2.3 Answers

2.3.1 Solution to Example

$$\frac{dy}{dx} = \frac{1}{y^4}$$

Invert both sides

$$\frac{dx}{dy} = y^4$$

Integrate both sides with respect to y

$$\int \frac{dx}{dy}\, dy = \int y^4\, dy$$

$$x = \frac{y^5}{5} + c$$

$$5x = y^5 + 5c$$

$$y^5 = 5x - 5c$$

$$y^5 = 5x + A$$

$$y = (5x + A)^{\frac{1}{5}} \qquad \text{is the general solution}$$

Given that $y = 3$ when $x = 49$,

$$3^5 = 5 \times 49 + A$$

$$\therefore A = -2$$

Thus;

$$y = (5x - 2)^{\frac{1}{5}} \qquad \text{is particular solution}$$

[6 marks]

2.3.2 Solutions to Exercise

Answer 1

$$\frac{dy}{dx} = \frac{1}{y}$$

Invert both sides

$$\frac{dx}{dy} = y$$

Integrate both sides with respect to y

$$\int \frac{dx}{dy}\, dy = \int y\ dy$$

$$x = \frac{y^2}{2} + c$$

$$y = \sqrt{2x + A} \quad \text{is the general solution}$$

Given that $y = 3$ when $x = 4$,

$$y = \sqrt{2x + 1} \quad \text{is the particular solution}$$

[5 marks]

Answer 2

$$\frac{dy}{dx} = y$$

Invert both sides

$$\frac{dx}{dy} = \frac{1}{y}$$

Integrate both sides with respect to y

$$\int \frac{dx}{dy}\, dy = \int \frac{1}{y}\ dy$$

$$x = ln|y| + c$$

$$|y| = e^{x-c}$$

$$|y| = e^{-c} e^{x}$$

$$|y| = B e^{x} \quad \text{for } B \in \mathbb{R},\ B > 0$$

$$y = A e^{x} \quad \text{for } A \in \mathbb{R}$$

[5 marks]

Answer 3

$$\frac{dy}{dx} = y + 2$$

Invert both sides

$$\frac{dx}{dy} = \frac{1}{y + 2}$$

Integrate both sides with respect to y

$$\int \frac{dx}{dy}\, dy = \int \frac{1}{y + 2}\, dy$$

$$x = ln\left| y + 2 \right| + c$$

$$\left| y + 2 \right| = e^{x - c}$$

$$= e^{-c} e^{x}$$

$$= B e^{x} \qquad \text{for } B \in \mathbb{R}, B > 0$$

$$y + 2 = A e^{x} \qquad \text{for } A \in \mathbb{R}$$

$$y = A e^{x} - 2$$

[5 marks]

Answer 4

$$\frac{dy}{dx} = cos^2 y$$

Invert both sides

$$\frac{dx}{dy} = \frac{1}{cos^2 y}$$

$$\frac{dx}{dy} = sec^2 y$$

Integrate both sides with respect to y

$$\int \frac{dx}{dy}\, dy = \int sec^2 y\, dy$$

$$x = tan\, y + c$$

$$x - c = tan\, y$$

$$\therefore \quad y = arctan\,(x + A) \qquad \text{is the general solution}$$

Given that $y = \frac{\pi}{4}$ when $x = 7$

$$tan\left(\frac{\pi}{4} \right) = 7 + A$$

$$A = -6$$

$$\therefore \quad y = arctan\,(x - 6) \quad \text{is the particular solution}$$

[7 marks]

Answer 5

(i)

$$\frac{dy}{dx} = \frac{1}{2y - 8}$$

Invert both sides

$$\frac{dx}{dy} = 2y - 8$$

Integrate both sides with respect to y

$$\int \frac{dx}{dy}\, dy = \int 2y - 8 \; dy$$

$$x = y^2 - 8y + c \qquad \text{is the general solution}$$

Given that $y = 5$ when $x = 4$,

$$4 = 5^2 - 8 \times 5 + c$$

$$\therefore \; c = 19$$

$$x = y^2 - 8y + 19 \qquad \text{is the particular solution}$$

[4 marks]

(ii)

$$x = y^2 - 8y + 19$$

$$x = (y - 4)^2 - 16 + 19$$

$$x - 3 = (y - 4)^2$$

$$y = 4 + \sqrt{x - 3}$$

[3 marks]

Note that $y = 4 - \sqrt{x - 3}$ cannot be a particular solution

Answer 6

(a)

$$\frac{1}{[(4-x)(2-x)]} = \frac{A}{4-x} + \frac{B}{2-x}$$

Multiply through by the red bracket

$$\frac{1\,[\,\dots\dots\,]}{[(4-x)(2-x)]} = \frac{A\,[\,\dots\dots\,]}{4-x} + \frac{B\,[\,\dots\dots\,]}{2-x}$$

$$1 = A(2-x) + B(4-x)$$

$$\text{Let } x = 2 \;:\; 1 = 2B \;\Rightarrow\; B = \frac{1}{2}$$

$$\text{Let } x = 4 \;:\; 1 = -2A \;\Rightarrow\; A = -\frac{1}{2}$$

$$\therefore \frac{1}{(4-x)(2-x)} = -\frac{1}{2(4-x)} + \frac{1}{2(2-x)}$$

[2 marks]

(b)

$$\frac{dx}{dt} = k(4-x)(2-x)$$

$$\frac{dt}{dx} = \frac{1}{k\,(4-x)(2-x)}$$

$$\int \frac{dt}{dx}\,dx = \int \frac{1}{k\,(4-x)(2-x)}\,dx$$

$$\int dt = \frac{1}{2k}\int -\frac{1}{(4-x)} + \frac{1}{(2-x)}\,dx$$

$$2k\int dt = \int (-1)\,\frac{1}{4-x}\,dx - \int (-1)\,\frac{1}{2-x}\,dx$$

$$2kt + c = ln|4-x| - ln|2-x|$$

$$2kt + c = ln\left|\frac{4-x}{2-x}\right|$$

As $0 \leqslant x < 2$ the modulus signs are not needed.

$$e^{2kt+c} = \frac{4-x}{2-x} \;\Rightarrow\; e^{2kt}e^{c} = \frac{4-x}{2-x} \;\Rightarrow\; A\,e^{2kt} = \frac{4-x}{2-x}$$

Given that when $t = 0$, $x = 0$, the constant A can be found; $A = 2$

$$2\,e^{2kt}(2-x) = 4-x$$

$$4\,e^{2kt} - 2x\,e^{2kt} = 4-x$$

$$x - 2x\,e^{2kt} = 4 - 4\,e^{2kt}$$

$$x\left(1 - 2\,e^{2kt}\right) = 4 - 4\,e^{2kt}$$

$$x = \frac{4 - 4\,e^{2kt}}{1 - 2\,e^{2kt}} \qquad \square$$

[7 marks]

(c) With $A = 2$, $k = 0.1$ and $x = 1$,

$$A\,e^{2kt} = \frac{4 - x}{2 - x}$$

becomes,

$$2\,e^{0.2\,t} = 3$$

$$e^{0.2\,t} = \frac{3}{2}$$

take the natural log of both sides,

$$0.2\,t = ln\left(\frac{3}{2}\right)$$

$$t = 5\,ln\left(\frac{3}{2}\right)$$

$$= 2.03 \text{ seconds}$$

[2 marks]