A-Level Pure Mathematics
Year 2

THE BINOMIAL THEOREM REVISITED

									1									
								1		1								
							1		2		1							
						1		3		3		1						
					1		4		6		4		1					
				1		5		10		10		5		1				
			1		6		15		20		15		6		1			
		1		7		21		35		35		21		7		1		
	1		8		28		56		70		56		28		8		1	
1		9		36		84		126		126		84		36		9		1

Lesson 1

A-Level Pure : The Binomial Theorem : Year 2

1.1 What Was Covered In Year 1

We looked at Binomial Expansion in Year 1.
Before taking the topic further, let's recall some of the techniques employed.

1.2 Example

(i) Use Pascal's triangle to expand the brackets;

$$(2 + 3x)^4$$

Give each term in your expansion in as simple a form as possible.

$$\begin{aligned}(2 + 3x)^4 = \;& \left(\quad\right) \times \left(\quad\right) \times \left(\quad\right) \\ + \;& \left(\quad\right) \times \left(\quad\right) \times \left(\quad\right) \\ + \;& \left(\quad\right) \times \left(\quad\right) \times \left(\quad\right) \\ + \;& \left(\quad\right) \times \left(\quad\right) \times \left(\quad\right) \\ + \;& \left(\quad\right) \times \left(\quad\right) \times \left(\quad\right)\end{aligned}$$

(ii) By using your part **(i)** answer, or otherwise, expand the brackets of;

$$(2 + 3\sqrt{x})^4$$

1.3 Exercise

Question 1

(i) Use Pascal's triangle to expand the brackets;

$$(5 - 2x)^3$$

Give each term in your expansion in as simple a form as possible.

The following may be of help in setting out your working....

$$\begin{aligned}(5 - 2x)^3 = \; & (\quad) \times (\quad) \times (\quad) \\ & + (\quad) \times (\quad) \times (\quad) \\ & + (\quad) \times (\quad) \times (\quad) \\ & + (\quad) \times (\quad) \times (\quad)\end{aligned}$$

(ii) By using your part **(i)** answer, or otherwise, expand the brackets of;

$$(5 - 2\sqrt{x})^3$$

Question 2

Expand the brackets

(i)

$$(11 - x)^4$$

(ii)

$$(10 + x^4)^5$$

Question 3

(i) Expand the brackets, simplifying each term;

$$\left(x + \frac{5}{x}\right)^4$$

(ii) Which is the term that is independent in x ?

(iii) What is the coefficient on the term in x^2 ?

Question 4

(i) Expand the brackets

$$\left(1 + \frac{x}{2}\right)^4$$

In Year 2, it becomes crucial to first adjust the question so that the bracket contains the number 1 as its first term.

(ii) Prove that,

$$(4 + 2x)^4 = 256\left(1 + \frac{x}{2}\right)^4$$

(iii) Hence, using your part **(i)** answer, expand the following;

$$(4 + 2x)^4$$

Question 5

(i) Given that,

$$(6 + 9x^2)^5 = k(1 + ax^2)^5$$

State the value of *k*. and the value of *a*.

(ii) Expand the following in ascending powers of *x*, where *a* has the value determined in part **(i)**

$$(1 + ax^2)^5$$

(iii) Use your previous answers to expand the brackets of

$$(6 + 9x^2)^5$$

Question 6

By using Pascal's Triangle, or otherwise, expand the brackets;

$$\left(\frac{1}{3x^2}+x^3\right)^4$$

1.4 Answers

1.4.1 Solutions (To 1.2 Example)

(i)

$$\begin{aligned}(2 + 3x)^4 = {} & (1) \times (2)^4 \times (3x)^0 \\ & + (4) \times (2)^3 \times (3x)^1 \\ & + (6) \times (2)^2 \times (3x)^2 \\ & + (4) \times (2)^1 \times (3x)^3 \\ & + (1) \times (2)^0 \times (3x)^4 \\ = {} & 16 + 96x + 216x^2 + 216x^3 + 81x^4\end{aligned}$$

(ii)

$$(2 + 3\sqrt{x})^4 = 16 + 96\sqrt{x} + 216x + 216\left(\sqrt{x}\right)^3 + 81x^2$$

1.4.2 Solutions (To 1.3 Exercise)

Answer 1
(i)

$$\begin{aligned}(5 - 2x)^3 = {} & (1) \times (5)^3 \times (-2x)^0 \\ & + (3) \times (5)^2 \times (-2x)^1 \\ & + (3) \times (5)^1 \times (-2x)^2 \\ & + (1) \times (5)^0 \times (-2x)^3 \\ & = 125 - 150x + 60x^2 - 8x^3\end{aligned}$$

(ii)

$$= 125 - 150\sqrt{x} + 60x - 8\left(\sqrt{x}\right)^3$$

Answer 2
(i)

$$\begin{aligned}(11 - x)^4 = {} & (1) \times (11)^4 \times (-x)^0 \\ & + (4) \times (11)^3 \times (-x)^1 \\ & + (6) \times (11)^2 \times (-x)^2 \\ & + (4) \times (11)^1 \times (-x)^3 \\ & + (1) \times (11)^0 \times (-x)^4 \\ & = 14641 - 5324x + 726x^2 - 44x^3 + x^4\end{aligned}$$

