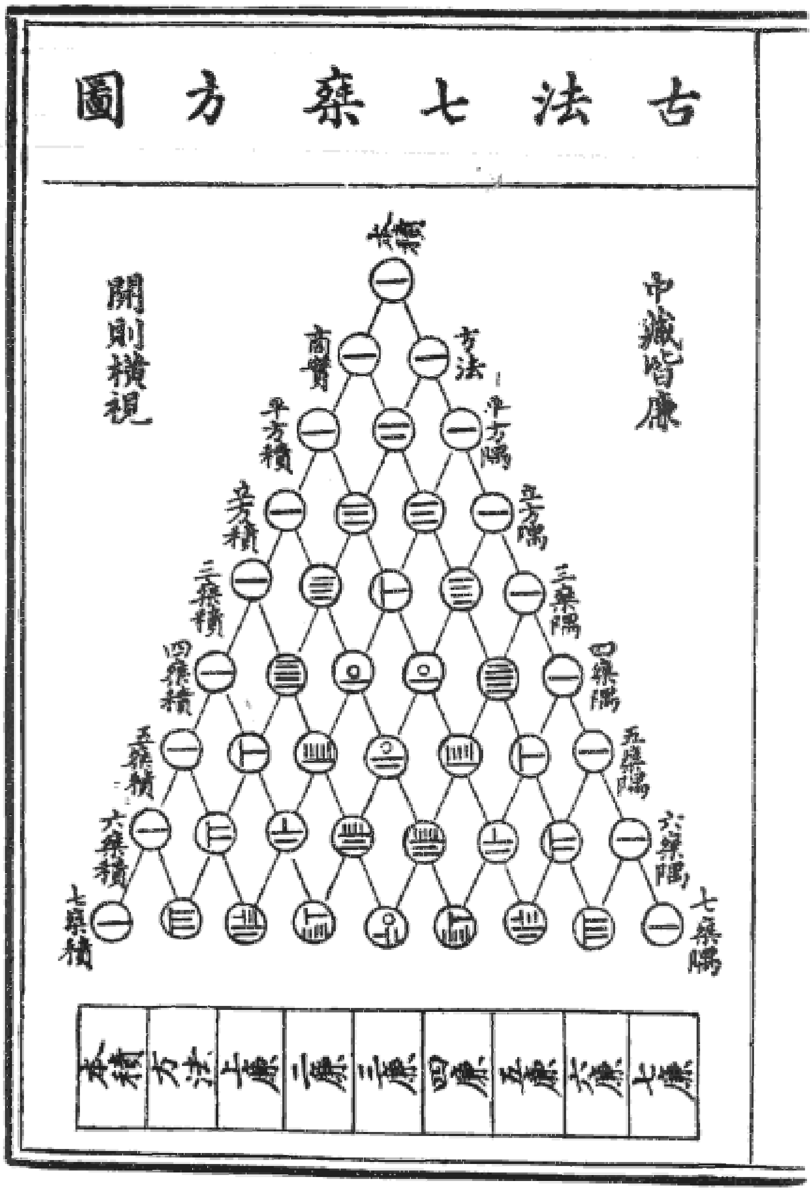
BINOMIAL EXPANSION

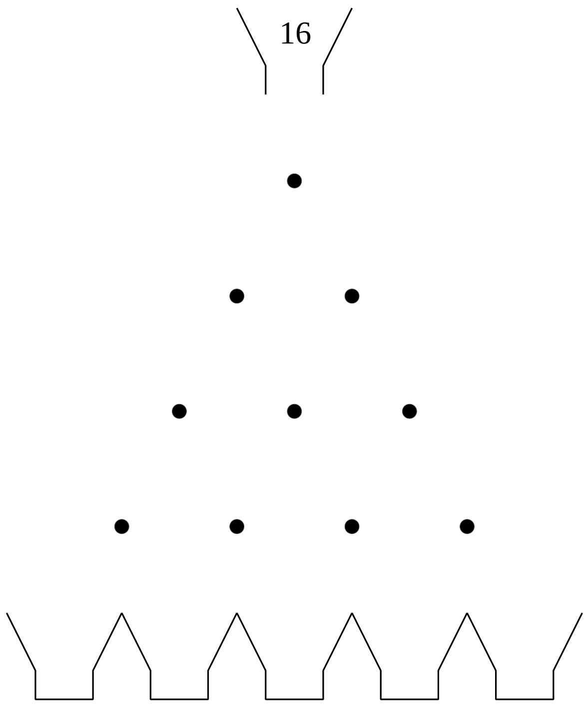


Lesson 1

Additional Mathematics
A-Level Pure Mathematics : Year 1
Binomial Expansion

1.1 Pinball

Sixteen steel ball bearings are released into a mathematical pinball machine.
The balls roll down the inclined surface, ending up in baskets at the bottom.
At each pin it is equally probable that a ball will pass to the left or to the right.
Work out how many ball bearings end up in each of the bottom baskets.



Teaching Video http://www.NumberWonder.co.uk/v9062/1.mp4

[6 marks]

1.2 Distribution On Average

Mathematicians would call the result “the distribution of the balls in the baskets”.
It's an “on average” result.
To achieve it in reality, the 16 steel ball bearings would be run through the machine many, many, times and the results averaged.

1.3 Exercise

Marks Available : 30

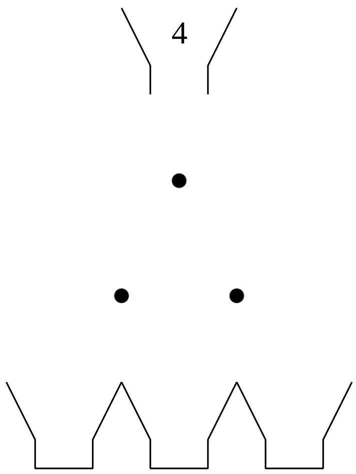
Question 1

Four steel ball bearings are released into a pinball machine.

At each pin it is equally probable that a ball will pass to the left or to the right.

(i) Above each pin write the total number of balls arriving at that pin

(ii) Work out how many ball bearings end up in each of the bottom baskets.



[2 marks]

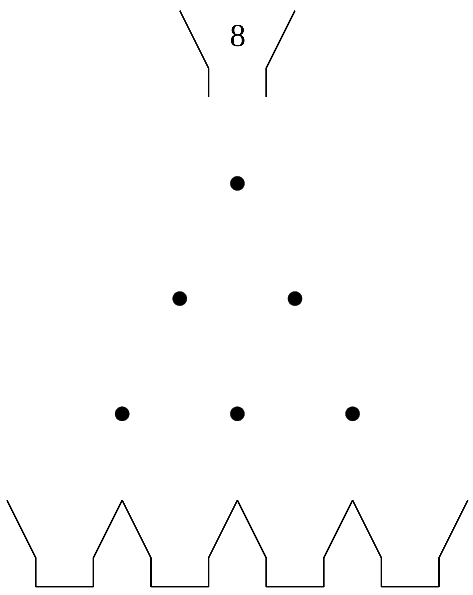
Question 2

Eight steel ball bearings are released into a pinball machine.

At each pin it is equally probable that a ball will pass to the left or to the right.

(i) Above each pin write the total number of balls arriving at that pin

(ii) Work out how many ball bearings end up in each of the bottom baskets.



[4 marks]

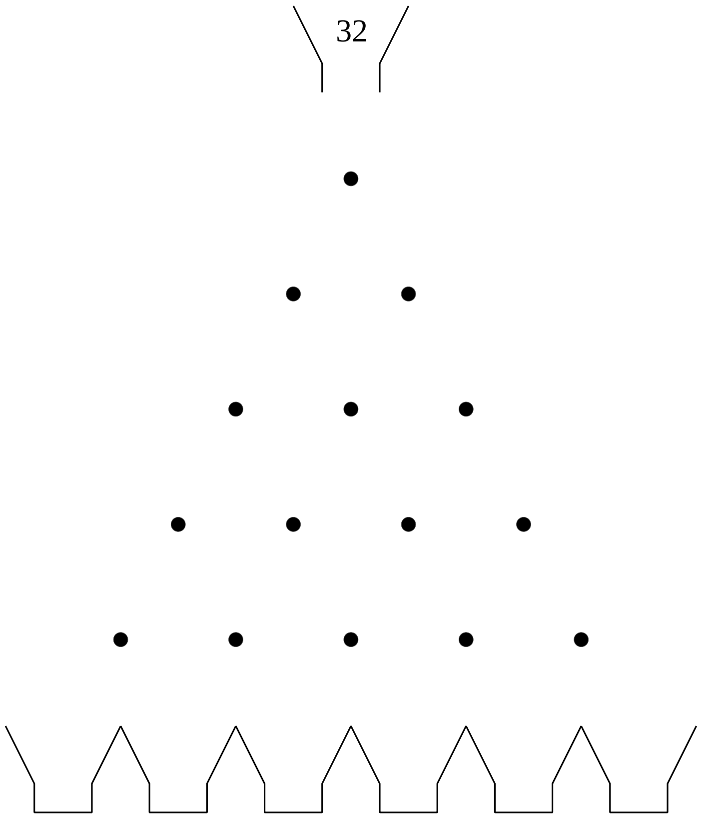
Question 3

Thirty-two steel ball bearings are released into a pinball machine.

At each pin it is equally probable that a ball will pass to the left or to the right.

(i) Above each pin write the total number of balls arriving at that pin

(ii) Work out how many ball bearings end up in each of the bottom baskets.



[8 marks]

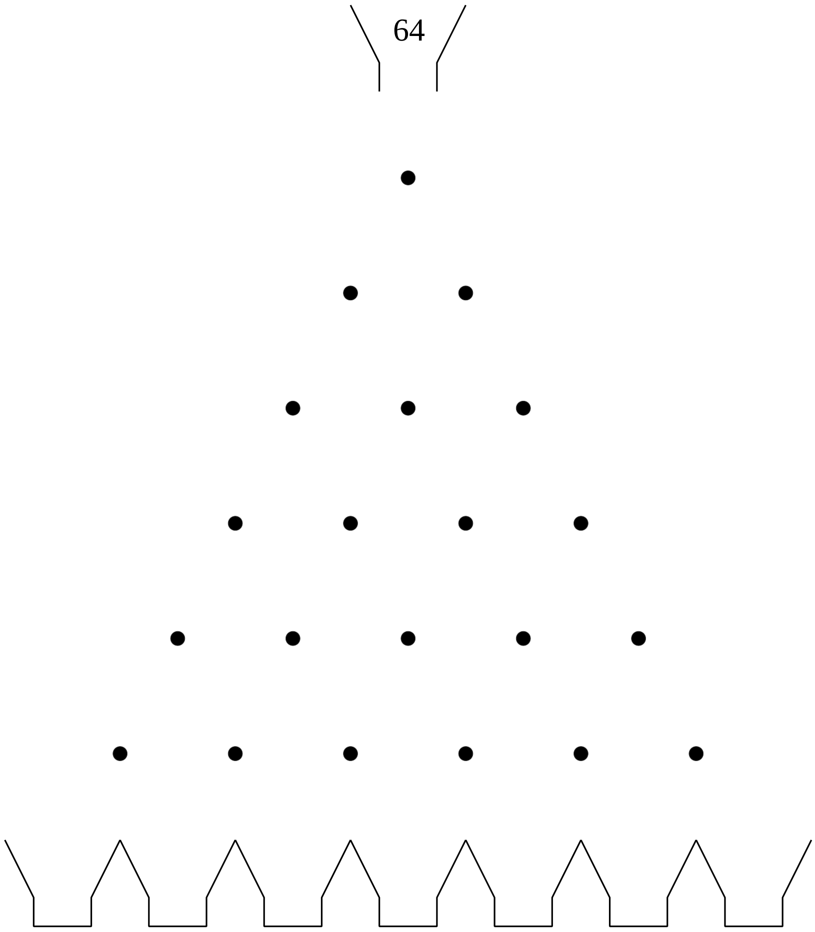
Question 4

Use the table below to summarize your results

Starting Number of Balls	Distribution of Balls in Baskets
4	
8	
16	1 4 6 4 1
32	

[4 marks]

Question 5



(i) Write down a prediction of the distribution that would result, on average, if 64 steel ball bearings were run through the above machine.

[2 marks]

(ii) **Optional**
If you feel it would be helpful is checking your part (i) answer, work through the intricate details of what happens at each pin.

[0 marks]

Question 6
Investigate expanding the following sequence of brackets.
To help you get started the first three results are given.
There is a connection with the pinball machines !

$(1 + x)^0 = 1$

$(1 + x)^1 = 1 + x$

$(1 + x)^2 = 1 + 2x + x^2$

$(1 + x)^3 =$

$(1 + x)^4 =$

$(1 + x)^5 =$

$(1 + x)^6 =$

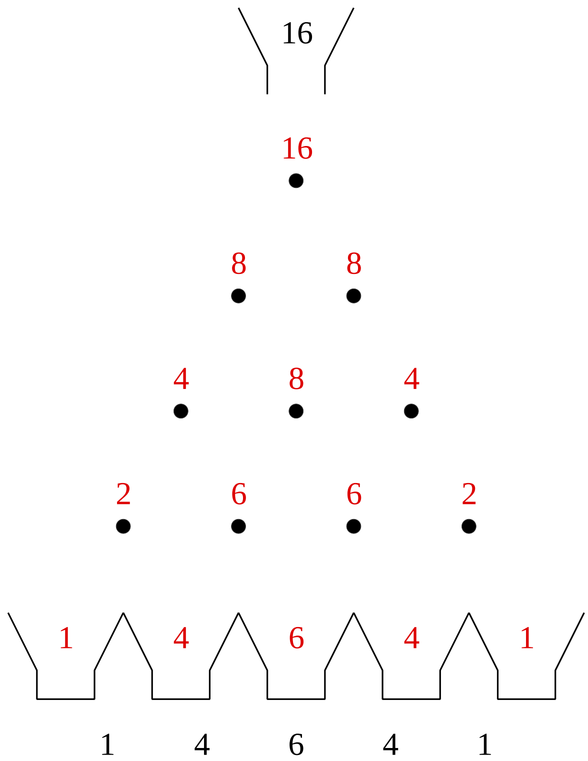
[6 marks]

Question 7
Find out about “Pascal's Triangle” and briefly explain its relevance to the pinball machines of this lesson and the expansion of $(1 + x)^n$

[4 marks]

1.4 Answers

1.4.1 Solution to 1.1 Example (Teaching Video)

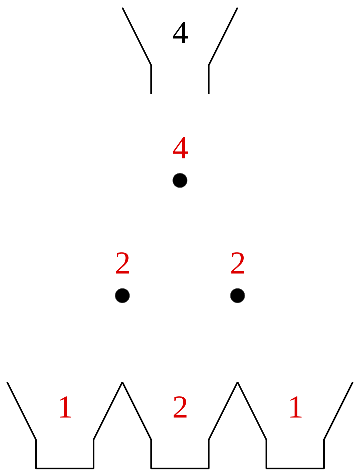


[6 marks]

1.4.2 Solution to 1.3 Exercise

Answer 1

(i) The number of balls arriving at each pin is,



[1 mark]

