

4.5 Answers

A-Level Pure Mathematics : Year 2

Applications of Trigonometry

4.5.1 Solution to 4.2 Example (Teaching Video)

$$2 \cos \left(3x + \frac{\pi}{2} \right) = 1$$

$$\cos \left(3x + \frac{\pi}{2} \right) = \frac{1}{2}$$

$$3x + \frac{\pi}{2} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}, \frac{19\pi}{3}, \dots$$

$$3x + \frac{3\pi}{6} = \frac{2\pi}{6}, \frac{10\pi}{6}, \frac{14\pi}{6}, \frac{22\pi}{6}, \frac{26\pi}{6}, \frac{34\pi}{6}, \frac{38\pi}{6}, \dots$$

$$3x = \dots, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{31\pi}{6}, \frac{35\pi}{6}, \dots$$

$$x = \dots, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}, \dots$$

[5 marks]

4.5.2 Solutions to 4.4 Exercise

Answer 1

$$2 \cos \left(x + \frac{2\pi}{3} \right) = \sqrt{2}$$

$$\cos \left(x + \frac{2\pi}{3} \right) = \frac{\sqrt{2}}{2}$$

$$x + \frac{2\pi}{3} = \dots, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \dots$$

$$x + \frac{8\pi}{12} = \dots, \frac{3\pi}{12}, \frac{21\pi}{12}, \frac{27\pi}{12}, \frac{45\pi}{12}, \dots$$

$$x = \frac{21\pi - 8\pi}{12}, \frac{27\pi - 8\pi}{12}$$

$$= \frac{13\pi}{12}, \frac{19\pi}{12}$$

[5 marks]

Answer 2

$$\sqrt{3} \tan \left(2x + \frac{\pi}{4} \right) = 3$$

$$\tan \left(2x + \frac{\pi}{4} \right) = \frac{3}{\sqrt{3}}$$

$$\tan \left(2x + \frac{\pi}{4} \right) = \sqrt{3}$$

$$2x + \frac{\pi}{4} = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \frac{13\pi}{3}, \dots$$

$$2x + \frac{3\pi}{12} = \frac{4\pi}{12}, \frac{16\pi}{12}, \frac{28\pi}{12}, \frac{40\pi}{12}, \frac{52\pi}{12}, \dots$$

$$2x = \frac{\pi}{12}, \frac{13\pi}{12}, \frac{25\pi}{12}, \frac{37\pi}{12}, \frac{49\pi}{12}, \dots$$

$$x = \frac{\pi}{24}, \frac{13\pi}{24}, \frac{25\pi}{24}, \frac{37\pi}{24}, \frac{49\pi}{24}, \dots$$

[6 marks]**Answer 3****(a)**

$$3 \sin (x + 45^\circ) = 2$$

$$\sin (x + 45) = \frac{2}{3}$$

$$x + 45 = 41.8, 138.2, 401.8, \dots$$

$$x = 93.2^\circ, 356.8^\circ$$

[4 marks]**(b)**

$$2 \sin^2 x + 2 = 7 \cos x$$

$$2 (1 - \cos^2 x) + 2 = 7 \cos x$$

$$2 \cos^2 x + 7 \cos x - 4 = 0$$

$$(2 \cos x - 1) (\cos x + 4) = 0$$

$$\text{Either } \cos x = \frac{1}{2} \text{ or } \cos x = -4$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3} \quad \text{no solutions}$$

(As decimals to 3 decimal places, 1.047, 5.236)

[6 marks]