(ii)

$$\begin{aligned}(10 + x^4)^5 &= (1) \times (10)^5 \times \left(x^4\right)^0 \\ &+ (5) \times (10)^4 \times \left(x^4\right)^1 \\ &+ (10) \times (10)^3 \times \left(x^4\right)^2 \\ &+ (10) \times (10)^2 \times \left(x^4\right)^3 \\ &+ (5) \times (10)^1 \times \left(x^4\right)^4 \\ &+ (1) \times (10)^0 \times \left(x^4\right)^5 \\ &= 100\,000 + 50\,000\,x^4 + 10\,000\,x^8 + 1\,000\,x^{12} + 50\,x^{16} + x^{20}\end{aligned}$$

Answer 3

(i)

$$\begin{aligned}\left(x + \frac{5}{x}\right)^4 &= (1) \times (x)^4 \times \left(\frac{5}{x}\right)^0 \\ &+ (4) \times (x)^3 \times \left(\frac{5}{x}\right)^1 \\ &+ (6) \times (x)^2 \times \left(\frac{5}{x}\right)^2 \\ &+ (4) \times (x)^1 \times \left(\frac{5}{x}\right)^3 \\ &+ (1) \times (x)^0 \times \left(\frac{5}{x}\right)^4 \\ &= x^4 + 20\,x^2 + 150 + \frac{500}{x^2} + \frac{625}{x^4}\end{aligned}$$

(ii) 150 or "3rd term"

(iii) 20

Answer 4

(i)

$$\left(1 + \frac{x}{2}\right)^4 = (1) \times (1)^4 \times \left(\frac{x}{2}\right)^0$$

$$+ (4) \times (1)^3 \times \left(\frac{x}{2}\right)^1$$

$$+ (6) \times (1)^2 \times \left(\frac{x}{2}\right)^2$$

$$+ (4) \times (1)^1 \times \left(\frac{x}{2}\right)^3$$

$$+ (1) \times (1)^0 \times \left(\frac{x}{2}\right)^4$$

$$= 1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{16}x^4$$

(ii)

$$(4 + 2x)^4 = [(4 + 2x)]^4$$

$$= \left[4\left(1 + \frac{x}{2}\right)\right]^4$$

$$= 4^4\left(1 + \frac{x}{2}\right)^4$$

$$= 256\left(1 + \frac{x}{2}\right)^4$$

(iii)

$$(4 + 2x)^4 = 256\left(1 + \frac{x}{2}\right)^4$$

$$= 256\left(1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{16}x^4\right)$$

$$= 256 + 512x + 384x^2 + 128x^3 + 16x^4$$

Answer 5

(i)

$$(6+9x^2)^5 = [\,(6+9x^2)\,]^5$$

$$= \left[6\left(1+\frac{3}{2}x^2\right)\right]^5$$

$$= 6^5\left(1+\frac{3}{2}x^2\right)^5$$

$$= 7776\left(1+\frac{3}{2}x^2\right)^5$$

$$\therefore\ k = 7776 \qquad \textit{and} \qquad a = \frac{3}{2}$$

(ii)

$$\left(1+\frac{3}{2}x^2\right)^5 = (1)\times(1)^5\times\left(\frac{3}{2}x^2\right)^0$$

$$+\,(5)\times(1)^4\times\left(\frac{3}{2}x^2\right)^1$$

$$+\,(10)\times(1)^3\times\left(\frac{3}{2}x^2\right)^2$$

$$+\,(10)\times(1)^2\times\left(\frac{3}{2}x^2\right)^3$$

$$+\,(5)\times(1)^1\times\left(\frac{3}{2}x^2\right)^4$$

$$+\,(1)\times(1)^0\times\left(\frac{3}{2}x^2\right)^5$$

$$= 1+\frac{15}{2}x^2+\frac{45}{2}x^4+\frac{135}{4}x^6+\frac{405}{16}x^8+\frac{243}{32}x^{10}$$

(iii)

$$(6+9x^2)^5 = 7776\left(1+\frac{3}{2}x^2\right)^5$$

$$= 7776\left(1+\frac{15}{2}x^2+\frac{45}{2}x^4+\frac{135}{4}x^6+\frac{405}{16}x^8+\frac{243}{32}x^{10}\right)$$

$$= 7776+58320x^2+174\,960x^4+262\,440x^6+196\,830x^8+59049x^{10}$$

Answer 6

$$\left(\frac{1}{3x^2} + x^3\right)^4 = (1) \times \left(\frac{1}{3x^2}\right)^4 \times (x^3)^0$$

$$+ (4) \times \left(\frac{1}{3x^2}\right)^3 \times (x^3)^1$$

$$+ (6) \times \left(\frac{1}{3x^2}\right)^2 \times (x^3)^2$$

$$+ (4) \times \left(\frac{1}{3x^2}\right)^1 \times (x^3)^3$$

$$+ (1) \times \left(\frac{1}{3x^2}\right)^0 \times (x^3)^4$$

$$= \frac{1}{81x^8} + \frac{4}{27x^3} + \frac{2}{3}x^2 + \frac{4}{3}x^7 + x^{12}$$

Lesson 2

2.1 Pascals' Triangle

The first few entries in Pascal's Triangle are;

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4	1					
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	**28**	8	1	
1	9	36	84	126	126	84	36	9	1

In general, entries in the triangle are referred to by describing which row and column they are in. However, there are a couple of catches in addition to the columns being askew.