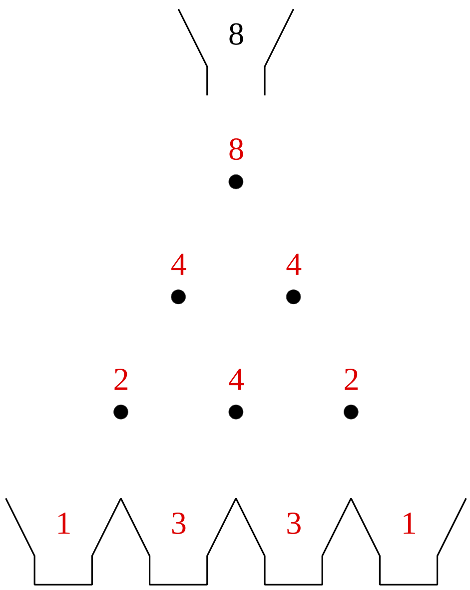
(ii) The distribution of the balls across the baskets is,

1 2 1

[1 mark]

Answer 2

(i) The number of balls arriving at each pin is,



[2 marks]

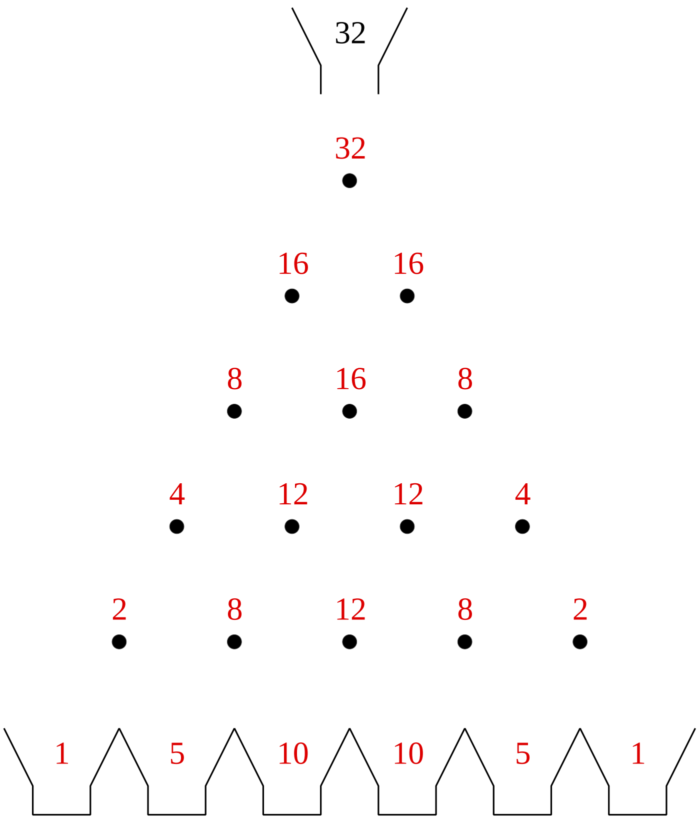
(ii) The distribution of the balls across the baskets is,

1 3 3 1

[2 marks]

Answer 3

(i) The number of balls arriving at each pin is,



[6 marks]

(ii) The distribution of the balls across the baskets is,

1 5 10 10 5 1

[2 marks]

Answer 4

Starting Number of Balls	Distribution of Balls in Baskets
4	1 2 1
8	1 3 3 1
16	1 4 6 4 1
32	1 5 10 10 5 1

[4 marks]

Answer 5

The distribution of the balls across the baskets will be,

1 6 15 20 15 6 1

[2 marks]

Answer 6

$(1 + x)^0 = 1$

$(1 + x)^1 = 1 + 1x$

$(1 + x)^2 = 1 + 2x + 1x^2$

$(1 + x)^3 = 1 + 3x + 3x^2 + 1x^3$

$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + 1x^4$

$(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + 1x^5$

$(1 + x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + 1x^6$

[6 marks]

Answer 7

Pascal's Triangle is an array of numbers.
Any non-edge number is the sum of the number immediately above it to the left and the number immediately above it to the right.
The numbers in the array are called the “binomial coefficients”

Each row gives the coefficients of the powers of x in the expansion of $(1 + x)^n$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

[4 marks]

Lesson 2

Additional Mathematics
A-Level Pure Mathematics : Year 1
Binomial Expansion

2.1 Making Use Of Pascal's Triangle

Pascal's Triangle, initially seeming curiosity, turns out to be extraordinarily useful. It is a hub at which many topics converge; probability and algebra initially and, later on, trigonometry and complex numbers.

For example, it is **the way** to expand the brackets of $(1 + x)^5$

Teaching Video: http://www.NumberWonder.co.uk/v9062/2.mp4

[3 marks]

2.2 Exercise

Marks Available : 50

Question 1

Here is Pascal's Triangle up to Row 9

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4	1					
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	
1	9	36	84	126	126	84	36	9	1

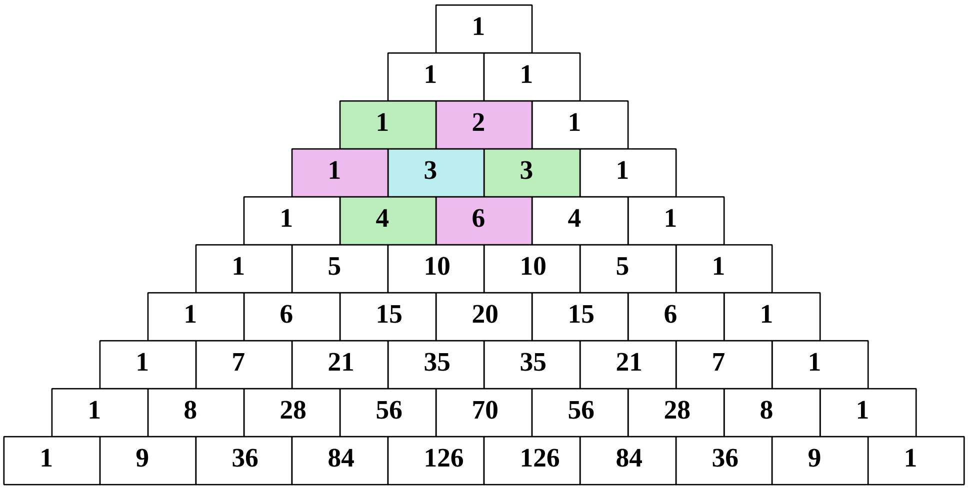
Using Pascal's Triangle, or otherwise, expand the brackets of $(1 + x)^8$

[3 marks]

Question 2

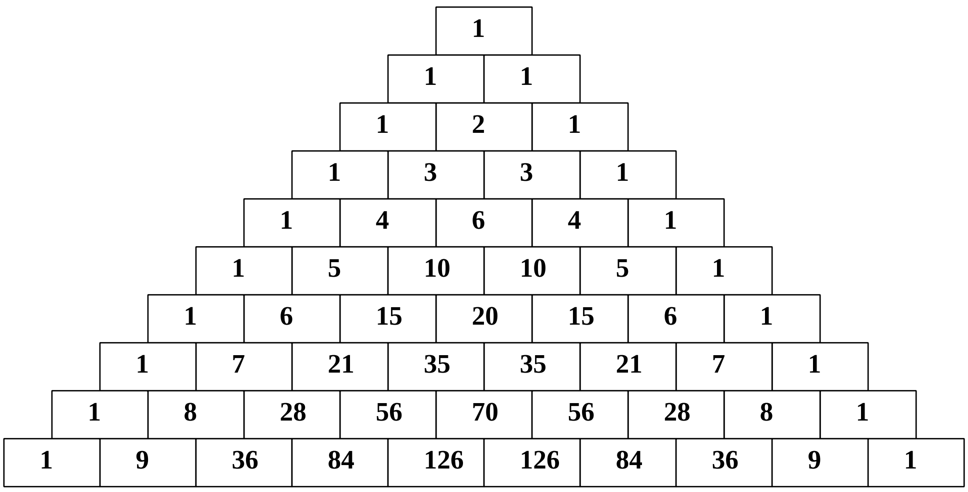
Pick a number in the diagonal that goes, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...
Colour it blue.
Colour the six bricks around it in alternating green and red.
Multiply together the three red numbers and, separately, multiply together the three green numbers.
Now add together the blue, the red product and the green product.
The number you get is always of a certain type.
What type of number do you always get ?

For Example



$$3 + 6 \times 1 \times 2 + 4 \times 1 \times 3 = 27$$

A copy of Pascal's Triangle for you to do a few on...



[6 marks]

Question 3

Using Pascal's Triangle, or otherwise, expand the brackets of $(1 + x)^{10}$

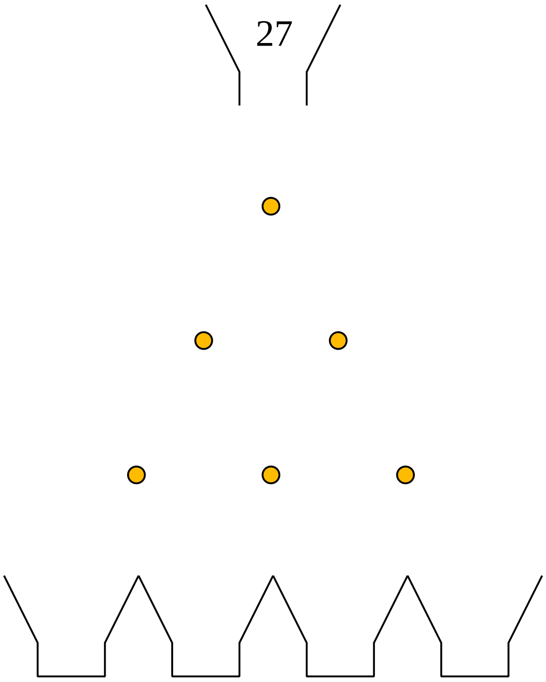
[4 marks]

Question 4

Twenty-seven steel ball bearings are released into a pinball machine.
At each pin the probability a ball will go left is half that of it going right.

(i) Above each pin write the total number of balls arriving at that pin

(ii) Show that the resulting distribution in the bottom baskets is 1 6 12 8



[4 marks]

Question 5

On the following copy of Pascal's Triangle, highlight **all** the Triangular Numbers.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

[3 marks]

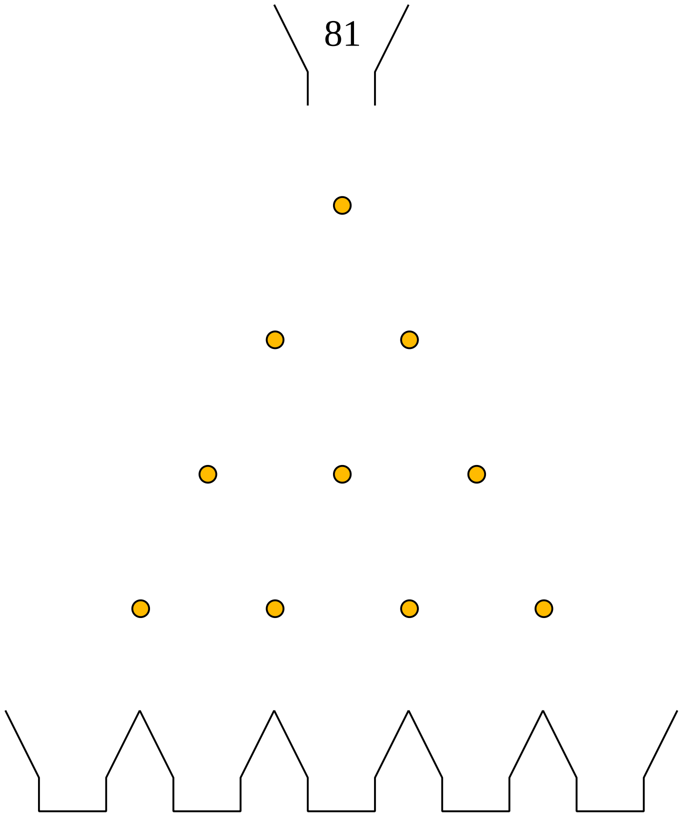
Question 6

Eighty-one steel ball bearings are released into a pinball machine.

At each pin the probability a ball will go left is half that of it going right.

(i) Above each pin write the total number of balls arriving at that pin

(ii) Work out how many ball bearings end up in each of the bottom baskets.



[6 marks]

Question 7

You are given that

$$(1-x)^0 = 1$$
$$(1-x)^1 = 1 - 1x$$
$$(1-x)^2 = 1 - 2x + 1x^2$$
$$(1-x)^3 = 1 - 3x + 3x^2 - 1x^3$$
$$(1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + 1x^4$$

Using Pascal's Triangle, or otherwise, expand the brackets of $(1-x)^7$

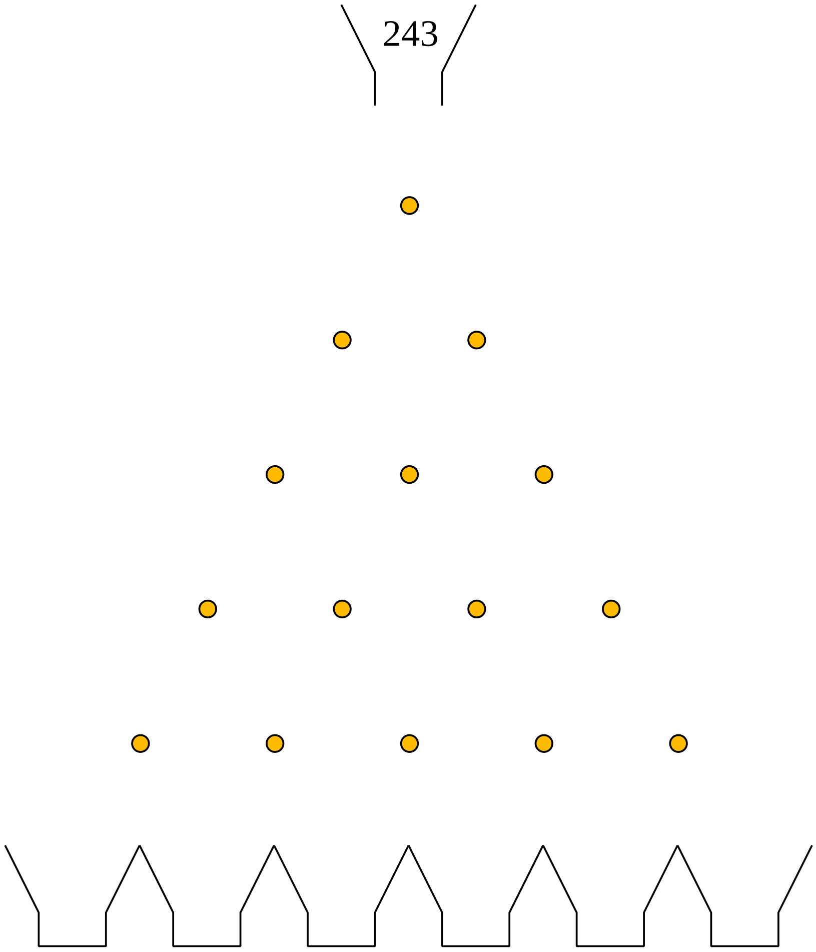
[3 marks]

Question 8

Two Hundred and Forty-Three steel ball bearings are released into a pinball machine.
At each pin the probability a ball will go left is half that of it going right.

(i) Above each pin write the total number of balls arriving at that pin.

