly different ways.
The following examination question often puzzles newcomers to this topic.

C1 Examination question from May 2014, Q5.

A sequence of numbers a_1, a_2, a_3, ... is defined by,

$$a_{n+1} = 5a_n - 3, \qquad n \geqslant 1$$

Given that $a_2 = 7$,

(a) find the value of a_1

[2 marks]

(b) Find the value of,

$$\sum_{r=1}^{4} a_r$$

[3 marks]

Teaching Video : http://www.NumberWonder.co.uk/Video/v9049(3a).mp4

3.4 Exercise

Question 1

C1 Examination question from January 2013, Q4

A sequence $u_1, u_2, u_3, \ldots$ satisfies

$$u_{n+1} = 2u_n - 1, \qquad n \geqslant 1$$

Given that $u_2 = 9$,

(a) find the value of u_3 and the value of u_4

[2 marks]

(b) Evaluate

$$\sum_{r=1}^{4} u_r$$

[3 marks]

Question 2

C1 Examination question from May 2010, Q5

A sequence of positive numbers is defined by

$$a_{n+1} = \sqrt{a_n^2 + 3}, \qquad n \geqslant 1$$

$$a_1 = 2$$

(a) Find a_2 and a_3 leaving your answers in surd form

[2 marks]

(b) Show that $a_5 = 4$

[2 marks]

Question 3

C1 Examination question from May 2006, Q7

An athlete prepares for a race by completing a practice run on each of 11 consecutive days. On each day after the first day, he runs further than he ran on the previous day. The lengths of his 11 practice runs form an arithmetic sequence with first term a km and common difference d km.

He runs 9 km on the 11th day, and he runs a total of 77 km over the 11 day period.

Find the value of a and the value of d.

[7 marks]

Question 4

C1 Examination question from January 2010, Q7

Jill gave money to a charity over a 20-year period, from year 1 to Year 20 inclusive. She gave £150 in year 1, £160 in Year 2, £170 in year 3, and so on, so that the amounts of money she gave each year formed an arithmetic sequence.

(a) Find the amount of money she gave in Year 10.

[2 marks]

(b) Calculate the total amount of money she gave over the 20-year period.

[3 marks]

Kevin also gave money to the charity over the same 20-year period.

He gave £A in Year 1 and the amounts of money he gave each year increased, forming an arithmetic sequence with common difference £30.
The total amount of money that Kevin gave over the 20-year period was **twice** the total amount of money that Jill gave.

(c) Calculate the value of A

[4 marks]

Question 5

C1 Examination question from May 2006, Q4

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 3$$

$$a_{n+1} = 3a_n - 5, \qquad n \geqslant 1$$

(a) Find the value of a_2 and the value of a_3

[2 marks]

(b) Calculate the value of

$$\sum_{r=1}^{5} a_r$$

[3 marks]

Question 6

C1 Examination question from May 2014, Q8

In the year 2000 a shop sold 150 computers. Each year the shop sold 10 more computers than the year before, so that the shop sold 160 computers in 2001, 170 computers in 2002, and so on forming an arithmetic sequence.

(a) Show that the shop sold 220 computers in 2007.

[2 marks]

(b) Calculate the total number of computers the shop sold from 2000 to 2013 inclusive.

[3 marks]

In the year 2000, the selling price of each computer was £900. the selling price fell by £20 each year, so that in 2001 the selling price was £880, in 2002 the selling price was £860, and so on forming an arithmetic sequence.

(c) In a particular year, the selling price of each computer in £s was equal to three times the number of computers the shop sold in that year.
By forming and solving an equation, find the year in which this occurred.

[4 marks]

Question 7

C1 Examination question from June 2009, Q7

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k$$

$$a_{n+1} = 2a_n - 7, \qquad n \geqslant 1$$

where k is a constant.

(a) Write down an expression for a_2 in terms of k

[1 mark]

(b) Show that

$$a_3 = 4k - 21$$

[2 marks]

Given that

$$\sum_{r=1}^{4} a_r = 43$$

(c) find the value of k

[4 marks]

Question 8

C1 Examination question from January 2008, Q11

The first term of an arithmetic sequence is 30 and the common difference is -1.5

(a) Find the value of the 25th term.

[2 marks]

The r^{th} term of the sequence is 0.

(b) Find the value of r

[2 marks]

The sum of the first n terms of the sequence is S_n

(c) Find the largest positive value of S_n

[3 marks]

3.5 Answers

3.5.1 Solution (Example)

(a)

$$a_2 = 5a_1 - 3$$
$$7 = 5a_1 - 3$$
$$5a_1 = 10$$
$$a_1 = 2$$

(b)

$$\sum_{r=1}^{4} a_r = a_1 + a_2 + a_3 + a_4$$
$$= 2 + 7 + 32 + 157$$
$$= 198$$

3.5.2 Solution (Exercise 3.2)

Answer 1

(a)

$$u_3 = 17, \qquad u_4 = 33$$

(b)

$$u_2 = 2u_1 - 1$$
$$9 = 2u_1 - 1$$
$$2u_1 = 10$$
$$u_1 = 5$$
$$\sum_{r=1}^{4} u_r = u_1 + u_2 + u_3 + u_4$$
$$= 5 + 9 + 17 + 33$$
$$= 64$$

Answer 2

(a)

$$a_2 = \sqrt{7} \qquad a_3 = \sqrt{10}$$

(b)

$$a_4 = \sqrt{(\sqrt{10})^2 + 3} = \sqrt{13}$$
$$a_5 = \sqrt{(\sqrt{13})^2 + 3} = \sqrt{16}$$
$$= 4$$

