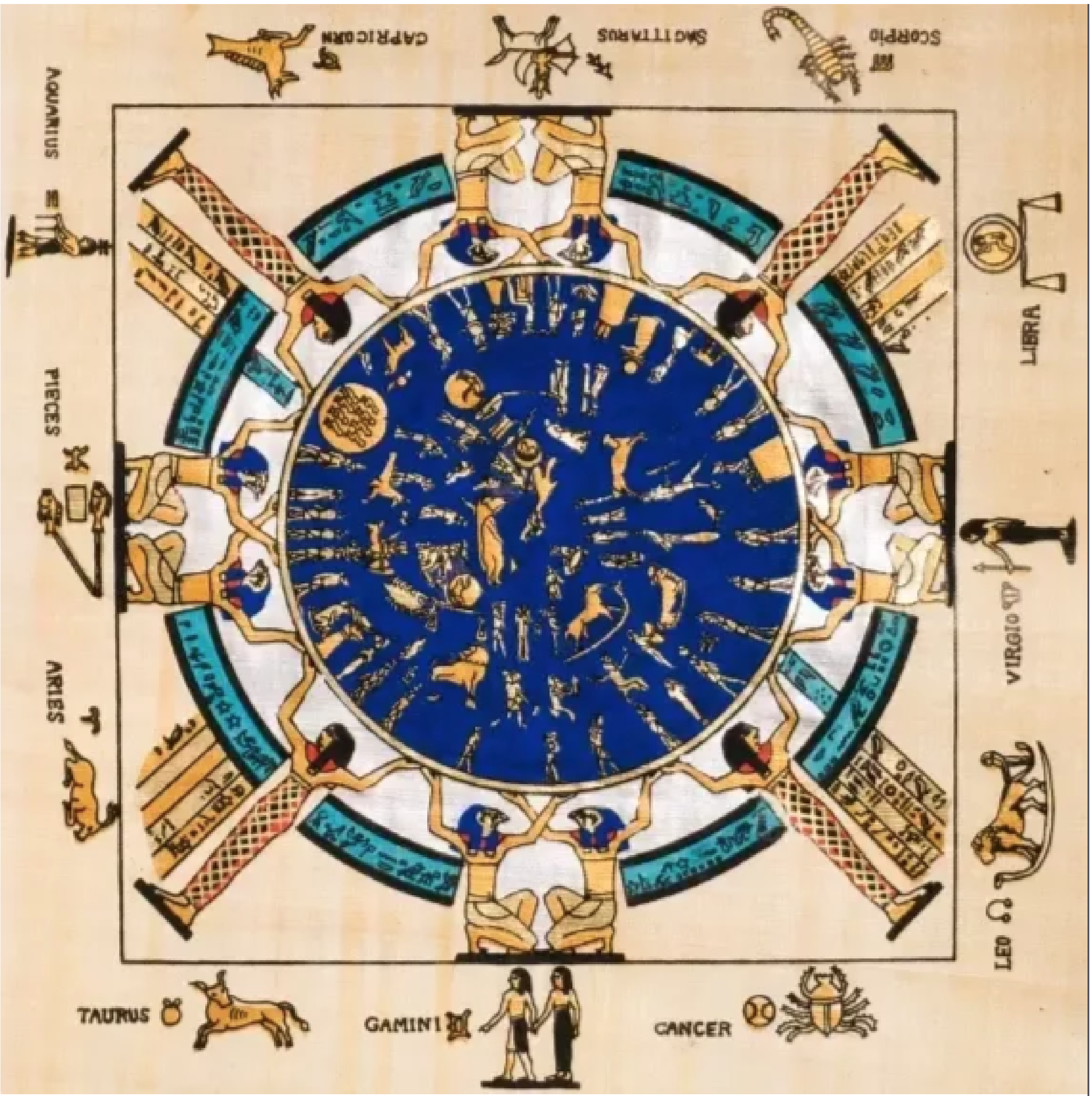
A-Level Pure Mathematics
Year 2

Applications of Trigonometry



On ancient Egyptian calendars, there were 360 days in the year.
Five days of festivities followed the end of each calendar year before the start of the next.
Possibly, this is the origin of there being 360° in a circle

1.1 Measuring Angles In Degrees

Ancient Persian (Iranian) and Egyptian calendars had 12 months of 30 days in a year. It's speculated that this is how our division of a circle into 360 degrees arose, with each day allocated 1 degree. The Persians and Egyptians appreciated the mathematical properties of 360, particularly its abundance of factors.

The 24 Factors of 360 :
1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360

Every six years, the Persians added a 13^{th} month to keep their calendar synchronized with the seasons, whereas the Egyptians had five additional days of festivities between the end of one year and the start of the next.

1.2 Measuring Angles In Radians

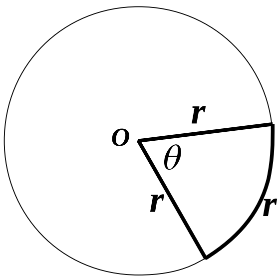
From the mathematician's point of view, it turns out that to divide a circle into 360 parts is not the most natural. Instead, it makes more mathematical sense to break it into 2π parts. These mathematical parts are called radians.

$$one\ radian \quad = \quad 57.29577951...\ degrees$$

Many mathematical properties, formulae and theorems are far more elegantly presented when the angles involved are expressed in radians, rather than degrees. Some of these will be encountered soon.

1.3 The Definition of a Radian

One radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.



Angle θ is 1 radian

1.4 Two Elegant Formulae

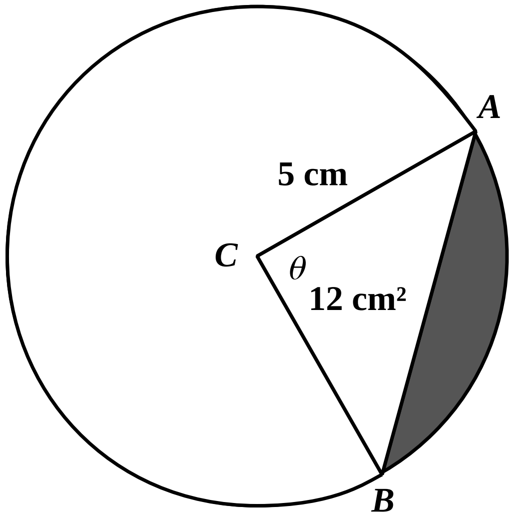
Arc Length and Sector Area

$$Arc\ length \quad = \quad r\,\theta^c$$

$$Sector\ area \quad = \quad \frac{1}{2}\,r^2\,\theta^c$$

1.5 Example

The circle shown, centre C, has radius 5 cm



The area of $\triangle ABC$ is 12 cm^2. Find, to three decimal places,

(i) the angle marked θ, in radians,

(ii) the area of the segment shaded,

(iii) the perimeter of the shaded segment.

Teaching Video : http://www.NumberWonder.co.uk/v9057/1a.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9057/1b.mp4 (Part 2)

<== Part 1 Part 2 ==>

After watching the teaching video write out a solution to the question.

☞

[7 marks]

1.6 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

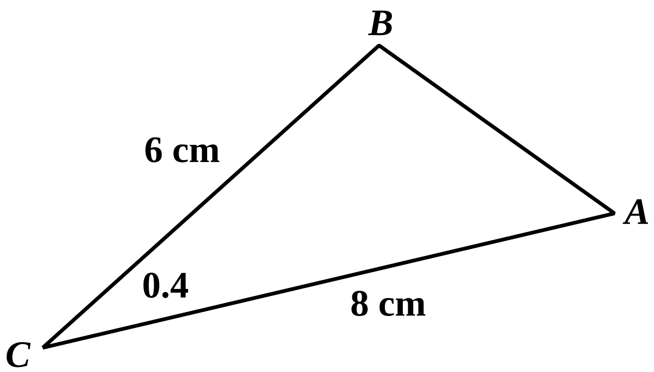
Put your calculator into RADIAN mode

Find, accurate to four decimal places, the value of,

(i) $\sin 1.5^c$ **(ii)** $\cos 0.8^c$

[2 marks]

Question 2



(i) Determine the area of $\triangle ABC$, to 2 decimal places, by using the formula,

$$Area\ \triangle \ = \ \frac{1}{2}\, a\, b \sin C^c$$

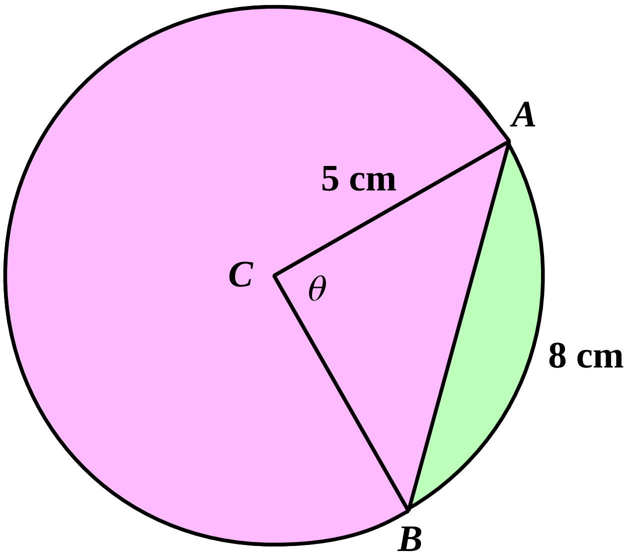
[2 marks]

(ii) Find the length of AB, to 2 decimal places, by using the formula,

$$c^2 = a^2 + b^2 - 2\, a\, b \cos C^c$$

[2 marks]

Question 3



(i) Find, in radians, the angle marked θ

[2 marks]

(ii) Find the area of the sector *ABC*

[2 marks]

(iii) Find, to 2 decimal places, the area of ΔABC

[2 marks]

(iv) Find the area of the minor segment, shaded green

[2 mark]

(v) Find the area of the major segment, shaded purple

[2 marks]

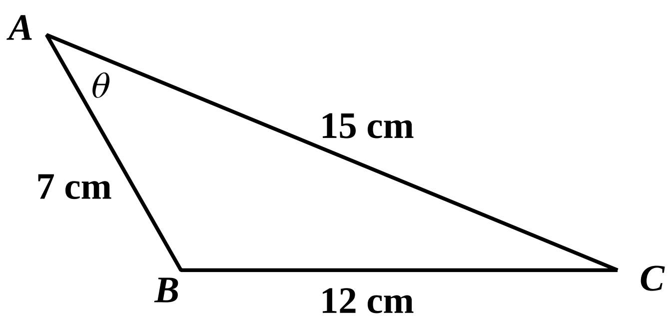
Question 4
Put your calculator into RADIAN mode.
Work out the acute angle, in radians and correct to three significant figures, of,
(i) $arccos\ 0.9$ **(ii)** $arcsin\ 0.92$

[2 marks]

Question 5
Convert the following angles from degrees into radians.
Write each answer in terms of π
(i) $30°$ **(ii)** $45°$ **(iii)** $90°$

[3 marks]

Question 6



(i) Determine, in radians, the angle marked θ by using the formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c}$$

[3 marks]