(ii) Work out how many ball bearings end up in each of the bottom baskets.



[11 marks]

Question 9

Use the table below to summarize your results from Q4, Q6 and Q8

Starting Number of Balls	Distribution of Balls in Baskets
27	1 6 12 8
81	
243	

[4 marks]

Teaching Interlude

The pattern summarized in the table is not so easy to spot.
There is another triangular array of numbers that results but it turns out it's not needed.
Before looking at why it's not needed observe that there is connection with expanding the brackets of $(1 + 2x)^n$ and the pinball machine ball distributions.
For example, with $n = 3$,

$$(1 + 2x)^3 = 1 + 6x + 12x^2 + 8x^3$$

where the 1 6 12 8 was the distribution from the pinball machine of question 4.

Clearly, as mathematicians we'd like an easy way to expand expressions such as $(1 + 2x)^3$ and the good news is that we only need Pascal's Triangle to do it.

Here is how,

- Use Pascal's Triangle for $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$
- Replace the x in that with $(2x)$

Thus, $$(1 + 2x)^3 = 1 + 3(2x) + 3(2x)^2 + (2x)^3$$
$$= 1 + 6x + 12x^2 + 8x^3$$

And there is the "1 6 12 8"

Question 10

Show how to use the above method to expand $(1 + 2x)^4$
This should match the distribution from question 6

[3 marks]

Question 11

Show how to use the above method to expand $(1 + 2x)^5$
This should match the distribution from question 8

[3 marks]

2.3 Answers

2.3.1 Solution to 2.2 Making Use of Pascal's Triangle (Teaching Video)

To expand the brackets of $(1 + x)^5$ use Row 5 of Pascal's Triangle.
Note that Row 5 is actually the 6th row, but the 1st row is Row Zero.

1									
1	**1**								
1	**2**	**1**							
1	**3**	**3**	**1**						
1	**4**	**6**	**4**	**1**					
1	**5**	**10**	**10**	**5**	**1**				
1	**6**	**15**	**20**	**15**	**6**	**1**			
1	**7**	**21**	**35**	**35**	**21**	**7**	**1**		
1	**8**	**28**	**56**	**70**	**56**	**28**	**8**	**1**	
1	**9**	**36**	**84**	**126**	**126**	**84**	**36**	**9**	**1**

$$(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

[3 marks]

2.3.2 Solutions to 2.2 Exercise

Answer 1

$$(1 + x)^8 = 1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$$

[3 marks]

Answer 2

The result is always a cube number.

[6 marks]

Out of interest, here is an A-Level standard proof.
A more accessible version of this (for GCSE students) should be easy to formulate as it only involves triangular numbers, counting numbers and the number 1.

$$n^3 = \binom{n+1}{2}\binom{n-1}{1}\binom{n}{0} + \binom{n+1}{1}\binom{n}{2}\binom{n-1}{0} + \binom{n}{1}$$

Proof

$$\binom{n+1}{2}\binom{n-1}{1}\binom{n}{0} = \frac{(n+1)n}{2} \times (n-1) \times 1$$

$$= \frac{(n+1)n(n-1)}{2}$$

$$\binom{n+1}{1}\binom{n}{2}\binom{n-1}{0} = (n+1) \times \frac{n(n-1)}{2} \times 1$$

$$= \frac{(n+1)n(n-1)}{2}$$

The sum is thus,

$$\binom{n+1}{2}\binom{n-1}{1}\binom{n}{0} + \binom{n+1}{1}\binom{n}{2}\binom{n-1}{0} + \binom{n}{1}$$

$$= \frac{(n+1)n(n-1)}{2} + \frac{(n+1)n(n-1)}{2} + n$$

$$= (n+1)n(n-1) + n$$

$$= \left(n^2 - 1\right)n + n$$

$$= n^3 - n + n$$

$$= n^3 \qquad \square$$

Answer 3

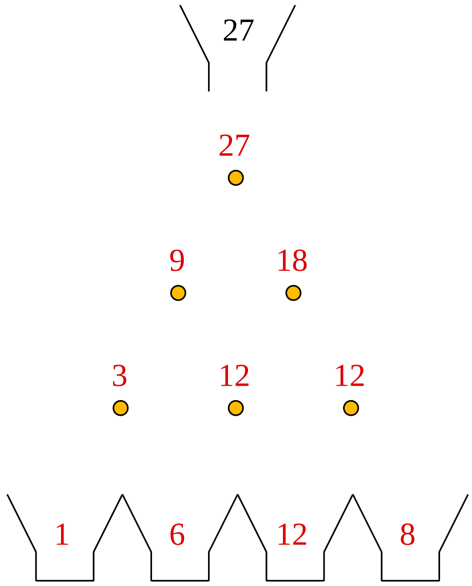
$$(1 + x)^{10}$$

$$= 1 + 10x + 45x^2 + 120x^3 + 210x^4 + 252x^5 + 210x^6 + 120x^7 + 45x^8 + 10x^9 + x^{10}$$

[4 marks]

Answer 4

(i) and **(ii)**



[4 marks]

Answer 5

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

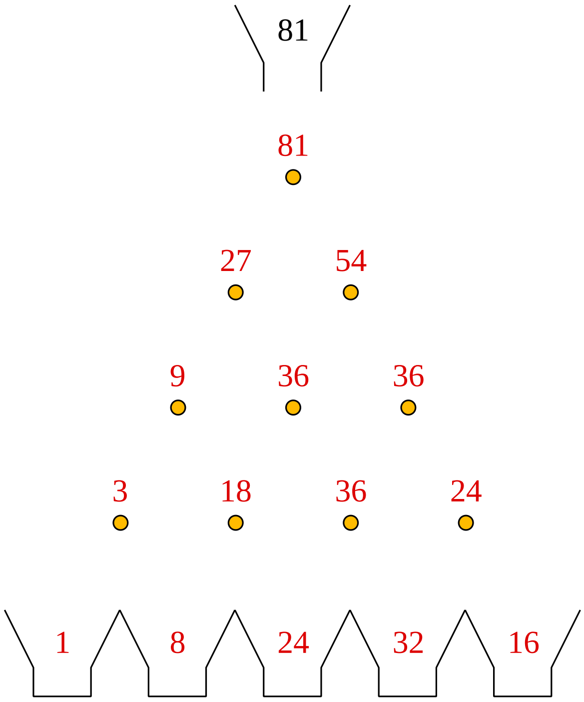
1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

[3 marks]

Answer 6



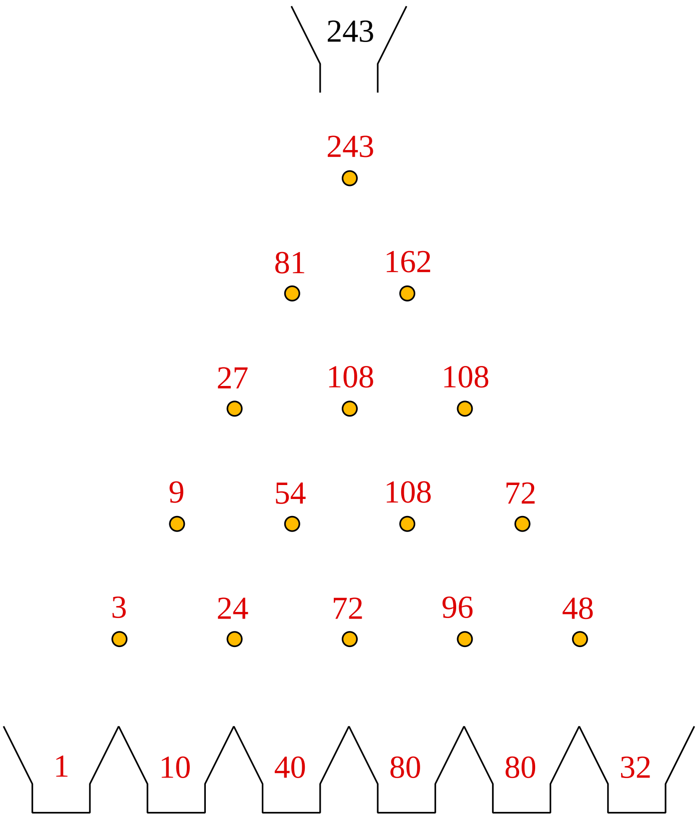
[6 marks]

Answer 7

$$(1 - x)^7 = 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$$

[3 marks]

Answer 8



[11 marks]

Answer 9

Starting Number of Balls	Distribution of Balls in Baskets
27	1 6 12 8
81	1 8 24 32 16
243	1 10 40 80 80 32

[4 marks]

Answer 10

$$(1 + 2x)^4 = 1 + 4(2x) + 6(2x)^2 + 4(2x)^3 + (2x)^4$$

$$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$$

And there is the "1 8 24 32 16"

[3 marks]

Answer 11

$$(1 + 2x)^5 = 1 + 5(2x) + 10(2x)^2 + 10(2x)^3 + 5(2x)^4 + (2x)^5$$

$$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$$

And there is the "1 10 40 80 80 32"

[3 marks]

Lesson 3

3.1 Pascal To The Max

Pascal's Triangle is turning out to be useful, not just in expanding the brackets of expressions of the form $(1 + x)^n$ for positive integer values of n, but also of expressions of the form $(1 + ax)^n$ where a is a constant.

For example, it is **the way** to expand the brackets of $(1 - 4x)^3$

Teaching Video: http://www.NumberWonder.co.uk/v9062/3.mp4

[4 marks]

3.2 Exercise

Marks Available : 40

Question 1

Expand the brackets, $(1 + 3x)^4$

[4 marks]

Question 2

Expand the brackets, $(1 - 2x)^5$

[4 marks]

Question 3

Pick a number in the diagonal that goes, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...
Colour it blue.
Colour the six bricks around it in alternating red, green, purple.
For each colour work out the difference between the two numbers of that colour
Add up these three colour differences.
The number you get is always of a certain type.
What type of number do you always get ?

For Example

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

(21 − 1) + (7 − 5) + (15 − 1) = 36

A copy of Pascal's Triangle for you to do a few on...

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

[4 marks]

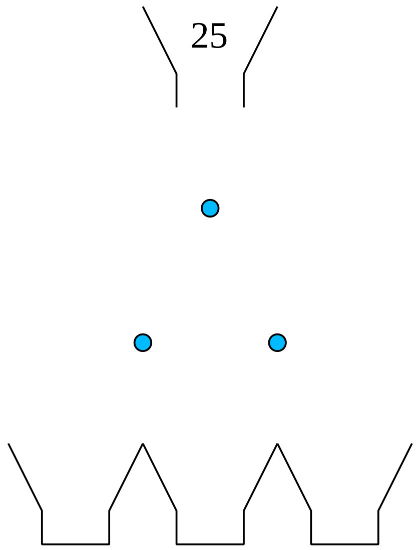
Question 4

Twenty-five steel ball bearings are released into a pinball machine.

At each pin the probability a ball will go left is $\frac{2}{5}$ and go right is $\frac{3}{5}$

(i) Above each pin write the total number of balls arriving at that pin

(ii) Show that the resulting distribution in the bottom baskets is 4 12 9



[4 marks]

Question 5

Expand $(1 + 6x)^7$ in ascending powers of x up to the term in x^3 and simplify.

Note :

- Ascending powers of x means $a + bx + cx^2 + dx^3 + ex^4 + \ldots$
- The constants $a, b, c, d, e, \ldots.$ are called the coefficients
- This question only wants $a + bx + cx^2 + dx^3 \Leftarrow$ STOP !
- Simplify means bracketed things like $(6x)^3$ in the answer will cost marks

[4 marks]

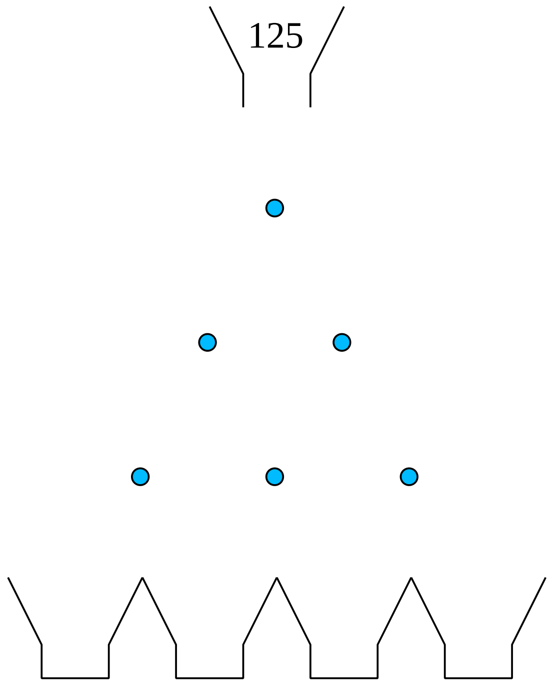
Question 6

Twenty-five steel ball bearings are released into a pinball machine.

At each pin the probability a ball will go left is $\frac{2}{5}$ and go right is $\frac{3}{5}$

(i) Above each pin write the total number of balls arriving at that pin.

(ii) Work out how many ball bearings end up in each of the bottom baskets.



[6 marks]

Question 7

Additional Mathematics Examination Question from June 2010, Q2 (OCR)

Expand $(1 - x)^{12}$ in ascending powers of x up to the term in x^3

Simplify your answer.

[3 marks]

Question 8

A-Level Examination Question from January 2019, C12, Q5 (Edexcel)

Find the first 4 terms, in ascending powers of *x*, of the expansion of

$$\left(1 - \frac{x}{2}\right)^8$$

Give each term in its simplest form

[4 marks]

Question 9

You are given that

$$\begin{aligned}
(x + y)^0 &= 1 \\
(x + y)^1 &= 1x + 1y \\
(x + y)^2 &= 1x^2 + 2xy + 1y^2 \\
(x + y)^3 &= 1x^3 + 3x^2y + 3xy^2 + 1y^3 \\
(x + y)^4 &= 1x^4 + 4x^3y + 6x^2y^2 - 4xy^3 + 1y^4
\end{aligned}$$

Using Pascal's Triangle, or otherwise, expand the brackets of $(x + y)^7$

[3 marks]

Question 10

Using the result that,

$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

expand the brackets of,

$$(2 + 3x)^3$$

It you get this correct, and got question 6 correct, the two answers will have the same distribution of balls in baskets as coefficients of x.

If they are not the same, one or the other or both are wrong !

[4 marks]

3.3 Answers

3.3.1 Solution to 3.1 Pascal To The Max (Teaching Video)

From Pascal's Triangle,

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3$$

with the x replaced with $(-4x)$ is obtained,

$$\begin{aligned}(1 - 4x)^3 &= 1 + 3(-4x) + 3(-4x)^2 + (-4x)^3 \\ &= 1 - 12x + 48x^2 - 64x^3\end{aligned}$$

[4 marks]

3.3.2 Solutions to 3.2 Exercise

Answer 1
From Pascal's Triangle,

$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

with the x replaced with $(3x)$ is obtained,

$$\begin{aligned}(1 + 3x)^4 &= 1 + 4(3x) + 6(3x)^2 + 4(3x)^3 + (3x)^4 \\ &= 1 + 12x + 54x^2 + 108x^3 + 81x^4\end{aligned}$$

[4 marks]

Answer 2
From Pascal's Triangle,

$$(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

with the x replaced with $(-2x)$ is obtained,

$$\begin{aligned}(1 - 2x)^5 &= 1 + 5(-2x) + 10(-2x)^2 + 10(-2x)^3 + 5(-2x)^4 + (-2x)^5 \\ &= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5\end{aligned}$$

[4 marks]

Answer 3
The result is always a square number.