Answer 3

$$S_n = \frac{n}{2}\{ a + L \} \qquad n \geqslant 1$$

$$77 = \frac{11}{2}\{ a + 9 \}$$

$$a + 9 = 14$$

$$a = 5$$

$$L_n = a + (n - 1) d \qquad n \geqslant 1$$

$$9 = 5 + (11 - 1) d$$

$$10d = 4$$

$$d = 0.4$$

Answer 4

(a)

$$L_n = a + (n - 1) d \qquad n \geqslant 1$$

$$L_{10} = 150 + (10 - 1) \times 10$$

$$= £240$$

(b)

$$S = \frac{n}{2}\{ 2a + (n - 1) d \} \qquad n \geqslant 1$$

$$= \frac{20}{2}\{ 2 \times 150 + (20 - 1) \times 10$$

$$= 10 \{ 300 + 190 \}$$

$$= £4900$$

(c)

$$S = \frac{n}{2}\{ 2a + (n - 1) d \} \qquad n \geqslant 1$$

$$2 \times 4900 = \frac{20}{2}\{ 2A + (20 - 1) \times 30 \}$$

$$9800 = 10 \{2A + 570 \}$$

$$2A = 980 - 570$$

$$A = £205$$

Answer 5

(a)

$$a_2 = 4, \qquad a_3 = 7$$

(b)

$$a_4 = 16 \qquad and \qquad a_5 = 43$$

$$\sum_{r=1}^{5} a_r = a_1 + a_2 + a_3 + a_4 + a_5$$

$$= 3 + 4 + 7 + 16 + 43$$

$$= 73$$

Answer 6

(a)

$$L_n = a + (n - 1)\,d \qquad n \geqslant 1$$

$$L_8 = 150 + (8 - 1) \times 10$$

$$= 150 + 70$$

$$= 220 \text{ computers}$$

(b)

$$S = \frac{n}{2}\{\,2a + (n - 1)\,d\,\} \qquad n \geqslant 1$$

$$= \frac{14}{2}\,\{\,2 \times 150 + (14 - 1) \times 10$$

$$= 7 \times \{\,300 + 130\,\}$$

$$= 3010$$

(c) The two arithmetic progressions are,

N° of computers : 150, 160, 170, 180, 190, ... , $\{10n + 140\}$, ...

Cost per computer : 900. 880, 860, 840, 820, ... , $\{920 - 20n\}$, ...

The stipulated condition is that,

$$920 - 20n = 3\,(10n + 140)$$

$$920 - 20n = 30n + 420$$

$$500 = 50n$$

$$n = 10 \text{ years}$$

$$i.e, \quad 2009$$

Answer 7

(a)

$$a_2 = 2\,a_1 - 7$$

$$a_2 = 2k - 7$$

(b)

$$a_3 = 2\,(\,2k - 7\,) - 7$$

$$= 4k - 14 - 7$$

$$= 4k - 21$$

(c)

$$\sum_{r=1}^{4} a_r = 43$$

$$a_1 + a_2 + a_3 + a_4 = 43$$

$$k + 2k - 7 + 4k - 21 + 8k - 49 = 43$$

$$15k - 77 = 43$$

$$15k = 120$$

$$k = 8$$

Answer 8

(a)

$$L_n = a + (\,n - 1\,)\,d \qquad n \geqslant 1$$

$$L_{25} = 30 + (\,25 - 1\,) \times (-1.5)$$

$$= -6$$

(b)

$$L_n = a + (\,n - 1\,)\,d \qquad n \geqslant 1$$

$$0 = 30 + (\,r - 1\,) \times (-1.5)$$

$$0 = 30 - 1.5r + 1.5$$

$$1.5r = 31.5$$

$$r = 21$$

(c)

$$S = \frac{n}{2}\{\,a + L\,\} \qquad n \geqslant 1$$

$$= \frac{21}{2}\{\,30 + 0\,\}$$

$$= 10.5 \times 30$$

$$= 315$$

Lesson 4

A-Level Pure Mathematics
Sequences & Series : Year 2

4.1 Harder Examination Questions

Question 1

C1 Examination question from January 2012, Q4

A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1$$

$$x_{n+1} = a x_n + 5, \qquad n \geqslant 1$$

where a is a constant.

(a) Write down an expression for x_2 in terms of a

[1 mark]

(b) Show that

$$x_3 = a^2 + 5a + 5$$

[2 marks]

Given that $x_3 = 41$

(c) find the possible values of a

[3 marks]

Question 2

C1 Examination question from January 2012, Q9

A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1 :

Salary in Year 1 is £P

Salary increases by £($2T$) each year, forming an arithmetic sequence

Scheme 2 :

Salary in Year 1 is £(P + 1800)

Salary increases by £T each year, forming an arithmetic sequence

(a) Show that the **total** earned under salary Scheme 1 for the 10-year period is

$$£(10P + 90T)$$

[2 marks]

For the 10-year period, the **total** earned is the same for both salary schemes.

(b) Find the value of T

[4 marks]

For this value of T, the salary in Year 10 under Salary Scheme 2 is £29 850

(c) Find the value of P

[3 marks]

Question 3

C1 Examination question from May 2013, Q4

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 4$$

$$a_{n+1} = k(a_n + 2), \qquad n \geqslant 1$$

where k is a constant.

