4.4 Answers

4.4.1 Solutions (To 4.2 Worked Example)

$$\frac{x^2 + 3}{[(1 - x)(1 + x)^2]} = \frac{A}{(1 - x)} + \frac{B}{(1 + x)} + \frac{C}{(1 + x)^2}$$

Multiply through by the square bracketed expression...

$$\frac{(x^2 + 3)\ [\dots]}{[(1 - x)(1 + x)^2]} = \frac{A\ [\dots]}{(1 - x)} + \frac{B\ [\dots]}{(1 + x)} + \frac{C\ [\dots]}{(1 + x)^2}$$

$$x^2 + 3 = A(1 + x)^2 + B(1 - x)(1 + x) + C(1 - x)$$

$$Let\ x = -1\ :\ 4 = 2C \quad \therefore C = 2$$

$$Match\ coefficients\ of\ x^2\ :\ A = 1$$

$$Let\ x = 1\ :\ 4 = 4 + 2B \quad \therefore B = 0$$

Thus

$$\frac{x^2 + 3}{(1 - x)(1 + x)^2} = \frac{1}{(1 - x)} + \frac{2}{(1 + x)^2}$$

The Binomial Theorem now needs to be applied separately to each partial fraction;

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$\begin{aligned}(1 - x)^{-1} &= 1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{6}(-x)^3 + \dots \\ &= 1 + x + x^2 + x^3 + \dots\end{aligned}$$

$$\begin{aligned}2(1 + x)^{-2} &= 2\left[1 + (-2)(x) + \frac{(-2)(-3)}{2}(x)^2 + \frac{(-2)(-3)(-4)}{6}(x)^3 + \dots\right] \\ &= 2 - 4x + 6x^2 - 8x^3 + \dots\end{aligned}$$

Thus...

$$\frac{x^2 + 3}{(1 - x)(1 + x)^2} = 3 - 3x + 7x^2 - 7x^3 + \dots$$

$$valid\ for\ |x| < 1$$

4.4.2 Solutions (To 4.3 Exercise)

Answer 1

$$\frac{1}{[\,(1+3x)\,(1+2x)\,]} = \frac{A}{(1+3x)} + \frac{B}{(1+2x)}$$

Multiply through by the square bracketed expression...

$$\frac{1\,[\,...\,]}{[\,(1+3x)\,(1+2x)\,]} = \frac{A\,[\,...\,]}{(1+3x)} + \frac{B\,[\,...\,]}{(1+2x)}$$

$$1 = A\,(1+2x) + B\,(1+3x)$$

$$Let\ x = -\frac{1}{2} \ :\ 1 = B\left(-\frac{1}{2}\right) \quad \therefore B = -2$$

$$Let\ x = -\frac{1}{3} \ :\ 1 = A\left(\frac{1}{3}\right) \quad \therefore A = 3$$

Thus

$$\frac{1}{(1+3x)\,(1+2x)} = \frac{3}{(1+3x)} - \frac{2}{(1+2x)}$$

The Binomial Theorem now needs to be applied separately to each partial fraction;

$$(1+x)^n = 1 + nx + \frac{n\,(n-1)}{2\,!}x^2 + \frac{n\,(n-1)\,(n-2)}{3\,!}x^3 + \ ...$$

$$3\,(1+3x)^{-1} = 3\left[1 + (-1)\,(3x) + \frac{(-1)\,(-2)}{2}\,(3x)^2 + \frac{(-1)\,(-2)\,(-3)}{6}\,(3x)^3 + \ ...\right]$$

$$= 3 - 9x + 27x^2 - 81x^3 + \ ...$$

$$-2\,(1+2x)^{-1} = -2\left[1 + (-1)\,(2x) + \frac{(-1)(-2)}{2}\,(2x)^2 + \frac{(-1)(-2)(-3)}{6}\,(2x)^3 + \ ...\right]$$

$$= -2 + 4x - 8x^2 + 16x^3 + \ ...$$

Thus...

$$\frac{1}{(1+3x)\,(1+2x)} = 1 - 5x + 19x^2 - 65x^3 + \ ...$$

$$valid\ for\ \left|x\right| < \frac{1}{3}$$

Answer 2

$$\frac{14 + 5x}{[\ (1 + x)\ (2 - x)\ ]} = \frac{A}{(1 + x)} + \frac{B}{(2 - x)}$$

Multiply through by the square bracketed expression...

$$\frac{(14 + 5x)\ [\ ...\ ]}{[\ (1 + x)\ (2 - x)\ ]} = \frac{A\ [\ ...\ ]}{(1 + x)} + \frac{B\ [\ ...\ ]}{(2 - x)}$$

$$14 + 5x = A\ (2 - x) + B\ (1 + x)$$

$$Let\ x = 2\ :\ 24 = B\ (3) \quad \therefore B = 8$$

$$Let\ x = -1\ :\ 9 = A\ (3) \quad \therefore A = 3$$

Thus

$$\frac{1}{(1 + x)\ (2 - x)} = \frac{3}{(1 + x)} + \frac{8}{(2 - x)}$$

The Binomial Theorem now needs to be applied separately to each partial fraction;

$$(1 + x)^n = 1 + nx + \frac{n\ (n - 1)}{2\,!} x^2 + \frac{n\ (n - 1)\ (n - 2)}{3\,!} x^3 + \ ...$$