(ii) Find the area of $\triangle ABC$ by using the formula,

$$Area\ \triangle = \frac{1}{2}\, b\, c \sin A^c$$

[2 marks]

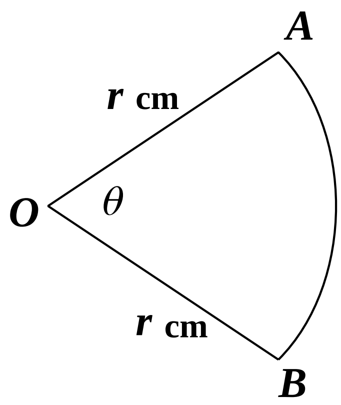
Question 7

A-Level Examination Question from June 2018, Q3

The diagram shows a sector AOB of a circle with centre O and radius r cm

The angle AOB is θ radians

The area of the sector AOB is 11 cm^2



Given that the perimeter of the sector is 4 times the length of the arc AB, find the exact value of r

[4 marks]

1.7 Answers

1.7.1 Solution to 1.5 Example (Teaching Video)

(i)

$$Area\ \Delta = \frac{1}{2}\, a\, b \sin C$$

$$12 = \frac{1}{2} \times 5 \times 5 \times \sin\theta$$

$$\sin\theta = \frac{12 \times 2}{5 \times 5}$$

$$= 0.9600$$

$$\theta = \arcsin 0.9600$$

$$= 1.287^{c}$$

(ii) Area Segment Shaded = Area Sector *ABC* – Area ΔABC

$$= \frac{1}{2} r^2 \theta - 12$$

$$= \frac{1}{2} \times 5^2 \times 1.287 - 12$$

$$= 4.088 \text{ cm}^2$$

(iii) Perimeter Segment Shaded = Arc Length *AB* + Length *AB*

To get Length *AB* use The Cosine Rule,

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$= 5^2 + 5^2 - 2 \times 5 \times 5 \times \cos 1.287$$

$$= 25 + 25 - 14.000$$

$$= 36$$

$$\therefore\ c = 6 \text{ cm}$$

$$Arc\ Length = r\theta$$

$$= 5 \times 1.287$$

$$= 6.435 \text{ cm}$$

$$Perimeter = 12.435 \text{ cm}$$

[7 marks]

1.7.2 Solutions to 1.6 Exercise

Answer 1

(i) 0.9975 **(ii)** 0.6967

[2 marks]

Answer 2

(i)
$$Area\ \Delta = \frac{1}{2}\, a\, b \sin C^c$$
$$= \frac{1}{2} \times 6 \times 8 \times \sin 0.4$$
$$= 9.35 \text{ cm}^2$$

[2 marks]

(ii)
$$c^2 = a^2 + b^2 - 2\, a\, b \cos C$$
$$= 6^2 + 8^2 - 2 \times 6 \times 8 \times \cos 0.4$$
$$= 36 + 64 - 88.422$$
$$= 11.578$$
$$c = 3.40 \text{ cm}$$

[2 marks]

Answer 3

(i)
$$Arc\ Length = r\,\theta$$
$$8 = 5\theta$$
$$\theta = 1.6^c$$

[2 marks]

(ii)
$$Area\ Sector = \frac{1}{2} r^2 \theta$$
$$= \frac{1}{2} \times 5^2 \times 1.6$$
$$= 20 \text{ cm}^2$$

[2 marks]

(iii)
$$Area\ \Delta = \frac{1}{2}\, a\, b \sin C$$
$$= \frac{1}{2} \times 5 \times 5 \times \sin 1.6$$
$$= 12.49 \text{ cm}^2$$

[2 marks]

(iv) $Area\ Minor\ Segment = 20 - 12.49 = 7.51 \text{ cm}^2$

[2 marks]

(v) $Area\ Major\ Segment = Area\ \text{O} - 7.51 = 5^2\pi - 7.51 = 71.03 \text{ cm}^2$

[2 marks]

Answer 4

(i) 0.451^c **(ii)** 1.168^c

[2 marks]

Answer 5

(i) $\frac{\pi}{6}$ **(ii)** $\frac{\pi}{4}$ **(iii)** $\frac{\pi}{2}$

[3 marks]

Answer 6

(i)

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2\,b\,c} \\ &= \frac{15^2 + 7^2 - 12^2}{2 \times 15 \times 7} \\ &= \frac{130}{210} \\ &= 0.6190 \\ A &= \arccos 0.6190 \\ &= 0.9033^c \end{aligned}$$

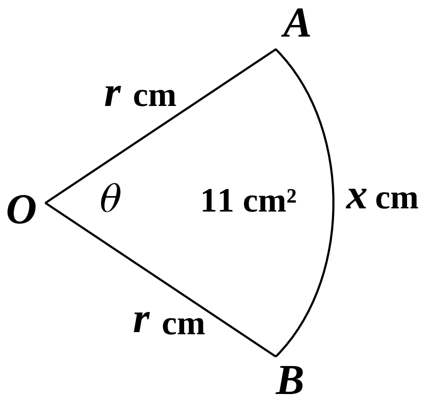
[3 marks]

(ii)

$$\begin{aligned} Area\,\triangle &= \frac{1}{2}\, b\, c \sin A^c \\ &= \frac{1}{2} \times 15 \times 7 \times \sin 0.9033 \\ &= 41.2 \text{ cm}^2 \end{aligned}$$

[2 marks]

Answer 7



$$x + 2r = 4x \Rightarrow r = 1.5x$$

$$Arc\ Length,\ x = r\theta \Rightarrow r = \frac{x}{\theta}$$

Combining these two results, $1.5x = \dfrac{x}{\theta} \Rightarrow \theta = \dfrac{2}{3}$

$$Area\ Sector,\ 11 = \frac{1}{2}r^2 \times \frac{2}{3} \Rightarrow r = \sqrt{33}\ \text{cm}$$

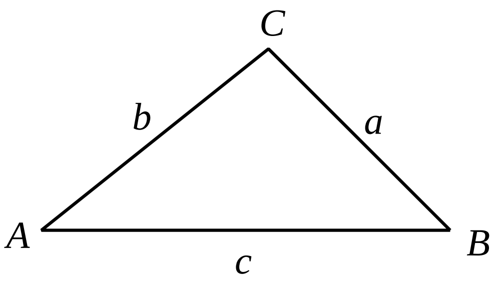
[4 marks]

Lesson 2

2.1 Arc Length and Sector Area Problems

All of our Non-Right Angled Triangle Formulae work equally well whether the units of angle measurement are Radians or Degrees.

Non-Right Angled Triangle Trigonometric Formulae



The Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Known : *One side length & two (and $\therefore$ three) angles*
Seeking : *Any side length*

The Upside Down Sine Rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Known : *Two side lengths & an excluded angle*
Seeking : *The other excluded angle*

The Cosine Rule

$$a^2 = b^2 + c^2 - 2\,b\,c\cos A$$

Known : *Two side lengths & the included angle*
Seeking : *The third side length*

The Reversed Cosine Rule

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c}$$

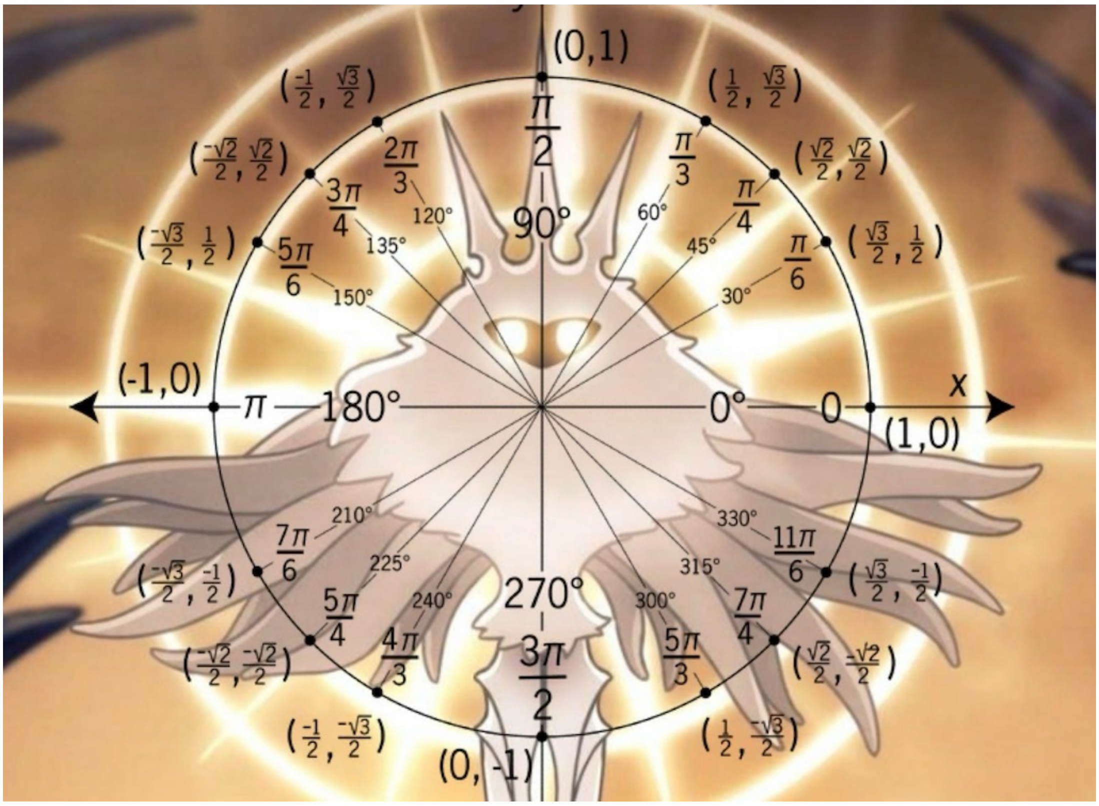
Known : *Three side lengths*
Seeking : *Any angle*

Useful Area of a Triangle Formula

$$Area\,\triangle = \frac{1}{2}\,a\,b\sin C$$

2.2 Radian, Degrees Conversion Diagram

The following diagram may be of use in getting a feel for the relationship between radians and degrees. Mathematically the key fact to memorise is that $\pi^c = 180°$



2.3 Video Support

Most of the questions in Exercise 2.4 come with a video solution from “Exam Solutions”. It's time consuming to watch lots of video's and you are not expected to do so. After trying a question, should you need help, then watch the video. From experience, students find Question 5(b) tricky.

2.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 45

Here is a reminder of the formulae for Arc Length and Sector Area.
Angles MUST be in radians.