(a) Write down an expression for a_2 in terms of k

[1 mark]

Given that

$$\sum_{i=1}^{3} a_i = 2$$

(b) find the two possible values of k

[6 marks]

Question 4

C1 Examination question from January 2008, Q7

A sequence is given by:

$$x_1 = 1$$

$$x_{n+1} = x_n (p + x_n)$$

where p is a constant ($p \neq 0$).

(a) Find x_2 in terms of p

[1 mark]

(b) Show that

$$x_3 = 1 + 3p + 2p^2$$

[2 marks]

Given that $x_3 = 1$

(c) find the value of p

[3 marks]

(d) write down the value of x_{2008}

[2 marks]

Question 5

C1 Examination question from January 2013, Q7

Lewis played a game of space invaders.
He scored points for each spaceship that he captured.

Lewis scored 140 points for capturing his first spaceship.

He scored 160 points for capturing his second spaceship, 180 points for capturing his third spaceship, and so on.

The number of points scored for capturing each successive spaceship formed an arithmetic sequence.

(a) Find the number of points that Lewis scored for capturing his 20th spaceship.

[2 marks]

(b) Find the total number of points Lewis scored for capturing his first 20 spaceships.

[3 marks]

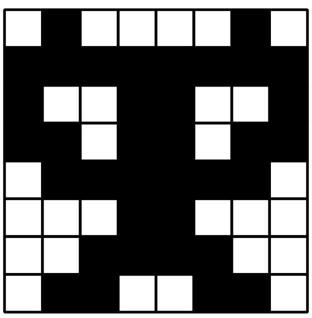
Sian played an adventure game,
She scored points for each dragon that she captured.
The number of points that Sian scored for capturing each successive dragon formed an arithmetic sequence.

Sian captured n dragons and the total number of points that she scored for capturing all n dragons was 8500

Given that Sian scored 300 points for capturing her first dragon and then 700 points for capturing her nth dragon,

(c) find the value of n

[3 marks]



Question 6

C1 Examination question from May 2007, Q8

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k$$

$$a_{n+1} = 3a_n + 5, \qquad n \geqslant 1$$

where k is a positive integer.

(a) Write down an expression for a_2 in terms of k

[1 mark]

(b) Show that $a_3 = 9k + 20$

[2 marks]

(c) **(i)** Find $\sum_{r=1}^{4} a_r$ in terms of k.

(ii) Show that $\sum_{r=1}^{4} a_r$ is divisible by 10.

[4 marks]

Question 7

C1 Examination question from January 2011, Q6

An arithmetic sequence has first term a and common difference d.
The sum of the first 10 terms of the sequence is 162.

(a) Show that $10a + 45d = 162$

[2 marks]

Given also that the sixth term of the sequence is 17

(b) write down a second equation in a and d

[1 mark]

(c) find the value of a and the value of d

[4 marks]

Question 8

C1 Examination question from May 2010, Q9

A farmer has a pay scheme to keep fruit pickers working throughout the 30 day season. He pays £a for their first day, £($a + d$) for their second day, £($a + 2d$) for their third day, and so on, thus increasing the daily payment by £d for each extra day they work.

A picker who works for all 30 days will earn £40.75 on the final day.

(a) Use this information to form an equation in a and d

[2 marks]

A picker who works for all 30 days will earn a total of £1005

(b) Show that $15 (a + 40.75) = 1005$

[2 marks]

(c) Hence find the value of a and the value of d

[4 marks]

Question 9

C1 Examination question from May 2011, Q5

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k$$

$$a_{n+1} = 5a_n + 3, \qquad n \geqslant 1$$

where k is a positive integer.

(a) Write down an expression for a_2 in terms of k

[1 mark]

(b) Show that $a_3 = 25k + 18$

[2 marks]

(c) **(i)** Find $\sum_{r=1}^{4} a_r$ in terms of k, in its simplest form.

(ii) Show that $\sum_{r=1}^{4} a_r$ is divisible by 6.

[4 marks]

Question 10

C1 Examination question from May 2011, Q9

(a) Calculate the sum of all the even numbers from 2 to 100 inclusive,

$$2 + 4 + 6 + \ldots\ldots + 100$$

[3 marks]

(b) In the arithmetic series

$$k + 2k + 3k + \ldots\ldots + 100$$

k is a positive integer and k is a factor of 100

(i) Find, in terms of k, an expression for the number of terms in this series

(ii) Show that the sum of this series is

$$50 + \frac{5000}{k}$$

[4 marks]

(c) Find, in terms of k, the 50th term of the arithmetic sequence

$$(2k + 1), \ (4k + 4), \ (6k + 7), \ ......$$

giving your answer in its simplest form

[2 marks]

4.2 Solutions (Exercise 4.1)

Answer 1

(a)

$$x_2 = a + 5$$

(b)

$$x_3 = a\,(a + 5) + 5$$
$$= a^2 + 5a + 5$$

(c)

$$41 = a^2 + 5a + 5$$
$$a^2 + 5a - 36 = 0$$
$$(a + 9)\,(a - 4) = 0$$
$$\text{Either } x = -9 \quad \text{or} \quad x = 4$$

Answer 2

(a)

$$S = \frac{n}{2}\{\,2a + (n - 1)\,d\,\} \qquad\qquad n \geqslant 1$$
$$= \frac{10}{2}\{\,2 \times P + (10 - 1) \times 2T$$
$$= 5\,\{\,2P + 18T\,\}$$
$$= £\,(10P + 90T) \qquad \square$$

