10.4 Answers

Additional Mathematics
A-Level Pure Mathematics : Year 1
Binomial Expansion

10.4.1 Solution to 10.2 Example (Teaching Video)

(i)

$$\left(2+\frac{x}{4}\right)^8 = 1\times(2)^8\times\left(\frac{x}{4}\right)^0 + 8\times(2)^7\times\left(\frac{x}{4}\right)^1 + 28\times(2)^6\times\left(\frac{x}{4}\right)^2 + \ldots$$

$$= 256 + 256x + 112\,x^2 + \ldots$$

(six terms terms with higher powers of x "thrown away")

[4 marks]

(ii) We require,

$$2+\frac{x}{4} = 2.075$$

$$\frac{x}{4} = 0.075$$

$$x = 0.3$$

$$(2.075)^8 = \left(2+\frac{0.3}{4}\right)^8$$

$$= 256 + 256\times 0.3 + 112\times 0.3^2 + \ldots$$

$$= 342.88$$ which is 343 to the nearest integer

Incidentally :

$2.075^8 = 343.6725208...$ (direct from a calculator)
which is confirmation that, even having throwing away six terms, an approximately correct answer was obtained.

So, when x is small, throwing away most of the terms containing higher powers of x is a valid thing to do, as a reasonable approximation to the exact answer is still obtained.

A small amount of accuracy has been sacrificed for a big reduction in complexity.

[4 marks]

10.4.2 Solution to 10.3 Exercise

Answer 1

(i)

$$\left(3 + \frac{x}{9}\right)^{11} = 1 \times (3)^{11} \times \left(\frac{x}{9}\right)^{0}$$
$$+ 11 \times (3)^{10} \times \left(\frac{x}{9}\right)^{1}$$
$$+ 55 \times (3)^{9} \times \left(\frac{x}{9}\right)^{2}$$
$$+ \ldots$$
$$= 177147 + 72171x + 13365x^2 + \ldots \text{ (9 terms thrown away)}$$

[4 marks]

(ii) We require,

$$3 + \frac{x}{9} = 3.01$$
$$\frac{x}{9} = 0.01$$
$$x = 0.09$$

$$(3.01)^{11} = \left(3 + \frac{0.09}{9}\right)^{11}$$
$$= 177147 + 72171 \times 0.09 + 13365 \times 0.09^2 + \ldots$$
$$= 183750.6465 \text{ which is 183751 to 6 sig fig}$$

Incidentally :

$$3.01^{11} = 183751.7363... \text{ (direct from calculator)}$$

which is confirmation that, even having throwing away nine terms, an approximately correct answer was obtained.

[4 marks]

Answer 2

It's possible to simply target the required term.
However, here's the full expansion,

$$\left(\frac{x}{2}+\frac{4}{x}\right)^4 = 1\times\left(\frac{x}{2}\right)^4\times\left(\frac{4}{x}\right)^0 + 4\times\left(\frac{x}{2}\right)^3\times\left(\frac{4}{x}\right)^1 + 6\times\left(\frac{x}{2}\right)^2\times\left(\frac{4}{x}\right)^2 + 4\times\left(\frac{x}{2}\right)^1\times\left(\frac{4}{x}\right)^3 + 1\times\left(\frac{x}{2}\right)^0\times\left(\frac{4}{x}\right)^4$$

$$= \frac{x^4}{16}+\frac{x^2}{2}+24+\frac{128}{x^2}+\frac{256}{x^4}$$

The constant term is the 24

[4 marks]

Answer 3

(i) $(1-3x)^5 = 1\times(1)^5\times(-3x)^0 + 5\times(1)^4\times(-3x)^1 + 10\times(1)^3\times(-3x)^2 + 10\times(1)^2\times(-3x)^3 + \ldots$

$$(1-3x)^5 = 1-15x+90x^2-270x^3+\ldots$$

[4 marks]

(ii)

$$(1+x)(1-3x)^5 = (1+x)\left(1-15x+90x^2-270x^3+\ldots\right)$$
$$= 1-15x+90x^2-270x^3+\ldots$$
$$+\quad x-15x^2+\quad 90x^3-270x^4+\ldots$$
$$= 1-14x+75x^2-180x^3+\ldots$$
$$\approx 1-14x \qquad \square$$

[4 marks]