[4 marks]

Out Of Interest : A Proof

This could be proven using the formula for Triangular Numbers.

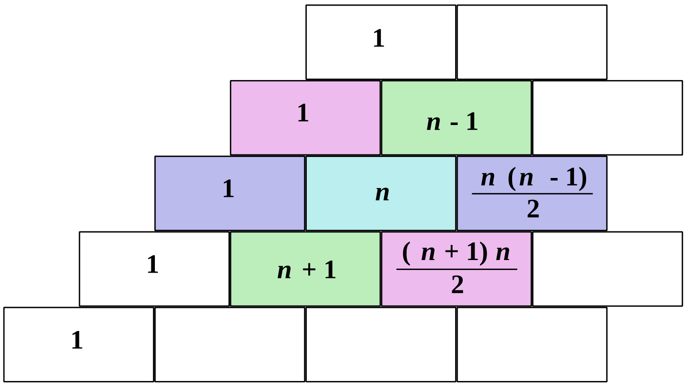
$$T_{ROW} = \frac{(Row)(Row - 1)}{2}$$

In Row 6, for example,

$$T_6 = \frac{6 \times 5}{2} = 15$$

Which is correct : 1 6 15 20 15 6 1

In general



$$\left(\frac{(n+1)n}{2} - 1\right) + \left((n+1) - (n-1)\right) + \left(\frac{n(n-1)}{2} - 1\right)$$

$$= \frac{n^2 + n - 2}{2} + \frac{4}{2} + \frac{n^2 - n - 2}{2}$$

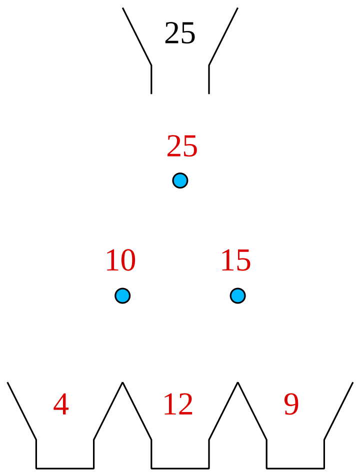
$$= \frac{n^2 + n - 2 + 4 + n^2 - n - 2}{2}$$

$$= \frac{2n^2}{2}$$

$$= n^2 \qquad \square$$

Answer 4

(i) and **(ii)**



[4 marks]

Answer 5

From Pascal's Triangle,

$$(1 + x)^7 = 1 + 7x + 21x^2 + 35x^3 + \ldots$$

with the x replaced with $(6x)$ is obtained,

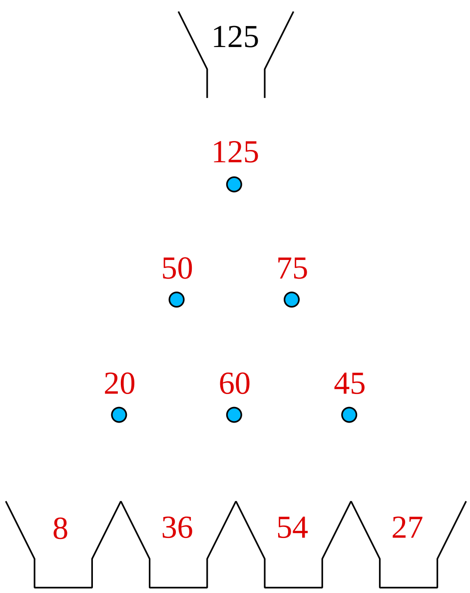
$$(1 + 6x)^7 = 1 + 7(6x) + 21(6x)^2 + 35(6x)^3 + \ldots$$

$$= 1 + 42x + 756x^2 + 7560x^3 + \ldots$$

[4 marks]

Answer 6

(i) and **(ii)**



[6 marks]

Answer 7

The sign will alternate,

$$1 - 12x + 66x^2 - 220x^3 + \ldots$$

[3 marks]

Answer 8

From Pascal's Triangle,

$$(1 + x)^8 = 1 + 8x + 28x^2 + 56x^3 + \ldots$$

with the x replaced with $\left(-\dfrac{x}{2}\right)$ is obtained,

$$\left(1 - \frac{x}{2}\right)^8 = 1 + 8\left(-\frac{x}{2}\right) + 28\left(-\frac{x}{2}\right)^2 + 56\left(-\frac{x}{2}\right)^3 + \ldots$$

$$= 1 - 4x + 7x^2 - 7x^3 + \ldots$$

[4 marks]

Answer 9

$$(x + y)^7$$

$$= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

[3 marks]

Answer 10

$$(2 + 3x)^3 = 2^3 + 3(2)^2(3x) + 3(2)(3x)^2 + (3x)^3$$

$$= 8 + 36x + 54x^2 + 27x^3$$

[4 marks]

Lesson 4

Additional Mathematics
A-Level Pure Mathematics : Year 1
Binomial Expansion

4.1 Slay Those Brackets

When expanding brackets using Pascal's Triangle the working can become cluttered thus making it easy to make an error, and also hard to tack one down. However, there is a robust and reliable way to set out the calculations. That's the focus of this lesson.

4.2 Example

Expand the brackets of $(3 - 2x)^4$

The Solution :

Teaching Video: http://www.NumberWonder.co.uk/v9062/4.mp4

$$
\begin{array}{llllllll}
(3 - 2x)^4 = & & 1 & \times & (\quad) & \times & (\quad) \\
& + & 4 & \times & (\quad) & \times & (\quad) \\
& + & 6 & \times & (\quad) & \times & (\quad) \\
& + & 4 & \times & (\quad) & \times & (\quad) \\
& + & 1 & \times & (\quad) & \times & (\quad)
\end{array}
$$

Simplifying,

$$(3 - 2x)^4 =$$

[4 marks]

4.3 Exercise

Marks Available : 30

Question 1

Use the template below to help expand the brackets of $(5 + 3x)^3$

$$
\begin{aligned}
(5 + 3x)^3 = \quad & 1 \times (\quad) \times (\quad) \\
+ \quad & 3 \times (\quad) \times (\quad) \\
+ \quad & 3 \times (\quad) \times (\quad) \\
+ \quad & 1 \times (\quad) \times (\quad)
\end{aligned}
$$

Simplifying,

$$(5 + 3x)^3 =$$

[4 marks]

Question 2

Use the template below to help expand the brackets of $(2 - 5x)^5$

$$
\begin{aligned}
(2 - 5x)^5 = \quad & 1 \times (\quad) \times (\quad) \\
+ \quad & 5 \times (\quad) \times (\quad) \\
+ \quad & 10 \times (\quad) \times (\quad) \\
+ \quad & 10 \times (\quad) \times (\quad) \\
+ \quad & 5 \times (\quad) \times (\quad) \\
+ \quad & 1 \times (\quad) \times (\quad)
\end{aligned}
$$

Simplifying,

$$(2 - 5x)^5 =$$

[5 marks]

Question 3

Expand the brackets of $\left(4 + x^2\right)^3$

[4 marks]

Question 4

(i) Expand the brackets and simplify the terms;

$$\left(x + \frac{1}{x}\right)^4$$

[4 marks]

(ii) In the expansion, if a term is without any positive integer power of *x* then that term is said to be "independent of *x*"
Which is the term of the expansion which is *independent of x* ?

[1 mark]

Question 5

On your calculator find the button marked ${}^{n}C_{r}$

It's often, but not always, above the multiplication button.

Use this button to work out the following,

${}^{4}C_{0}$ ${}^{4}C_{1}$ ${}^{4}C_{2}$ ${}^{4}C_{3}$ ${}^{4}C_{4}$

Hopefully, you have now realised that Pascal's Triangle is in your calculator !

[2 marks]

Question 6

A-Level Examination Question from January 2015, C12, Q4(a) (Edexcel)

Find the first 4 terms in ascending powers of x of the binomial expansion of

$$\left(2 + \frac{x}{4}\right)^{10}$$

giving each term in its simplest form

[5 marks]

Question 7

Additional Mathematics Exam Question from June 2018, Paper 1, Q9(i) (Cambridge)

Find the first 3 terms in the expansion of $\left(2x - \frac{1}{16x}\right)^8$ in descending powers of x

[5 marks]

4.4 Answers

4.4.1 Solution to 4.3 Example (Teaching Video)

$$
\begin{aligned}
(3-2x)^4 = \; & 1 \times (3)^4 \times (-2x)^0 \\
+ \; & 4 \times (3)^3 \times (-2x)^1 \\
+ \; & 6 \times (3)^2 \times (-2x)^2 \\
+ \; & 4 \times (3)^1 \times (-2x)^3 \\
+ \; & 1 \times (3)^0 \times (-2x)^4
\end{aligned}
$$

So,

$$(3-2x)^4 = 81 - 216x + 216x^2 - 96x^3 + 16x^4$$

[4 marks]

2.5.3 Solutions to 2.4 Exercise

Answer 1

$$
\begin{aligned}
(5+3x)^3 = \; & 1 \times (5)^3 \times (3x)^0 \\
+ \; & 3 \times (5)^2 \times (3x)^1 \\
+ \; & 3 \times (5)^1 \times (3x)^2 \\
+ \; & 1 \times (5)^0 \times (3x)^3
\end{aligned}
$$

Simplifying,

$$(5+3x)^3 = 125 + 225x + 135x^2 + 27x^3$$

[4 marks]

Answer 2

$$
\begin{aligned}
(2-5x)^5 = \; & 1 \times (2)^5 \times (-5x)^0 \\
+ \; & 5 \times (2)^4 \times (-5x)^1 \\
+ \; & 10 \times (2)^3 \times (-5x)^2 \\
+ \; & 10 \times (2)^2 \times (-5x)^3 \\
+ \; & 5 \times (2)^1 \times (-5x)^4 \\
+ \; & 1 \times (2)^0 \times (-5x)^5
\end{aligned}
$$

Simplifying,

$$(2-5x)^5 = 32 - 400x + 2000x^2 - 5000x^3 + 6250x^4 - 3125x^5$$

[5 marks]

Answer 3

$$\left(4 + x^2\right)^3 = 1 \times (4)^3 \times \left(x^2\right)^0$$
$$+ \ 3 \times (4)^2 \times \left(x^2\right)^1$$
$$+ \ 3 \times (4)^1 \times \left(x^2\right)^2$$
$$+ \ 1 \times (4)^0 \times \left(x^2\right)^3$$

Simplifying,

$$\left(4 + x^2\right)^3 = 64 + 48x^2 + 12x^4 + x^6$$

[4 marks]

Answer 4

(i)

$$\left(x + \frac{1}{x}\right)^4 = 1 \times (x)^4 \times \left(\frac{1}{x}\right)^0$$
$$+ \ 4 \times (x)^3 \times \left(\frac{1}{x}\right)^1$$
$$+ \ 6 \times (x)^2 \times \left(\frac{1}{x}\right)^2$$
$$+ \ 4 \times (x)^1 \times \left(\frac{1}{x}\right)^3$$
$$+ \ 1 \times (x)^0 \times \left(\frac{1}{x}\right)^4$$

Simplifying,

$$\left(x + \frac{1}{x}\right)^4 = x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$$

[4 marks]

(ii) The term which is independent of x is the 6

[1 mark]

Answer 5

$^4C_0 = 1$ $^4C_1 = 4$ $^4C_2 = 6$ $^4C_3 = 4$ $^4C_4 = 1$

[2 marks]

Answer 6

$^{10}C_0 = 1 \qquad {}^{10}C_1 = 10 \qquad {}^{10}C_2 = 45 \qquad {}^{10}C_3 = 120$

$$\begin{aligned}\left(2 + \frac{x}{4}\right)^{10} = \quad & 1 \times (2)^{10} \times \left(\frac{x}{4}\right)^0 \\ + \; & 10 \times (2)^9 \times \left(\frac{x}{4}\right)^1 \\ + \; & 45 \times (2)^8 \times \left(\frac{x}{4}\right)^2 \\ + \; & 120 \times (2)^7 \times \left(\frac{x}{4}\right)^3 \\ + \; & \ldots\end{aligned}$$

Simplifying,

$$\left(2 + \frac{x}{4}\right)^4 = 1024 + 1280x + 720\,x^2 + 240\,x^3 + \ldots$$

[5 marks]

Answer 7

$$\begin{aligned}\left(2x - \frac{1}{16x}\right)^8 = \quad & 1 \times (2x)^8 \times \left(-\frac{1}{16x}\right)^0 \\ + \; & 8 \times (2x)^7 \times \left(-\frac{1}{16x}\right)^1 \\ + \; & 28 \times (2x)^6 \times \left(-\frac{1}{16x}\right)^2 \\ + \; & \ldots\end{aligned}$$

Simplifying,

$$\left(2x - \frac{1}{16x}\right)^8 = 256\,x^8 - 64\,x^6 + 7\,x^4 + \ldots$$

[5 marks]

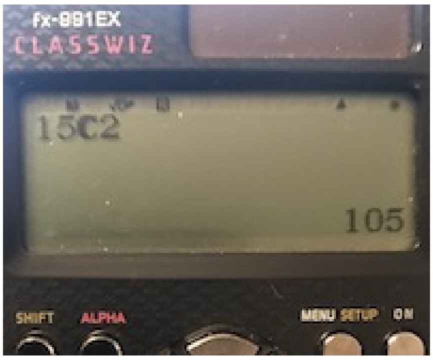
Lesson 5

Additional Mathematics
A-Level Pure Mathematics : Year 1
Binomial Expansion

5.1 Homework

Marks Available : 23

Question 1



Check that your calculator gives $^{15}C_0 = 1, \quad ^{15}C_1 = 15$ and $^{15}C_2 = 105$

(i) Use your calculator to write down the value of,

$^{15}C_3 \qquad ^{15}C_4 \qquad ^{15}C_5$

[3 marks]

(ii) Hence write out the first six terms of the the expansion of,

$$(1 + x)^{15}$$

[2 marks]

Question 2

A-Level Examination Question from January 2006, P2, Q1(a) (Edexcel)

Write down the binomial expansion, in ascending powers of x of $(1 + 6x)^4$

[3 marks]

Question 3

A-Level Examination Question from January 2008, C2, Q3(a) (Edexcel)

Find the first 4 terms of the expansion of $\left(1 + \frac{x}{2}\right)^{10}$ giving each term in its simplest form.

Hint :

$$\left(1 + \frac{x}{2}\right)^{10} = {}^{10}C_0 \times (1)^{10} \times \left(\frac{x}{2}\right)^0$$
$$+ {}^{10}C_1 \times (1)^9 \times \left(\frac{x}{2}\right)^1$$
$$+ {}^{10}C_2 \times (1)^8 \times \left(\frac{x}{2}\right)^2$$
$$+ {}^{10}C_3 \times (1)^7 \times \left(\frac{x}{2}\right)^3$$
$$+ \ldots$$

[4 marks]

Question 4

A-Level Examination Question from January 2010, C2, Q1 (Edexcel)

Find the first 3 terms, in ascending powers of *x*, of binomial expansion of

$$(3 - x)^6$$

and simplify each term.

