3.5 Answers

3.5.1 Solutions (To 3.3 Worked Example)

The big issue in this question is the number 2 in the position where the Binomial Theorem requires the number 1

$$(2 + 3x)^{-3} = \left[2\left(1 + \frac{3x}{2} \right) \right]^{-3}$$
$$= \frac{1}{8}\left(1 + \frac{3x}{2} \right)^{-3}$$

Ignoring the eighth for the time being...

The Binomial Theorem

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$\ldots \textit{with } n = -3 \textit{ and } x \textit{ replaced with } \frac{3x}{2} \textit{ yields} \ldots$$

$$\left(1 + \frac{3x}{2} \right)^{-3} = 1 + (-3)\left(\frac{3x}{2} \right) + \frac{(-3)(-4)}{2}\left(\frac{3x}{2} \right)^2 + \frac{(-3)(-4)(-5)}{6}\left(\frac{3x}{2} \right)^3$$
$$= 1 - \frac{9x}{2} + \frac{27x^2}{2} - \frac{135x^3}{4} + \ldots$$

Now, don't forget about that ignored eighth !

$$= \frac{1}{8}\left(1 + \frac{3x}{2} \right)^{-3} = \frac{1}{8} - \frac{9x}{16} + \frac{27x^2}{16} - \frac{135x^3}{32} + \ldots$$

$$\textit{valid for } |x| < \frac{2}{3}$$

3.5.2 Solutions (To 3.3 Exercise)

Answer 1

The number 3 is in the position where the Binomial Theorem requires the number 1

$$= (3 + 2x)^{-3} = \left[3\left(1 + \frac{2x}{3} \right) \right]^{-3}$$

$$= \frac{1}{27} \left(1 + \frac{2x}{3} \right)^{-3}$$

Ignoring the twenty-seventh for the time being...

The Binomial Theorem

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots$$

$$\ldots \textit{with } n = -3 \textit{ and } x \textit{ replaced with } \frac{2x}{3} \textit{ yields...}$$

$$\left(1 + \frac{2x}{3} \right)^{-3} = 1 + (-3) \left(\frac{2x}{3} \right) + \frac{(-3)\ (-4)}{2} \left(\frac{2x}{3} \right)^2 + \frac{(-3)\ (-4)\ (-5)}{6} \left(\frac{2x}{3} \right)^3$$

$$= 1 - 2x + \frac{8x^2}{3} - \frac{80x^3}{27} + \ldots$$

Now, don't forget about that ignored twenty-seventh !

$$\frac{1}{27} \left(1 + \frac{2x}{3} \right)^{-3} = \frac{1}{27} - \frac{2x}{27} + \frac{8x^2}{81} - \frac{80x^3}{729} + \ldots$$

$$\textit{valid for } |x| < \frac{3}{2}$$

Answer 2

(a)

The number 8 is in the position where the Binomial Theorem requires the number 1

$$= (8 - 3x)^{\frac{1}{3}} = \left[8\left(1 - \frac{3x}{8}\right) \right]^{\frac{1}{3}}$$

$$= 2\left(1 - \frac{3x}{8}\right)^{\frac{1}{3}}$$

Ignoring the two for the time being...

The Binomial Theorem

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$\ldots \textit{with } n = \frac{1}{3} \textit{ and } x \textit{ replaced with } -\frac{3x}{8} \textit{ yields...}$$

$$\left(1 - \frac{3x}{8}\right)^{\frac{1}{3}} =$$

$$1 + \left(\frac{1}{3}\right)\left(-\frac{3x}{8}\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}\left(-\frac{3x}{8}\right)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}\left(-\frac{3x}{8}\right)^3$$

$$= 1 - \frac{x}{8} - \frac{x^2}{64} - \frac{5x^3}{1536} + \ldots$$

Now, don't forget about that ignored two !

$$2\left(1 - \frac{3x}{8}\right)^{\frac{1}{3}} = 2 - \frac{x}{4} - \frac{x^2}{32} - \frac{5x^3}{768} + \ldots$$

$$\textit{valid for } |x| < \frac{8}{3}$$

(b)

The method is to substitute $x = 0.1$ into the part (a) binomial expansion to get...

$$7.7^{\frac{1}{3}} \approx 2 - \frac{0.1}{4} - \frac{0.1^2}{32} - \frac{5 \times 0.1^3}{768}$$

$$= 1.97468099\ldots$$

$(\textit{which is very close to the value a calculator would give directly, of } 1.974680822)$

Answer 3

(a)

The number 2 is in the position where the Binomial Theorem requires the number 1

$$\frac{1}{(2 - 5x)^2} = (2 - 5x)^{-2}$$

$$= \left[2\left(1 - \frac{5x}{2}\right) \right]^{-2}$$

$$= \frac{1}{4}\left(1 - \frac{5x}{2}\right)^{-2}$$

Ignoring the quarter for the time being...

The Binomial Theorem

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ \ldots$$

$$\ldots \textit{ with } \ n = -2 \ \textit{ and } \ x \ \textit{ replaced with } \ -\frac{5x}{2} \textit{ yields}\ldots$$

$$\left(1 - \frac{5x}{2}\right)^{-2} = 1 + (-2)\left(-\frac{5x}{2}\right) + \frac{(-2)(-3)}{2}\left(-\frac{5x}{2}\right)^2 + \ldots$$

$$= 1 + 5x + \frac{75x^2}{4} + \ldots$$

Now, don't forget about that ignored quarter !

$$\frac{1}{4}\left(1 - \frac{5x}{2}\right)^{-2} = \frac{1}{4} + \frac{5x}{4} + \frac{75x^2}{16} + \ldots$$

$$\textit{valid for } \ |x| < \frac{2}{5}$$

(b) $k = -3$

(c)

$$A = \frac{45}{8} \qquad \textit{or} \qquad 5.625$$

Answer 4

(a)

The number 4 is in the position where the Binomial Theorem requires the number 1

$$\frac{1}{\sqrt{(4 - 3x)}} = (4 - 3x)^{-\frac{1}{2}}$$

$$= \left[4\left(1 - \frac{3x}{4}\right) \right]^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1 - \frac{3x}{4}\right)^{-\frac{1}{2}}$$

Ignoring the half for the time being...

The Binomial Theorem

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ ...$$

$$\textit{... with } \ n = -\frac{1}{2} \ \textit{ and } \ x \ \textit{ replaced with } \ -\frac{3x}{4} \textit{ yields...}$$

$$\left(1 - \frac{3x}{4}\right)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)\left(-\frac{3x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{3x}{4}\right)^2 + \ ...$$

$$= 1 + \frac{3x}{8} + \frac{27x^2}{128} + \ ...$$

Now, don't forget about that ignored half !

$$\frac{1}{2}\left(1 - \frac{3x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2} + \frac{3x}{16} + \frac{27x^2}{256} + \ ...$$

$$\textit{valid for } \ |x| < \frac{4}{3}$$

(b)

$$(x + 8)\left(\frac{1}{2} + \frac{3x}{16} + \frac{27x^2}{64} + \ ...\right) = 4 + 2x + \frac{33}{32}x^2 + \ ...$$