6.4 Answers

Additional Mathematics
A-Level Pure Mathematics : Year 1
Binomial Expansion

6.4.1 Solution to 6.2 Example (Teaching Video)

The model for spinning a coin three times is, $(H + T)^3$

$$\begin{aligned}(H+T)^3 = \quad & 1 \times (H)^3 \times (T)^0 \\ + \quad & 3 \times (H)^2 \times (T)^1 \\ + \quad & 3 \times (H)^1 \times (T)^2 \\ + \quad & 1 \times (H)^0 \times (T)^3\end{aligned}$$

Simplifying, $(H + T)^3 = 1H^3T^0 + 3H^2T^1 + 3H^1T^2 + 1H^0T^3$

The term corresponding to two heads and one tail is coloured red.

$$\begin{aligned}\mathrm{P}(2\ heads,\ 1\ tail) &= 3H^2T^1 \\ &= 3 \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right) \\ &= \frac{4}{9}\end{aligned}$$

[4 marks]

6.4.2 Solutions to 6.3 Exercise

Answer 1

The model for spinning a coin four times is, $(H + T)^4$

$$\begin{aligned}(H+T)^4 = \quad & 1 \times (H)^4 \times (T)^0 \\ + \quad & 4 \times (H)^3 \times (T)^1 \\ + \quad & 6 \times (H)^2 \times (T)^2 \\ + \quad & 4 \times (H)^1 \times (T)^3 \\ + \quad & 1 \times (H)^0 \times (T)^4\end{aligned}$$

Simplifying,

$$(H + T)^4 = 1H^4T^0 + 4H^3T^1 + 6H^2T^2 + 4H^1T^3 + 1H^0T^4$$

The term corresponding to two heads and two tails is coloured red.

$$\begin{aligned}\mathrm{P}(2\ heads,\ 2\ tails) &= 6H^2T^2 \\ &= 6 \times \left(\frac{3}{4}\right)^2 \times \left(\frac{1}{4}\right)^2 \\ &= \frac{27}{128} \qquad \text{(About 0.211)}\end{aligned}$$

[4 marks]

Answer 2

The model for spinning a coin six times is, $(H + T)^6$

$$
\begin{aligned}
(H + T)^6 = \quad & 1 \times (H)^6 \times (T)^0 \\
+ \quad & 6 \times (H)^5 \times (T)^1 \\
+ \quad & 15 \times (H)^4 \times (T)^2 \\
+ \quad & 20 \times (H)^3 \times (T)^3 \\
+ \quad & 15 \times (H)^2 \times (T)^4 \\
+ \quad & 6 \times (H)^1 \times (T)^5 \\
+ \quad & 1 \times (H)^0 \times (T)^6
\end{aligned}
$$

Simplifying,

$$(H + T)^6$$

$$= 1H^6T^0 + 6H^5T^1 + 15H^4T^2 + 20H^3T^3 + 15H^2T^4 + 6H^1T^5 + 1H^0T^6$$

The term corresponding to four heads and two tails is coloured red.

$$
\begin{aligned}
\mathrm{P}(4\,heads,\ 2\,tails) &= 15H^4T^2 \\
&= 15 \times \left(\frac{1}{3}\right)^4 \times \left(\frac{2}{3}\right)^2 \\
&= \frac{20}{243} \qquad \text{(About 0.082)}
\end{aligned}
$$

[4 marks]

Answer 3

The model for spinning a coin three times is, $(H + T)^3$

$$
\begin{aligned}
(H + T)^3 = \quad & 1 \times (H)^3 \times (T)^0 \\
+ \quad & 3 \times (H)^2 \times (T)^1 \\
+ \quad & 3 \times (H)^1 \times (T)^2 \\
+ \quad & 1 \times (H)^0 \times (T)^3
\end{aligned}
$$