Arc Length and Sector Area

$$Arc\ length \ = \ r\,\theta^c$$

$$Sector\ area \ = \ \frac{1}{2} r^2 \theta^c$$

Question 1

A-Level Examination Question from the C2 Paper, January 2011, Q2

In the triangle ABC, $AB = 11$ cm, $BC = 7$ cm and $CA = 8$ cm

(a) Find the size of angle C, giving your answer in radians to 3 significant figures

[3 marks]

(b) Find the area of triangle ABC

Give your answer in cm^2 to 3 significant figures

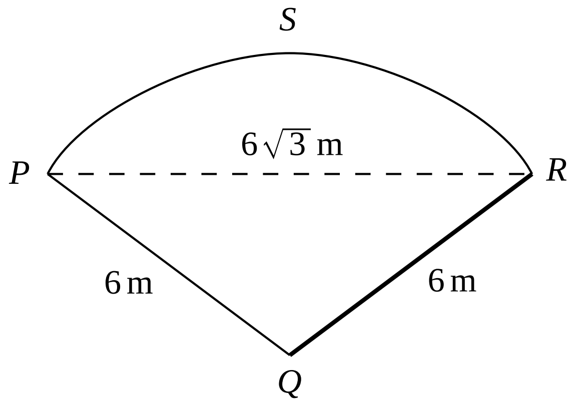
[3 marks]

Need help with Question 1 ?

http://www.NumberWonder.co.uk/v9057/2a.mp4

Question 2

A-Level Examination Question from the C2 Paper, January 2007, Q9



Shown is a plan of a patio.

The patio *PQRS* is in the shape of a sector of a circle with centre *Q* and radius 6 m.

Given that the length of the straight line *PR* is $6\sqrt{3}$ m,

(a) find the exact size of angle *PQR* in radians

[3 marks]

(b) Show that the area of the patio *PQRS* is 12π m^2

[2 marks]

(c) Find the exact area of the triangle *PQR*

[2 marks]

(d) Find, in m^2 to 1 decimal place, the area of the segment *PRS*

[2 marks]

(e) Find, in m to 1 decimal place, the perimeter of the patio *PQRS*

[2 marks]

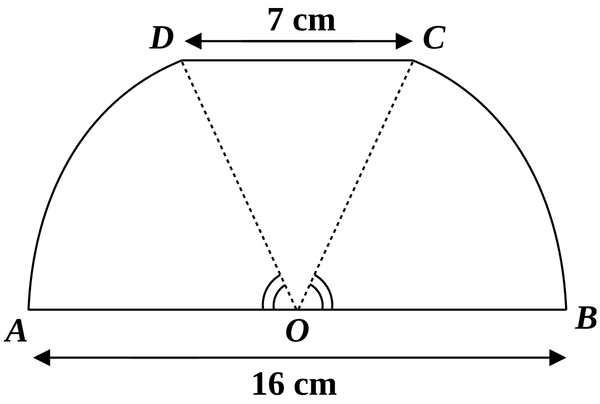
Need help with Question 2 ?
http://www.NumberWonder.co.uk/v9057/2b.mp4 (Part (a))
http://www.NumberWonder.co.uk/v9057/2c.mp4 (Part (b))
http://www.NumberWonder.co.uk/v9057/2d.mp4 (Part (c))
http://www.NumberWonder.co.uk/v9057/2e.mp4 (Part (d))
http://www.NumberWonder.co.uk/v9057/2f.mp4 (Part (e))

Part (a) Part (b) Part (c)

Part (d) Part (e)

Question 3

A-Level Examination Question from the C2 Paper, May 2015, Q4



Shown is a sketch design for a scraper blade.

The blade *AOBCDA* consists of an isosceles triangle *COD* joined along its equal sides to sectors *OBC* and *ODA* of a circle with centre *O* and radius 8 cm.

Angles *AOD* and *BOC* are equal.

AOB is a straight line and is parallel to the line *DC*.

DC has length 7 cm.

(a) Show that the angle *COD* is 0.906 radians, correct to 3 significant figures.

[2 marks]

(b) Find the perimeter of *AOBCDA*, giving your answer to 3 significant figures.

[3 marks]

(c) Find the area of *AOBCDA*, giving your answer to 3 significant figures.

[3 marks]

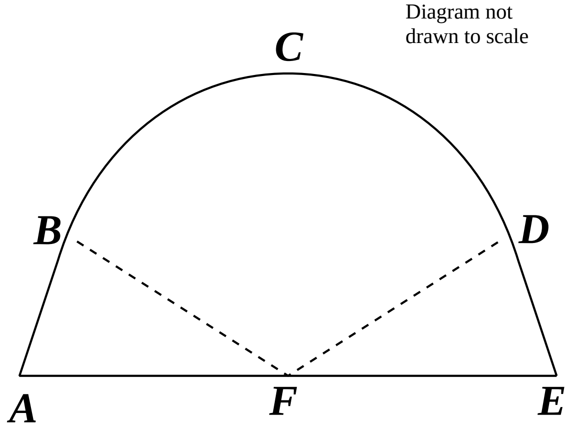
Need help with Question 3 ?
http://www.NumberWonder.co.uk/v9057/2g.mp4 (Part (a))
http://www.NumberWonder.co.uk/v9057/2h.mp4 (Part (b))
http://www.NumberWonder.co.uk/v9057/2i.mp4 (Part (c))

Part (a) Part (b) Part (c)

Question 4

A-Level Examination Question from the C2 Paper, June 2017, Q4

Diagram not drawn to scale



The sketch represents the cross-section of a large tent, *ABCDEF*

AB and *DE* are line segments of equal length

Angle *FAB* and angle *DEF* are equal

F is the midpoint of the straight line *AE* and *FC* is perpendicular to *AE*

BCD is an arc of a circle of radius 3.5 m with centre at *F*

It is given that

$$AF = FE = 3.7 \text{ m}$$

$$BF = FD = 3.5 \text{ m}$$

$$\text{angle } BFD = 1.77 \text{ radians}$$

Find

(a) the length of the arc *BCD* in metres to 2 decimal places

[2 marks]

(b) the area of the sector *FBCD* in m^2 to 2 decimal places

[2 marks]

(c) the total area of the cross-section on the tent in m^2 to 2 decimal places

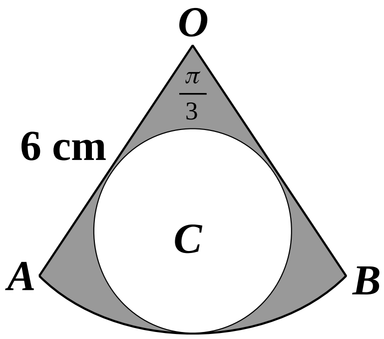
[4 marks]

Need help with Question 4 ?
http://www.NumberWonder.co.uk/v9057/2j.mp4 (All Parts)

All Parts

Question 5

A-Level Examination Question from the C2 Paper, May 2011, Q5



The shape shown is a pattern for a pendant

It consists of a sector OAB of a circle centre O, of radius 6 cm, and angle $AOB = \dfrac{\pi}{3}$

The circle C, inside the sector, touches the two straight edges, OA and OB and the arc AB as shown.

Find

(a) the area of the sector OAB

[2 marks]

(b) the radius of the circle C

[3 marks]

The region outside the circle *C* and inside the sector *OAB* is shown shaded.

(c) Find the area of the shaded region

[2 marks]

Need help with Question 5 ?

http://www.NumberWonder.co.uk/v9057/2k.mp4 (Part (a))

http://www.NumberWonder.co.uk/v9057/2l.mp4 (Part (b))

http://www.NumberWonder.co.uk/v9057/2m.mp4 (Part (c))

Part (a) Part (b) Part (c)

Question 6

(i) Sketch a circle of radius r, with a chord PQ added.
Within the sector POQ, of angle θ, shade the sector between chord PQ and arc PQ and label its area, A

[2 marks]

(ii) By considering the formula for the area of a sector and the area of a triangle show that

$$A = \frac{1}{2} r^2 (\theta - \sin\theta)$$

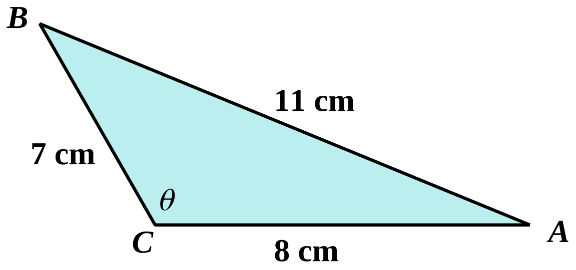
[3 marks]

2.5 Answers

2.5.1 Solutions to 2.4 Exercise

Answer 1

(a) Sketch of the triangle,



$$\cos C = \frac{a^2 + b^2 - c^2}{2\,a\,b}$$

$$= \frac{7^2 + 8^2 - 11^2}{2 \times 7 \times 8}$$

$$= -\frac{1}{14}$$

$$C = 1.64^c$$

[3 marks]

(b)

$$Area\,\triangle = \frac{1}{2}\,ab\sin C$$

$$= \frac{1}{2} \times 7 \times 8 \times \sin 1.6423$$

$$= 27.9\text{ cm}^2$$

[3 marks]

Answer 2

(a)

$$\cos Q = \frac{p^2 + r^2 - q^2}{2pr}$$

$$= \frac{6^2 + 6^2 - (6\sqrt{3})^2}{2 \times 6 \times 6}$$

$$= -\frac{1}{2}$$

$$C = \frac{2\pi}{3} \text{ radians *exact*}$$

[3 marks]

(b)

$$Area\ Sector = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2} \times 6^2 \times \frac{2\pi}{3}$$

$$= 12\pi \text{ m}^2$$

[2 marks]

(c)

$$Area\ \Delta = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} \times 6 \times 6 \times \sin\left(\frac{2\pi}{3}\right)$$

$$= 18 \times \frac{\sqrt{3}}{2}$$

$$= 9\sqrt{3} \text{ m}^2$$

[2 marks]

(d)

$$Area\ Segment = Area\ Sector - Area\ \Delta$$

$$= 12\pi - 9\sqrt{3}$$

$$= 22.1 \text{ m}^2$$

[2 marks]

(e)

$$Perimeter\ Sector = 6 + 6 + Arc\ Length$$

$$= 12 + r\theta$$

$$= 12 + 6 \times \frac{2\pi}{3}$$

$$= 24.6 \text{ m}$$

[2 marks]

Answer 3

(a) In $\triangle COD$,

$$\cos O = \frac{d^2 + c^2 - o^2}{2 \times d \times c}$$

$$= \frac{8^2 + 8^2 - 7^2}{2 \times 8 \times 8}$$

$$= \frac{79}{128}$$

$O = 0.906$ radians to 3 significant figures ☐

[2 marks]

(b)

$$\angle COB = \angle DOA = x$$

$$\therefore\ 2x + 0.906 = \pi \text{ as } AOB \text{ is a straight line}$$

$$x = 1.118 \text{ radians}$$

$$Arc\ Length\ BC = Arc\ Length\ AD = r\theta$$

$$8 \times 1.118 = 8.942 \text{ m}$$

$$Total\ Perimeter = 16 + 8.942 + 7 + 8.942$$

$$= 40.9 \text{ cm}$$

[3 marks]

(c)

$$Area\ Sector\ COB = Area\ Sector\ DOA = \frac{1}{2} r^2 \theta$$

$$\frac{1}{2} \times 8^2 \times 1.118 = 35.776 \text{ m}^2$$

$$Area\ \Delta = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} \times 8 \times 8 \times \sin 0.906$$

$$= 25.185 \text{ m}^2$$

$$Total\ Area = 35.776 + 25.185 + 35.776$$

$$= 96.7 \text{ m}^2$$

[3 marks]