Answer 4

(i)

$$
\begin{aligned}
&{}^2C_0 + 2\,{}^2C_1 + 4\,{}^2C_2 \\
&= 1 + 2 \times 2 + 4 \times 1 \\
&= 9 \qquad \text{which is } 3^2
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&{}^3C_0 + 2\,{}^3C_1 + 4\,{}^3C_2 + 8\,{}^3C_3 \\
&= 1 + 2 \times 3 + 4 \times 3 + 8 \times 1 \\
&= 27 \qquad \text{which is } 3^3
\end{aligned}
$$

(iii)

$$
\begin{aligned}
&{}^4C_0 + 2\,{}^4C_1 + 4\,{}^4C_2 + 8\,{}^4C_3 + 16\,{}^4C_4 \\
&= 1 + 2 \times 4 + 4 \times 6 + 8 \times 4 + 16 \times 1 \\
&= 81 \qquad \text{which is } 3^4
\end{aligned}
$$

(iv)

$$
\begin{aligned}
&{}^5C_0 + 2\,{}^5C_1 + 4\,{}^5C_2 + 8\,{}^5C_3 + 16\,{}^5C_4 + 32\,{}^5C_5 \\
&= 1 + 2 \times 5 + 4 \times 10 + 8 \times 10 + 16 \times 5 + 32 \times 1 \\
&= 243 \qquad \text{which is } 3^5
\end{aligned}
$$

This works because,

$$(1 + x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

and with $x = 2$ this becomes

$$3^n = \sum_{r=0}^{n} {}^nC_r\, 2^r$$

[2, 2, 2, 2 marks]

Answer 5

Targeting the relevant term,

$$
\begin{aligned}
\left(\frac{x^2}{2} - \frac{2}{x}\right)^9 &= \ldots + {}^9C_6 \times \left(\frac{x^2}{2}\right)^3 \times \left(-\frac{2}{x}\right)^6 + \ldots \\
&= \ldots + 84 \times \frac{x^6}{8} \times \frac{64}{x^6} + \ldots \\
&= \ldots + 672 + \ldots
\end{aligned}
$$

So the constant term is 672

[5 marks]

Answer 6

(i)

$$(1-5x)^{30} = 1 \times (1)^{30} \times (-5x)^0 + 30 \times (1)^{29} \times (-5x)^1 + 435 \times (1)^{28} \times (-5x)^2 + 4060 \times (1)^{27} \times (-5x)^3 + \ldots$$

So,

$$(1-5x)^{30} = 1 - 150x + 10875x^2 - 507\,500x^3 + \ldots$$

[4 marks]

(ii) We require that,

$$1 - 5x = 0.995$$

$$5x = 0.005$$

$$x = 0.001$$

$$(0.995)^{30} = (1 - 5 \times 0.001)^{30}$$

$$= 1 - 150 \times 0.001 + 10875 \times 0.001^2 - 507\,500 \times 0.001^3 + \ldots$$

$$= 1 - 0.15 + 0.010875 - 0.000\,507\,5 + \ldots$$

$$\approx 0.860\,367\,5$$

That is, $0.860\,368$ to six decimal places

[3 marks]

(iii)

$0.995^{30} = 0.860\,384$ direct from calculator

$$\%\,error = \frac{0.860\,384 - 0.860\,368}{0.860\,384} \times 100$$

$$= 0.001\,859\,\%$$

i.e. $0.002\,\%$

[3 marks]

Answer 7

(i) $(1 + \delta)^3 = 1 + 3\delta + 3\delta^2 + \delta^3$

[2 marks]

(ii) If δ is small then, in comparison to the number 1, δ^2 and δ^3 will be negligible. If the terms $3\delta^2$ and δ^3 are "thrown away" what remains is;

$$(1 + \delta)^3 = 1 + 3\delta \qquad \square$$

[1 mark]

(iii)

$$\begin{aligned} x^3 - 0.9x - 0.206 &= 0 \\ (1 + \delta)^3 - 0.9(1 + \delta) - 0.206 &= 0 \\ 1 + 3\delta - 0.9 - 0.9\delta - 0.206 &= 0 \\ 2.1\delta - 0.106 &= 0 \\ 2.1\delta &= 0.106 \\ \delta &= 0.05 \end{aligned}$$

$\therefore$ the root is 1.05 to 3 significant figures

[4 marks]