4.3 Answers

4.3.1 Solution to Example

$$\frac{dm}{dt} = -3m$$

This is a Type Two differential equation with the *y* replaced with *m*, and the *x* with *t*.

$$\frac{dm}{dt} = -3m$$

Invert both sides

$$\frac{dt}{dm} = -\frac{1}{3m}$$

Integrate both sides with respect to *m*

$$\int \frac{dt}{dm}\, dm = -\frac{1}{3} \int \frac{1}{m}\, dm$$

$$t = -\frac{1}{3} \ln|m| + c$$

As mass cannot be negative the modulus signs can be dropped.

$$-3t = \ln m - 3c$$

$$\ln m = -3t + 3c$$

$$m = e^{-3t+3c}$$

$$m = e^{-3t}\, e^{3c}$$

$$m = A e^{-3t} \qquad \text{is the general solution.}$$

But $m = 12$ when $t = 0$, $\therefore A = 12$

$$m = 12\, e^{-3t} \qquad \text{is the particular solution.}$$

That is, $A = 12$, $k = 3$

[6 marks]

4.3.2 Solutions to Exercise

Answer 1

$$\frac{dy}{dx} = \frac{e^{2x}}{y}$$

Type 3 : Separate the variables

$$y\frac{dy}{dx} = e^{2x}$$

Integrate both sides with respect to x

$$\int y\frac{dy}{dx}\,dx = \int e^{2x}\,dx$$

$$\int y \ dy = \int e^{2x}\,dx$$

$$\frac{y^2}{2} = \frac{e^{2x}}{2} + c$$

$$y^2 = e^{2x} + A \quad \text{is the general solution}$$

Given that $y = 3$ when $x = 0 \Rightarrow 3^2 = 1 + A \qquad \therefore\ A = 8$

$$y^2 = e^{2x} + 8 \quad \text{is the particular solution}$$

[6 marks]

Answer 2

$$\frac{dy}{dx} + csc^2\,3x = 0$$

$$\frac{dy}{dx} = -\,csc^2\,3x$$

Type 1 : Integrate both sides with respect to x

$$\int \frac{dy}{dx}\,dx = \frac{1}{3}\int 3\left(-\,csc^2\,3x\right)\,dx$$

$$\int 1\,dy = \frac{1}{3}\int 3\left(-\,csc^2\,3x\right)\,dx$$

$$y = \frac{1}{3}\cot 3x + c \qquad \text{is the general solution}$$

Given that $y = 1$ when $x = \frac{\pi}{4} \Rightarrow$

$$1 = \frac{1}{3\tan\left(\frac{3\pi}{4}\right)} + c \Rightarrow 1 = \frac{1}{-\,3} + c \qquad \therefore\ c = \frac{4}{3}$$

$$y = \frac{1}{3}\cot 3x + \frac{4}{3} \qquad \text{is the particular solution}$$

[6 marks]

Answer 3

(i)

$$\frac{dP}{dt} = K\,P$$

$$5 = K \times 50 \qquad \therefore\ K = 0.1$$

$$\frac{dP}{dt} = 0.1\,P$$

[2 marks]

(ii)

$$\frac{dP}{dt} = 0.1\,P$$

Type 2 : Invert both sides

$$\frac{dt}{dP} = \frac{10}{P}$$

Integrate both sides with respect to P

$$\int \frac{dt}{dP}\ dP = \int \frac{10}{P}\,dP$$

$$\int 1\,dt = 10 \int \frac{1}{P}\ dP$$

$$t = 10\,ln\left|P\right| + c$$

$$0.1\,t = ln\,P + 0.1\,c \qquad P,\ \text{population, is always positive}$$

$$ln\,P = 0.1\,t - 0.1\,c$$

$$P = e^{0.1\,t - 0.1\,c}$$

$$P = e^{0.1\,t}\ e^{-0.1\,c}$$

$$P = A\,e^{0.1\,t} \quad \text{is the general solution}$$

But $P = 50$ when $t = 0$, $\therefore\ A = 50$

$$P = 50\,e^{0.1\,t} \quad \text{is the particular solution}$$

[6 marks]

(iii) When $t = 24$, $P = 551$ rabbits.