(b)

$$\frac{10}{2}\{\,2 \times (P + 1800) + (10 - 1) \times T\,\} = 10P + 90T$$
$$5\,\{\,2P + 3600 + 9T\,\} = 10P + 90T$$
$$10P + 18000 + 45T = 10P + 90T$$
$$45T = 18000$$
$$T = 400$$

(c)

$$L_n = a + (n - 1)\,d \qquad\qquad n \geqslant 1$$
$$29850 = P + 1800 + (10 - 1) \times 400$$
$$29850 = P + 5400$$
$$P = 24450$$

Answer 3

(a)

$$a_2 = 6k$$

(b)

$$\begin{aligned} a_3 &= k\,(6k + 2) \\ &= 6k^2 + 2k \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{3} a_i &= 2 \\ a_1 + a_2 + a_3 &= 2 \\ 4 + 6k + 6k^2 + 2k &= 2 \\ 6k^2 + 8k + 2 &= 0 \\ 3k^2 + 4k + 1 &= 0 \\ (3k + 1)\,(k + 1) &= 0 \end{aligned}$$

$$\therefore\ k = -\frac{1}{3} \ \ or \ \ k = -1$$

Answer 4

(a)

$$\begin{aligned} x_2 &= 1 \times (p + 1) \\ &= p + 1 \end{aligned}$$

(b)

$$\begin{aligned} x_3 &= (p + 1)\,(p + (p + 1)) \\ &= (p + 1)\,(2p + 1) \\ &= 2p^2 + 3p + 1 \\ &= 1 + 3p + 2p^2 \qquad \square \end{aligned}$$

(c)

$$\begin{aligned} 1 + 3p + 2p^2 &= 1 \\ 2p^2 + 3p &= 0 \\ p\,(2p + 3) &= 0 \end{aligned}$$

Either $p = 0$ (Reject as told $p \neq 0$)

*o*or $p = -\dfrac{3}{2}$

(d)

As $x_1 = x_2$ the sequence has period 2

$$\therefore x_{2008} = x_2 = -0.5$$

Answer 5

(a)

$$L_n = a + (n - 1)\,d \qquad n \geqslant 1$$

$$L_{20} = 140 + (20 - 1) \times 20$$

$$= 520 \text{ points}$$

(b)

$$S_n = \frac{n}{2}\{a + L\} \qquad n \geqslant 1$$

$$S_{20} = \frac{20}{2}\{140 + 520\}$$

$$= 6600 \text{ points}$$

(c)

$$S_n = \frac{n}{2}\{a + L\} \qquad n \geqslant 1$$

$$8500 = \frac{n}{2}\{300 + 700\}$$

$$17000 = n\{1000\}$$

$$n = 17 \text{ dragons}$$

Answer 6

(a)

$$a_2 = 3k + 5$$

(b)

$$a_3 = 3(3k + 5) + 5$$

$$= 9k + 15 + 5$$

$$= 9k + 20 \qquad \square$$

(c) **(i)**

$$\sum_{r=1}^{4} a_r = a_1 + a_2 + a_3 + a_4$$

$$= k + 3k + 5 + 9k + 20 + 27k + 65$$

$$= 40k + 90$$

(ii)

$$\sum_{r=1}^{4} a_r = 40k + 90$$

$$= 10(4k + 9)$$

which is divisible by 10 $\square$

Answer 7

(a)

$$S = \frac{n}{2}\{ 2a + (n - 1)\, d \} \qquad n \geqslant 1$$

$$162 = \frac{10}{2}\{2a + 9d\}$$

$$10a + 45d = 162 \qquad \square$$

(b)

$$L_n = a + (n - 1)\, d \qquad n \geqslant 1$$

$$17 = a + 5d$$

(c)

$$\begin{array}{lr} & 10a + 50d = 170 \\ \text{SUBTRACT} & 10a + 45d = 162 \\ \hline & 5d = 8 \\ & d = 1.6 \\ & \therefore\ a = 17 - 5d \\ & = 9 \end{array}$$

Answer 8

(a)

$$L_n = a + (n - 1)\, d \qquad n \geqslant 1$$

$$40.75 = a + 29d$$

(b)

$$S_n = \frac{n}{2}\{ a + L \} \qquad n \geqslant 1$$

$$1005 = \frac{30}{2}\{ a + 40.75 \}$$

$$15\,(a + 40.75) = 1005 \qquad \square$$

(c)

$$67 = a + 40.5$$

$$a = £26.25$$

$$\therefore\ d = \frac{40.75 - 26.25}{29}$$

$$d = 0.5$$

Answer 9

(a)

$$a_2 = 5k + 3$$

(b)

$$\begin{aligned} a_3 &= 5\,(5k + 3) + 3 \\ &= 25k + 15 + 3 \\ &= 25k + 18 \end{aligned}$$

(c) **(i)**

$$\begin{aligned} \sum_{r=1}^{4} a_i &= a_1 + a_2 + a_3 + a_4 \\ &= k + 5k + 3 + 25k + 18 + 125k + 93 \\ &= 156k + 114 \end{aligned}$$

(ii)

$$\begin{aligned} \sum_{r=1}^{4} a_i &= 156k + 114 \\ &= 6\,(26k + 19) \end{aligned}$$

whisch is divisible by 6 □

Answer 10

(a) The sequence is in Arithmetic Progression...

$$a = 2, \quad d = 2, \quad L = 100$$

$$\begin{aligned} L_n &= a + (n - 1)\,d \qquad n \geqslant 1 \\ 100 &= 2 + (n - 1) \times 2 \\ 98 &= 2n - 2 \\ 2n &= 100 \\ n &= 50 \qquad \text{which is, perhaps, obvious} \end{aligned}$$

$$\begin{aligned} S_n &= \frac{n}{2}\,\{a + L\} \qquad n \geqslant 1 \\ &= \frac{50}{2}\,\{2 + 100\} \\ &= 2550 \end{aligned}$$