$$3(1 + x)^{-1} = 3\left[\ 1 + (-1)\ (x) + \frac{(-1)\ (-2)}{2}\ (x)^2 + \frac{(-1)\ (-2)\ (-3)}{6}\ (x)^3 + \ ...\ \right]$$

$$= 3 - 3x + 3x^2 - 3x^3 + \ ...$$

$$+8\ [\ (2 - x)\ ]^{-1} = 8\left[\ 2\left(1 - \frac{x}{2}\right)\ \right]^{-1}$$

$$= 8 \times 2^{-1} \left(1 - \frac{x}{2}\right)^{-1}$$

$$= 4\left[1 + (-1)\ \left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2}\left(-\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{6}\left(-\frac{x}{2}\right)^3 + \ ...\ \right]$$

$$= 4 + 2x + x^2 + \frac{1}{2}x^3 + \ ...$$

Thus...

$$\frac{14 + 5x}{(1 + x)\ (2 - x)} = 7 - x + 4x^2 - \frac{5}{2}x^3 + \ ...$$

$$valid\ for\ |x| < 1$$

Answer 3

(a)

$$\frac{3x - 1}{[\ (1 - 2x)^2\ ]} = \frac{A}{(1 - 2x)} + \frac{B}{(1 - 2x)^2}$$

Multiply through by the square bracket,

$$3x - 1 = A\,(1 - 2x) + B$$

Let $x = 0.5$ to get $0.5 = B$

Match coefficients of terms in x : $3 = -\,2\,A \;\Rightarrow\; A = -\,1.5$

(b) From part (a) we have that,

$$f\,(x) = \frac{(-1.5)}{(1 - 2x)} + \frac{0.5}{(1 - 2x)^2}$$

$$= (-1.5)\,(1 - 2x)^{-1} + 0.5\,(1 - 2x)^{-2}$$

$$= (-1.5)\left(1 + (-1)(-2x) + \frac{(-1)(-2)}{2}(-2x)^2 + \frac{(-1)(-2)(-3)}{3 \times 2}(-2x)^3 + \ldots\right)$$

$$+ (0.5)\left(1 + (-2)(-2x) + \frac{(-2)(-3)}{2}(-2x)^2 + \frac{(-2)(-3)(-4)}{3 \times 2}(-2x)^3 + \ldots\right)$$

$$= (-1.5)\left(1 + 2x + 4x^2 + 8x^3 + \ldots\right) + 0.5\left(1 + 4x + 12x^2 + 32x^3\right)$$

$$= -1.5 - 3x - 6x^2 - 12x^3 + \ldots + 0.5 + 2x + 6x^2 + 16x^3 + \ldots$$

$$= -1 - x + 4x^3 + \ldots$$

Answer 4

(a)

$$\frac{27x^2 + 32x + 16}{[(3x+2)^2(1-x)]} = \frac{A}{(3x+2)} + \frac{B}{(3x+2)^2} + \frac{C}{(1-x)}$$

Multiply through by the square bracketed expression...

$$27x^2 + 32x + 16 = A(3x+2)(1-x) + B(1-x) + C(3x+2)^2$$

Let $x = 1$: $\quad 75 = 25\,C \quad \Rightarrow \quad C = 3$

Let $x = -\frac{2}{3}$:

$$12 - \frac{64}{3} + 16 = B \times \frac{5}{3}$$

$$\frac{20}{3} = B \times \frac{5}{3}$$

$$B = 4$$

Match up coefficients of x^2 :

$$27 = -3A + 9C$$

$$but\ C = 3 \qquad \therefore\ A = 0 \qquad \square$$

(b) From (a) we have that,

$$f(x) = \frac{4}{(3x+2)^2} + \frac{3}{(1-x)}$$

$$= 4\,[3x+2]^{-2} + 3\,[1-x]^{-1}$$

$$= 4\left[2\left(\frac{3}{2}x + 1\right)\right]^{-2} + 3[1-x]^{-1}$$

$$= \left(1 + \frac{3}{2}x\right)^{-2} + 3\,[1-x]^{-1}$$

$$= 1 + (-2)\left(\frac{3x}{2}\right) + \frac{(-2)(-3)}{2}\left(\frac{3x}{2}\right)^2 + \ldots$$

$$+3\left[1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \ldots\right]$$

$$= 1 - 3x + \frac{27}{4}x^2 + \ldots + 3 + 3x + 3x^2 + \ldots$$

$$= 4 + 0x + \frac{39}{4}x^2 + \ldots$$

(c) True value is,

$$f\,(0.2) \;=\; \frac{4}{(\,3 \times 0.2 \,+\, 2\,)^2} \;+\; \frac{3}{(\,1 \,-\, 0.2\,)}$$

$$= \frac{4 \times 5^2}{13^2} + \frac{3}{0.8}$$

$$= \frac{2935}{676}$$

Estimate is,

$$f\,(0.2) \;=\; 4 \,+\, 0x \,+\, \frac{39}{4}\,0.2^{\,2} \,+\, \ldots$$

$$= \frac{439}{100}$$

$$= 4.39$$

$$\%\,error \;=\; \frac{4.39 \,-\, \frac{2935}{676}}{\frac{2935}{676}} \times 100$$

$$= 1.1120954\,\%$$

$$i.e.\;\; 1.11\,\%$$