3.3 Answers

3.3.1 Solution to Example

$$\frac{dy}{dx} = \frac{y^2}{x}$$

Separate the variables

$$\frac{1}{y^2}\,\frac{dy}{dx} = \frac{1}{x}$$

Integrate both sides with respect to x

$$\int y^{-2}\,\frac{dy}{dx}\,dx = \int \frac{1}{x}\,dx$$

$$-y^{-1} = ln|x| + c$$

$$-y = \frac{1}{ln|x| + c}$$

$$y = \frac{1}{-ln|x| - c}$$

$$y = \frac{1}{A - ln|x|} \quad \text{is the general solution}$$

Given that $y = 4$ when $x = 1$

$$4 = \frac{1}{A - ln\,1}$$

$$\therefore\ A = 0.25$$

Thus;

$$y = \frac{1}{0.25 - ln|x|} \quad \text{is the particular solution}$$

[6 marks]

3.3.2 Solutions to Exercise

Answer 1

$$\frac{dy}{dx} = 6xy^2$$

Separate the variables

$$\frac{1}{y^2}\,\frac{dy}{dx} = 6x$$

Integrate both sides with respect to x

$$\int y^{-2}\,\frac{dy}{dx}\,dx = 6\int x\;dx$$

$$-y^{-1} = 3x^2 + c$$

$$-y = \frac{1}{3x^2 + c}$$

$$y = \frac{1}{-3x^2 - c}$$

$$y = \frac{1}{A - 3x^2} \qquad \text{is the general solution}$$

Given that $y = 0.2$ when $x = 1$

$$0.2 = \frac{1}{A - 3}$$

$$\therefore\; A = 8$$

Thus;

$$y = \frac{1}{8 - 3x^2} \qquad \text{is the particular solution}$$

[6 marks]

Answer 2

$$\cos^2 x \; \frac{dy}{dx} - \frac{1}{y^2} = 0$$

Separate the variables

$$\cos^2 x \; \frac{dy}{dx} = \frac{1}{y^2}$$

$$y^2 \frac{dy}{dx} = \frac{1}{\cos^2 x}$$

Integrate both sides with respect to x

$$\int y^2 \frac{dy}{dx}\, dx = \int \sec^2 x \; dx$$

$$\int y^2 \; dy = \int \sec^2 x \; dx$$

$$\frac{y^3}{3} = \tan x + c$$

$$y^3 = 3 \tan x + 3c$$

$$y = (3 \tan x + A)^{\frac{1}{3}} \quad \text{is the general solution}$$

Given that $y = 2$ when $x = \dfrac{\pi}{4}$

$$2 = \left(3 \tan\left(\frac{\pi}{4}\right) + A\right)^{\frac{1}{3}}$$

$$\therefore \; A = 5$$

Thus $\quad y = (3 \tan x + 5)^{\frac{1}{3}}$ is the particular solution

[6 marks]

Answer 3

$$\int y \; dy = 3 \int x^2 \, dx$$

$$\frac{y^2}{2} = \frac{3x^3}{3} + c$$

$$y^2 = 2x^3 + k$$

But $y = 3$ when $x = 2, \quad \therefore \; k = -7$

$$y^2 = 2x^3 - 7$$

[4 marks]

Answer 4

$$\frac{dy}{dx} = e^y \, e^x$$

$$\frac{1}{e^y} \frac{dy}{dx} = e^x$$

$$e^{-y} \frac{dy}{dx} = e^x$$

$$\int e^{-y} \, dy = \int e^x \, dx$$

$$-e^{-y} = e^x + c$$

$$\text{But } y = 0 \text{ when } x = 0, \quad \therefore \ c = -2$$

$$e^{-y} + e^x = 2$$

[4 marks]

Answer 5

(i) Acceleration

[1 mark]

(ii) 10 ms^{-2}

[1 mark]

(iii) 50 ms^{-1}

[1 mark]

(iv)

$$\frac{dv}{dt} = 0.2 \, (50 - v)$$

$$\frac{1}{50 - v} \frac{dv}{dt} = 0.2$$

[2 marks]

(v)

$$\int \frac{1}{50 - v} \, dv = \int 0.2 \, dt$$

$$-\ln\left| 50 - v \right| = 0.2\,t + c$$

$$\ln\left| 50 - v \right| = -0.2\,t + k$$

As the terminal velocity is 50 ms^{-1} from part (iii), v is always less than 50 and so the modulus signs can be dropped.

$$50 - v = A\,e^{-0.2\,t}$$

$$v = 50 - A\,e^{-0.2\,t}$$

$$\text{But } v = 0 \text{ when } t = 0, \quad \therefore \ A = 50$$

$$v = 50 - 50\,e^{-0.2\,t}$$

$$v = 50 \left(1 - e^{-0.2\,t} \right)$$

[5 marks]

Answer 6

(i) $$\frac{1}{v\,(1+v)} = \frac{1}{v} - \frac{1}{1+v}$$

[2 marks]

(ii)

$$\frac{dv}{dt} = -\left(v + v^2\right)$$

$$\frac{1}{v+v^2}\frac{dv}{dt} = -1$$

$$\int \frac{1}{v(1+v)}\,dv = -\int 1\,dt$$

$$\int \frac{1}{v} - \frac{1}{1+v}\,dv = -\int 1\,dt$$

$$ln|v| - ln|1+v| = -t + c$$

$$ln\left|\frac{v}{1+v}\right| = -t + c$$

As we're told $v > 0$ the modulus signs can be dropped

$$\frac{v}{1+v} = e^{-t+c}$$

$$= e^{-t}e^{c}$$

$$= A\,e^{-t}$$

But $v = 1$ when $t = 0$, $\therefore\ A = \frac{1}{2}$

$$\frac{2v}{1+v} = e^{-t}$$

[6 marks]

(iii)

$$2v = e^{-t} + v\,e^{-t}$$

$$2v - v\,e^{-t} = e^{-t}$$

Multiply through by e^{t}

$$2v\,e^{t} - v = 1$$

$$v\left(2e^{t} - 1\right) = 1$$

$$v = \frac{1}{2e^{-t} - 1}$$

[2 marks]