Simplifying,

$$(H + T)^3 = 1H^3T^0 + 3H^2T^1 + 3H^1T^2 + 1H^0T^3$$

$$\mathrm{P}(3H,\ 0T) = 1 \times 0.2^3 = 0.008$$

$$\mathrm{P}(2H,\ 1T) = 3 \times 0.2^2 \times 0.8 = 0.096$$

$$\mathrm{P}(1H,\ 2T) = 3 \times 0.2 \times 0.8^2 = 0.384$$

$$\mathrm{P}(0H,\ 3T) = 1 \times 0.8^3 = 0.512 \qquad \Leftarrow \text{ *** Three tails is most likely ***}$$

[5 marks]

Answer 4

The model for spinning a coin nine times is, $(H + T)^9$

$$\begin{aligned}(H + T)^9 = \quad & 1 \times (H)^9 \times (T)^0 \\ + \quad & 9 \times (H)^8 \times (T)^1 \\ + \quad & 36 \times (H)^7 \times (T)^2 \\ + \quad & \ldots\end{aligned}$$

Simplifying,

$$(H + T)^9 = 1H^9T^0 + 9H^8T^1 + 36H^7T^2 + \ldots$$

The term corresponding to seven heads and two tails is coloured red.

$$\begin{aligned}\mathrm{P}(7\,heads,\ 2\,tails) &= 36H^7T^2 \\ &= 36 \times \left(\frac{1}{2}\right)^7 \times \left(\frac{1}{2}\right)^2 \\ &= \frac{9}{128} \qquad \text{(About 0.070)}\end{aligned}$$

The percentage probability is 7%

[4 marks]

Answer 5

The model for spinning a coin six times is, $(H + T)^6$

$$\begin{aligned}(H + T)^6 = \quad & 1 \times (H)^6 \times (T)^0 \\ + \quad & 6 \times (H)^5 \times (T)^1 \\ + \quad & 15 \times (H)^4 \times (T)^2 \\ + \quad & \ldots\end{aligned}$$

Simplifying,

$$(H + T)^6 = 1H^6T^0 + 6H^5T^1 + 15H^4T^2 + \ldots$$

$$\begin{aligned}\mathrm{P}(H > T) &= 1 \times \left(\frac{1}{2}\right)^6 + 6\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right) + 15\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2 \\ &= 22 \times \left(\frac{1}{2}\right)^6 \\ &= \frac{22}{64} \\ &= \frac{11}{32} \\ &= 0.34375 \qquad \text{which is about a third}\end{aligned}$$

[5 marks]

Answer 6

(i) 0.4

[1 mark]

(ii) The model for spinning a coin ten times is, $(H + T)^{10}$

Passing the driving test can be likened to *heads*, failing it likened to *tails*

Thus,

$$(P + F)^{10} = 1 \times (P)^{10} \times (F)^0 + 10 \times (P)^9 \times (F)^1 + 45 \times (P)^8 \times (F)^2 + \ldots$$

Simplifying,

$$(P + F)^{10} = 1\,P^{10}F^0 + 10\,P^9F^1 + 45\,P^8F^2 + \ldots$$

The term corresponding to eight passing (and two failing) is coloured red.

$$\mathrm{P}(8\,pass,\ 2\,fail) = 45\,P^8F^2$$
$$= 45 \times (0.6)^8 \times (0.4)^2$$
$$= 0.12093$$

The percentage probability is 12%

[5 marks]

Answer 7

The model for spinning a coin twelve times is, $(H + T)^{12}$

Getting a question correct can be likened to *heads*, getting it wrong it likened to *tails*

Thus,

$$(C + W)^{12} = 1 \times (C)^{12} \times (W)^0 + 12 \times (C)^{11} \times (W)^1 + \ldots$$

Simplifying,

$$(C + W)^{12} = 1\,C^{12}W^0 + 12\,C^{11}W^1 + \ldots$$

The first two term corresponding to “less than two questions wrong”

$$\mathrm{P}(12\ correct \text{ or } 11\ correct) = 1 \times 0.25^{12} + 12 \times 0.25^{11} \times 0.75$$
$$= (0.25)^{12} + 12 \times (0.25)^{11} \times 0.75$$
$$= 0.000\,002\,205$$

[5 marks]

Answer 8

The model for spinning a coin ten times is, $(H + T)^{10}$

Being Right-Handed can be likened to *heads*, being Left-Handed likened to *tails*