[4 marks]

Question 5

A-Level Examination Question from October 2016, C12, Q5 (Edexcel)

(a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of,

$$\left(3 - \frac{ax}{2}\right)^5$$

where a is a positive constant. Give each term in its simplest form.

[4 marks]

Given that, in the expansion, the coefficient of x is equal to the coefficient of x^3

(b) find the exact value of a in its simplest form

[3 marks]

5.2 Answers to 5.1 Homework

Answer 1

(i)

$^{15}C_3 = 455 \qquad ^{15}C_4 = 1365 \qquad ^{15}C_5 = 3003$

[3 marks]

(ii) $(1 + x)^{15} = 1 + 15x + 105x^2 + 455x^3 + 1365x^4 + 3003x^5$

[2 marks]

Answer 2

$$
\begin{aligned}
(1 + 6x)^4 = \; & 1 \times (1)^4 \times (6x)^0 \\
+ \; & 4 \times (1)^3 \times (6x)^1 \\
+ \; & 6 \times (1)^2 \times (6x)^2 \\
+ \; & 4 \times (1)^1 \times (6x)^3 \\
+ \; & 1 \times (1)^0 \times (6x)^4
\end{aligned}
$$

Simplifying,

$$(1 + 6x)^4 = 1 + 24x + 216x^2 + 864x^3 + 1296x^4$$

[4 marks]

Answer 3

$$
\begin{aligned}
\left(1 + \frac{x}{2}\right)^{10} = \; & 1 \times (1)^{10} \times \left(\frac{x}{2}\right)^0 \\
& + 10 \times (1)^9 \times \left(\frac{x}{2}\right)^1 \\
& + 45 \times (1)^8 \times \left(\frac{x}{2}\right)^2 \\
& + 120 \times (1)^7 \times \left(\frac{x}{2}\right)^3 \\
& + \ldots
\end{aligned}
$$

Simplifying,

$$\left(1 + \frac{x}{2}\right)^{10} = 1 + 5x + \frac{45}{4}x^2 + 15x^3 + \ldots$$

[4 marks]

Answer 4

$$(3 - x)^6 = 1 \times (3)^6 \times (-x)^0$$
$$+ 6 \times (3)^5 \times (-x)^1$$
$$+ 15 \times (3)^4 \times (-x)^2$$
$$+ \ ...$$

Simplifying,

$$(3 - x)^6 = 729 - 1458x + 1215\,x^2 - \ ...$$

[4 marks]

Answer 5

(a)

$$\left(3 - \frac{ax}{2}\right)^5 = 1 \times (3)^5 \times \left(-\frac{ax}{2}\right)^0$$
$$+ 5 \times (3)^4 \times \left(-\frac{ax}{2}\right)^1$$
$$+ 10 \times (3)^3 \times \left(-\frac{ax}{2}\right)^2$$
$$+ 10 \times (3)^2 \times \left(-\frac{ax}{2}\right)^3$$
$$+ \ ...$$

Simplifying,

$$\left(3 - \frac{ax}{2}\right)^5 = 243 - \frac{405}{2}ax + \frac{135}{2}\,a^2\,x^2 - \frac{45}{4}\,a^3\,x^3 + \ ...$$

[4 marks]

(b)

$$-\ \frac{45}{4}\,a^3 = -\,\frac{405}{2}\,a$$

Multiply through by (-4)

$$45\,a^3 = 810\,a$$

$a^2 = 18$ OK to divide by a as“a is a positive constant”

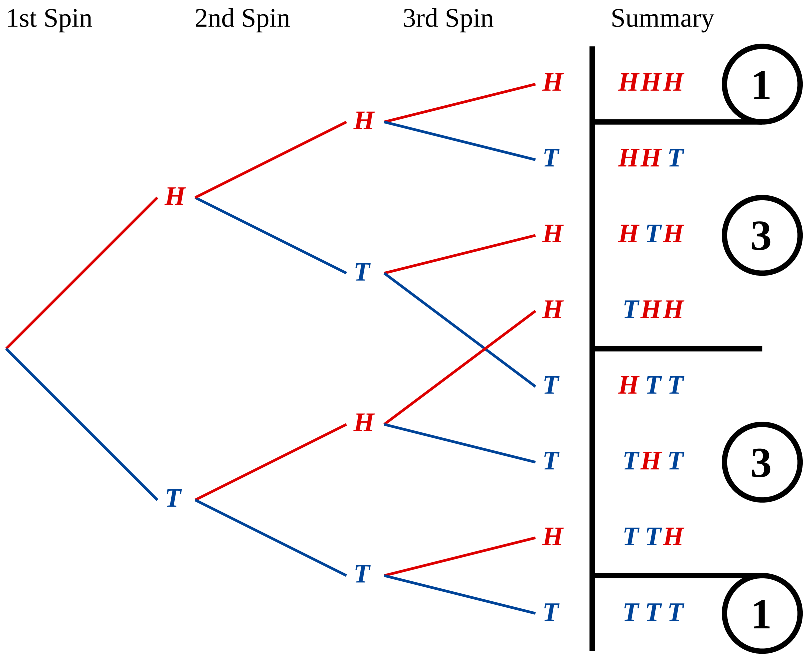
$$a = 3\sqrt{2}$$

[3 marks]

Lesson 6

6.1 Spinning Coins

A coin is spun three times
Each time it may spin heads, *H*, or tails, *T*
Here is a tree diagram that shows all the possible outcomes,



Of particular interest is the summary where there is,

- 1 way of spinning three heads and zero tails
- 3 ways of spinning two heads and one tail (if the order does not matter)
- 3 ways of spinning one head and two tails (if the order does not matter)
- 1 way of spinning zero heads and three tails

It is not a coincidence that these numbers are Row 3 of Pascal's Triangle.
The implication is that Pascal's Triangle is going to be very useful in tackling probability questions beyond GCSE.

From the fact that $(x + y)^3 = 1x^3y^0 + 3x^2y^1 + 3x^1y^2 + 1x^0y^3$
with x replaced with H and y replaced with T we get,

$$(H + T)^3 = 1H^3T^0 + 3H^2T^1 + 3H^1T^2 + 1H^0T^3$$

This is an mathematical algebraic model for spinning a coin three times !

The term $3H^2T^1$ tells you that there are three ways of spinning two heads and one tail (if the order in which the heads and tails occur does not matter)

6.2 Example

A coin is biased such that, when it is spun, the probability of it showing heads is twice the probability of it showing tails;

In other words, $$P(H) = \frac{2}{3}$$ $$P(T) = \frac{1}{3}$$

When spun three times what is the probability that two heads and one tail occur ?

http://www.NumberWonder.co.uk/v9062/6.mp4

[4 marks]

6.3 Exercise

Marks Available : 50

Question 1

A coin is biased such that, when it is spun, the probability of it showing heads is three times the probability of it showing tails;

In other words, $$P(H) = \frac{3}{4}$$ $$P(T) = \frac{1}{4}$$

When spun four times what is the probability that two heads and two tails occur ? Give your answer as an exact fraction.

[4 marks]

Question 2

A coin is biased such that, when it is spun, the probability of it showing heads is half the probability of it showing tails;

In other words,

$$P(H) = \frac{1}{3}$$

$$P(T) = \frac{2}{3}$$

When spun six times what is the probability that four heads and two tails occur ?
Give your answer correct to three decimal places.

[4 marks]

Question 3

A coin is biased such that

$$P(heads) = 0.2$$

$$P(tails) = 0.8$$

It is spun three times.
What is the most likely number of tails to occur ?
Back up your answer with mathematical calculations.

[5 marks]

Question 4
A fair coin is spun nine times.
What is the percentage probability that exactly seven heads and two tails are spun ?

[4 marks]

Question 5
A fair coin is spun six times.
Show that the probability that more heads are spun than tails is roughly a third.
Show your working.

[5 marks]

Question 6

When a learner driver takes their practical driving test, experience suggests that the probability of them passing is 0.6

(i) What is the probability of them failing ?

[1 mark]

(ii) At a test centre, ten students take their driving test in a day.
What is the percentage probability that exactly eight pass ?

[5 marks]

Question 7

In a multiple choice test each question presents four options, only one of which is correct. In a twelve question test what is the probability that a person, picking answers at random, gets less than two questions wrong ?

Give your (very small) answer correct to three significant figures.

[5 marks]

Question 8

Additional Mathematics Examination Question from June 2018, Q9 (a) (OCR)

The proportion of people who are left-handed is 20%

For a group of 10 students chosen at random, use the binomial distribution to find the probability that

(i) no student is left handed

[2 marks]

(ii) exactly 4 students are left handed

[3 marks]

Question 9

Additional Mathematics Examination Question from June 2013, Q6 (OCR)

Amanda throws 3 fair dice.

What is the probability that,

(i) exactly 2 sixes are thrown

[3 marks]

(ii) at least 1 six is thrown ?

[3 marks]

Question 10

Additional Mathematics Examination Question from June 2017, Q8 (OCR)

Four ordinary six-sided dice are rolled.

Find the probability that at least 2 sixes are obtained.

[6 marks]

6.4 Answers

6.4.1 Solution to 6.2 Example (Teaching Video)

The model for spinning a coin three times is, $(H + T)^3$

$$
\begin{aligned}
(H + T)^3 = \quad & 1 \times (H)^3 \times (T)^0 \\
+ \; & 3 \times (H)^2 \times (T)^1 \\
+ \; & 3 \times (H)^1 \times (T)^2 \\
+ \; & 1 \times (H)^0 \times (T)^3
\end{aligned}
$$

Simplifying, $\quad (H + T)^3 = 1\,H^3\,T^0 + 3\,H^2\,T^1 + 3\,H^1\,T^2 + 1\,H^0\,T^3$

The term corresponding to two heads and one tail is coloured red.

$$
\begin{aligned}
\mathrm{P}(2\ heads,\ 1\ tail) &= 3\,H^2\,T^1 \\
&= 3 \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right) \\
&= \frac{4}{9}
\end{aligned}
$$

[4 marks]

6.4.2 Solutions to 6.3 Exercise

Answer 1

The model for spinning a coin four times is, $(H + T)^4$

$$
\begin{aligned}
(H + T)^4 = \quad & 1 \times (H)^4 \times (T)^0 \\
+ \; & 4 \times (H)^3 \times (T)^1 \\
+ \; & 6 \times (H)^2 \times (T)^2 \\
+ \; & 4 \times (H)^1 \times (T)^3 \\
+ \; & 1 \times (H)^0 \times (T)^4
\end{aligned}
$$

Simplifying,

$$(H + T)^4 = 1\,H^4\,T^0 + 4\,H^3\,T^1 + 6\,H^2\,T^2 + 4\,H^1\,T^3 + 1\,H^0\,T^4$$

The term corresponding to two heads and two tails is coloured red.

$$
\begin{aligned}
\mathrm{P}(2\ heads,\ 2\ tails) &= 6\,H^2\,T^2 \\
&= 6 \times \left(\frac{3}{4}\right)^2 \times \left(\frac{1}{4}\right)^2 \\
&= \frac{27}{128} \qquad \text{(About 0.211)}
\end{aligned}
$$

[4 marks]

Answer 2

The model for spinning a coin six times is, $(H + T)^6$

$$
\begin{aligned}
(H + T)^6 = \quad & 1 \times (H)^6 \times (T)^0 \\
+ \; & 6 \times (H)^5 \times (T)^1 \\
+ \; & 15 \times (H)^4 \times (T)^2 \\
+ \; & 20 \times (H)^3 \times (T)^3 \\
+ \; & 15 \times (H)^2 \times (T)^4 \\
+ \; & 6 \times (H)^1 \times (T)^5 \\
+ \; & 1 \times (H)^0 \times (T)^6
\end{aligned}
$$

Simplifying,

$$(H + T)^6$$
$$= 1\,H^6\,T^0 + 6\,H^5\,T^1 + 15\,H^4\,T^2 + 20\,H^3\,T^3 + 15\,H^2\,T^4 + 6\,H^1\,T^5 + 1\,H^0\,T^6$$

The term corresponding to four heads and two tails is coloured red.

$$
\begin{aligned}
\mathrm{P}(4\,heads,\ 2\,tails) &= 15\,H^4\,T^2 \\
&= 15 \times \left(\frac{1}{3}\right)^4 \times \left(\frac{2}{3}\right)^2 \\
&= \frac{20}{243} \qquad \text{(About 0.082)}
\end{aligned}
$$

[4 marks]

Answer 3

The model for spinning a coin three times is, $(H + T)^3$

$$
\begin{aligned}
(H + T)^3 = \quad & 1 \times (H)^3 \times (T)^0 \\
+ \; & 3 \times (H)^2 \times (T)^1 \\
+ \; & 3 \times (H)^1 \times (T)^2 \\
+ \; & 1 \times (H)^0 \times (T)^3
\end{aligned}
$$

Simplifying,

$$(H + T)^3 = 1\,H^3\,T^0 + 3\,H^2\,T^1 + 3\,H^1\,T^2 + 1\,H^0\,T^3$$

$\mathrm{P}(3H,\ 0T) = 1 \times 0.2^3 = 0.008$

$\mathrm{P}(2H,\ 1T) = 3 \times 0.2^2 \times 0.8 = 0.096$

$\mathrm{P}(1H,\ 2T) = 3 \times 0.2 \times 0.8^2 = 0.384$

$\mathrm{P}(0H,\ 3T) = 1 \times 0.8^3 = 0.512$ $\Leftarrow$ *** Three tails is most likely ***

[5 marks]

Answer 4

The model for spinning a coin nine times is, $(H + T)^9$

$$
\begin{aligned}
(H+T)^9 = & \quad 1 \times (H)^9 \times (T)^0 \\
& + 9 \times (H)^8 \times (T)^1 \\
& + 36 \times (H)^7 \times (T)^2 \\
& + \ldots
\end{aligned}
$$

Simplifying,

$$(H + T)^9 = 1\,H^9\,T^0 + 9\,H^8\,T^1 + \color{red}{36\,H^7\,T^2} + \ldots$$

The term corresponding to seven heads and two tails is coloured red.

$$
\begin{aligned}
P(7\ heads,\ 2\ tails) &= 36\,H^7\,T^2 \\
&= 36 \times \left(\frac{1}{2}\right)^7 \times \left(\frac{1}{2}\right)^2 \\
&= \frac{9}{128} \qquad \text{(About 0.070)}
\end{aligned}
$$

The percentage probability is 7%

[4 marks]

Answer 5

The model for spinning a coin six times is, $(H + T)^6$

$$
\begin{aligned}
(H+T)^6 = & \quad 1 \times (H)^6 \times (T)^0 \\
& + 6 \times (H)^5 \times (T)^1 \\
& + 15 \times (H)^4 \times (T)^2 \\
& + \ldots
\end{aligned}
$$

Simplifying,

$$(H + T)^6 = 1\,H^6\,T^0 + 6\,H^5\,T^1 + 15\,H^4\,T^2 + \ldots$$

$$
\begin{aligned}
P(H > T) &= 1 \times \left(\frac{1}{2}\right)^6 + 6\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right) + 15\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2 \\
&= 22 \times \left(\frac{1}{2}\right)^6 \\
&= \frac{22}{64} \\
&= \frac{11}{32} \\
&= 0.34375 \qquad \text{which is about a third}
\end{aligned}
$$