[1 mark]

Answer 4

(a)

$$\frac{1 \times [\ ...\ ]}{[\ P(5 - P)\ ]} = \frac{A \times [\ ...\ ]}{P} + \frac{B \times [\ ...\ ]}{5 - P}$$

$$1 = A\,(5 - P) + B\,(P)$$

Let $P = 0 \Rightarrow A = \frac{1}{5}$ and let $P = 5 \Rightarrow B = \frac{1}{5}$

$$\therefore\ \frac{1}{P\,(5 - P)} = \frac{1}{5}\left(\frac{1}{P} + \frac{1}{5 - P}\right)$$

[3 marks]

(b)

$$\frac{dP}{dt} = \frac{1}{15} P(5 - P)$$

Type 2 : Invert both sides

$$\frac{dt}{dP} = 15\left(\frac{1}{P(5-P)}\right)$$

$$\frac{dt}{dP} = 15 \times \frac{1}{5}\left(\frac{1}{P} + \frac{1}{5-P}\right)$$

Integrate both sides with respect to P

$$\int \frac{dt}{dP}\, dP = 3\int \frac{1}{P} + \frac{1}{5-P}\, dP$$

$$\int 1\, dt = 3\int \frac{1}{P} + \frac{1}{5-P}\, dP$$

$$\frac{1}{3}t = ln|P| - ln|5-P| + K$$

$$-\frac{1}{3}t = ln(5-P) - ln(P) - K \qquad P \text{ is always positive}$$

$$K - \frac{1}{3}t = ln\left(\frac{5-P}{P}\right)$$

$$e^{K-\frac{1}{3}t} = \frac{5-P}{P}$$

$$e^{K}e^{-\frac{1}{3}t} = \frac{5}{P} - 1$$

$$\frac{5}{P} = 1 + e^{K}e^{-\frac{1}{3}t}$$

$$\frac{P}{5} = \frac{1}{1 + e^{K}e^{-\frac{1}{3}t}}$$

$$P = \frac{5}{1 + c\,e^{-\frac{1}{3}t}} \quad \text{is the general solution}$$

But $P = 1$ when $t = 0$, $\therefore\ c = 4$

$$P = \frac{5}{1 + 4e^{-\frac{1}{3}t}}$$

[8 marks]

(c)

As $t \to \infty$, $e^{-\frac{1}{3}t} \to 0$ and so $P \to 5$

As P is in thousands the population $\to 5000$ □

[1 mark]

Answer 5

(a)

$$\frac{1}{4 - x^2} = \frac{1}{(2 + x)(2 - x)}$$

$$\frac{1 \times [\,...\,]}{[\,(2 + x)(2 - x)\,]} = \frac{A \times [\,...\,]}{(2 + x)} + \frac{B \times [\,...\,]}{(2 - x)}$$

$$1 = A(2 - x) + B(2 + x)$$

Let $P = 2$ to get $B = \dfrac{1}{4}$

Let $P = -2$ to get $A = \dfrac{1}{4}$

$$\int \frac{1}{4 - x^2}\, dx = \frac{1}{4} \int \left(\frac{1}{2 + x} + \frac{1}{2 - x} \right) dx$$

$$= \frac{1}{4} \left(ln|2 + x| - ln|2 - x| \right) + c$$

$$= \frac{1}{4}\, ln \left| \frac{2 + x}{2 - x} \right| + c$$

[4 marks]