(b) **(i)**

$$a = k, \quad d = k, \quad L = 100$$

$$L_n = a + (n - 1) d \qquad n \geqslant 1$$

$$100 = k + (n - 1) k$$

$$100 = k + nk - k$$

$$n = \frac{100}{k}$$

(ii)

$$S_n = \frac{n}{2} \{ a + L \} \qquad n \geqslant 1$$

$$= \frac{100}{2k} \{ k + 100 \}$$

$$= 50 + \frac{5000}{k}$$

(c)

$$a = 2k + 1 \qquad d = 2k + 3$$

$$L_n = a + (n - 1) d \qquad n \geqslant 1$$

$$= 2k + 1 + (50 - 1) (2k + 3)$$

$$= 2k + 1 + 49 (2k + 3)$$

$$= 2k + 1 + 98k + 147$$

$$= 100k + 148$$

Lesson 5

A-Level Pure Mathematics
Sequences & Series : Year 2

5.1 TEST

Algebraically, an Arithmetic Progression is a number sequence of the form;

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad \ldots, \quad a + (n - 1)\, d$$

where a is the initial term
d is the common difference
n is number of terms.

The n^{th} term, L_n, is given by

$$L_n = a + (n - 1)\, d \qquad n \geqslant 1$$

The sum of an AP is given by

$$S = \frac{n}{2}\{ a + L \} \qquad n \geqslant 1$$

n times the average of the first and last terms

Or

$$S = \frac{n}{2}\{ 2a + (n - 1)\, d \} \qquad n \geqslant 1$$

Question 1

A sequence of numbers, A, is described by the formula:

$$A_n = 4n - 3 \qquad n \geqslant 1$$

(i) Write out the first six terms in the sequence.

[2 marks]

(ii) Is this sequence in Arithmetic Progression ?
Give a reason for your answer.

[2 marks]

Question 2

A series of numbers, B, is described in sigma notation as:

$$\sum_{1}^{\infty} n^2 - n$$

(i) Write out the first six terms in the series.

[2 marks]

(ii) Is this series in Arithmetic Progression ?
Give a reason for your answer.

[2 marks]

Question 3
In question 1 we had a ***sequence*** of numbers, and in question 2 a ***series*** of numbers. What is the difference between a sequence and a series of numbers ?

[2 marks]

Question 4
How many terms are in this arithmetic series ?

$$8 + 14 + 20 + \ldots + 164$$

[4 marks]

Question 5
Find the sum of this arithmetic series :

$$123 + 112 + 101 + \ldots \ldots + 13$$

[7 marks]

Question 6

The sum S of an arithmetic series is given by

$$S = \sum_{1}^{40} (4 + 3r)$$

(i) Write down the first six terms of the series.

[1 mark]

(ii) Find the common difference of the series.

[1 mark]

(iii) Calculate the value of S.

[4 marks]

Question 7

The fifth term of an arithmetic series is 26 and the eleventh term is 44.

(i) Find the first term and the common difference.

[7 marks]

(ii) List the first twelve terms of the arithmetic progression.

[1 mark]

Question 8

C1 Examination question from June 2008, Q7

Sue is training for a marathon. her training includes a run every Saturday starting with a run of 5 km on the first Saturday. Each Saturday she increases the length of her run from the previous Saturday by 2 km.

(a) Show that on the 4th Saturday of training she runs 11 km.

[1 mark]

(b) Find an expression, in terms of n, for the length of her training run on the n^{th} Saturday.

[2 marks]

(c) Show that the total distance she runs on Saturdays in n weeks of training is $n (n + 4)$ km.

[3 marks]

On the n^{th} Saturday Sue runs 43 km.

(d) Find the value of n.

[2 marks]

(e) Find the total distance, in km, Sue runs on Saturdays in n weeks of training.

[2 marks]

Question 9

C1 Examination question from January 2006, Q7

On Alice's 11th birthday she started to receive an annual allowance. The first annual allowance was £500 and on each following birthday the allowance was increased by £200.

(a) Show that, immediately after her 12th birthday, the total of the allowances that Alice had received was £1200.

[1 mark]

(b) Find the amount of Alice's annual allowance on her 18th birthday.

[2 marks]

(c) Find the total of the allowances that Alice had received up to and including her 18th birthday.

[3 marks]

When the total of the allowances that Alice had received reached £32 000 the allowance stopped.

(d) Find how old Alice was when she received her last allowance.

[7 marks]

Question 10
This final question is a 'challenge' question; hard but not impossible !

An arithmetic progression contains only positive integer terms.
The sum of the first three terms is 51.
The sum of the last four terms is 332.

(i) Show that only two arithmetic progressions satisfy these conditions.

(ii) List the two arithmetic progressions.

[8 marks]

5.2 Solutions (Test 5.1)

Answer 1

(i)

$$1,\ 5,\ 9,\ 13,\ 17,\ 21,\ \ldots$$

(ii) Yes, $a = 1$ and $d = 4$

Answer 2

(i) The series is,

$$0 + 2 + 6 + 12 + 20 + 30 + \ldots$$

So the terms of the series are,

$$0,\ 2,\ 6,\ 12,\ 20,\ 30,\ \ldots$$

(ii) No, the differences between successive terms are not all the same.