[5 marks]

Answer 6

(i) 0.4

[1 mark]

(ii) The model for spinning a coin ten times is, $(H + T)^{10}$

Passing the driving test can be likened to *heads*, failing it likened to *tails*

Thus,

$$
\begin{aligned}
(P + F)^{10} = \quad & 1 \times (P)^{10} \times (F)^{0} \\
+ \quad & 10 \times (P)^{9} \times (F)^{1} \\
+ \quad & 45 \times (P)^{8} \times (F)^{2} \\
+ \quad & \ldots
\end{aligned}
$$

Simplifying,

$$(P + F)^{10} = 1\,P^{10}\,F^{0} + 10\,P^{9}\,F^{1} + 45\,P^{8}\,F^{2} + \ldots$$

The term corresponding to eight passing (and two failing) is coloured red.

$$
\begin{aligned}
\mathrm{P}(\,8\,pass,\;2\,fail\,) &= 45\,P^{8}\,F^{2} \\
&= 45 \times (0.6)^{8} \times (0.4)^{2} \\
&= 0.12093
\end{aligned}
$$

The percentage probability is 12%

[5 marks]

Answer 7

The model for spinning a coin twelve times is, $(H + T)^{12}$

Getting a question correct can be likened to *heads*, getting it wrong it likened to *tails*

Thus,

$$
\begin{aligned}
(C + W)^{12} = \quad & 1 \times (C)^{12} \times (W)^{0} \\
+ \quad & 12 \times (C)^{11} \times (W)^{1} \\
+ \quad & \ldots
\end{aligned}
$$

Simplifying,

$$(C + W)^{12} = 1\,C^{12}\,W^{0} + 12\,C^{11}\,W^{1} + \ldots$$

The first two term corresponding to “less than two questions wrong”

$$
\begin{aligned}
\mathrm{P}(\,12\;correct\text{ or }11\;correct\,) &= 1 \times 0.25^{12} + 12 \times 0.25^{11} \times 0.75 \\
&= (0.25)^{12} + 12 \times (0.25)^{11} \times 0.75 \\
&= 0.000\,002\,205
\end{aligned}
$$

[5 marks]

Answer 8

The model for spinning a coin ten times is, $(H + T)^{10}$

Being Right-Handed can be likened to *heads*, being Left-Handed likened to *tails*

Thus,

$$
\begin{aligned}
(R + L)^{10} = \; & 1 \times (R)^{10} \times (L)^{0} \\
+ \; & 10 \times (R)^{9} \times (L)^{1} \\
+ \; & 45 \times (R)^{8} \times (L)^{2} \\
+ \; & 120 \times (R)^{7} \times (L)^{3} \\
+ \; & 210 \times (R)^{6} \times (L)^{4} \\
+ \; & \ldots
\end{aligned}
$$

Simplifying,

$$(R + L)^{10} = 1\,R^{10}L^{0} + 10\,R^{9}L^{1} + 45\,R^{8}L^{2} + 120\,R^{7}L^{3} + 210\,R^{6}L^{4} + \ldots$$

(i) The term corresponding to no student is left handed is coloured green.

$$
\begin{aligned}
P(10R,\ 0L) &= 1\,R^{10}L^{0} \\
&= 1 \times (0.8)^{10} \times (0.2)^{0} \\
&= 0.107
\end{aligned}
$$

[2 marks]

(ii) The term corresponding to exactly 4 students are left handed is coloured red.

$$
\begin{aligned}
P(6R,\ 4L) &= 210\,R^{6}L^{4} \\
&= 210 \times (0.8)^{6} \times (0.2)^{4} \\
&= 0.088
\end{aligned}
$$

[3 marks]

Answer 9

The model for spinning a coin three times is, $(H + T)^3$

Throwing a six can be likened to *heads*, throwing something else likened to *tails*

Thus,

$$\begin{array}{llllllll} (S+E)^3 = & & 1 & \times & (S)^3 & \times & (E)^0 \\ & + & 3 & \times & (S)^2 & \times & (E)^1 \\ & + & 3 & \times & (S)^1 & \times & (E)^2 \\ & + & 1 & \times & (S)^0 & \times & (E)^3 \end{array}$$

Simplifying,

$$(S + E)^3 = 1\,S^3 E^0 + 3\,S^2 E^1 + 3\,S^1 E^2 + 1\,S^0 E^3$$

(i) The term corresponding to exactly 2 sixes are thrown is coloured green.

$$\begin{aligned} P(2\,S,\ 1\,E) &= 3\,S^2 E^1 \\ &= 3 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^1 \\ &= \frac{5}{72} \qquad \text{(About 0.0694)} \end{aligned}$$

[3 marks]

(ii) The terms corresponding to at least 1 six is thrown are coloured green or red.

$$\begin{aligned} P(\text{At least 1 six}) &= 1 - 1\,S^0 E^3 \\ &= 1 - 1 \times \left(\frac{1}{6}\right)^0 \times \left(\frac{5}{6}\right)^3 \\ &= \frac{91}{216} \qquad \text{(About 0.421)} \end{aligned}$$

[3 marks]

Answer 10

The model for spinning a coin four times is, $(H + T)^4$

Throwing a six can be likened to *heads*, throwing something else likened to *tails*

Thus,

$$
\begin{aligned}
(S + E)^4 = & \quad 1 \times (S)^4 \times (E)^0 \\
& + 4 \times (S)^3 \times (E)^1 \\
& + 6 \times (S)^2 \times (E)^2 \\
& + 4 \times (S)^1 \times (E)^3 \\
& + 1 \times (S)^0 \times (E)^4
\end{aligned}
$$

Simplifying,

$$(S + E)^4 = 1\,S^4 E^0 + 4\,S^3 E^1 + 6\,S^2 E^2 + 4\,S^1 E^3 + 1\,S^0 E^4$$

The terms corresponding to at least 2 sixes are coloured red.

$$
\begin{aligned}
\text{P (at least 2 sixes)} &= 1\,S^4 E^0 + 4\,S^3 E^1 + 6\,H^2 T^2 \\
&= \left(\frac{1}{6}\right)^4 + 4\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^1 + 6 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2 \\
&= \frac{1}{1296} + \frac{20}{1296} + \frac{150}{1296} \\
&= \frac{171}{1296} \\
&= \frac{19}{144} \qquad \text{(About 0.132)}
\end{aligned}
$$

[6 marks]

Lesson 7

Additional Mathematics
A-Level Pure Mathematics : Year 1
Binomial Expansion

7.1 Investigating Binomial Probability

Binomial means “two state”: A real life “two state” situation is the flipping of a coin.

Suppose a biased coin is weighted with a probability of 0.36 of landing tails.
It is flipped 8 times. What is the probability that it lands tails exactly twice ?

To answer this question we begin by wanting to know how many different ways 2 tails and 6 heads can occur.

Method 1 : List them and observe the list has 28 entries
Takes time and is tricky to get correct.

Method 2 : Write out Pascal's Triangle
Observe that Row 8, Column 2 is 28
Less time but a tedious job for just one number

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

Method 3 : Use a calculator

$$^{8}C_{2} = 28$$

Having got the 28...
Each of the 28 items in the list will be a multiplication of $0.36 \times 0.36 \times 0.64 \times 0.64 \times 0.64 \times 0.64 \times 0.64 \times 0.64$
(with those same numbers occurring in different orders)
Thus the quick calculation is;

$$28 \times 0.36^2 \times 0.64^6$$
$$= 28 \times 0.1296 \times 0.0687$$
$$= 0.249 \quad \text{(Give answers to 3 decimal places)}$$

So, when this biased coin is flipped eight times, there is a 25% probability it will land tails exactly twice.

H H H H H H T T

H H H H H T H T
H H H H H T T H

H H H H T H H T
H H H H T H T H
H H H H T T H H

H H H T H H H T
H H H T H H T H
H H H T H T H H
H H H T T H H H

H H T H H H H T
H H T H H H T H
H H T H H T H H
H H T H T H H H
H H T T H H H H

H T H H H H H T
H T H H H H T H
H T H H H T H H
H T H H T H H H
H T H T H H H H
H T T H H H H H

T H H H H H H T
T H H H H H T H
T H H H H T H H
T H H H T H H H
T H H T H H H H
T H T H H H H H
T T H H H H H H

7.2 Exercise

Marks Available : 60

Question 1

Here is Pascal's Triangle, left justified.

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

(i) Circle the entry in Row 7, Column 5

Notice this is in the 8th row and the 6th column !

[1 mark]

(ii) Use your calculator to determine $^{7}C_{5}$

*This should be your part **(i)** answer*

[1 mark]

Question 2

Here is Pascal's Triangle, left justified.

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4	1					
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	
1	9	36	84	126	126	84	36	9	1

(i) Circle the entry in Row 4, Column 0

[1 mark]

(ii) Circle the entry in Row 5, Column 5

[1 mark]

(iii) Circle the entry in Row 9, Column 3

[1 mark]

(iv) Find a solution pair (n, r) to

$$^{n}C_{r} = 70$$

[1 mark]

Question 3

Use your calculator to determine ${}^{13}C_5$

[1 mark]

Question 4

A coin is flipped 6 times.

(i) Describe which calculator buttons you would press in order to determine the number of ways exactly 2 tails could be obtained ?

[1 mark]

(ii) Write out enough of Pascal's Triangle so that you can then draw a circle around the number in it, corresponding to your part (i) answer

[2 marks]

Question 5

A coin is flipped 20 times.

In how many ways can exactly 13 tails be obtained ?

HINT : Use a calculator.

[2 marks]

Question 6

A coin is flipped 40 times.

In how many ways can exactly 5 heads be obtained ?

[2 marks]

Question 7

A biased coin is weighted such that it has a probability of 0.45 of landing tails.
It is flipped 8 times. What is the probability that it lands tails exactly thrice ?
Give your answer to 3 decimal places.

[4 marks]

Question 8

A typist has a probability of 0.99 of typing each letter in a sentence correctly.
What is the percentage probability of exactly two mistakes in a sentence containing 180 letters, if mistakes are made at random ? Give your answer to 3 decimal places.

[4 marks]

Question 9

In a box of Smarties™ there are eight different colours which normally occur in equal proportions. Sebastian is given 24 Smarties™, and blue ones are his favourite.
Assume these come from a very large box.

(i) How many blue Smarties™ would he expect to get (on average) ?

[1 mark]

(ii) What is the probability that he gets this number ?

[4 marks]

(iii) What is the probability that he gets fewer than expected ?

[4 marks]

(iv) What is the probability that he gets more than expected ?

[1 mark]

(v) Explain why the assumption was made that the 24 Smarties™ given to Sebastian came from a very large box ?

[2 marks]

Question 10

(i) Determine ${}^{6}C_{2}$

[1 mark]

(ii) Determine ${}^{6}C_{4}$

[1 mark]

(iii) Explain with the help of Pascal's Triangle why ${}^{6}C_{2} = {}^{6}C_{4}$

[2 marks]

(iv) Explain, in your own words, why ${}^{n}C_{0} = {}^{n}C_{n}$

[2 marks]

(v) Explain, in your own words, why ${}^{n}C_{2} = {}^{n}C_{n-2}$

[2 marks]

(vi) What formula can be written down for ${}^{n}C_{m}$ that generalises the observations made above ?

[2 marks]

The next question is about obtaining the probability distribution curve for a simple coin flipping situation.

Question 11

A biased coin is weighted such that it has a probability of 0.4 of landing tails.
It is flipped 6 times.

(i) Show that the probability of exactly 4 tails being obtained is 0.138.
That is, about 14% probable.

[2 marks]

(ii) Work out the probability of exactly 0, 1, 2, 3, 5 and 6 tails being obtained.
Present your solutions in the table below.

N° of tails	0	1	2	3	4	5	6
Probability					14%		

[8 marks]

(iii) Present your table of results as an accurate bar chart.

[6 marks]

7.3 Answers

Answer 1

(i)

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

[1 mark]

(ii) $^{7}C_5 = 21$

[1 mark]

Answer 2

(i), **(ii)** and **(iii)**

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4	1					
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	
1	9	36	84	126	126	84	36	9	1

[1, 1, 1 marks]

(iv) $^{8}C_4 = 70$ (Other answers are possible)

[1 mark]

Answer 3

$^{13}C_5 = 1287$

[1 mark]

Answer 4

(i) $^{6}C_{2} \; = \; 15$

(ii)

$$\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
1 & 5 & 10 & 10 & 5 & 1 & \\
1 & 6 & 15 & 20 & 15 & 6 & 1
\end{array}$$

[1, 1, 2 marks]

Answer 5

$^{20}C_{13} \; = \; 77\,520$

[2 marks]

Answer 6

$^{40}C_{5} \; = \; 658\,008$

[2 marks]

Answer 7

$^{8}C_{3} \times 0.45^{3} \times 0.55^{5} \; = \; 0.257$

[4 marks]

Answer 8

$^{180}C_{2} \times 0.99^{178} \times 0.01^{2} \; = \; 0.269$

That is, 26.9%

[4 marks]

Answer 9

(i) 3

[1 mark]

(ii) $^{24}C_3 \times \left(\frac{1}{8}\right)^3 \times \left(\frac{7}{8}\right)^{21} = 0.239$

[4 marks]

(iii)

$$^{24}C_2 \times \left(\frac{1}{8}\right)^2 \times \left(\frac{7}{8}\right)^{22}$$

$$+ \, ^{24}C_1 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{7}{8}\right)^{23}$$

$$+ \, ^{24}C_0 \times \left(\frac{1}{8}\right)^0 \times \left(\frac{7}{8}\right)^{24}$$

$$= 0.229 + 0.139 + 0.041 = 0.409$$

[4 marks]

(iv) 0.352

[1 mark]

(v) The “very large box” means that the probability of getting any given colour remains constant as Smarties™ are taken out of the box and eaten. Yum !

[2 marks]

Answer 10

(i) and **(ii)** $^6C_2 = {}^6C_4 = 15$

[1, 1 mark]

(iii) Pascal's triangle is symmetrical about the central coefficients

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

[2 marks]

(iv) $^nC_0$ (which is the left outside edge) $= {}^nC_n$ (which is the right outside edge)

[2 marks]

(v) $^nC_2$ (Δ-number in column 1) $= {}^nC_{n-2}$ (Δ-number in column $n - 1$)

[2 marks]

(vi) $^nC_m = {}^nC_{n-m}$ for $m \leqslant n$

[2 marks]

Answer 11

(i)

$$P(4\text{ tails}) = {}^6C_4 \times 0.4^4 \times (1 - 0.4)^{6-4}$$
$$= 15 \times 0.4^4 \times 0.6^2$$
$$= 0.138 \quad \text{i.e. About } 14\% \quad \square$$

[2 marks]

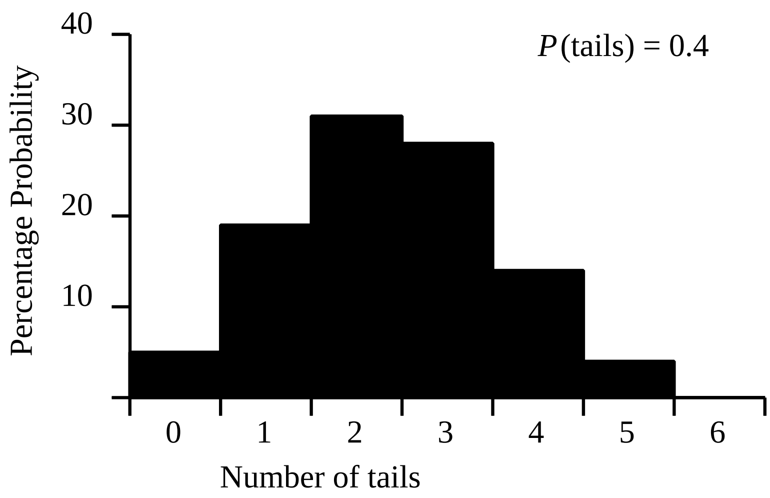
(ii)

N° of tails	0	1	2	3	4	5	6
Probability	5%	19%	31%	28%	14%	4%	0%

Rounding errors mean we've got 101% but as the table is being used to produce a bar chart, let's not worry about it !