(b)

$$\frac{dx}{dt} = 10\left(4 - x^2\right)$$

Type 2 : Invert both sides

$$\frac{dt}{dx} = \frac{1}{10(4 - x^2)}$$

Integrate both sides with respect to x

$$\int 1\; dt = \frac{1}{10} \int \frac{1}{4 - x^2}\, dx$$

$$t = \frac{1}{40}\, ln \left| \frac{2 + x}{2 - x} \right| + \frac{c}{10}$$

As $0 \leqslant x < 2$ the modulus signs can be dropped

$$40\,t = ln \left(\frac{2 + x}{2 - x} \right) + 4c$$

$$e^{40\,t} = \frac{A\,(2 + x)}{(2 - x)}$$

As $x = 0$ when $t = 0$, $A = 1$

$$e^{40\,t}\,(2 - x) = 2 + x$$

$$2\,e^{40\,t} - 2 = x\,e^{40\,t} + x$$

$$x = \frac{2\left(e^{40\,t} - 1\right)}{\left(e^{40\,t} + 1\right)}$$

[6 marks]

Answer 6

(a) $u = 4 - \sqrt{h} \Leftrightarrow u = 4 - h^{\frac{1}{2}}$ $\left(\text{Also } \sqrt{h} = 4 - u \right)$

$$\Rightarrow \frac{du}{dh} = -\frac{1}{2} h^{-\frac{1}{2}}$$

$$= -\frac{1}{2\sqrt{h}}$$

$$= -\frac{1}{2(4 - u)}$$

$$= \frac{1}{2(u - 4)}$$

$$\int \frac{1}{4 - \sqrt{h}} \, dh = \int \frac{1}{u} 2(u - 4) \, du$$

$$= 2 \int \frac{u - 4}{u} \, du$$

$$= 2 \int 1 - \frac{4}{u} \, du$$

$$= 2 [u - 4 \, ln|u|] + c$$

$$= 2u - 8 \, ln|u| + c$$

$$= -8 \, ln|4 - \sqrt{h}| + 2\left(4 - \sqrt{h} \right) + c$$

$$= -8 \, ln|4 - \sqrt{h}| + 8 - 2\sqrt{h} + c$$

$$= -8 \, ln|4 - \sqrt{h}| - 2\sqrt{h} + k \qquad \square$$

[6 marks]

(b) It's in the nature of a tree to increase in height so the rate of change of height will need to always be positive.

$$\therefore \frac{dh}{dt} \geqslant 0$$

$$\frac{t^{0.25} (4 - \sqrt{h})}{20} \geqslant 0$$

The time t can never be negative so we require that,

$$\left(4 - \sqrt{h} \right) \geqslant 0$$

$$-\sqrt{h} \geqslant -4$$

$$\sqrt{h} \leqslant 4$$

$$0 \leqslant h \leqslant 16 \text{ metres}$$

[2 marks]

(c) Solving the differential equation,

$$\frac{dh}{dt} = \frac{t^{0.25}\,(4 - \sqrt{h}\,)}{20}$$

Separate the variables, keeping part (a) in mind,

$$\frac{1}{4 - \sqrt{h}}\,\frac{dh}{dt} = \frac{1}{20}\,t^{0.25}$$

Integrate both sides with respect to t

$$\int \frac{1}{4 - \sqrt{h}}\,dh = \frac{1}{20}\int t^{0.25}\,dt$$

$$-\,8\,ln\left|4 - \sqrt{h}\,\right| - 2\sqrt{h} + k_1 = \frac{1}{20}\,\frac{t^{1.25}}{1.25} + k_2$$

$$t^{1.25} = -200\,ln\left|4 - \sqrt{h}\,\right| - 50\sqrt{h} + c$$

Using the initial conditions, that $h = 1$ when $t = 0$,

$$0 = -200\,ln\,3 - 50 + c$$

$$c = 200\,ln\,3 + 50$$

$$\therefore\; t^{1.25} = -200\,ln\left|4 - \sqrt{h}\,\right| - 50\sqrt{h} + 200\,ln\,3 + 50$$

When $h = 12$, $t^{1.25} = -200\,ln\left|4 - \sqrt{12}\,\right| - 50\sqrt{12} + 200\,ln\,3 + 50$

$$= 221.280$$

$$\therefore\; t = 75.2 \text{ years}$$

[7 marks]