Answer 3

A sequences is a list of numbers with successive terms separated by commas.
A series is a list of numbers with addition signs between successive terms.

Answer 4

$$\begin{aligned} L_n &= a + (n - 1)\,d \qquad n \geqslant 1 \\ 164 &= 8 + (n - 1) \times 6 \\ 164 &= 8 + 6n - 6 \\ 6n &= 162 \\ n &= 27 \text{ terms} \end{aligned}$$

Answer 5

$$\begin{aligned} L_n &= a + (n - 1)\,d \qquad n \geqslant 1 \\ 13 &= 123 + (n - 1) \times (-11) \\ 13 &= 123 - 11n + 11 \\ 11n &= 121 \\ n &= 11 \text{ terms} \end{aligned}$$

$$\begin{aligned} S &= \frac{n}{2}\{ a + L \} \qquad n \geqslant 1 \\ &= \frac{11}{2}\{ 123 + 13 \} \\ &= 748 \end{aligned}$$

Answer 6

(i)

$$S = \sum_{1}^{40} (4 + 3r)$$

$$= 7 + 10 + 13 + 16 + 19 + 22 + \ldots$$

So the first six terms are,

7, 10, 13, 16, 19, 22

(ii) $d = 3$

(iii) $a = 7 \quad n = 40$

$$S = \frac{n}{2}\{ 2a + (n - 1)\, d \} \qquad n \geqslant 1$$

$$= \frac{40}{2}\{ 2 \times 7 + 39 \times 3 \}$$

$$= 2620$$

Answer 7

(i)

$$a + 10d = 44$$

SUBTRACT $\quad a + 4d = 26$

$$6d = 18$$

$$d = 3 \qquad \therefore\ a = 14$$

(ii)

14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, ...

Answer 8

(a)

$$L_n = a + (n - 1)\, d \qquad n \geqslant 1$$

$$= 5 + (4 - 1) \times 2$$

$$= 11 \text{ km} \qquad \square$$

(b)

$$L_n = 5 + (n - 1)\, 2$$

$$= 3 + 2n$$

(c)

$$S = \frac{n}{2}\{ 2a + (n - 1)\, d \} \qquad n \geqslant 1$$

$$= \frac{n}{2}\{ 2 \times 5 + (n - 1) \times 2 \}$$

$$= n\,(n + 4) \qquad \square$$

(d)

$$3 + 2n = 43$$
$$n = 20$$

(e)

$$n (n + 4) = 20 \times 24$$
$$= 480 \text{ km}$$

Answer 9

(a)

$$S = \frac{n}{2} \{ 2a + (n - 1) d \} \qquad n \geqslant 1$$
$$= \frac{2}{2} \{ 2 \times 500 + (2 - 1) \times 200 \}$$
$$= £1200 \qquad \square$$

(b)

$$L_n = a + (n - 1) d \qquad n \geqslant 1$$
$$L_{18} = 500 + (8 - 1) \times 200$$
$$= £1900$$

(c)

$$S = \frac{n}{2} \{ 2a + (n - 1) d \} \qquad n \geqslant 1$$
$$= \frac{8}{2} \{ 2 \times 500 + (8 - 1) \times 200 \}$$
$$= £9600$$

(d)

$$S = \frac{n}{2} \{ 2a + (n - 1) d \} \qquad n \geqslant 1$$
$$32000 = \frac{n}{2} \{ 2 \times 500 + (n - 1) \times 200 \}$$
$$64000 = n \{ 1000 + 200n - 200 \}$$
$$64000 = 800n + 200 n^2$$
$$n^2 + 4n - 320 = 0 \qquad \text{divide through by 200}$$
$$(n + 2)^2 - 324 = 0 \qquad \text{completing the square}$$
$$(n + 2)^2 = 324$$
$$n + 2 = \pm 18$$
$$n = 16 \qquad (\text{Reject } n = -20)$$
$$\therefore \text{ Alice is aged } 26$$

Answer 10

The sum of the first three terms,

$$a + a + d + a + 2d = 51$$
$$3a + 3d = 51$$
$$a + d = 17$$
$$\therefore d \leqslant 16 \qquad \text{as sequence only has positive integer terms}$$

The sum of the last four terms,

$$a + (n-4)d + a + (n-3)d + a + (n-2)d + a + (n-1)d = 332$$
$$4a + 4nd - 10d = 332$$
$$2a + 2nd - 5d = 166$$

Substitute $a = 17 - d$ to get,

$$2(17 - d) + 2nd - 5d = 166$$
$$2nd - 7d = 132$$
$$d(2n - 7) = 2 \times 2 \times 3 \times 11$$

Work through the possibilities,

d	2	4	6	11	12
$2n$ - 7	66	33	22	12	11
n	36.5	20	14.5	9.5	9

Notice that for n to be an integer, $2n$ - 7 has to be odd

As n must be an integer, two possibilities need to be listed,

13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61, 65, 69, 73, 77, 81, 85, 89

5, 17, 29, 41, 53, 65, 77, 89, 101

6.1 Primes &Arithmetic Progressions

Some Arithmetic Progressions contain no primes,

$$AP\{4,10\} \ : \ 4, \ 14, \ 24, \ 34, \ 44, \ 54, \ 64, \ 74, \ 84, \ 94, \ \ldots$$

Some contain only an initial term that is prime,

$$AP\{5,10\} \ : \ \mathbf{5}, \ 15, \ 25, \ 35, \ 45, \ 55, \ 65, \ 75, \ 85, \ 95, \ \ldots$$