The first row is termed Row 0, and the letter n is used for the Row number.
The first column is termed Column 0, and the letter r is used for the Column number.

The description can be written in two forms;

$$\binom{n}{r} = {}^{n}C_{r}$$

So, the highlighted cell which contains the number 28 is in Row 8, Column 6.

Check that you can extract the 28 from your calculator,

$$\binom{8}{6} = {}^{8}C_{6} = 28$$

There is also a formula for calculating the entries in the triangle,

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

The factorials generate big numbers and you may find that your calculator can't handle the numbers for even modest values of n
70! is the first factorial that my Casio fx-991 EX Classwiz can't handle.

2.2 Example

Find an expression in terms of *a* that contains no factorials for

$$^{a+6}C_{a+4}$$

where *a* is an unknown integer constant.

2.3 Exercise

Question 1

Use your calculator to work out;

(i)

$$7\,!$$

(ii)

$$0\,!$$

Question 2

Without using a calculator, determine the exact value of

(i)

$$\frac{100\,!}{98\,!}$$

(ii)

$$\frac{100\,!}{2\,!\,(100-2)\,!}$$

Question 3

Find an expression in terms of *a* that contains no factorials for

$$^{a+11}C_{a+9}$$

where *a* is an unknown integer constant.

Question 4
Use your calculator to determine,

(i)

$$\binom{11}{7}$$

(ii)

$$^{14}C_5$$

Question 5
From our Year 1 course we have that

$$\begin{aligned}(a + bx)^n = &\binom{n}{0} \times a^n \times (bx)^0 \\ &+ \binom{n}{1} \times a^{n-1} \times (bx)^1 \\ &+ \binom{n}{2} \times a^{n-2} \times (bx)^2 \\ &+ \binom{n}{3} \times a^{n-3} \times (bx)^3 \\ &+ \ldots\end{aligned}$$

We are now going to assume $a = b = 1$ in the above and replace the Pascal Triangle coefficients using

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Thus, determine the first four terms of The Binomial Expansion of $(1 + x)^n$

$$(1 + x)^n = 1 + nx +$$

Question 6

The Binomial Theorem states that...

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ ...$$

... provided $|x| < 1$

and, amazingly, this version is valid for n being any integer, even a negative.

Use the binomial theorem to find the first four terms in the expansion of...

$$\frac{1}{(1 + 6x)}$$

... and state the values of x for which your expansion is valid.

Question 7
The Binomial Theorem states that...

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ \ldots$$

... provided $|x| < 1$

and, incredibly, this version is is valid for all rational numbers.

Use the binomial theorem to find the first four terms in the expansion of...

$$\sqrt{1+3x}$$

... and state the values of x for which your expansion is valid.

Question 8
The Binomial Theorem states that...

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

... provided $|x| < 1$

Use the binomial theorem to find the first four terms in the expansion of...

$$\frac{1}{(1-x)^2}$$

... and state the values of x for which your expansion is valid.

Question 9

The Binomial Theorem states that...

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ ...$$

... provided $|x| < 1$

Use the binomial theorem to find the first four terms in the expansion of...

$$\frac{1}{(3 + x)}$$

... and state the values of x for which your expansion is valid.

2.4 Answers

2.4.1 Solutions (To 2.2 Example)

$$\text{Using} \quad {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$\text{we get} \quad {}^{a+6}C_{a+4} = \frac{(a+6)!}{(a+4)!\,((a+6)-(a+4))!}$$

$$= \frac{(a+6)\,(a+5)\,(a+4)!}{(a+4)!\,(a+6-a-4))!}$$

$$= \frac{(a+6)\,(a+5)}{2}$$

2.4.2 Solutions (To 2.3 Exercise)

Answer 1

(i) 5040 **(ii)** 1

Answer 2

(i)

$$\frac{100!}{98!} = \frac{100 \times 99 \times 98!}{98!} = 9900$$

(ii)

$$\frac{100!}{2!\,(100-2)!} = \frac{100!}{2!\,(98)!} = \frac{1}{2!} \times 9900 = 4950$$

Answer 3

$$\text{Using} \quad {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$\text{we get} \quad {}^{a+11}C_{a+9} = \frac{(a+11)!}{(a+9)!\,((a+11)-(a+9))!}$$

$$= \frac{(a+11)\,(a+10)\,(a+9)!}{(a+9)!\,(a+11-a-9))!}$$

$$= \frac{(a+11)\,(a+10)}{2}$$

Answer 4

(i) **(ii)**

$$\binom{11}{7} = 330$$

$$^{14}C_5 = 2002$$

Answer 5

$$\begin{aligned}(a + bx)^n = &\binom{n}{0} \times a^n \times (bx)^0 \\ &+ \binom{n}{1} \times a^{n-1} \times (bx)^1 \\ &+ \binom{n}{2} \times a^{n-2} \times (bx)^2 \\ &+ \binom{n}{3} \times a^{n-3} \times (bx)^3 \\ &+ \ldots\end{aligned}$$

becomes....