Thus,

$$
\begin{aligned}
(R + L)^{10} = \quad & 1 \times (R)^{10} \times (L)^{0} \\
+ \quad & 10 \times (R)^{9} \times (L)^{1} \\
+ \quad & 45 \times (R)^{8} \times (L)^{2} \\
+ \quad & 120 \times (R)^{7} \times (L)^{3} \\
+ \quad & 210 \times (R)^{6} \times (L)^{4} \\
+ \quad & \ldots
\end{aligned}
$$

Simplifying,

$$(R + L)^{10} = 1\,R^{10}\,L^{0} + 10\,R^{9}\,L^{1} + 45\,R^{8}\,L^{2} + 120\,R^{7}\,L^{3} + 210\,R^{6}\,L^{4} + \ldots$$

(i) The term corresponding to no student is left handed is coloured green.

$$
\begin{aligned}
\mathrm{P}(10\,R,\ 0\,L) &= 1\,R^{10}\,L^{0} \\
&= 1 \times (0.8)^{10} \times (0.2)^{0} \\
&= 0.107
\end{aligned}
$$

[2 marks]

(ii) The term corresponding to exactly 4 students are left handed is coloured red.

$$
\begin{aligned}
\mathrm{P}(6\,R,\ 4\,L) &= 210\,R^{6}\,L^{4} \\
&= 210 \times (0.8)^{6} \times (0.2)^{4} \\
&= 0.088
\end{aligned}
$$

[3 marks]

Answer 9

The model for spinning a coin three times is, $(H + T)^3$

Throwing a six can be likened to *heads*, throwing something else likened to *tails*

Thus,

$$\begin{aligned}(S + E)^3 = \quad & 1 \times (S)^3 \times (E)^0 \\ + \quad & 3 \times (S)^2 \times (E)^1 \\ + \quad & 3 \times (S)^1 \times (E)^2 \\ + \quad & 1 \times (S)^0 \times (E)^3\end{aligned}$$

Simplifying,

$$(S + E)^3 = 1S^3E^0 + 3S^2E^1 + 3S^1E^2 + 1S^0E^3$$

(i) The term corresponding to exactly 2 sixes are thrown is coloured green.

$$\begin{aligned}\mathrm{P}(2S,\ 1E) &= 3S^2E^1 \\ &= 3 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^1 \\ &= \frac{5}{72} \qquad \text{(About 0.0694)}\end{aligned}$$

[3 marks]

(ii) The terms corresponding to at least 1 six is thrown are coloured green or red.

$$\begin{aligned}\mathrm{P}(\text{At least 1 six}) &= 1 - 1S^0E^3 \\ &= 1 - 1 \times \left(\frac{1}{6}\right)^0 \times \left(\frac{5}{6}\right)^3 \\ &= \frac{91}{216} \qquad \text{(About 0.421)}\end{aligned}$$

[3 marks]

Answer 10

The model for spinning a coin four times is, $(H + T)^4$

Throwing a six can be likened to *heads*, throwing something else likened to *tails*

Thus,

$$\begin{aligned}(S+E)^4 = {} & 1 \times (S)^4 \times (E)^0 \\ & + 4 \times (S)^3 \times (E)^1 \\ & + 6 \times (S)^2 \times (E)^2 \\ & + 4 \times (S)^1 \times (E)^3 \\ & + 1 \times (S)^0 \times (E)^4\end{aligned}$$

Simplifying,

$$(S+E)^4 = 1S^4E^0 + 4S^3E^1 + 6S^2E^2 + 4S^1E^3 + 1S^0E^4$$

The terms corresponding to at least 2 sixes are coloured red.

$$\begin{aligned}\text{P(at least 2 sixes)} &= 1S^4E^0 + 4S^3E^1 + 6H^2T^2 \\ &= \left(\frac{1}{6}\right)^4 + 4\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^1 + 6 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2 \\ &= \frac{1}{1296} + \frac{20}{1296} + \frac{150}{1296} \\ &= \frac{171}{1296} \\ &= \frac{19}{144} \qquad \text{(About 0.132)}\end{aligned}$$

[6 marks]