Some contain many primes,

$$AP\{9,10\} \ : \ 9, \ \mathbf{19}, \ \mathbf{29}, \ 39, \ 49, \ \mathbf{59}, \ 69, \ \mathbf{79}, \ \mathbf{89}, \ 99, \ \ldots$$

These three examples are amongst those depicted below,

$y = 0 + 10n$										
$y = 1 + 10n$		11		31	41		61	71		
$y = 2 + 10n$	2									
$y = 3 + 10n$	3	13	23		43	53		73	83	
$y = 4 + 10n$										
$y = 5 + 10n$	5									
$y = 6 + 10n$										
$y = 7 + 10n$	7	17		37	47		67			97
$y = 8 + 10n$										
$y = 9 + 10n$		19	29			59		79	89	

6.2 Proof

Theorem : No number in the arithmetic progression $AP\{4,10\}$ is prime.

Proof : All terms in $AP\{4,10\}$ are of the form.

$$y = 4 + 10n, \qquad for \ \ n = 0, \ 1, \ 2, \ 3, \ \ldots$$

$$y = 2\ (\ 2 + 5n\ )$$

This shows that these numbers are always divisible by 2.

The first term, 4, is not prime, and so no term in $AP\{4,10\}$ is prime $\square$

6.3 Trapping the Primes

The “ten wide”number grid shows that, with the exception of 2 and 5, all primes must be in one of $AP\{1,10\}$, $AP\{3,10\}$, $AP\{7,10\}$ or $AP\{9,10\}$.

When a prime except 2 or 5 is divided by 10 the remainder must be one of 1, 3, 7 or 9.

Also, when an integer is divided by 10 if the remainder is 0, 2, 4, 5, 6 or 8 then that number cannot be prime.

6.4 Exercise

Question 1
Prove the following theorem,
Theorem : No number in the arithmetic progression $AP\{6,10\}$ is prime.

Question 2
George says that he has a number that when divided by 10 has remainder 7 and so the number must be prime.
Give an example that proves George is wrong.

Question 3
Write down a prime number that is in $AP\{5,10\}$.

Question 4
Disprove the following theorem,
Theorem : No number in the arithmetic progression $AP\{2,10\}$ is prime.

Question 5
The “ten wide” number grid trapped the primes in four rows, with the exception of the prime numbers 2 and 5. Is “ten wide” the best “trapper of primes” ?
A “six wide” trapper is shown below.

$y = 0+6n$																	
$y = 1+6n$		7	13	19		31	37	43			61	67	73	79			97
$y = 2+6n$	2																
$y = 3+6n$	3																
$y = 4+6n$																	
$y = 5+6n$	5	11	17	23	29		41	47	53	59		71		83	89		101

With the exception of 2 and 3, in which two Arithmetic Progressions does this number grid suggest all of the primes may lie ?

Question 6
Prove the following theorem,
Theorem : No number in the arithmetic progression $AP\{4,6\}$ is prime.

Question 7
When a prime number, other than 2 or 3, is divided by 6 what must the remainder be one of ?

Question 8
A number, x, when divided by 6 has remainder 2.
What can be said about the nature of x ?

Question 9
Use the number grid below to produce a "seven wide" prime trapper.

$y = 0+7n$														
$y = 1+7n$														
$y = 2+7n$														
$y = 3+7n$														
$y = 4+7n$														
$y = 5+7n$														
$y = 6+7n$														

Is this a good prime trapper ?
Explain your answer.

Question 10
In 1837 Peter Dirichlet [1805-1859] proved that for any two positive coprime integers, a and d, there are infinitely many primes of the form $y = d + an$
Of the numbers 0, 1, 2, 3, 4, 5, 6, and 7, which are coprime to 8 ?

Question 11
Which *AP*s in an "eight wide" number grid will contain infinitely many primes ?

Question 12
Which *AP*s in an "nine wide" number grid will contain infinitely many primes ?

Question 13
Consider arithmetic progressions of the form $AP\{a,16\}$

(i) For what values of a will the sequence contain infinitely many primes ?

(ii) List 3 composite numbers in $AP\{3,16\}$

(iii) List three prime numbers in $AP\{3,16\}$

Question 14
For any given positive integer n, greater than 1, the number of integers coprime to n is given a special name, the Euler totient function, φ.
Thus for an "n wide" number grid the number of as for which $AP\{a,n\}$ contains an infinite number of primes is given by $\phi(n)$.
To work out $\phi(n)$ use the fact that,

$$\phi\left(p^m\right) = p^{m-1}(p - 1) \qquad where\ \ p\ \ is\ prime$$

For the "seven wide" number grid,

$$\begin{aligned}\phi\left(7^1\right) &= 7^{1-1}(7 - 1)\\ &= 7^0 \times 6\\ &= 6\end{aligned}$$

Thus the primes are trapped in six of the seven rows of the number grid.

For the "ten wide" number grid, we utilise the multiplicative property the ϕ-function.

$$\begin{aligned}\phi(10) &= \phi(2 \times 5)\\ &= \phi\left(2^1\right) \times \phi\left(5^1\right)\\ &= 2^0 \times 1 \times 5^0 \times 4\\ &= 4\end{aligned}$$

Thus the primes are trapped in four of the ten rows of the number grid.