Answer 4**(i)**

$$\begin{aligned}
 \csc 2x &= \frac{1}{\sin 2x} \\
 &= \frac{1}{2 \sin x \cos x} \\
 &= \frac{1}{2} \times \frac{1}{\sin x} \times \frac{1}{\cos x} \\
 &= \frac{1}{2} \csc x \sec x
 \end{aligned}$$

i.e. The value of λ is $\frac{1}{2}$

[3 marks]**(ii)**

$$\begin{aligned}
 3 \sec^2 \theta + 3 \sec \theta &= 2 \tan^2 \theta \\
 3 \sec^2 \theta + 3 \sec \theta &= 2 (\sec^2 \theta - 1) \\
 \sec^2 \theta + 3 \sec \theta + 2 &= 0 \\
 (\sec \theta + 1)(\sec \theta + 2) &= 0 \\
 \text{Either } \sec \theta &= -1 \text{ or } \sec \theta = -2 \\
 \cos \theta &= -1 \text{ or } \cos \theta = -\frac{1}{2} \\
 \theta &= \pi, \frac{2\pi}{3}, \frac{4\pi}{3}
 \end{aligned}$$

[6 marks]

Answer 5**(i)**

$$4 \csc^2 2\theta - \csc^2 \theta = \frac{4}{\sin^2 2\theta} - \frac{1}{\sin^2 \theta}$$

[2 marks]**(ii)**

$$\begin{aligned} &= \frac{4}{2^2 \sin^2 \theta \cos^2 \theta} - \frac{1}{\sin^2 \theta} \times \frac{\cos^2 \theta}{\cos^2 \theta} \\ &= \frac{1 - \cos^2 \theta}{\sin^2 \theta \cos^2 \theta} \\ &= \frac{\sin^2 \theta}{\sin^2 \theta \cos^2 \theta} \\ &= \frac{1}{\cos^2 \theta} \\ &= \sec^2 \theta \end{aligned}$$

[4 marks]**(iii)**

$$\begin{aligned} 4 \csc^2 2\theta - \csc^2 \theta &= 4 \\ \sec^2 \theta &= 4 \\ \sec \theta &= \pm 2 \\ \cos \theta &= \pm \frac{1}{2} \\ \theta &= \frac{\pi}{3}, \frac{2\pi}{3} \end{aligned}$$

[3 marks]

Answer 6**(i) (a)**

$$2 \tan x - \cot x = 5 \csc x$$

$$5 \csc x + \cot x - 2 \tan x = 0$$

$$\frac{5}{\sin x} + \frac{\cos x}{\sin x} - \frac{2 \sin x}{\cos x} = 0$$

multiply through by $\sin x \cos x$

$$5 \cos x + \cos^2 x - 2 \sin^2 x = 0$$

$$\cos^2 x + 5 \cos x - 2 (1 - \cos^2 x) = 0$$

$$3 \cos^2 x + 5 \cos x - 2 = 0$$

$$\therefore a = 3, b = 5 \text{ and } c = -2$$

[4 marks]**(b)**

$$2 \tan x - \cot x = 5 \csc x$$

$$3 \cos^2 x + 5 \cos x - 2 = 0$$

$$(3 \cos x - 1)(\cos x + 2) = 0$$

$$\text{Either } \cos x = \frac{1}{3} \text{ or } \cos x = -2$$

$$x = 1.231^\circ, 5.052^\circ \text{ or no solutions}$$

$$\therefore x = 1.23^\circ, 5.05^\circ$$

[4 marks]**(ii)**

$$\begin{aligned} \tan \theta + \cot \theta &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \\ &= \frac{\sin \theta}{\cos \theta} \times \frac{\sin \theta}{\sin \theta} + \frac{\cos \theta}{\sin \theta} \times \frac{\cos \theta}{\cos \theta} \\ &= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \\ &= \frac{1}{\sin \theta \cos \theta} \times \frac{2}{2} \\ &= \frac{2}{\sin 2\theta} \\ &= 2 \csc 2\theta \end{aligned}$$

Which is of the required form □

$$\text{i.e. } \lambda = 2$$

[4 marks]