[8 marks]

(iii)



[6 marks]

Lesson 8

8.1 The Binomial Theorem

Previously,[†] expanding the brackets of $(3 - 2x)^4$ was tackled.
Here is a reminder of the recommended method of setting out a solution;

$$\begin{aligned}(3 - 2x)^4 = \quad & 1 \times (3)^4 \times (-2x)^0 \\ + \; & 4 \times (3)^3 \times (-2x)^1 \\ + \; & 6 \times (3)^2 \times (-2x)^2 \\ + \; & 4 \times (3)^1 \times (-2x)^3 \\ + \; & 1 \times (3)^0 \times (-2x)^4\end{aligned}$$

$$\therefore \quad (3 - 2x)^4 = 81 - 216x + 216x^2 - 96x^3 + 16x^4$$

This method can be generalised to give The Binomial Theorem;

$$\begin{aligned}(a + b)^n = \; & {}^nC_0 \times (a)^n \times (b)^0 \\ + \; & {}^nC_1 \times (a)^{n-1} \times (b)^1 \\ + \; & {}^nC_2 \times (a)^{n-2} \times (b)^2 \\ + \; & {}^nC_3 \times (a)^{n-3} \times (b)^3 \\ + \; & \ldots \\ + \; & {}^nC_r \times (a)^{n-r} \times (b)^r \\ + \; & \ldots \\ + \; & {}^nC_{n-1} \times (a)^1 \times (b)^{n-1} \\ + \; & {}^nC_n \times (a)^0 \times (b)^n\end{aligned}$$

The Binomial Theorem (for integer n)

$$(a + b)^n = a^n + {}^nC_1 a^{n-1} b + \ldots + {}^nC_r a^{n-r} b^r + \ldots + b^n$$

This is provided to candidates in Additional Mathematics and A-Level exams

† In Lesson 4. Example 4.2, along with a Teaching Video solution

8.2 Exercise

Marks Available : 40

Question 1

Expand the brackets

$$(5x - x^2)^3 = \quad 1 \times (\quad) \times (\quad)$$

$$+ \quad 3 \times (\quad) \times (\quad)$$

$$+ \quad 3 \times (\quad) \times (\quad)$$

$$+ \quad 1 \times (\quad) \times (\quad)$$

So,

$$(5x - x^2)^3 =$$

[5 marks]

Question 2

In mathematics the pling symbol “!” means “factorial”

$5! = 5 \times 4 \times 3 \times 2 \times 1$

Check, on your calculator, that 5! = 120 (Using the special button marked “!”)

Work out

(i) 6! **(ii)** 10! **(iii)** 13!

[3 marks]

Question 3

My calculator gives an error message when I try to work out 94!

The answer is way too big for my calculator to handle.

What is the smallest number, x, for which your calculator cannot calculate $x!$?

[1 mark]

Question 4

Given calculator limitations, there is no point using one to work out

$$\frac{5000!}{4999!}$$

However, the answer, if you think about it, is easy to obtain using brain power.

What is the answer ?

[2 marks]

Question 5

Work out the following using a mixture of cunning and calculator

(i) $\dfrac{100!}{99\,!}$ **(ii)** $\dfrac{101!}{98!}$ **(iii)** $\dfrac{2021!}{2018!}$

[1, 2, 3 marks]

Question 6

Simplify

$$\frac{(\,n + 4\,)!}{(\,n + 1\,)!}$$

[3 marks]

Question 7

Simplify

$$\frac{(\,n + 1\,)!}{(\,n - 1\,)!}$$

[3 marks]

Question 8

The numbers in Pascal's Triangle are given by

$$^{n}C_{r} = \frac{n!}{r!\ (n - r)!}$$

Use this to derive a simplified expression for $^{n}C_{2}$

[4 marks]

Question 9

Expand the brackets;

$$(2 + x)\ (4 + 5x)^{3}$$

[8 marks]

Question 10

Further Mathematics Specimen Exam Paper 1, June 2020, Q16 (AQA)

The coefficient of the x^4 term in the expansion of $(2x + a)^6$ is 60

Work out the possible values of a

[5 marks]

8.3 Answers

Additional Mathematics
A-Level Pure Mathematics : Year 1
Binomial Expansion

Answer 1

$$\begin{aligned}(5x - x^2)^3 = \; & 1 \times (5x)^3 \times (-x^2)^0 \\ + \; & 3 \times (5x)^2 \times (-x^2)^1 \\ + \; & 3 \times (5x)^1 \times (-x^2)^2 \\ + \; & 1 \times (5x)^0 \times (-x^2)^3\end{aligned}$$

So,

$$(5x - x^2)^3 = 125x^3 - 75x^4 + 15x^5 - x^6$$

[5 marks]

Answer 2

(i) 720 **(ii)** 3 628 800 **(iii)** 6 227 020 800

[3 marks]

Answer 3

70! is a likely answer

(Biggest that works on CASIO fx-991EX ClassWiz is 69! = 1.7×10^{98})

[1 mark]

Answer 4

$$\frac{5000!}{4999!} = \frac{5000 \times 4999 \times 4998 \times \ldots \times 2 \times 1}{4999!}$$

$$= \frac{5000 \times 4999!}{4999!}$$

$$= 5000$$

[2 marks]

Answer 5

(i) 100 **(ii)** 999 900 **(iii)** 8 242 405 980

[1, 2, 3 marks]

Answer 6

$$\frac{(n+4)!}{(n+1)!} = \frac{(n+4)(n+3)(n+2)(n+1)!}{(n+1)!}$$

$$= (n+4)(n+3)(n+2)$$

[3 marks]

Answer 7

$$\frac{(n+1)!}{(n-1)!} = \frac{(n+1)(n)(n-1)!}{(n-1)!}$$

$$= (n+1)\,n$$

[3 marks]

Answer 8

$$ {}^{n}C_2 = \frac{n!}{2!\,(n-2)!} $$

$$ = \frac{n\,(n-1)\,(n-2)!}{2!\,(n-2)!} $$

$$ = \frac{n\,(n-1)}{2} $$

This is a formula for triangular numbers so the above is a proof that column 2 of Pascal's triangle will contain the triangular numbers.

[4 marks]

Answer 9

$$
\begin{aligned}
(4 + 5x)^3 = {} & 1 \times (4)^3 \times (5x)^0 \\
& + 3 \times (4)^2 \times (5x)^1 \\
& + 3 \times (4)^1 \times (5x)^2 \\
& + 1 \times (4)^0 \times (5x)^3
\end{aligned}
$$

So $\quad (4 + 5x)^3 = 64 + 240x + 300x^2 + 125x^3$

$$
\begin{aligned}
(2 + x)\,(4 + 5x)^3 &= (2 + x)\left(64 + 240x + 300x^2 + 125x^3\right) \\
&= 128 + 480x + 600x^2 + 250x^3 \\
&\qquad + 64x + 240x^2 + 300x^3 + 125x^4 \\
&= 128 + 544x + 840x^2 + 550x^3 + 125x^4
\end{aligned}
$$

[8 marks]

Answer 10

$$
\begin{aligned}
(2x + a)^6 = {} & 1 \times (2x)^6 \times (a)^0 \\
& + 6 \times (2x)^5 \times (a)^1 \\
& + 15 \times (2x)^4 \times (a)^2 \qquad \Leftarrow\Leftarrow\Leftarrow \\
& + \ldots
\end{aligned}
$$

$$ \therefore\ 240\,a^2 x^4 = 60\,x^4 $$

$$ a^2 = \frac{1}{4} $$

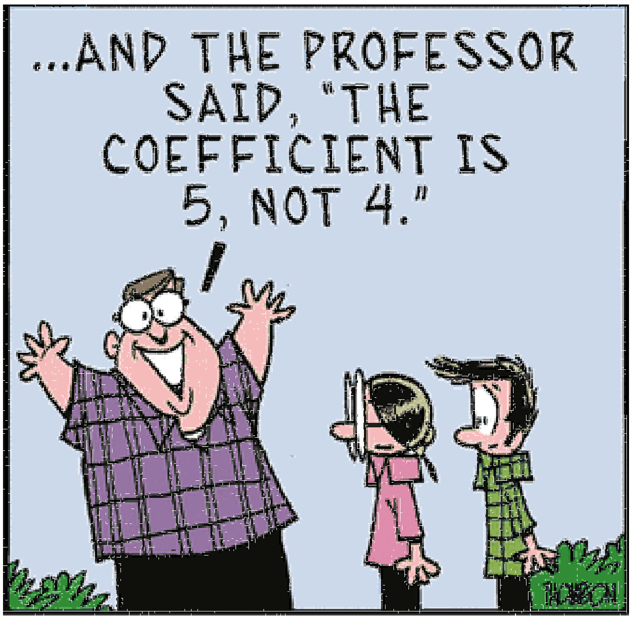
$$ a = \pm\frac{1}{2} $$

[5 marks]

Lesson 9

Additional Mathematics
A-Level Pure Mathematics : Year 1
Binomial Expansion

9.1 Coefficient Conundrums



When using the binomial theorem the expansion of the brackets typically leads to a series in ascending powers of x. The number in front of any given power of x is termed the coefficient of that power of x.
For example, consider the expansion;

$$(1 - x)^5 = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$$

- The coefficient of x^4 is 5
- The coefficient of x^3 is -10

Exam questions sometimes include a puzzle to do with the coefficients.

9.2 Example

When $(a + 3x)^3$ is expanded the coefficient of x is the same as that for x^3
What are the two possible values of the constant a ?

Teaching Video: http://www.NumberWonder.co.uk/v9062/9.mp4

[5 marks]

9.3 Exercise

Marks Available : 40

Question 1

When $(a + 5x)^4$ is expanded the coefficient of x is the same as that for x^3
What are the two possible values of the constant a ?

[5 marks]

Question 2

When $(2 + ax)^3$ is expanded the coefficient of x is the same as that for x^3
What are the two possible values of the constant a ?

[5 marks]

Question 3

A-Level Examination Question from May 2007, Paper C2, Q3 (Edexcel)

(a) Find the first four terms, in ascending powers of x, in the binomial expansion of $(1 + kx)^6$ where k is a non-zero constant.

[5 marks]

Given that, in this expansion, the coefficients of x and x^2 are equal, find

(b) the value of k

[2 marks]

(c) the coefficient of x^3

[1 mark]

Question 4

A-Level Examination Question from June 2009, Paper C2, Q2 (Edexcel)

(a) Find the first three terms, in ascending powers of x, of the binomial expansion of $(2 + kx)^7$ where k is a constant

Give each term in its simplest form

[5 marks]

Given that the coefficient of x^2 is 6 times the coefficient of x

(b) find the value of k

[2 marks]

Question 5

(a) Find the first three terms, in ascending powers of x, of the binomial expansion of $(5 + px)^{30}$ where p is a non-zero constant
There is no need to simplify the terms.

[2 marks]

(b) Given that in this expression the coefficient of x^2 is 29 times the coefficient of x find the value of p

[4 marks]

Question 6

AS Examination Question from May 2018, Q11 (Edexcel)

(a) Find the first 3 terms in ascending powers of x, of the binomial expansion of

$$\left(2 - \frac{x}{16}\right)^9$$

giving each term in its simplest form

[5 marks]

$$f(x) = (a + bx)\left(2 - \frac{x}{16}\right)^9$$

Given that the first two terms, in ascending powers of x, in the series expansion of $f(x)$ are 128 and $36x$,

(b) find the value of a

[2 marks]

(c) find the value of b

[2 marks]

9.4 Answers

9.4.1 Solution to the 9.2 Example (Teaching Video)

$$\begin{aligned}(a + 3x)^3 &= 1 \times (a)^3 \times (3x)^0 \\ &+ 3 \times (a)^2 \times (3x)^1 \\ &+ 3 \times (a)^1 \times (3x)^2 \\ &+ 1 \times (a)^0 \times (3x)^3 \\ \\ &= a^3 + 9a^2 x + 27ax^2 + 27x^3\end{aligned}$$

The question tells us that; $9a^2 = 27$

$$a^2 = 3$$

$$a = \pm\sqrt{3}$$

[5 marks]

9.4.2 Solutions to the 9.3 Exercise

Answer 1

$$\begin{aligned}(a + 5x)^4 &= 1 \times (a)^4 \times (5x)^0 \\ &+ 4 \times (a)^3 \times (5x)^1 \\ &+ 6 \times (a)^2 \times (5x)^2 \\ &+ 4 \times (a)^1 \times (5x)^3 \\ &+ 1 \times (a)^0 \times (5x)^4 \\ \\ &= a^4 + 20a^3 x + 150a^2 x^2 + 500ax^3 + 625x^4\end{aligned}$$

The question tells us that; $20a^3 = 500a$

$$a^2 = 25$$

$$a = \pm 5$$

[5 marks]

Answer 2

$$
\begin{aligned}
(2 + ax)^3 &= 1 \times (2)^3 \times (ax)^0 \\
&\quad + 3 \times (2)^2 \times (ax)^1 \\
&\quad + 3 \times (2)^1 \times (ax)^2 \\
&\quad + 1 \times (2)^0 \times (ax)^3 \\
&= 8 + 12a\,x + 6\,a^2\,x^2 + a^3\,x^3
\end{aligned}
$$

The question tells us that; $a^3 = 12a$

$$
\begin{aligned}
a^2 &= 12 \\
a &= \pm\sqrt{12} \\
a &= \pm 2\sqrt{3}
\end{aligned}
$$

[5 marks]

Answer 3

(a)

$$
\begin{aligned}
(1 + kx)^6 &= 1 \times (1)^6 \times (kx)^0 \\
&\quad + 6 \times (1)^5 \times (kx)^1 \\
&\quad + 15 \times (1)^4 \times (kx)^2 \\
&\quad + 20 \times (1)^3 \times (kx)^3 \\
&\quad + \ldots \\
&= 1 + 6k\,x + 15\,k^2\,x^2 + 20\,k^3\,x^3 + \ldots
\end{aligned}
$$

[5 marks]

(b)

$$15k^2 = 6k$$

$$15k = 6 \qquad \text{no division by zero as told } k \neq 0$$

$$k = \frac{2}{5}$$

[2 marks]

(c)

$$
\begin{aligned}
20\,k^3 &= 20 \times \left(\frac{2}{5}\right)^3 \\
&= \frac{32}{25}
\end{aligned}
$$

[1 mark]