Answer 4

(a)

$$Arc\ Length = r\,\theta$$
$$= 3.5 \times 1.77$$
$$= 6.20 \text{ m}$$

[2 marks]

(b)

$$Area\ Sector = \frac{1}{2} r^2\,\theta$$
$$= \frac{1}{2} \times 3.5^2 \times 1.77$$
$$= 10.84 \text{ m}^2$$

[2 marks]

(c)

$$\angle AFB = \angle EFD = x$$
$$2x + 1.77 = \pi \quad \text{as } AE \text{ is a straight line}$$
$$x = 0.6858$$
$$Area\ \triangle AFB = Area\ \triangle EFD = \frac{1}{2}\, ab \sin C$$
$$\frac{1}{2} \times 3.7 \times 3.5 \times \sin 0.6858 = 4.101 \text{ m}^2$$
$$Total\ Area = 4.101 + 10.84 + 4.101$$
$$= 19.04 \text{ m}^2$$

[4 marks]

Answer 5

(a)

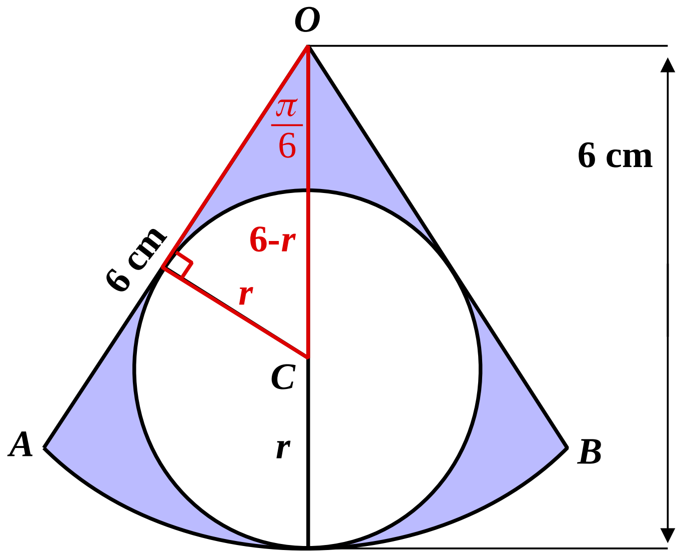
$$Area\ Sector = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2} \times 6^2 \times \frac{\pi}{3}$$

$$= 6\pi \text{ m}^2 \qquad (18.85 \text{ m}^2)$$

[2 marks]

(b)



$$sin\left(\frac{\pi}{6}\right) = \frac{r}{6 - r}$$

$$\frac{1}{2} = \frac{r}{6 - r}$$

$$6 - r = 2r$$

$$3r = 6$$

$$r = 2 \text{ cm}$$

[3 marks]

(c)

$$Area\ Shaded = Area\ Sector - Area\ Circle$$

$$= \frac{1}{2} r_s^{\ 2} \theta - \pi r_c^{\ 2}$$

$$= 6\pi - \pi \times 2^2$$

$$= 6\pi - 4\pi$$

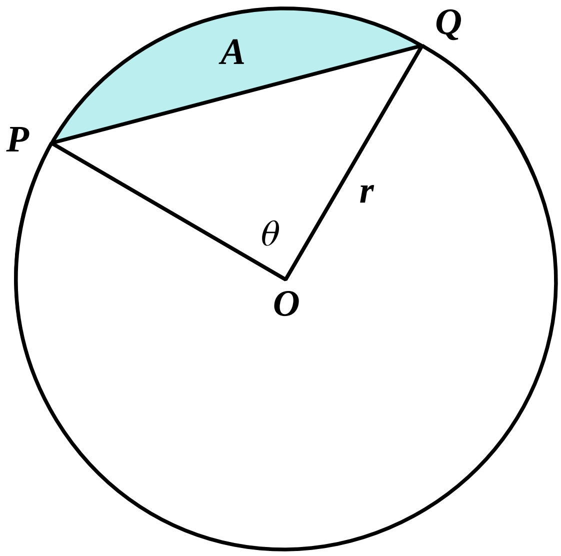
$$= 2\pi$$

$$= 6.28 \text{ m}^2$$

[2 marks]

Answer 6

(i)



[2 marks]

(ii)

$$Area\ A = Area\ Sector - Area\ \triangle$$
$$= \frac{1}{2} r^2 \theta - \frac{1}{2} \times r \times r \times \sin \theta$$
$$= \frac{1}{2} r^2 (\theta - \sin \theta)$$

[3 marks]

3.1 Conversions Between Radians and Degrees

The key relationship between radians and degrees is the fact that

$$\pi^c = 180°$$

3.1.1 Radians $\rightarrow$ Degrees

If both sides of the key relationship are divided by π we get that;

$$1^c = \left(\frac{180}{\pi}\right)^o \qquad \Rightarrow \qquad D = \frac{R \times 180}{\pi}$$

Thus, to convert radians $\rightarrow$ degrees, multiply by 180 and divide by π

3.1.2 Degrees $\rightarrow$ Radians

If both sides of the key relationship are divided by 180 we get that;

$$\left(\frac{\pi}{180}\right)^c = 1° \qquad \Rightarrow \qquad R = \frac{D \times \pi}{180}$$

Thus, to convert degrees $\rightarrow$ radians, multiply by π and divide by 180

3.1.3 Example

Convert these small angles, given in radians, into degrees to one decimal place

θ radians	0.1	0.2	0.3	0.4	0.5
θ degrees					

[5 marks]

3.2 An Investigation

Complete the following table, where θ is measured in radians.
Work to 3 decimal places

θ^c	0.1	0.2	0.3	0.4	0.5
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$1 - \frac{\theta^2}{2}$					

[5 marks]

3.3 Small Angle Approximations

The table suggests that when θ is small and measured in radians

- $\sin\theta \approx \theta$
- $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$
- $\tan\theta \approx \theta$

By 'small' is meant either “relatively close to zero” or “much less that whatever else is around”. The table suggests that when the angle has a size of more than half a radian (about 30 degrees) the approximation is becoming too inaccurate to be of practical use.

The angles at which the relative error exceeds 1% are;

0.244^c or 14° for the $\sin\theta$ approximation

0.664^c or 38° for the $\cos\theta$ approximation

and 0.176^c or 10° for the $\tan\theta$ approximation

Although these small angle approximations are useful in some situations, they are not a substitute for trigonometric manipulations that provide exact solutions to problems regardless of the size of the angle.

3.4 Example

(i) When θ is small and measured in radians, show that

$$\frac{\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta}{\theta} \approx 10$$

(ii) When $\theta = 0.1^c$ (about 6°) what is the percentage error introduced by using the small angle approximation for $\sin\theta$?

Teaching Video : http://www.NumberWonder.co.uk/v9057/3.mp4

What the video.
Write out the solution.

☞

[6 marks]

3.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

(i) When θ is small and measured in radians, show that

$$\frac{4\sin 3\theta - \tan 2\theta}{2\theta} \approx 5$$

[2 marks]

(ii) When $\theta = 0.1^c$ (about 6°) what is the percentage error introduced by using the small angle approximations ?

[3 marks]

Question 2

(i) When θ is small and measured in radians, show that

$$\frac{\cos 4\theta - 1}{\theta \sin 2\theta} \approx -4$$

[3 marks]

(ii) When $\theta = 0.1^c$ (about 6°) what is the percentage error introduced by using the small angle approximations ?

[3 marks]

Question 3

A-Level Paper 1 Examination Question from June 2018, Q1

Given that θ is small and measured in radians, use the small angle approximations to find an approximate value of

$$\frac{1 - \cos 4\theta}{2\theta \sin 3\theta}$$

[3 marks]

Question 4

(i) Show that, when θ is small and measured in radians,

$$\cos 2\theta - \sin^2 3\theta \approx 1 - 11\,\theta^2$$

[2 marks]

(ii) Explain why, if θ is small in relation to the number 1, the term involving θ^2 can be ignored.

[1 mark]

(iii) Hence determine the percentage error in using the approximate value of $\cos 2\theta - \sin^2 3\theta$ when $\theta = 0.1^c$

[2 marks]

Question 5

(i) Show that, when θ is small and measured in radians,

$$\frac{4\,cos\,3\theta \;-\; 2 \;+\; 5\,sin\,\theta}{1 \;-\; sin\,2\theta}$$

can be written as

$$9\theta \;+\; 2$$

[3 marks]

(ii) When $\theta = 0.2^{c}$ (about 11°) what is the percentage error introduced by using the small angle approximations ?

[2 marks]

Question 6

(i) Use the small angle approximations to derive an expression for

$$cos\,\theta \;+\; tan^2\,\theta$$

when θ is small and measured in radians.

[2 marks]

(ii) Hence explain why the small angle approximations give that

$$cos\,\theta \;+\; tan^2\,\theta \;\approx\; 1$$

when θ is small and measured in radians

[1 mark]

Question 7

(i) Use the small angle approximations to solve the equation

$$5 \sin 2\theta + \tan 3\theta = 0.39$$

[2 marks]

(ii) In view of your part **(i)** answer, was the use of the small angle approximations valid ?
Explain your answer.

[2 marks]

Question 8

(i) Show that, when θ is small and measured in radians,

$$\sin 5\theta - \cos 2\theta + \tan 2\theta$$

can be approximated by an equation of the form

$$a\,\theta^2 + b\,\theta + c$$

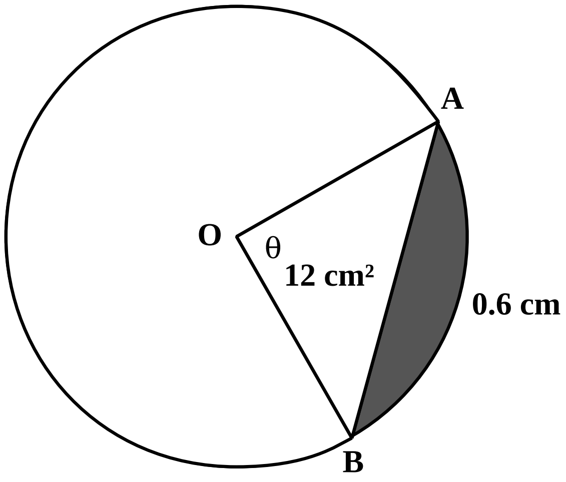
where *a, b* and *c* are integer constants to be found

[3 marks]

(ii) Jason claims that the approximation will have a value of *c* when θ is small because the terms in θ^2 and θ can be ignored.
By NOT ignoring these terms show that Jason is wrong when $\theta = 0.1^c$

[1 mark]

Question 9



(i) Determine the size of angle θ, making use of the small angle approximations where appropriate.