$$\begin{aligned}(1 + x)^n = &\frac{n!}{0!\,(n-0)!} \times 1^n \times (x)^0 \\ &+ \frac{n!}{1!\,(n-1)!} \times 1^{n-1} \times (x)^1 \\ &+ \frac{n!}{2!\,(n-2)!} \times 1^{n-2} \times (x)^2 \\ &+ \frac{n!}{3!\,(n-3)!} \times 1^{n-3} \times (x)^3 \\ &+ \ldots\end{aligned}$$

which simplifies to...

$$\begin{aligned}(1 + x)^n = &\ 1 \\ &+ n\,x \\ &+ \frac{n(n-1)}{2!}x^2 \\ &+ \frac{n(n-1)(n-2)}{3!}x^3 \\ &+ \ldots\end{aligned}$$

Answer 6

$$\frac{1}{(1+6x)} = (1+6x)^{-1}$$

The Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

... with $n = -1$ *and* x *replaced with* $6x$ *yields...*

$$(1+6x)^{-1} = 1 + (-1)(6x) + \frac{(-1)(-2)}{2}(6x)^2 + \frac{(-1)(-2)(-3)}{6}(6x)^3$$

$$= 1 - 6x + 36x^2 - 216x^3 + \ldots$$

$$\text{valid for } |x| < \frac{1}{6}$$

Answer 7

$$\sqrt{1+3x} = (1+3x)^{\frac{1}{2}}$$

The Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

... with $n = \frac{1}{2}$ *and* x *replaced with* $3x$ *yields...*

$$(1+3x)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)(3x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(3x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(3x)^3$$

$$= 1 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{27}{16}x^3 + \ldots$$

$$\text{valid for } |x| < \frac{1}{3}$$

Answer 8

$$\frac{1}{(1-x)^2} = (1-x)^{-2}$$

The Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

... with $n = -2$ *and* x *replaced with* $-x$ *yields...*

$$(1-x)^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)}{2}(-x)^2 + \frac{(-2)(-3)(-4)}{6}(-x)^3$$

$$= 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$\text{valid for } |x| < 1$$

Answer 9
There is an issue in this question with the number 3 being in the position where the Binomial Theorem requires the number 1

$$\frac{1}{(3+x)} = (3+x)^{-1}$$
$$= \left[3\left(1+\frac{x}{3}\right)\right]^{-1}$$
$$= \frac{1}{3}\left(1+\frac{x}{3}\right)^{-1}$$

Ignoring the third for the time being...

The Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$\ldots \textit{with } n = -1 \textit{ and } x \textit{ replaced with } \frac{x}{3} \textit{ yields...}$$

$$\left(1+\frac{x}{3}\right)^{-1} = 1 + (-1)\left(\frac{x}{3}\right) + \frac{(-1)\ (-2)}{2}\left(\frac{x}{3}\right)^2 + \frac{(-1)\ (-2)\ (-3)}{6}\left(\frac{x}{3}\right)^3$$
$$= 1 - \frac{x}{3} + \frac{x^2}{9} - \frac{x^3}{27} + \ldots$$

Now, don't forget about that ignored third !

$$\frac{1}{3}\left(1+\frac{x}{3}\right)^{-1} = \frac{1}{3} - \frac{x}{9} + \frac{x^2}{27} - \frac{x^3}{81} + \ldots$$

$$\textit{valid for } |x| < 3$$

Lesson 3

3.1 The Binomial Theorem & Infinite Series Expansion

The Binomial Theorem provides a powerful means of expanding the brackets of expressions such as;

$$\frac{1}{(1+x)} \qquad (1+7x)^{0.5} \qquad (2+3x)^{-4} \qquad \frac{6}{\sqrt{5x+6}}$$

The expansion is typically into an infinite series which will converge provided x is sufficiently small.

3.2 The Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

The number 1, prior to bracket expansion, is crucial.

The series converges provided $|x| < 1$.

3.3 Worked Example

C4 examination question from January 2013, Q1

Given

$$f(x) = (2+3x)^{-3} \qquad |x| < \frac{2}{3}$$

find the binomial expansion of $f(x)$, in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction.

[5 marks]

3.4 Exercise

Question 1

C4 examination question from June 2007, Q1

$$f(x) = (3 + 2x)^{-3} \qquad |x| < \frac{3}{2}$$

Find the binomial expansion of $f(x)$, in ascending powers of x, as far as the term in x^3
Give each coefficient as a simplified fraction.

[5 marks]

Question 2

C4 examination question from January 2008, Q2

(a) Use the binomial theorem to expand

$$(8 - 3x)^{\frac{1}{3}} \qquad |x| < \frac{8}{2}$$

in ascending powers of x, up to and including the term in x^3

Give each term as a simplified fraction.

[5 marks]

(b) Use your expansion, with a suitable value of x, to obtain an approximation to $\sqrt[3]{7.7}$. Give your answer to 7 decimal places.

[2 marks]

Question 3

C4 examination question from January 2012, Q3

(a) Expand

$$\frac{1}{(2-5x)^2} \qquad |x| < \frac{2}{5}$$

in ascending powers of x, up to and including the term in x^2

Give each term as a simplified fraction.

[5 marks]

Given that the binomial expansion of

$$\frac{2+kx}{(2-5x)^2}, \qquad |x| < \frac{2}{5}$$

is

$$\frac{1}{2} + \frac{7}{4}x + Ax^2 + \ \ldots$$

(b) find the value of the constant k

[2 marks]

(c) find the value of the constant A

[2 marks]

Question 4

C4 examination question from June 2008, Q5

(a) Expand

$$\frac{1}{\sqrt{(4 - 3x)}} \qquad |\,x\,| < \frac{4}{3}$$

in ascending powers of x, up to and including the term in x^2

Simplify each term.