Use Euler's totient function, ϕ, to work out the number of rows the primes are trapped in when the number grid has the following widths.
Use the facts that,

$$\phi(p^m) = p^{m-1}(p-1) \qquad \textit{where } p \textit{ is prime}$$

$$\phi(p^m q^n) = \phi(p^m) \times \phi(q^n) \qquad \textit{where } p \textit{ and } q \textit{ are coprime}$$

(i) 11

(ii) 6

(iii) 8

(iv) 30

Question 15

Now that you know about Euler's totient function, ϕ, you may like to use it to think about and investigate which widths of number grid are best at trapping the primes into relatively tight spaces. For each width what percentage of the available number of rows are the primes restricted to ?

6.5 Solutions (to 6.4 Exercise)

Answer 1

All terms in $AP\{6,10\}$ are of the form.

$$y = 6 + 10n, \qquad for \quad n = 0, 1, 2, 3, \ldots$$

$$y = 2\,(3 + 5n)$$

This shows that these numbers are always divisible by 2.

The first term, 6, is not prime, and so no term in $AP\{6,10\}$ is prime ☐

Answer 2

$AP\{7,10\}$ = 7, 17, 27, 37, 47, 57, 67, 77, 87, 97, ...

So any of the numbers

- 27 (divides by 3)
- 57 (divides by 3 and 19)
- 77 (divides by 7 and 11)
- 87 (divides by 3 and 29)

... prove George is wrong.

Other bigger numbers could also be used as answers.

Answer 3

$AP\{5,10\}$ = 5, 15, 25, 35, 45, 55, 65, 75. 85, 95, ...

The only prime is 5

Answer 4

$AP\{2,10\}$ = 2, 12, 22, 32, 42, 52, 62, 72, 82, 92, ...

The first number, 2, is prime

Answer 5

$AP\{1,6\}$ and $AP\{5,6\}$

Answer 6

All terms in $AP\{4,6\}$ are of the form.

$$y = 4 + 6n, \qquad for \quad n = 0, 1, 2, 3, \ldots$$

$$y = 2\,(2 + 3n)$$

This shows that these numbers are always divisible by 2.

The first term, 4, is not prime, and so no term in $AP\{4,6\}$ is prime ☐

Answer 7

Either 1 or 5

Answer 8

x is in $AP\{2,6\}$ which has terms which all divide by 2

The only prime is 2 so either $x = 2$ or it is composite.

Answer 9

$y = 0 + 7n$		7													
$y = 1 + 7n$					29		43				71				
$y = 2 + 7n$	2			23		37						79			
$y = 3 + 7n$	3		17		31				59		73				101
$y = 4 + 7n$		11						53		67					
$y = 5 + 7n$	5		19				47		61				89		
$y = 6 + 7n$		13				41						83		97	

It's not a good prime trapper as, other than 7, the primes can be in any six of the seven rows.

Answer 10

1, 3, 5 and 7 are coprime to 8

Answer 11

$AP\{1,8\}$, $AP\{3,8\}$, $AP\{5,8\}$ and $AP\{7,8\}$ will each contain infinitely many primes.

Answer 12

$AP\{1,9\}$, $AP\{2,9\}$, $AP\{4,9\}$, $AP\{5,9\}$, $AP\{7,9\}$ and $AP\{8,9\}$ will each contain infinitely many primes.

Answer 13

(i) a = 1, 3, 5, 7, 9, 11, 13, 15

$AP\{3,16\}$ = 3, 19, 35, 51, 67, 83, 99, 115, 131, ...

(ii) 35, 51, 99, 115, ...

(iii) 3, 19, 67, 83, 131, ...

Answer 14

(i)

$$\begin{aligned}\phi\left(11^1\right) &= 11^0(11 - 1) \\ &= 10\end{aligned}$$

(ii)

$$\begin{aligned}\phi(6) &= \phi(2 \times 3) \\ &= \phi\left(2^1\right) \times \phi\left(3^1\right) \\ &= 2^0 \times 1 \times 3^0 \times 2 \\ &= 2\end{aligned}$$

(iii)

$$\phi(8) = \phi\left(2^3\right) = 4$$

(iv)

$$\begin{aligned}\phi(30) &= \phi\left(2 \times 3 \times 5\right)\\ &= \phi(2) \times \phi(3) \times \phi(5)\\ &= 2^0 \times 1 \times 3^0 \times 2 \times 5^0 \times 4\\ &= 8\end{aligned}$$

Answer 15

Some values of the totient function are shown in the table below,

+	0	1	2	3	4	5	6	7	8	9
0	-	1	1	2	2	4	2	6	4	6
10	4	10	4	12	6	8	8	16	6	18
20	8	12	10	22	8	20	12	18	12	28
30	8	30	16	20	16	24	12	36	18	24
40	16	40	12	42	20	24	22	46	16	42
50	20	32	24	52	18	40	24	36	28	58
60	16	60	30	36	32	48	20	66	32	44
70	24	70	24	72	36	40	36	60	24	78
80	32	54	40	82	24	64	42	56	40	88
90	24	72	44	60	46	72	32	96	42	60
100	40	100	32	102	48	48	52	106	36	108
110	40	72	48	112	36	88	56	72	58	96
120	32	110	60	80	60	100	36	126	64	84
130	48	130	40	108	66	72	64	136	44	138
140	48	92	70	120	48	112	72	84	72	148
150	40	150	72	96	60	120	48	156	78	104
160	64	132	54	162	80	80	82	166	48	156
170	64	108	84	172	56	120	80	116	88	178
180	48	180	72	120	88	144	60	160	92	108
190	72	190	64	192	96	96	84	196	60	198

Good traps are number grids of width 6 on 33% and 30 on 27%

A quick search found 420 on 23% and 4620 on 21%