Answer 7**(i)**

$$R \sin(\theta - \alpha) = R \sin \theta \cos \alpha - R \cos \theta \sin \alpha$$

which is required to be $2 \sin \theta - 1.5 \cos \theta$

$$\therefore R \cos \alpha = 2 \quad \text{and} \quad R \sin \alpha = 1.5$$

Solving these two equations simultaneously by division,

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{1.5}{2}$$

$$\alpha = \arctan\left(\frac{1.5}{2}\right)$$

$$\alpha = 0.6435^\circ$$

$$\begin{aligned} R &= \sqrt{2^2 + 1.5^2} \\ &= 2.5 \end{aligned}$$

Conclusion :

$$2 \sin \theta - 1.5 \cos \theta = 2.5 \sin(\theta - 0.6435^\circ)$$

[3 marks]

(b) **(i)** 2.5
 (ii)

$$\sin(\theta - 0.6435) = 0$$

$$\theta - 0.6435 = \frac{\pi}{2}$$

$$\theta = 2.2143^\circ$$

[3 marks]

(c)

$$\begin{aligned} H &= 6 + 2 \sin\left(\frac{4\pi t}{25}\right) - 1.5 \cos\left(\frac{4\pi t}{25}\right) \\ &= 6 + 2.5 \sin\left(\frac{4\pi t}{25} - 0.6435\right) \end{aligned}$$

The maximum value of H is thus, 8.5 and it occurs when,

$$\sin\left(\frac{4\pi t}{25} - 0.6435\right) = 1$$

$$\left(\frac{4\pi t}{25} - 0.6435\right) = \frac{\pi}{2}$$

$$\frac{4\pi t}{25} = 2.2143^\circ$$

$$t = 4.4052 \text{ hours}$$

[3 marks]

(d) The question amounts to solving this equation,

$$6 + 2 \sin \left(\frac{4\pi t}{25} \right) - 1.5 \cos \left(\frac{4\pi t}{25} \right) = 7$$

$$6 + 2.5 \sin \left(\frac{4\pi t}{25} - 0.6435 \right) = 7$$

$$2.5 \sin \left(\frac{4\pi t}{25} - 0.6435 \right) = 1$$

$$\sin \left(\frac{4\pi t}{25} - 0.6435 \right) = 0.4$$

$$\frac{4\pi t}{25} - 0.6435 = 0.4115, 2.7301, 6.6947, 9.0133, \dots$$

$$\frac{4\pi t}{25} = 1.0550, 3.3736, 7.3382, 9.6568, \dots$$

$$t = 2.0989, 6.7116, 14.5988, 19.2116, \dots$$

$$t = 2.06 \text{ pm}, 6.43 \text{ pm}$$

[6 marks]

Answer 8**(a)**

$$\begin{aligned}
\text{LHS} &= \sec 2A + \tan 2A \\
&= \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A} \\
&= \frac{1 + \sin 2A}{\cos 2A} \\
&= \frac{1 + 2 \sin A \cos A}{\cos^2 A - \sin^2 A} \\
&= \frac{\cos^2 A + \sin^2 A + 2 \sin A \cos A}{(\cos A - \sin A)(\cos A + \sin A)} \\
&= \frac{(\cos A + \sin A)^2}{(\cos A - \sin A)(\cos A + \sin A)} \\
&= \frac{\cos A + \sin A}{\cos A - \sin A} \\
&= \text{RHS } \square
\end{aligned}$$

[5 marks]**(b)**

$$\begin{aligned}
\sec 2A + \tan 2A &= \frac{1}{2} \\
\frac{\cos A + \sin A}{\cos A - \sin A} &= \frac{1}{2} \\
2 \cos A + 2 \sin A &= \cos A - \sin A \\
3 \sin A &= -\cos A \\
\frac{\sin A}{\cos A} &= -\frac{1}{3} \\
\tan A &= -\frac{1}{3} \\
A &= 2.8198, 5.9614 \\
\therefore A &= 2.820, 5.961
\end{aligned}$$

[4 marks]