[5 marks]

(b) Hence, or otherwise, find the first 3 terms in the expansion of

$$\frac{x + 8}{\sqrt{(4 - 3x)}}$$

as a series in ascending powers of x

[4 marks]

3.5 Answers

3.5.1 Solutions (To 3.3 Worked Example)

The big issue in this question is the number 2 in the position where the Binomial Theorem requires the number 1

$$(2 + 3x)^{-3} = \left[2\left(1 + \frac{3x}{2} \right) \right]^{-3}$$
$$= \frac{1}{8} \left(1 + \frac{3x}{2} \right)^{-3}$$

Ignoring the eighth for the time being...

The Binomial Theorem

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ ...$$

$$...\ with\ \ n = -3\ \ and\ \ x\ \ replaced\ with\ \frac{3x}{2}\ yields...$$

$$\left(1 + \frac{3x}{2} \right)^{-3} = 1 + (-3) \left(\frac{3x}{2} \right) + \frac{(-3)\ (-4)}{2} \left(\frac{3x}{2} \right)^2 + \frac{(-3)\ (-4)\ (-5)}{6} \left(\frac{3x}{2} \right)^3$$
$$= 1 - \frac{9x}{2} + \frac{27x^2}{2} - \frac{135x^3}{4} + \ ...$$

Now, don't forget about that ignored eighth !

$$= \frac{1}{8} \left(1 + \frac{3x}{2} \right)^{-3} = \frac{1}{8} - \frac{9x}{16} + \frac{27x^2}{16} - \frac{135x^3}{32} + \ ...$$

$$valid\ for\ \left| x \right| < \frac{2}{3}$$

3.5.2 Solutions (To 3.3 Exercise)

Answer 1

The number 3 is in the position where the Binomial Theorem requires the number 1

$$= (3 + 2x)^{-3} = \left[3\left(1 + \frac{2x}{3}\right) \right]^{-3}$$

$$= \frac{1}{27}\left(1 + \frac{2x}{3}\right)^{-3}$$

Ignoring the twenty-seventh for the time being...

The Binomial Theorem

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ ...$$

$$\textit{... with } n = -3 \textit{ and } x \textit{ replaced with } \frac{2x}{3} \textit{ yields...}$$

$$\left(1 + \frac{2x}{3}\right)^{-3} = 1 + (-3)\left(\frac{2x}{3}\right) + \frac{(-3)\ (-4)}{2}\left(\frac{2x}{3}\right)^2 + \frac{(-3)\ (-4)\ (-5)}{6}\left(\frac{2x}{3}\right)^3$$

$$= 1 - 2x + \frac{8x^2}{3} - \frac{80x^3}{27} + \ ...$$

Now, don't forget about that ignored twenty-seventh !

$$\frac{1}{27}\left(1 + \frac{2x}{3}\right)^{-3} = \frac{1}{27} - \frac{2x}{27} + \frac{8x^2}{81} - \frac{80x^3}{729} + \ ...$$

$$\textit{valid for } |x| < \frac{3}{2}$$

Answer 2

(a)

The number 8 is in the position where the Binomial Theorem requires the number 1

$$= (8 - 3x)^{\frac{1}{3}} = \left[8\left(1 - \frac{3x}{8}\right) \right]^{\frac{1}{3}}$$

$$= 2\left(1 - \frac{3x}{8}\right)^{\frac{1}{3}}$$

Ignoring the two for the time being...

The Binomial Theorem

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$\ldots\ with\ \ n = \frac{1}{3}\ \ and\ \ x\ \ replaced\ with\ \ -\frac{3x}{8}\ yields\ldots$$

$$\left(1 - \frac{3x}{8}\right)^{\frac{1}{3}} =$$

$$1 + \left(\frac{1}{3}\right)\left(-\frac{3x}{8}\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}\left(-\frac{3x}{8}\right)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}\left(-\frac{3x}{8}\right)^3$$

$$= 1 - \frac{x}{8} - \frac{x^2}{64} - \frac{5x^3}{1536} + \ldots$$

Now, don't forget about that ignored two !

$$2\left(1 - \frac{3x}{8}\right)^{\frac{1}{3}} = 2 - \frac{x}{4} - \frac{x^2}{32} - \frac{5x^3}{768} + \ldots$$

$$valid\ for\ \left|x\right| < \frac{8}{3}$$

(b)

The method is to substitute $x = 0.1$ into the part (a) binomial expansion to get...

$$7.7^{\frac{1}{3}} \approx 2 - \frac{0.1}{4} - \frac{0.1^2}{32} - \frac{5 \times 0.1^3}{768}$$

$$= 1.97468099\ldots$$

$(which\ is\ very\ close\ to\ the\ value\ a\ calculator\ would\ give\ directly,\ \ of\ 1.974680822)$

Answer 3

(a)

The number 2 is in the position where the Binomial Theorem requires the number 1

$$\frac{1}{(2 - 5x)^2} = (2 - 5x)^{-2}$$

$$= \left[2\left(1 - \frac{5x}{2}\right) \right]^{-2}$$

$$= \frac{1}{4}\left(1 - \frac{5x}{2}\right)^{-2}$$

Ignoring the quarter for the time being...