[4 marks]

(ii) Justify the use of the small angle approximations in deriving your part (i) answer

[1 marks]

3.6 Answers

3.6.1 Solutions to 3.1.3 Introductory Example

θ radians	0.1	0.2	0.3	0.4	0.5
θ degrees	5.7	11.5	17.2	22.9	28.6

[5 marks]

3.6.2 Solutions to 3.2 Investigation

θ^c	0.1	0.2	0.3	0.4	0.5
$\sin\theta$	0.100	0.199	0.296	0.389	0.479
$\cos\theta$	0.995	0.980	0.955	0.921	0.878
$\tan\theta$	0.100	0.203	0.309	0.423	0.546
$1 - \frac{\theta^2}{2}$	0.995	0.980	0.955	0.920	0.875

[5 marks]

3.6.3 Solutions to 3.4 Example

(i)

$$\begin{aligned} \text{LHS} &= \frac{\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta}{\theta} \\ &\approx \frac{\theta + 2\theta + 3\theta + 4\theta}{\theta} \\ &= \frac{10\theta}{\theta} \\ &= 10 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

(ii) when $\theta = 0.1$,

$$\frac{\sin 0.1 + \sin 0.2 + \sin 0.3 + \sin 0.4}{0.1} = 9.834$$

$$\%age\ error = \frac{10 - 9.834}{9.834} \times 100 = 1.7\,\%$$

[3 marks]

3.6.4 Solutions to 3.5 Exercise

Answer 1

(i)

$$\begin{aligned} \text{LHS} &= \frac{4\ sin\,3\theta\ -\ tan\,2\theta}{2\theta} \\ &\approx \frac{4\ \times\ 3\theta\ -\ 2\theta}{2\theta} \\ &= \frac{10\theta}{2\theta} \\ &= 5 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[2 marks]

(ii) when $\theta = 0.1$,

$$\frac{4\,sin\,0.3\ -\ tan\,0.2}{0.2}\ =\ 4.899$$

$$\%age\ error\ =\ \frac{5\ -\ 4.899}{4.899}\ \times\ 100\ =\ 2.1\ \%$$

[3 marks]

Answer 2

(i)

$$\begin{aligned} \text{LHS} &= \frac{cos\,4\theta\ -\ 1}{\theta\ sin\,2\theta} \\ &\approx \frac{\left(1\ -\ \frac{(4\theta)^2}{2}\right)\ -\ 1}{2\ \theta^2} \\ &= \frac{1\ -\ 8\theta^2\ -\ 1}{2\theta^2} \\ &= -\frac{8\ \theta^2}{2\ \theta^2} \\ &= -4 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

(ii) when $\theta = 0.1$,

$$\frac{cos\,0.4\ -\ 1}{0.1\ \times\ sin\,0.2}\ =\ -3.973$$

$$\%age\ error\ =\ \frac{-\ 4\ +\ 3.973}{-\ 3.973}\ \times\ 100\ =\ 0.68\ \%$$

[3 marks]

Answer 3

$$\frac{1 - \cos 4\theta}{2\theta \sin 3\theta} \approx \frac{1 - \left(1 - \frac{(4\theta)^2}{2}\right)}{2\theta \times 3\theta}$$

$$= \frac{8\,\theta^2}{6\,\theta^2}$$

$$= \frac{4}{3}$$

[3 marks]

Answer 4

(i)

$$\text{LHS} = \cos 2\theta - \sin^2 3\theta$$

$$\approx \left(1 - \frac{(2\theta)^2}{2}\right) - (3\theta)^2$$

$$= 1 - 2\,\theta^2 - 9\,\theta^2$$

$$= 1 - 11\,\theta^2$$

$$= \text{RHS} \qquad \square$$

[2 marks]

(ii) When θ is small, θ^2 is negligible in comparison with the number 1

[1 mark]

(iii) when $\theta = 0.1^c$

$$\sin^2 3\theta - \cos 2\theta = 0.8927343853$$

$$\%age\ error = \frac{1 - 0.8927343853}{0.8927343853} \times 100 = 12\,\%$$

[2 marks]

Answer 5

(i)

$$\text{LHS} = \frac{4\cos 3\theta - 2 + 5\sin\theta}{1 - \sin 2\theta}$$

$$\approx \frac{4\left(1 - \frac{(3\theta)^2}{2}\right) - 2 + 5\theta}{1 - 2\theta}$$

$$= \frac{4 - 18\,\theta^2 - 2 + 5\theta}{1 - 2\theta}$$

$$= \frac{2 + 5\theta - 18\,\theta^2}{1 - 2\theta}$$

$$= \frac{(2 + 9\theta)(1 - 2\theta)}{(1 - 2\theta)}$$

$$= 9\theta + 2$$

$$= \text{RHS} \qquad \square$$

[3 marks]

(ii) when $\theta = 0.2$

$$\frac{4\,cos\,0.6 \;-\; 2 \;+\; 5\,sin\,0.2}{1 \;-\; sin\,0.4} \;=\; 3.758201847$$

$$\%age\ error \;=\; \frac{3.8 \;-\; 3.758201847}{3.758201847} \times 100 \;=\; 1.11\ \%$$

[2 marks]

Answer 6

(i)

$$cos\,\theta \;+\; tan^2\,\theta$$

$$\approx \;\left(1 \;-\; \frac{\theta^2}{2}\right) \;+\; \theta^2$$

$$=\; 1 \;+\; \frac{\theta^2}{2}$$

(ii) When θ is small and measured in radians the term in θ^2 can be ignored and so the required approximation is obtained.

i.e. $\quad cos\,\theta \;+\; tan^2\,\theta \;\approx\; 1$

[2, 1 marks]

Answer 7

(i)

$$5\,sin\,2\theta \;+\; tan\,3\theta \;=\; 0.39$$

for small values of θ becomes.....

$$5 \times 2\theta \;+\; 3\theta \;=\; 0.39$$

$$13\theta \;=\; 0.39$$

$$\theta \;=\; 0.03^{\,c}$$

(ii) Use of the small angle approximations was valid for this solution as it is small relative to the period of the trigonometric waveform under consideration.

[2, 2 marks]

Answer 8

(i)

$$sin\,5\theta \;-\; cos\,2\theta \;+\; tan\,2\theta$$

$$\approx\; 5\theta \;-\; \left(1 \;-\; \frac{(2\theta)^2}{2}\right) \;+\; 2\theta$$

$$=\; 2\,\theta^2 \;+\; 7\theta \;-\; 1$$

[3 marks]

(ii) Jason claim that the approximation will have value -1 however, when $\theta = 0.1$,

$$2 \times 0.1^2 + 7 \times 0.1 - 1 = -0.28$$

$\therefore$ Jason is incorrect

In general it's incorrect to ignore a term in θ, because θ then has to be so small for the approximation to be valid that it's no longer useful.

[1 mark]

The Edexcel A-Level Pure Mathematics textbook has this wrong on page 134 where the incorrect Example 21 inspired this question

Answer 9

(i)

$$Arc\ Length = r\,\theta$$

$$0.6 = r\,\theta$$

$$r = \frac{0.6}{\theta}$$

$$Area\ \Delta = \frac{1}{2}\,a\,b\,sin\,C$$

$$12 = \frac{1}{2}\,r^2\,sin\,\theta$$

$$24 = \left(\frac{0.6}{\theta}\right)^2 sin\,\theta$$

using the small angle approximation

$$24 = \frac{0.36}{\theta}$$

$$\therefore \theta = 0.015^c$$

[4 marks]

(ii) When θ is small we're effectively saying that the area of the triangle is roughly the same as the area of the sector, and that the area of the segment is negligible.

[1 mark]

4.1 Trig Equations In Radians

For GCSE many formulae were learnt, such as The Sine Rule and The Cosine Rule. As was seen in Lesson 2, these formulae work just as well whether radians are in use or degrees. Similarly, all of the techniques learnt in the Year 1 A-Level course to solve trigonometric equations work equally well in radians or degrees.

In the A-Level examination, a questions may be presented with the angles in radians along with an expectation that the answer will be given in radians. Such a question can be tackled entirely in radians or, at the start, the angles converted to degrees, the problem then tackled, and the answers in degrees convert to radians. It's crucial not to underestimate how many answers there are in the given interval; the key step where care is needed is the point at which you take *arcsin*, *arccos* or *arctan.*

4.2 Example

Solve the following equation over the interval $0 \leqslant x \leqslant 2\pi$

Give exact answers in terms of π

$$2\cos\left(3x + \frac{\pi}{2}\right) = 1$$

Teaching Video : http://www.NumberWonder.co.uk/v9057/4.mp4

Watch the video.
Write out a solution.

☞

[5 marks]

4.3 Exact values table

	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Not Defined
	0^c	$\frac{\pi}{6}^c$	$\frac{\pi}{4}^c$	$\frac{\pi}{3}^c$	$\frac{\pi}{2}^c$

4.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 75

Question 1

Solve the following equation over the interval $0 \leqslant x \leqslant 2\pi$

Give exact answers in terms of π

$$2\cos\left(x + \frac{2\pi}{3}\right) = \sqrt{2}$$

[5 marks]

Question 2

Solve the following equation over the interval $0 \leqslant x \leqslant 2\pi$

Give exact answers in terms of π

$$\sqrt{3}\ tan\left(2x + \frac{\pi}{4}\right) = 3$$

[6 marks]

Question 3

A-Level C2 Examination question from May 2011, Q7.

(a) Solve for $0 \leqslant x < 360°$, giving your answers in degrees to 1 decimal place,

$$3 \sin (x + 45°) = 2$$

[4 marks]

(b) Find, for $0 \leqslant x < 2\pi$, all the solutions of

$$2 \sin^2 x + 2 = 7 \cos x$$

giving your answers in radians.

You must show clearly how you obtained your answers.

[6 marks]

Question 4

A-Level C3 (R) Examination question from June 2013, Q6

(i) Use an appropriate double angle formula to show that

$$cosec\ 2x = \lambda\ cosec\ x\ sec\ x$$

and state the value of the constant λ

[3 marks]

(ii) Solve, for $0 \leqslant \theta < 2\pi$, the equation

$$3\ sec^2\ \theta + 3\ sec\ \theta = 2\ tan^2\ \theta$$

You must show all your working. Give your answers in terms of π

[6 marks]

Question 5

A-Level C3 Examination question from June 2012, Q5

(i) Express

$$4\,cosec^2\,2\theta \;-\; cosec^2\,\theta$$

in terms of $sin\,\theta$ and $cos\,\theta$

[2 marks]

(ii) Hence show that

$$4\,cosec^2\,2\theta \;-\; cosec^2\,\theta \;=\; sec^2\,\theta$$

[4 marks]

(iii) Hence or otherwise solve, for $0 \leqslant \theta < \pi$

$$4\,cosec^2\,2\theta \;-\; cosec^2\,\theta \;=\; 4$$

giving your answers in terms of π

[3 marks]

Question 6

A-Level C3 (R) Examination question from June 2014, Q3

(i) **(a)** Show that

$$2\tan x - \cot x = 5\operatorname{cosec} x$$

may be written in the form

$$a\cos^2 x + b\cos x + c = 0$$

stating the values of the constants a, b and c

[4 marks]

(b) Hence solve, for $0 \leqslant x < 2\pi$ the equation

$$2\tan x - \cot x = 5\operatorname{cosec} x$$

giving your answers to 3 significant figures

[4 marks]

(ii) Show that

$$\tan\theta + \cot\theta \equiv \lambda\operatorname{cosec} 2\theta \qquad \theta \neq \frac{n\pi}{2},\ n \in \mathbb{Z}$$

stating the value of the constant λ

[4 marks]