Answer 4

(a) $(2 + kx)^7 = 1 \times (2)^7 \times (kx)^0$

$+ \quad 7 \times (2)^6 \times (kx)^1$

$+ \quad 21 \times (2)^5 \times (kx)^2$

$+ \quad 35 \times (2)^4 \times (kx)^3$

$+ \quad ...$

$$= 128 + 448k\,x + 672\,k^2\,x^2 + 560\,k^3\,x^3 + ...$$

(only the first three terms are required)

[5 marks]

(b) $672\,k^2 = 6 \times 448k$

divide both sides by 672

$$k^2 = 4k$$

$$k^2 - 4k = 0$$

$$k (k - 4) = 0$$

Either $k = 0$ or $k = 4$

(I suspect the $k = 0$ answer was not intended by the question setter)

[2 marks]

Answer 5

(a) $(5 + px)^{30} = 1 \times (5)^{30} \times (px)^0$

$+ \quad 30 \times (5)^{29} \times (px)^1$

$+ \quad 435 \times (5)^{28} \times (px)^2$

$+ \quad 4060 \times (5)^{27} \times (px)^3$

$+ \quad ...$

$$= 5^{30} + 5^{29} \times 30p\,x + 5^{28} \times 435\,p^2\,x^2 + 5^{27} \times 4060\,p^3\,x^3 + ...$$

(only the first three terms required)

[2 marks]

(b) $5^{28} \times 435\,p^2 = 29 \times 5^{29} \times 30p$

Divide through by 5^{29} and also by p as we're told $p \neq 0$

$$87p = 29 \times 30$$

$$p = 10$$

[4 marks]

Answer 6

(a)

$$\left(2-\frac{x}{16}\right)^9 = 1\times(2)^9\times\left(-\frac{x}{16}\right)^0$$
$$+\ 9\times(2)^8\times\left(-\frac{x}{16}\right)^1$$
$$+\ 36\times(2)^7\times\left(-\frac{x}{16}\right)^2$$
$$+\ \ldots$$

$$= 512 - 144x + 18x^2 + \ldots$$

$$(a+bx)\left(2-\frac{x}{16}\right)^9 = (a+bx)\ (512 - 144x + 18x^2 + \ldots)$$
$$= 512a - 144ax + 512bx + \ldots$$

[5 marks]

(b)

$$512a = 128$$
$$a = \frac{1}{4}$$

[2 marks]

(c)

$$-144a + 512b = 36$$
$$-36 + 512b = 36$$
$$b = \frac{72}{512}$$
$$b = \frac{9}{64}$$

[2 marks]

Lesson 10

Additional Mathematics
A-Level Pure Mathematics : Year 1
Binomial Expansion

10.1 Put It In The Bin

When a mathematical expression starts to get complicated, one of the ways to reduce the complexity is to think carefully about which parts of the expression are dominant and which, if any, are so insignificant they can be thrown away.

Suppose we have a variable x which, although unknown, is likely to be small, say, definitely positive and less than 1.

Suppose $x = 0.1$ in which case:

$$0.1^0 = 1$$
$$0.1^1 = 0.1$$
$$0.1^2 = 0.01$$
$$0.1^3 = 0.001$$
$$0.1^4 = 0.0001$$
$$0.1^5 = 0.00001$$
$$0.1^6 = 0.000001$$

The implication of this observation is that, if x is sufficiently small, large powers of x can be ignored without making much difference to an answer that is only required to a few significant figures.

Here is the mathematical way of making that observation:

The Powers Of Small x Rule

$$\text{If} \quad -1 < x < 1, \ \text{then} \ x^n \to 0 \ \text{ as } n \to \infty$$

10.2 Example

(i) Find the first three terms of the binomial expansion, in ascending powers of x, of the expression $\left(2 + \frac{x}{4}\right)^8$

(ii) Use your expansion to estimate the value of 2.075^8 giving your answer to the nearest integer.

Teaching Video: http://www.NumberWonder.co.uk/v9062/10.mp4

[4, 3 marks]

10.3 Exercise

Marks Available : 50

Question 1

(i) Find the first three terms of the binomial expansion, in ascending powers of x, of the expression $\left(3 + \frac{x}{9}\right)^{11}$

[4 marks]

(iii) Use your expansion to estimate the value of 3.01^{11} giving your answer to six significant figures.

[4 marks]

Question 2

Find the constant term (the term independent of x) in the expansion of

$$\left(\frac{x}{2} + \frac{4}{x}\right)^4$$

[4 marks]

Question 3

(i) Find the first four terms, in ascending powers of x, of the binomial expansion of $(1 - 3x)^5$

Give each term in its simplest form

[4 marks]

(b) If x is small, so that x^2 and higher powers can be ignored, show that

$$(1 + x)(1 - 3x)^5 \approx 1 - 14x$$

[4 marks]

Question 4

There are many fascinating relationships hidden in Pascal's Triangle.
This question explores one of these

(i) Work out ${}^{2}C_{0} + 2\,{}^{2}C_{1} + 4\,{}^{2}C_{2}$

[2 marks]

(ii) Work out ${}^{3}C_{0} + 2\,{}^{3}C_{1} + 4\,{}^{3}C_{2} + 8\,{}^{3}C_{3}$

[2 marks]

(iii) Guess the answer, then work out

$${}^{4}C_{0} + 2\,{}^{4}C_{1} + 4\,{}^{4}C_{2} + 8\,{}^{4}C_{3} + 16\,{}^{4}C_{4}$$

[2 marks]

(iv) Write out the next calculation in this sequence and predict the answer ahead of working it out

[2 marks]

Question 5

Find the constant term (the term independent of x) in the expansion of

$$\left(\frac{x^2}{2} - \frac{2}{x} \right)^9$$

[5 marks]

Question 6

$$f(x) = (1 - 5x)^{30}$$

(i) Find the first four terms, in ascending powers of x, in the binomial expansion of $f(x)$

[4 marks]

(ii) Use your answer to part (i) to estimate the value of 0.995^{30} giving your answer to 6 decimal places

[3 marks]

(iii) Use your calculator to evaluate 0.995^{30} and calculate the percentage error in your answer to part (ii)

[3 marks]

Question 7

Additional Mathematics Examination Question from June 2016, Q8 (OCR)

(i) Write down the binomial expansion of $(1 + \delta)^3$

[2 marks]

(ii) Hence explain why, if δ is small, $(1 + \delta)^3 \approx 1 + 3\delta$

[$\approx$ means "is approximately equal to"]

[1 mark]

You are given that the equation $x^3 - 0.9x - 0.206 = 0$ has a root very close to $x = 1$

(iii) Substitute $x = 1 + \delta$ into the equation and use the approximation in part (ii) to find an estimate of this root, correct to 3 significant figures. Show all your working.

[4 marks]

10.4 Answers

10.4.1 Solution to 10.2 Example (Teaching Video)

(i)

$$\left(2 + \frac{x}{4}\right)^8 = 1 \times (2)^8 \times \left(\frac{x}{4}\right)^0$$
$$+ \ 8 \times (2)^7 \times \left(\frac{x}{4}\right)^1$$
$$+ \ 28 \times (2)^6 \times \left(\frac{x}{4}\right)^2$$
$$+ \ \ldots$$
$$= 256 + 256x + 112\,x^2 + \ldots$$

(six terms terms with higher powers of x “thrown away”)

[4 marks]

(ii) We require,

$$2 + \frac{x}{4} = 2.075$$
$$\frac{x}{4} = 0.075$$
$$x = 0.3$$

$$(2.075)^8 = \left(2 + \frac{0.3}{4}\right)^8$$
$$= 256 + 256 \times 0.3 + 112 \times 0.3^2 + \ldots$$
$$= 342.88 \qquad \text{which is 343 to the nearest integer}$$

Incidentally :

$2.075^8 = 343.6725208\ldots$ (direct from a calculator)

which is confirmation that, even having throwing away six terms, an approximately correct answer was obtained.

So, when x is small, throwing away most of the terms containing higher powers of x is a valid thing to do, as a reasonable approximation to the exact answer is still obtained.

A small amount of accuracy has been sacrificed for a big reduction in complexity.

[4 marks]

10.4.2 Solution to 10.3 Exercise

Answer 1

(i)

$$\left(3 + \frac{x}{9}\right)^{11} = 1 \times (3)^{11} \times \left(\frac{x}{9}\right)^{0}$$

$$+ 11 \times (3)^{10} \times \left(\frac{x}{9}\right)^{1}$$

$$+ 55 \times (3)^{9} \times \left(\frac{x}{9}\right)^{2}$$

$$+ \ldots$$

$$= 177147 + 72171x + 13365\,x^2 + \ldots \text{ (9 terms thrown away)}$$

[4 marks]

(ii) We require,

$$3 + \frac{x}{9} = 3.01$$

$$\frac{x}{9} = 0.01$$

$$x = 0.09$$

$$(3.01)^{11} = \left(3 + \frac{0.09}{9}\right)^{11}$$

$$= 177147 + 72171 \times 0.09 + 13365 \times 0.09^2 + \ldots$$

$$= 183750.6465 \quad \text{which is 183751 to 6 sig fig}$$

Incidentally :

$$3.01^{11} = 183751.7363\ldots \text{ (direct from calculator)}$$

which is confirmation that, even having throwing away nine terms, an approximately correct answer was obtained.

[4 marks]

Answer 2

It's possible to simply target the required term.
However, here's the full expansion,

$$\left(\frac{x}{2} + \frac{4}{x} \right)^4 = 1 \times \left(\frac{x}{2}\right)^4 \times \left(\frac{4}{x}\right)^0$$
$$+ 4 \times \left(\frac{x}{2}\right)^3 \times \left(\frac{4}{x}\right)^1$$
$$+ 6 \times \left(\frac{x}{2}\right)^2 \times \left(\frac{4}{x}\right)^2$$
$$+ 4 \times \left(\frac{x}{2}\right)^1 \times \left(\frac{4}{x}\right)^3$$
$$+ 1 \times \left(\frac{x}{2}\right)^0 \times \left(\frac{4}{x}\right)^4$$

$$= \frac{x^4}{16} + \frac{x^2}{2} + 24 + \frac{128}{x^2} + \frac{256}{x^4}$$

The constant term is the 24

[4 marks]

Answer 3

(i) $(1 - 3x)^5 = 1 \times (1)^5 \times (-3x)^0$

$+ 5 \times (1)^4 \times (-3x)^1$

$+ 10 \times (1)^3 \times (-3x)^2$

$+ 10 \times (1)^2 \times (-3x)^3$

$+ \ldots$

$$(1 - 3x)^5 = 1 - 15x + 90x^2 - 270x^3 + \ldots$$

[4 marks]

(ii)

$$(1 + x)(1 - 3x)^5 = (1 + x)\left(1 - 15x + 90x^2 - 270x^3 + \ldots\right)$$
$$= 1 - 15x + 90x^2 - 270x^3 + \ldots$$
$$+ x - 15x^2 + 90x^3 - 270x^4 + \ldots$$
$$= 1 - 14x + 75x^2 - 180x^3 + \ldots$$
$$\approx 1 - 14x \qquad \square$$

[4 marks]

Answer 4

(i)

$$^{2}C_0 + 2\,^{2}C_1 + 4\,^{2}C_2$$
$$= 1 + 2 \times 2 + 4 \times 1$$
$$= 9 \qquad \text{which is } 3^2$$

(ii)

$$^{3}C_0 + 2\,^{3}C_1 + 4\,^{3}C_2 + 8\,^{3}C_3$$
$$= 1 + 2 \times 3 + 4 \times 3 + 8 \times 1$$
$$= 27 \qquad \text{which is } 3^3$$

(iii)

$$^{4}C_0 + 2\,^{4}C_1 + 4\,^{4}C_2 + 8\,^{4}C_3 + 16\,^{4}C_4$$
$$= 1 + 2 \times 4 + 4 \times 6 + 8 \times 4 + 16 \times 1$$
$$= 81 \qquad \text{which is } 3^4$$

(iv)

$$^{5}C_0 + 2\,^{5}C_1 + 4\,^{5}C_2 + 8\,^{5}C_3 + 16\,^{5}C_4 + 32\,^{5}C_5$$
$$= 1 + 2 \times 5 + 4 \times 10 + 8 \times 10 + 16 \times 5 + 32 \times 1$$
$$= 243 \qquad \text{which is } 3^5$$

This works because,

$$(1 + x)^n = \sum_{r=0}^{n} {}^{n}C_r\, x^r$$

and with $x = 2$ this becomes

$$3^n = \sum_{r=0}^{n} {}^{n}C_r\, 2^r$$

[2, 2, 2, 2 marks]

Answer 5

Targeting the relevant term,

$$\left(\frac{x^2}{2} - \frac{2}{x} \right)^9 = \ldots + {}^{9}C_6 \times \left(\frac{x^2}{2} \right)^3 \times \left(-\frac{2}{x} \right)^6 + \ldots$$
$$= \ldots + 84 \times \frac{x^6}{8} \times \frac{64}{x^6} + \ldots$$
$$= \ldots + 672 + \ldots$$

So the constant term is 672

[5 marks]

Answer 6

(i)

$$
\begin{aligned}
(1 - 5x)^{30} = \; & 1 \times (1)^{30} \times (-5x)^0 \\
& + 30 \times (1)^{29} \times (-5x)^1 \\
& + 435 \times (1)^{28} \times (-5x)^2 \\
& + 4060 \times (1)^{27} \times (-5x)^3 \\
& + \ldots
\end{aligned}
$$

So,

$$(1 - 5x)^{30} = 1 - 150x + 10875\,x^2 - 507\,500\,x^3 + \ldots$$

[4 marks]

(ii) We require that,

$$
\begin{aligned}
1 - 5x &= 0.995 \\
5x &= 0.005 \\
x &= 0.001
\end{aligned}
$$

$$
\begin{aligned}
(0.995)^{30} &= (1 - 5 \times 0.001)^{30} \\
&= 1 - 150 \times 0.001 + 10875 \times 0.001^2 - 507\,500 \times 0.001^3 + \ldots \\
&= 1 - 0.15 + 0.010875 - 0.000\,507\,5 + \ldots \\
&\approx 0.860\,367\,5
\end{aligned}
$$

That is, $0.860\,368$ to six decimal places

[3 marks]

(iii)

$0.995^{30} = 0.860\,384$ direct from calculator

$$\%\,error = \frac{0.860\,384 - 0.860\,368}{0.860\,384} \times 100$$

$$= 0.001\,859\,\%$$

i.e. $0.002\,\%$

[3 marks]

Answer 7

(i) $(1+\delta)^3 = 1 + 3\delta + 3\delta^2 + \delta^3$

[2 marks]

(ii) If δ is small then, in comparison to the number 1, δ^2 and δ^3 will be negligible. If the terms $3\delta^2$ and δ^3 are "thrown away" what remains is;

$$(1+\delta)^3 = 1 + 3\delta \qquad \square$$

[1 mark]

(iii)

$$\begin{aligned} x^3 - 0.9x - 0.206 &= 0 \\ (1+\delta)^3 - 0.9(1+\delta) - 0.206 &= 0 \\ 1 + 3\delta - 0.9 - 0.9\delta - 0.206 &= 0 \\ 2.1\delta - 0.106 &= 0 \\ 2.1\delta &= 0.106 \\ \delta &= 0.05 \end{aligned}$$

$\therefore$ the root is 1.05 to 3 significant figures

[4 marks]