The Binomial Theorem

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ \ldots$$

$$\ldots \textit{ with } \ n = -2 \ \textit{ and } \ x \ \textit{ replaced with } \ -\frac{5x}{2} \textit{ yields...}$$

$$\left(1 - \frac{5x}{2}\right)^{-2} = 1 + (-2)\left(-\frac{5x}{2}\right) + \frac{(-2)(-3)}{2}\left(-\frac{5x}{2}\right)^2 + \ldots$$

$$= 1 + 5x + \frac{75x^2}{4} + \ldots$$

Now, don't forget about that ignored quarter !

$$\frac{1}{4}\left(1 - \frac{5x}{2}\right)^{-2} = \frac{1}{4} + \frac{5x}{4} + \frac{75x^2}{16} + \ldots$$

$$\textit{valid for } \left| x \right| < \frac{2}{5}$$

(b) $k = -3$

(c)

$$A = \frac{45}{8} \qquad \textit{or} \qquad 5.625$$

Answer 4

(a)

The number 4 is in the position where the Binomial Theorem requires the number 1

$$\frac{1}{\sqrt{(4 - 3x)}} = (4 - 3x)^{-\frac{1}{2}}$$

$$= \left[4\left(1 - \frac{3x}{4}\right) \right]^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1 - \frac{3x}{4}\right)^{-\frac{1}{2}}$$

Ignoring the half for the time being...

The Binomial Theorem

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$$

$$\ldots \textit{with} \;\; n = -\frac{1}{2} \;\; \textit{and} \;\; x \;\; \textit{replaced with} \;\; -\frac{3x}{4} \; \textit{yields...}$$

$$\left(1 - \frac{3x}{4}\right)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)\left(-\frac{3x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{3x}{4}\right)^2 + \ldots$$

$$= 1 + \frac{3x}{8} + \frac{27x^2}{128} + \ldots$$

Now, don't forget about that ignored half !

$$\frac{1}{2}\left(1 - \frac{3x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2} + \frac{3x}{16} + \frac{27x^2}{256} + \ldots$$

$$\textit{valid for} \; |x| < \frac{4}{3}$$

(b)

$$(x + 8)\left(\frac{1}{2} + \frac{3x}{16} + \frac{27x^2}{64} + \ldots\right) = 4 + 2x + \frac{33}{32}x^2 + \ldots$$

4.1 The Binomial Theorem & Partial Fractions

Earlier work with Partial Fractions allows us to cope with finding the Binomial Expansion of more difficult expressions.

4.2 Example

$$f(x) = \frac{x^2 + 3}{(1 - x)(1 + x)^2} \qquad x \neq \pm 1$$

Express $f(x)$ as partial fractions, and so obtain a cubic approximation for $f(x)$
State the range of values of x for which the expansion is valid

4.3 Exercise

Question 1

$$g(x) = \frac{1}{(1 + 3x)(1 + 2x)} \qquad x \neq -\frac{1}{3}, \quad -\frac{1}{2}$$

Express $g(x)$ as partial fractions, and so obtain a cubic approximation for $g(x)$
State the range of values of x for which the expansion is valid

Question 2

$$h(x) = \frac{14 + 5x}{(1 + x)(2 - x)} \qquad x \neq -1,\ 2$$

Express $h(x)$ as partial fractions, and so obtain a cubic approximation for $h(x)$.
State the range of values of x for which the expansion is valid.

Question 3

C4 Examination Question from June 2006, Q2

$$f(x) = \frac{3x - 1}{(1 - 2x)^2} \qquad |x| < \frac{1}{2}$$

Given that, for $x \neq \frac{1}{2}$

$$\frac{3x - 1}{(1 - 2x)^2} = \frac{A}{(1 - 2x)} + \frac{B}{(1 - 2x)^2}$$

where A and B are constants,

(a) find the values of A and B

[3 marks]

(b) Hence, or otherwise, find the series expansion of $f(x)$ in ascending powers of x up to and including the term in x^3 simplifying each term

[6 marks]

Question 4

C4 Examination Question from January 2009, Q3

$$f(x) = \frac{27x^2 + 32x + 16}{(3x+2)^2(1-x)} \qquad |x| < \frac{2}{3}$$

Given that $f(x)$ can be expressed in the form

$$f(x) = \frac{A}{(3x+2)} + \frac{B}{(3x+2)^2} + \frac{C}{(1-x)}$$

(a) find the values of B and C and show that $A = 0$

[4 marks]

(b) Hence, or otherwise, find the series expansion of $f(x)$ in ascending powers of x up to and including the term in x^2 simplifying each term

[6 marks]

(c) Find the percentage error made in using the series expansion in part **(b)** to estimate the value of $f(0.2)$. Give your answer to 2 significant figures.