Question 7

A-Level C3 Examination question from June 2010, Q7

(i) Express

$$2\,sin\,\theta \;-\; 1.5\,cos\,\theta$$

in the form

$R\,sin\,(\,\theta\,-\,\alpha\,)$ where $R\,>\,0$ and $0\,<\,\alpha\,<\,\dfrac{\pi}{2}$

Give the value of α to 4 decimal places

[3 marks]

(b) **(i)** Find the maximum value of

$$2\,sin\,\theta \;-\; 1.5\,cos\,\theta$$

(ii) Find the value of θ, for $0 \leqslant \theta < \pi$ at which this maximum occurs

[3 marks]

Tom models the height of sea water, H metres, on a particular day by the equation

$$H = 6 + 2\,sin\left(\frac{4\pi t}{25}\right) - 1.5\,cos\left(\frac{4\pi t}{25}\right) \qquad 0 \leqslant t < 12$$

where t hours is the number of hours after midday

(c) Calculate the maximum value of H predicted by this model and the value of t, to 2 decimal places, when this maximum occurs

[3 marks]

(d) Calculate, to the nearest minute, the times when the height of sea water is predicted, by this model, to be 7 metres

[6 marks]

Question 8

A-Level C3 Examination question from June 2015, Q8

(a) Prove that

$$sec\,2A + tan\,2A \equiv \frac{cos\,A + sin\,A}{cos\,A - sin\,A} \qquad A \neq \frac{(2n + 1)\,\pi}{4} \quad n \in \mathbb{Z}$$

[5 marks]

(b) Hence solve, for $0 \leqslant A < 2\pi$

$$sec\,2A + tan\,2A = \frac{1}{2}$$

Give your answers to 3 decimal places

[4 marks]

4.5 Answers

4.5.1 Solution to 4.2 Example (Teaching Video)

$$
\begin{aligned}
2\cos\left(3x + \frac{\pi}{2}\right) &= 1 \\
\cos\left(3x + \frac{\pi}{2}\right) &= \frac{1}{2} \\
3x + \frac{\pi}{2} &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}, \frac{19\pi}{3}, \ldots \\
3x + \frac{3\pi}{6} &= \frac{2\pi}{6}, \frac{10\pi}{6}, \frac{14\pi}{6}, \frac{22\pi}{6}, \frac{26\pi}{6}, \frac{34\pi}{6}, \frac{38\pi}{6}, \ldots \\
3x &= \ldots, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{31\pi}{6}. \frac{35\pi}{3}, \ldots \\
x &= \ldots, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}. \frac{35\pi}{18}, \ldots
\end{aligned}
$$

[5 marks]

4.5.2 Solutions to 4.4 Exercise

Answer 1

$$
\begin{aligned}
2\cos\left(x + \frac{2\pi}{3}\right) &= \sqrt{2} \\
\cos\left(x + \frac{2\pi}{3}\right) &= \frac{\sqrt{2}}{2} \\
x + \frac{2\pi}{3} &= \ldots, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \ldots \\
x + \frac{8\pi}{12} &= \ldots, \frac{3\pi}{12}, \frac{21\pi}{12}, \frac{27\pi}{12}, \frac{45\pi}{12}, \ldots \\
x &= \frac{21\pi - 8\pi}{12}, \frac{27\pi - 8\pi}{12} \\
&= \frac{13\pi}{12}, \frac{19\pi}{12}
\end{aligned}
$$

[5 marks]

Answer 2

$$\sqrt{3}\ tan\left(2x + \frac{\pi}{4}\right) = 3$$

$$tan\left(2x + \frac{\pi}{4}\right) = \frac{3}{\sqrt{3}}$$

$$tan\left(2x + \frac{\pi}{4}\right) = \sqrt{3}$$

$$2x + \frac{\pi}{4} = \frac{\pi}{3},\ \frac{4\pi}{3},\ \frac{7\pi}{3},\ \frac{10\pi}{3},\ \frac{13\pi}{3},\ \ldots$$

$$2x + \frac{3\pi}{12} = \frac{4\pi}{12},\ \frac{16\pi}{12},\ \frac{28\pi}{12},\ \frac{40\pi}{12},\ \frac{52\pi}{12},\ \ldots$$

$$2x = \frac{\pi}{12},\ \frac{13\pi}{12},\ \frac{25\pi}{12},\ \frac{37\pi}{12},\ \frac{49\pi}{12},\ \ldots$$

$$x = \frac{\pi}{24},\ \frac{13\pi}{24},\ \frac{25\pi}{24},\ \frac{37\pi}{24},\ \frac{49\pi}{24},\ \ldots$$

[6 marks]

Answer 3

(a)

$$3\,sin\,(\,x + 45°\,) = 2$$

$$sin\,(\,x + 45\,) = \frac{2}{3}$$

$$x + 45 = 41.8,\ 138.2,\ 401.8,\ \ldots$$

$$x = 93.2°,\ 356.8°$$

[4 marks]

(b)

$$2\,sin^2 x + 2 = 7\,cos\,x$$

$$2\left(1 - cos^2 x\right) + 2 = 7\,cos\,x$$

$$2\,cos^2 x + 7\,cos\,x - 4 = 0$$

$$(\,2cos\,x - 1\,)(\,cos\,x + 4\,) = 0$$

Either $cos\,x = \frac{1}{2}$ or $cos\,x = -4$

$x = \frac{\pi}{3},\ \frac{5\pi}{3}$ no solutions

(As decimals to 3 decimal places, 1.047, 5.236)

[6 marks]

Answer 4

(i)

$$\begin{aligned} csc\,2x &= \frac{1}{sin\,2x} \\ &= \frac{1}{2\,sin\,x\,cos\,x} \\ &= \frac{1}{2} \times \frac{1}{sin\,x} \times \frac{1}{cos\,x} \\ &= \frac{1}{2}\,csc\,x\,sec\,x \end{aligned}$$

i.e. The value of λ is $\frac{1}{2}$

[3 marks]

(ii)

$$3\,sec^2\,\theta + 3\,sec\,\theta = 2\,tan^2\,\theta$$

$$3\,sec^2\,\theta + 3\,sec\,\theta = 2\left(sec^2\,x - 1\right)$$

$$sec^2\,\theta + 3\,sec\,\theta + 2 = 0$$

$$(\,sec\,\theta + 1\,)(\,sec\,\theta + 2\,) = 0$$

Either $sec\,\theta = -1$ or $sec\,\theta = -2$

$cos\,\theta = -1$ or $cos\,\theta = -\frac{1}{2}$

$$\theta = \pi,\ \frac{2\pi}{3},\ \frac{4\pi}{3}$$

[6 marks]

Answer 5

(i)

$$4\,csc^2\,2\theta \;-\; csc^2\,\theta \;=\; \frac{4}{sin^2\,2\theta} \;-\; \frac{1}{sin^2\,\theta}$$

[2 marks]

(ii)

$$= \frac{4}{2^2\,sin^2\,\theta\,cos^2\,\theta} - \frac{1}{sin^2\,\theta} \times \frac{cos^2\,\theta}{cos^2\,\theta}$$

$$= \frac{1\;-\;cos^2\,\theta}{sin^2\,\theta\,cos^2\,\theta}$$

$$= \frac{sin^2\,\theta}{sin^2\,\theta\,cos^2\,\theta}$$

$$= \frac{1}{cos^2\,\theta}$$

$$= sec^2\,\theta$$

[4 marks]

(iii)

$$4\,csc^2\,2\theta \;-\; csc^2\,\theta \;=\; 4$$

$$sec^2\,\theta \;=\; 4$$

$$sec\,\theta \;=\; \pm\,2$$

$$cos\,\theta \;=\; \pm\,\frac{1}{2}$$

$$\theta \;=\; \frac{\pi}{3},\;\; \frac{2\pi}{3}$$

[3 marks]

Answer 6

(i) **(a)**

$$2\,tan\,x - cot\,x = 5\,csc\,x$$

$$5\,csc\,x + cot\,x - 2\,tan\,x = 0$$

$$\frac{5}{sin\,x} + \frac{cos\,x}{sin\,x} - \frac{2\,sin\,x}{cos\,x} = 0$$

multiply through by $sin\,x\,cos\,x$

$$5\,cos\,x + cos^2 x - 2\,sin^2 x = 0$$

$$cos^2 x + 5\,cos\,x - 2\left(1 - cos^2 x\right) = 0$$

$$3\,cos^2 x + 5\,cos\,x - 2 = 0$$

$$\therefore\ a = 3,\ b = 5 \text{ and } c = -2$$

[4 marks]

(b)

$$2\,tan\,x - cot\,x = 5\,csc\,x$$

$$3\,cos^2 x + 5\,cos\,x - 2 = 0$$

$$(\,3\,cos\,x - 1\,)\ (\,cos\,x + 2\,) = 0$$

$$\text{Either}\quad cos\,x = \frac{1}{3} \quad \text{or} \quad cos\,x = -2$$

$$x = 1.231^c,\ 5.052^c \quad \text{or} \quad \text{no solutions}$$

$$\therefore\ x = 1.23^c,\ 5.05^c$$

[4 marks]

(ii)

$$tan\,\theta + cot\,\theta = \frac{sin\,\theta}{cos\,\theta} + \frac{cos\,\theta}{sin\,\theta}$$

$$= \frac{sin\,\theta}{cos\,\theta} \times \frac{sin\,\theta}{sin\,\theta} + \frac{cos\,\theta}{sin\,\theta} \times \frac{cos\,\theta}{cos\,\theta}$$

$$= \frac{sin^2\,\theta + cos^2\,\theta}{sin\,\theta\,cos\,\theta}$$

$$= \frac{1}{sin\,\theta\,cos\,\theta} \times \frac{2}{2}$$

$$= \frac{2}{sin\,2\theta}$$

$$= 2\,csc\,2\theta$$

Which is of the required form ☐

i.e. $\lambda = 2$

[4 marks]

Answer 7

(i)

$$R\sin(\theta - \alpha) = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

which is required to be $2\sin\theta - 1.5\cos\theta$

$$\therefore R\cos\alpha = 2 \quad \text{and} \quad R\sin\alpha = 1.5$$

Solving these two equations simultaneously by division,

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{1.5}{2}$$

$$\alpha = \arctan\left(\frac{1.5}{2}\right)$$

$$\alpha = 0.6435^c$$

$$R = \sqrt{2^2 + 1.5^2}$$

$$= 2.5$$

Conclusion :

$$2\sin\theta - 1.5\cos\theta = 2.5\sin(\theta - 0.6435^c)$$

[3 marks]

(b) **(i)** 2.5

(ii)

$$\sin(\theta - 0.6435) = 0$$

$$\theta - 0.6435 = \frac{\pi}{2}$$

$$\theta = 2.2143^c$$

[3 marks]