[4 marks]

4.4 Answers

4.4.1 Solutions (To 4.2 Worked Example)

$$\frac{x^2 + 3}{[\,(1 - x)\,(1 + x)^2\,]} = \frac{A}{(1 - x)} + \frac{B}{(1 + x)} + \frac{C}{(1 + x)^2}$$

Multiply through by the square bracketed expression...

$$\frac{(x^2 + 3)\,[\,...\,]}{[\,(1 - x)\,(1 + x)^2\,]} = \frac{A\,[\,...\,]}{(1 - x)} + \frac{B\,[\,...\,]}{(1 + x)} + \frac{C\,[\,...\,]}{(1 + x)^2}$$

$$x^2 + 3 = A\,(1 + x)^2 + B\,(1 - x)\,(1 + x) + C\,(1 - x)$$

$$\mathit{Let}\; x = -1 \;:\; 4 = 2C \quad \therefore C = 2$$

$$\mathit{Match\ coefficients\ of}\; x^2 \;:\; A = 1$$

$$\mathit{Let}\; x = 1 \;:\; 4 = 4 + 2B \;\;\therefore B = 0$$

Thus

$$\frac{x^2 + 3}{(1 - x)\,(1 + x)^2} = \frac{1}{(1 - x)} + \frac{2}{(1 + x)^2}$$

The Binomial Theorem now needs to be applied separately to each partial fraction;

$$(1 + x)^n = 1 + nx + \frac{n\,(n - 1)}{2\,!}\,x^2 + \frac{n\,(n - 1)\,(n - 2)}{3\,!}\,x^3 + \;...$$

$$(1 - x)^{-1} = 1 + (-1)\,(-x) + \frac{(-1)\;(-2)}{2}\,(-x)^2 + \frac{(-1)\;(-2)\;(-3)}{6}\,(-x)^3 + ...$$

$$= 1 + x + x^2 + x^3 + ...$$

$$2\,(1 + x)^{-2} = 2\left[\,1 + (-2)\,(x) + \frac{(-2)\;(-3)}{2}\,(x)^2 + \frac{(-2)\;(-3)\;(-4)}{6}\,(x)^3 + ...\right]$$

$$= 2 - 4x + 6x^2 - 8x^3 + ...$$

Thus...

$$\frac{x^2 + 3}{(1 - x)\,(1 + x)^2} = 3 - 3x + 7x^2 - 7x^3 + ...$$

$$\mathit{valid\ for}\;|x| < 1$$

4.4.2 Solutions (To 4.3 Exercise)

Answer 1

$$\frac{1}{[\,(1+3x)\,(1+2x)\,]} = \frac{A}{(1+3x)} + \frac{B}{(1+2x)}$$

Multiply through by the square bracketed expression...

$$\frac{1\,[\,...\,]}{[\,(1+3x)\,(1+2x)\,]} = \frac{A\,[\,...\,]}{(1+3x)} + \frac{B\,[\,...\,]}{(1+2x)}$$

$$1 = A\,(1+2x) + B\,(1+3x)$$

$$Let\ x = -\frac{1}{2} \;:\; 1 = B\left(-\frac{1}{2}\right) \quad \therefore B = -2$$

$$Let\ x = -\frac{1}{3} \;:\; 1 = A\left(\frac{1}{3}\right) \quad \therefore A = 3$$

Thus

$$\frac{1}{(1+3x)\,(1+2x)} = \frac{3}{(1+3x)} - \frac{2}{(1+2x)}$$

The Binomial Theorem now needs to be applied separately to each partial fraction;

$$(1+x)^n = 1 + nx + \frac{n\,(n-1)}{2\,!}x^2 + \frac{n\,(n-1)\,(n-2)}{3\,!}x^3 + \;...$$

$$3\,(1+3x)^{-1} = 3\left[\,1 + (-1)\,(3x) + \frac{(-1)\,(-2)}{2}\,(3x)^2 + \frac{(-1)\,(-2)\,(-3)}{6}\,(3x)^3 + \;...\right]$$

$$= 3 - 9x + 27x^2 - 81x^3 + \;...$$

$$-2\,(1+2x)^{-1} = -2\left[\,1 + (-1)\,(2x) + \frac{(-1)(-2)}{2}\,(2x)^2 + \frac{(-1)(-2)(-3)}{6}\,(2x)^3 + \;...\right]$$

$$= -2 + 4x - 8x^2 + 16x^3 + \;...$$

Thus...

$$\frac{1}{(1+3x)\,(1+2x)} = 1 - 5x + 19x^2 - 65x^3 + \;...$$

$$valid\ for\ \left|x\right| < \frac{1}{3}$$

Answer 2

$$\frac{14 + 5x}{[\,(1+x)\,(2-x)\,]} = \frac{A}{(1+x)} + \frac{B}{(2-x)}$$

Multiply through by the square bracketed expression...

$$\frac{(14 + 5x)\ [\,...\,]}{[\,(1+x)\,(2-x)\,]} = \frac{A\,[\,...\,]}{(1+x)} + \frac{B\,[\,...\,]}{(2-x)}$$

$$14 + 5x = A\,(2-x) + B\,(1+x)$$

$$Let\,x = 2 \ :\ 24 = B\,(3) \quad \therefore B = 8$$

$$Let\,x = -1 \ :\ 9 = A\,(3) \quad \therefore A = 3$$

Thus

$$\frac{1}{(1+x)\,(2-x)} = \frac{3}{(1+x)} + \frac{8}{(2-x)}$$

The Binomial Theorem now needs to be applied separately to each partial fraction;

$$(1+x)^n = 1 + nx + \frac{n\,(n-1)}{2\,!}x^2 + \frac{n\,(n-1)\,(n-2)}{3\,!}x^3 + \ ...$$