(c)

$$H = 6 + 2\sin\left(\frac{4\pi t}{25}\right) - 1.5\cos\left(\frac{4\pi t}{25}\right)$$

$$= 6 + 2.5\sin\left(\frac{4\pi t}{25} - 0.6435\right)$$

The maximum value of H is thus, 8.5 and it occurs when,

$$\sin\left(\frac{4\pi t}{25} - 0.6435\right) = 1$$

$$\left(\frac{4\pi t}{25} - 0.6435\right) = \frac{\pi}{2}$$

$$\frac{4\pi t}{25} = 2.2143^c$$

$$t = 4.4052 \text{ hours}$$

[3 marks]

(d) The question amounts to solving this equation,

$$6 + 2\,sin\left(\frac{4\pi t}{25}\right) - 1.5\,cos\left(\frac{4\pi t}{25}\right) = 7$$

$$6 + 2.5\,sin\left(\frac{4\pi t}{25} - 0.6435\right) = 7$$

$$2.5\,sin\left(\frac{4\pi t}{25} - 0.6435\right) = 1$$

$$sin\left(\frac{4\pi t}{25} - 0.6435\right) = 0.4$$

$$\frac{4\pi t}{25} - 0.6435 = 0.4115,\ 2.7301,\ 6.6947,\ 9.0133,\ \ldots$$

$$\frac{4\pi t}{25} = 1.0550,\ 3.3736,\ 7.3382,\ 9.6568,\ \ldots$$

$$t = 2.0989,\ 6.7116,\ 14.5988,\ 19.2116,\ \ldots$$

t = 2.06 pm, 6.43 pm

[6 marks]

Answer 8

(a)

$$
\begin{aligned}
\text{LHS} &= sec\,2A + tan\,2A \\
&= \frac{1}{cos\,2A} + \frac{sin\,2A}{cos\,2A} \\
&= \frac{1 + sin\,2A}{cos\,2A} \\
&= \frac{1 + 2\,sin\,A\,cos\,A}{cos^2 A - sin^2 A} \\
&= \frac{cos^2 A + sin^2 A + 2\,sin\,A\,cos\,A}{(cos\,A - sin\,A)(cos\,A + sin\,A)} \\
&= \frac{(cos\,A + sin\,A)^2}{(cos\,A - sin\,A)\,(cos\,A + sin\,A)} \\
&= \frac{cos\,A + sin\,A}{cos\,A - sin\,A} \\
&= \text{RHS} \;\square
\end{aligned}
$$

[5 marks]

(b)

$$
\begin{aligned}
sec\,2A + tan\,2A &= \frac{1}{2} \\
\frac{cos\,A + sin\,A}{cos\,A - sin\,A} &= \frac{1}{2} \\
2\,cos\,A + 2\,sin\,A &= cos\,A - sin\,A \\
3\,sin\,A &= -\,cos\,A \\
\frac{sin\,A}{cos\,A} &= -\frac{1}{3} \\
tan\,A &= -\frac{1}{3} \\
A &= 2.8198,\;\; 5.9614 \\
\therefore\;\; A &= 2.820,\;\; 5.961
\end{aligned}
$$

[4 marks]

Lesson 5

A-Level Pure Mathematics : Year 2

Applications of Trigonometry

5.1 Revision

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

Convert the following angles, written in radians, into their degrees equivalent,

(i) $\frac{\pi}{2}$ **(ii)** $\frac{5\pi}{6}$ **(iii)** $\frac{13\pi}{12}$

[3 marks]

Question 2

Convert the following angles, written in degrees, into their radian equivalent.
Give exact answers in terms of π.

(i) $45°$ **(ii)** $15°$ **(iii)** $330°$

[3 marks]

Question 3

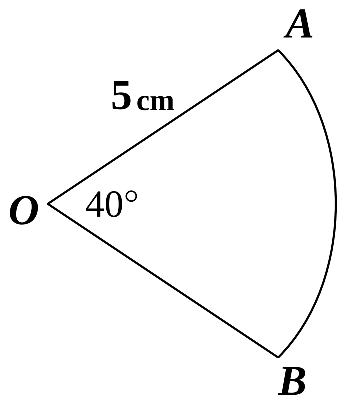
Solve the following equation over the interval $0 \leqslant x \leqslant 2\pi$
Give exact answers in terms of π

$$2\,cos\left(2x + \frac{\pi}{8}\right) = \sqrt{2}$$

[6 marks]

Question 4

A-Level Examination question from June 2019, Paper 2, Q3



The diagram shows a sector *AOB* of a circle with centre *O*, radius 5 cm and angle *AOB* = 40°.

The attempt of a student to find the area of the sector is shown below,

$$\begin{aligned} \textit{Area of sector} &= \frac{1}{2} r^2 \theta \\ &= \frac{1}{2} \times 5^2 \times 40 \\ &= 500 \text{ cm}^2 \end{aligned}$$

(a) Explain the error made by this student.

[1 mark]

(b) Write out a correct solution.

[2 marks]

Question 5

(i) When θ is small and measured in radians, use the small angle approximations to show that,

$$\frac{1 - \cos 3\theta}{\theta \tan 2\theta} \approx \frac{9}{4}$$

[4 marks]

(ii) When $\theta = 0.1^c$ (about 6°) what is the percentage error introduced by using the small angle approximations ?

[3 marks]

Question 6

A-Level Examination question from June 2018, Paper 2, Q7 (i)

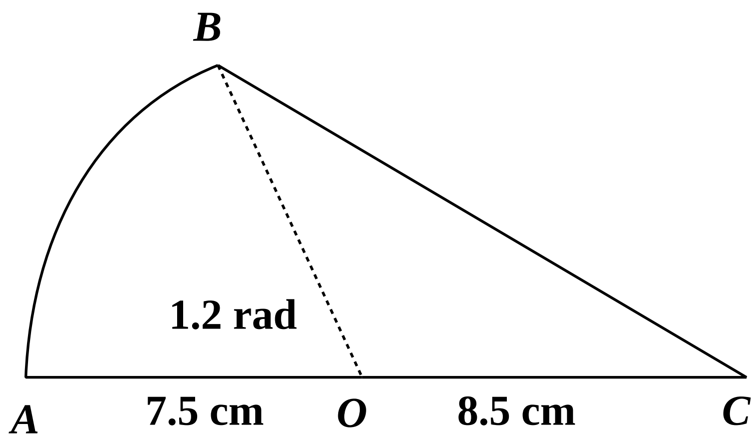
Solve the equation,

$$4\,sin\,x \;=\; sec\,x \qquad \text{for} \qquad 0 \leqslant x < \frac{\pi}{2}$$

[4 marks]

Question 7

A-Level Official Mock Examination Question from 2019, Paper 1, Q2



The shape *AOCBA*, shown, consists of a sector *AOB* of a circle centre *O* joined to a triangle *BOC*.

The points *A*, *O* and *C* lie on a straight line with *AO* = 7.5 cm and *OC* = 8.5 cm.

The size of angle *AOB* is 1.2 radians.

Find, in cm, the perimeter of *AOCBA*, giving your answer to one decimal place.

[5 marks]

Question 8

A-Level Examination question from June 2010, C3, Q3

(a) Express $5 \cos x - 3 \sin x$ in the form $R \cos (x + \alpha)$ where $R > 0$ and $0 < \alpha < \frac{1}{2} \pi$

[4 marks]

(b) Hence, or otherwise, solve the equation

$$5\cos x - 3\sin x = 4$$

for $0 \leqslant x < 2\pi$, giving your answers to 2 decimal places.

[5 marks]

Answer 1

(i) $90°$ **(ii)** $150°$ **(iii)** $195°$

[3 marks]

Answer 2

(i) $\dfrac{\pi}{4}$ **(ii)** $\dfrac{\pi}{12}$ **(iii)** $\dfrac{11\,\pi}{6}$

[3 marks]

Answer 3

$$2\,cos\left(2x + \frac{\pi}{8}\right) = \sqrt{2}$$

$$cos\left(2x + \frac{\pi}{8}\right) = \frac{\sqrt{2}}{2}$$

$$\left(2x + \frac{\pi}{8}\right) = \frac{\pi}{4},\ \frac{7\,\pi}{4},\ \frac{9\,\pi}{4},\ \frac{15\,\pi}{4}$$

$$2x = \frac{\pi}{8},\ \frac{13\,\pi}{8},\ \frac{17\,\pi}{8},\ \frac{29\,\pi}{8}$$

$$x = \frac{\pi}{16},\ \frac{13\,\pi}{16},\ \frac{17\,\pi}{16},\ \frac{29\,\pi}{16}$$

[6 marks]

Answer 4

(a) The student has used an angle measured in degrees in a formula that requires the angle to be in radians.

(b)

$$Area\ of\ sector = \frac{1}{2}\,r^2\,\theta$$

$$= \frac{1}{2} \times 5^2 \times 40 \times \frac{\pi}{180}$$

$$= \frac{25\,\pi}{9}$$

$$= 8.73\ \text{cm}^2$$

[1, 2 marks]

Answer 5

(i) $$\frac{1 - cos\,3\theta}{\theta\ tan\,2\theta} \approx \frac{1 - \left(1 - \frac{(3\theta)^2}{2}\right)}{\theta \times (2\theta)} = \frac{9\,\theta^2}{4\,\theta^2} = \frac{9}{4}$$

(ii) Exact value is 2.203320164

$$\%\ error = \frac{2.25 - 2.203320164}{2.203320164} \times 100$$

$$= 2.12\ \%$$

[4, 3 marks]

Answer 6

$$\begin{aligned} 4\,sin\,x &= sec\,x \\ 4\,sin\,x &= \frac{1}{cos\,x} \\ 4\,sin\,x\,cos\,x &= 1 \\ 2\,sin\,x\,cos\,x &= \frac{1}{2} \\ sin\,2x &= \frac{1}{2} \\ 2x &= arcsin\left(\frac{1}{2}\right) \\ 2x &= \frac{\pi}{6}, \frac{5\pi}{6} \\ x &= \frac{\pi}{12}, \frac{5\pi}{12} \end{aligned}$$

[4 marks]

Answer 7

Use the cosine rule to get length *BC*,

$$\begin{aligned} BC^2 &= 7.5^2 + 8.5^2 - 2 \times 7.5 \times 8.5 \times cos\,(\pi - 1.2) \\ &= 174.7 \\ \therefore\ BC &= 13.217 \end{aligned}$$

Use arc length formula to get the length of the arc *AB*,

$$\begin{aligned} Arc\ length\ AB &= 7.5 \times 1.2 \\ &= 9\ \text{cm} \end{aligned}$$

$$\begin{aligned} Perimeter &= 13.217 + 9 + 7.5 + 8.5 \\ &= 38.2\ \text{cm} \qquad \text{(one decimal place)} \end{aligned}$$

[5 marks]

Answer 8

(a)

$$\begin{aligned} 5\,cos\,x - 3\,sin\,x &= R\,cos\,(x + \alpha) \\ &= R\,cos\,x\,cos\,\alpha - R\,sin\,x\,sin\,\alpha \\ \therefore\ tan\,\alpha &= \frac{R\,sin\,\alpha}{R\,cos\,\alpha} \\ &= \frac{3}{5} \\ \alpha &= arctan\left(\frac{3}{5}\right) \\ &= 0.540 \text{ radians} \\ R &= \sqrt{3^2 + 5^2} \\ &= \sqrt{34} \\ &= 5.83 \end{aligned}$$

Thus,

$$5\,cos\,x - 3\,sin\,x = \sqrt{34}\ cos\,(x + 0.540^c)$$

[4 marks]

(b)

$$\begin{aligned} 5\,cos\,x - 3\,sin\,x &= 4 \\ \sqrt{34}\ cos\,(x + 0.540) &= 4 \\ cos\,(x + 0.540) &= 0.686 \\ x + 0.540 &= arccos\,0.686 \\ &= 0.815,\ 2\pi - 0.815 \\ x &= 0.275^c,\ 4.93^c \end{aligned}$$

[5 marks]

6.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

Convert the following angles, written in radians, into their degrees equivalent,

(i) $\dfrac{\pi}{3}$ **(ii)** $\dfrac{7\pi}{15}$ **(iii)** $\dfrac{17\pi}{12}$

[3 marks]

Question 2

Convert the following angles, written in degrees, into their radian equivalent.
Give exact answers in terms of π.