$$\begin{aligned} 3(1+x)^{-1} &= 3\left[\,1 + (-1)\ (x) + \frac{(-1)\ (-2)}{2}\ (x)^2 + \frac{(-1)\ (-2)\ (-3)}{6}\ (x)^3 + \ ...\right] \\ &= 3 - 3x + 3x^2 - 3x^3 + \ ... \end{aligned}$$

$$\begin{aligned} +8\ [\,(2-x)\,]^{-1} &= 8\left[\,2\left(1 - \frac{x}{2}\right)\,\right]^{-1} \\ &= 8 \times 2^{-1}\left(1 - \frac{x}{2}\right)^{-1} \\ &= 4\left[1 + (-1)\ \left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2}\left(-\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{6}\left(-\frac{x}{2}\right)^3 + \ ...\right] \\ &= 4 + 2x + x^2 + \frac{1}{2}x^3 + \ ... \end{aligned}$$

Thus...

$$\frac{14 + 5x}{(1+x)\,(2-x)} = 7 - x + 4x^2 - \frac{5}{2}x^3 + \ ...$$

$$valid\,for\ |x| < 1$$

Answer 3

(a)

$$\frac{3x - 1}{[\ (1 - 2x)^2\ ]} = \frac{A}{(1 - 2x)} + \frac{B}{(1 - 2x)^2}$$

Multiply through by the square bracket,

$$3x - 1 = A\ (1 - 2x) + B$$

Let $x = 0.5$ to get $0.5 = B$

Match coefficients of terms in x : $3 = -2A \Rightarrow A = -1.5$

(b) From part (a) we have that,

$$f\,(x) = \frac{(-1.5)}{(1 - 2x)} + \frac{0.5}{(1 - 2x)^2}$$

$$= (-1.5)\ (1 - 2x)^{-1} + 0.5\ (1 - 2x)^{-2}$$

$$= (-1.5)\left(1 + (-1)(-2x) + \frac{(-1)(-2)}{2}(-2x)^2 + \frac{(-1)(-2)(-3)}{3 \times 2}(-2x)^3 + ...\right)$$

$$+ (0.5)\left(1 + (-2)(-2x) + \frac{(-2)(-3)}{2}(-2x)^2 + \frac{(-2)(-3)(-4)}{3 \times 2}(-2x)^3 + ...\right)$$

$$= (-1.5)\left(1 + 2x + 4x^2 + 8x^3 + ...\right) + 0.5\left(1 + 4x + 12x^2 + 32x^3\right)$$

$$= -1.5 - 3x - 6x^2 - 12x^3 + ... + 0.5 + 2x + 6x^2 + 16x^3 + ...$$

$$= -1 - x + 4x^3 + ...$$

Answer 4
(a)

$$\frac{27x^2 + 32x + 16}{[(3x + 2)^2(1 - x)]} = \frac{A}{(3x + 2)} + \frac{B}{(3x + 2)^2} + \frac{C}{(1 - x)}$$

Multiply through by the square bracketed expression...

$$27x^2 + 32x + 16 = A(3x + 2)(1 - x) + B(1 - x) + C(3x + 2)^2$$

Let $x = 1$: $75 = 25C \quad \Rightarrow \quad C = 3$

Let $x = -\frac{2}{3}$:

$$12 - \frac{64}{3} + 16 = B \times \frac{5}{3}$$

$$\frac{20}{3} = B \times \frac{5}{3}$$

$$B = 4$$

Match up coefficients of x^2 :

$$27 = -3A + 9C$$

$$but\ C = 3 \qquad \therefore\ A = 0 \qquad \square$$

(b) From (a) we have that,

$$f(x) = \frac{4}{(3x + 2)^2} + \frac{3}{(1 - x)}$$

$$= 4[3x + 2]^{-2} + 3[1 - x]^{-1}$$

$$= 4\left[2\left(\frac{3}{2}x + 1\right)\right]^{-2} + 3[1 - x]^{-1}$$

$$= \left(1 + \frac{3}{2}x\right)^{-2} + 3[1 - x]^{-1}$$

$$= 1 + (-2)\left(\frac{3x}{2}\right) + \frac{(-2)(-3)}{2}\left(\frac{3x}{2}\right)^2 + \dots$$

$$+ 3\left[1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \dots\right]$$

$$= 1 - 3x + \frac{27}{4}x^2 + \dots + 3 + 3x + 3x^2 + \dots$$

$$= 4 + 0x + \frac{39}{4}x^2 + \dots$$

(c) True value is,

$$f\,(0.2) \;=\; \frac{4}{(\,3 \times 0.2 \,+\, 2\,)^2} \;+\; \frac{3}{(\,1 \,-\, 0.2\,)}$$

$$= \frac{4 \;\times 5^2}{13^2} \;+\; \frac{3}{0.8}$$

$$= \frac{2935}{676}$$

Estimate is,

$$f\,(0.2) \;=\; 4 \;+\; 0x \;+\; \frac{39}{4}\,0.2^{\,2} \;+\; \ldots$$

$$= \frac{439}{100}$$

$$= 4.39$$

$$\%\,error \;=\; \frac{4.39 \;-\; \frac{2935}{676}}{\frac{2935}{676}} \;\times\; 100$$

$$= 1.1120954\,\%$$

$$i.e.\;\; 1.11\,\%$$