(i) $30°$ **(ii)** $300°$ **(iii)** $75°$

[3 marks]

Question 3

Solve the following equation over the interval $0 \leqslant x \leqslant 2\pi$
Give exact answers in terms of π

$$2\, sin\left(2x - \frac{\pi}{6}\right) = \sqrt{3}$$

[5 marks]

Question 4

A-Level Specimen Examination Question 2019, Paper 2, Q2

(a) Given that θ is small, use the small angle approximation for $\cos\theta$ to show that,

$$1 + 4\cos\theta + 3\cos^2\theta \approx 8 - 5\theta^2$$

[3 marks]

Adele uses $\theta = 5°$ to test the approximation in part (a)

Adele's working is shown below,

> Using my calculator
>
> $$1 + 4\cos(5°) + 3\cos^2(5°) = 7.962 \quad \text{(3 decimal places)}$$
>
> Using the approximation $8 - 5\theta^2$ gives $8 - 5(5)^2 = -117$
>
> Therefore, $1 + 4\cos\theta + 3\cos^2\theta \approx 8 - 5\theta^2$ is not true for $\theta = 5°$

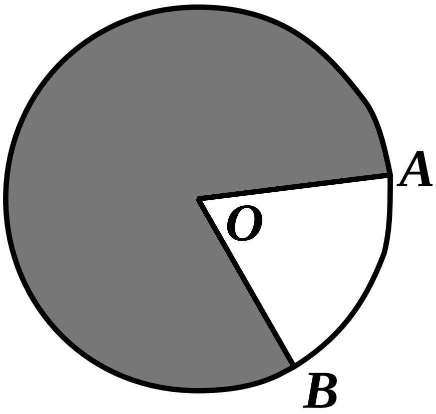
(b) **(i)** Identify the mistake made by Adele in her working

(ii) Show that $8 - 5\theta^2$ can be used to give a good approximation to $1 + 4\cos\theta + 3\cos^2\theta$ for an angle of size $5°$

[2 marks]

Question 5

A-Level Specimen Examination Question 2019, Paper 2, Q2



The diagram shows a circle with centre O.

The points A and B lie on the circumference of the circle.

The area of the major sector, shown shaded, is 135 cm^2.

The reflex angle AOB is 4.8 radians.

Find the exact length, in cm, of the minor arc AB, giving your answer in the form $a\pi + b$ where a and b are integers to be found.

[4 marks]

Question 6

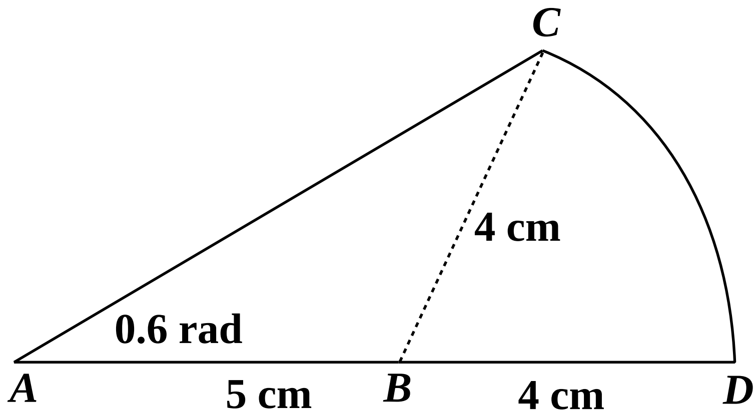
Solve the equation,

$$4\,cos\,x \;=\; csc\,x \qquad\qquad \text{for} \qquad 0 \;\leqslant\; x \;<\; \frac{\pi}{2}$$

[4 marks]

Question 7

A-Level Examination Question from January 2010, C2, Q4



The emblem shown consists of a triangle *ABC* joined to a sector *CBD* of a circle with radius 4 cm and centre *B*. The points *A*, *B* and *D* lie on a straight line with *AB* = 5 cm and *BD* = 4 cm. Angle *BAC* = 0.6 radians and *AC* is the longest side of the triangle *ABC*.

(a) Show that angle *ABC* = 1.76 radians, correct to 3 significant figures.

[4 marks]

(b) Find the area of the emblem

[3 marks]

Question 8

A-Level Examination question from June 2008, C3, Q2

$$f(x) = 5\cos x + 12\sin x$$

Given that $f(x) = R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$,

(a) find the value of R and the value of α to 3 decimal places

[4 marks]

(b) Hence solve the equation

$$5\,cos\,x \;+\; 12\,sin\,x \;=\; 6$$

for $0 \;<\; x \;<\; 2\pi$

[5 marks]

6.2 Answers

Answer 1

(i) 60° **(ii)** 84° **(iii)** 255°

[3 marks]

Answer 2

(i) $\dfrac{\pi}{6}$ **(ii)** $\dfrac{5\pi}{3}$ **(iii)** $\dfrac{5\pi}{12}$

[3 marks]

Answer 3

$$2\,sin\left(2x - \frac{\pi}{6}\right) = \sqrt{3}$$

$$sin\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\left(2x - \frac{\pi}{6}\right) = \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{7\pi}{3},\ \frac{8\pi}{3}$$

$$2x = \frac{\pi}{2},\ \frac{5\pi}{6},\ \frac{5\pi}{2},\ \frac{17\pi}{6}$$

$$x = \frac{\pi}{4},\ \frac{5\pi}{12},\ \frac{5\pi}{4},\ \frac{17\pi}{12}$$

[5 marks]

Answer 4

(a) METHOD 1

$$\begin{aligned} LHS &= 1 + 4\cos\theta + 3\cos^2\theta \\ &\approx 1 + 4\left(1 - \frac{\theta^2}{2}\right) + 3\left(1 - \frac{\theta^2}{2}\right)^2 \\ &= 1 + 4 - 2\theta^2 + 3\left(1 - \theta^2 + \frac{\theta^4}{4}\right) \\ &= 5 - 2\theta^2 + 3 - 3\theta^2 + \frac{3}{4}\theta^4 \\ &= 8 - 5\theta^2 + \frac{3}{4}\theta^4 \\ &\approx 8 - 5\theta^2 \quad \text{for small values of } \theta \\ &= RHS \quad \square \end{aligned}$$

[3 marks]

METHOD 2

$$\begin{aligned} LHS &= 1 + 4\cos\theta + 3\cos^2\theta \\ &= 1 + 4\cos\theta + 3(1 - \sin^2\theta) \\ &= 1 + 4\left(1 - \frac{\theta^2}{2}\right) + 3 - 3\theta^2 \\ &= 1 + 4 - 2\theta^2 + 3 - 3\theta^2 \\ &= 8 - 5\theta^2 \\ &= RHS \quad \square \end{aligned}$$

[3 marks]

(b) **(i)** Adele has used degrees where radians should be used.

(ii) $1 + 4\cos\left(5 \times \frac{\pi}{180}\right) + 3\cos^2\left(5 \times \frac{\pi}{180}\right) = 7.962$

$$8 - 5\theta^2 = 8 - 5 \times \left(5 \times \frac{\pi}{180}\right)^2 = 7.962$$

$\therefore$ $8 - 5\theta^2$ is a good approximation for $1 + 4\cos\theta + 3\cos^2\theta$ when $\theta = 5°$

[2 marks]

Answer 5

$$Area\ Sector = \frac{1}{2} r^2 \theta$$

$$135 = \frac{1}{2} \times r^2 \times 4.8$$

$$r^2 = \frac{225}{4}$$

$$r = \frac{15}{2}$$

$$(Minor)\ Arc\ Length\ AB = r\,\theta_{minor}$$

$$= 7.5 \times (2\pi - 4.8)$$

$$= 15\pi - 36 \text{ cm} \qquad \text{(It's about 11.1... cm)}$$

$$\therefore\ a = 15,\ b = -36$$

[4 marks]

Answer 6

$$4 \cos x = \csc x$$

$$4 \cos x = \frac{1}{\sin x}$$

$$4 \sin x \cos x = 1$$

$$2 \sin x \cos x = \frac{1}{2}$$

$$\sin 2x = \frac{1}{2}$$

$$2x = \arcsin\left(\frac{1}{2}\right)$$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$

[4 marks]

Answer 7
(a)
Use the sine rule to get angle *ACB*,

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\sin C = \frac{5 \times \sin 0.6}{4}$$

$$= 0.706$$

$$C = \arcsin 0.706$$

$$= 0.7835$$

Also 2.358 is a solution but is not needed as we're told $\angle ABC$ is the largest angle

$$\therefore \angle ABC = \pi - 0.6 - 0.7835$$

$$= 1.76, \quad \text{as expected}$$

[4 marks]

(b)

$$Sector\ Area = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2} \times 4^2 \times (\pi - 1.76)$$

$$= 11.1 \text{ cm}^2$$

$$Area\ \triangle ABC = \frac{1}{2} \times 5 \times 4 \times \sin 1.76$$

$$= 9.8$$

$$\therefore Area\ Emblem = 20.9 \text{ cm}^2$$

[3 marks]

Answer 8

(a)

$$\begin{aligned} 5\,cos\,x + 12\,sin\,x &= R\,cos\,(x - \alpha) \\ &= R\,cos\,x\,cos\,\alpha + R\,sin\,x\,sin\,\alpha \end{aligned}$$

$$\begin{aligned} \text{Now,}\quad tan\,\alpha &= \frac{R\,sin\,\alpha}{R\,cos\,\alpha} \\ &= \frac{12}{5} \\ \alpha &= arctan\left(\frac{12}{5}\right) \\ &= 1.176 \text{ radians} \\ R &= \sqrt{12^2 + 5^2} \\ &= 13 \end{aligned}$$

Thus,

$$5\,cos\,x + 12\,sin\,x = 13\,cos\,(x - 1.176^c)$$

[4 marks]

(b)

$$\begin{aligned} 5\,cos\,x + 12\,sin\,x &= 6 \\ 13\,cos\,(x - 1.176) &= 6 \\ cos\,(x - 1.176) &= 0.462 \\ x - 1.176 &= arccos\,0.462 \\ &= -1.091,\ 1.091 \\ x &= 0.085,\ 2.267 \end{aligned}$$

[5 marks]