5.4 Answers

5.4.1 Solution to 5.2 The Coffee Stain example

(a) **(i)** $C = 2\pi r$ **(ii)** $A = \pi r^2$

[2 marks]

(b) The question is asking for $\dfrac{dC}{dt}$ which is not immediately accessible.

Strategy : Find $\dfrac{dC}{dr}$ and then use $\dfrac{dC}{dt} = \dfrac{dC}{dr} \times \dfrac{dr}{dt}$

$$C = 2\pi r \Rightarrow \frac{dC}{dr} = 2\pi$$

$$\begin{aligned} \frac{dC}{dt} &= \frac{dC}{dr} \times \frac{dr}{dt} \\ &= 2\pi \times 0.04 \\ &= 0.251 \text{ mm s}^{-1} \end{aligned}$$

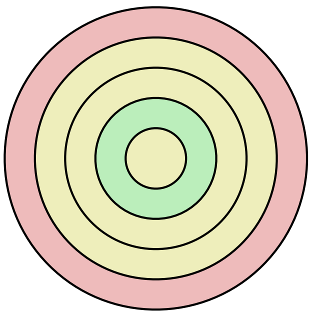
[3 marks]

(c) $A = \pi r^2 \Rightarrow \dfrac{dA}{dr} = 2\pi r$

$$\begin{aligned} \frac{dA}{dt} &= \frac{dA}{dr} \times \frac{dr}{dt} \\ &= 2\pi r \times 0.04 \\ &= 2\pi \times 8 \times 0.04 \qquad \text{when } r = 8 \text{ mm} \\ &= 2.01 \text{ mm}^2 \text{ s}^{-1} \end{aligned}$$

[3 marks]

(d) The major assumption is that the radius is growing at a constant rate. Each increase creates a concentric ring of the same width as previous rings but with an ever greater area, requiring ever larger amounts of coffee to stain. Compare the red area with the green in this diagram.



The modelling assumption may be realistic over a restricted time span, say $4 \leqslant t \leqslant 8$ seconds. Making the rate of increase in area the constant may be a more realistic modelling assumption or the rate of increase of radius inversely proportional to the radius better still.

[3 marks]

5.4.2 Solutions to 5.3 Exercise

Answer 1

(i) $V = L^3 \Rightarrow \dfrac{dV}{dL} = 3L^2$

Also, $L = \sqrt[3]{V}$ and so when $V = 8000$, $L = \sqrt[3]{8000} = 20$ mm

$$\frac{dL}{dt} = \frac{dL}{dV} \times \frac{dV}{dt}$$
$$= \frac{1}{3L^2} \times 2700$$
$$= \frac{1}{3 \times 20^2} \times 2700$$
$$= 2.25 \text{ mm h}^{-1}$$

Note that this rate is not constant, it's only the rate when $V = 8000$ mm^3

[4 marks]

(ii) Told to assume the volume melting is constant at 2700 mm^3 per hour

$$0 = 8000 - 2700\,t$$
$$t = 2.963$$
$$t = 2 \text{ hours and } 58 \text{ minutes}$$

[1 mark]

Answer 2

(i) Let A be the cross sectional area (a circle of radius x), in which case,

$A = \pi x^2 \Rightarrow \dfrac{dA}{dx} = 2\pi x$

Furthermore, $\dfrac{dx}{dt} = \dfrac{dx}{dA} \times \dfrac{dA}{dt}$

$$= \frac{1}{2\pi x} \times 0.032$$
$$= \frac{1}{2\pi \times 3} \times 0.032$$
$$= 0.00170 \text{ cm s}^{-1}$$

[4 marks]

(ii) Let V be volume of rod (a cylinder, radius x, length $1.5x$), in which case,

$V = \pi x^2 \times 1.5x = 1.5\pi x^3 \Rightarrow \dfrac{dV}{dx} = 4.5\pi x^2$

Furthermore, $\dfrac{dV}{dt} = \dfrac{dV}{dx} \times \dfrac{dx}{dt}$

$$= 4.5\pi \times 3^2 \times 0.00170$$
$$= 0.216 \text{ cm}^3 \text{ s}^{-1}$$

[4 marks]

Answer 3

Assembling the pieces...

$$V = \frac{1}{3}\pi h^2 (90 - h)$$

$$= 30\pi h^2 - \frac{1}{3}\pi h^3$$

$$\frac{dV}{dh} = 60\pi h - \pi h^2 \qquad \text{Told } \frac{dV}{dt} = 180 \text{ cm}^3 \text{ s}^{-1}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{60\pi h - \pi h^2} \times 180$$

$$= \frac{1}{60\pi \times 15 - \pi 15^2} \times 180 \quad \text{when } h = 15 \text{ cm}$$

$$= 0.085 \text{ cm s}^{-1}$$

[5 marks]

Answer 4

(a) $$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2 \qquad \text{Told } \frac{dV}{dt} = 3 \text{ cm}^3 \text{ s}^{-1}$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{4\pi r^2} \times 3$$

$$= \frac{1}{4\pi \times 4^2} \times 3$$

$$= 0.0149 \text{ cm s}^{-1}$$

[4 marks]

(b) $$S = 4\pi r^2 \Rightarrow \frac{dS}{dr} = 8\pi r$$

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$$

$$= 8\pi r \times 0.0149$$

$$= 8\pi \times 4 \times 0.0149$$

$$= 1.500 \text{ cm}^2 \text{ s}^{-1}$$

[2 marks]

Answer 5

(i) $V = \left(h^6 + 16\right)^{\frac{1}{2}} - 4 \Rightarrow \dfrac{dV}{dh} = \dfrac{1}{2}\left(h^6 + 16\right)^{-\frac{1}{2}} 6h^5$

$$= \frac{3h^5}{\sqrt{h^6 + 16}}$$

When $h = 2$, $\dfrac{dV}{dh} = \dfrac{3 \times 2^5}{\sqrt{2^6 + 16}}$

$$= \frac{96}{4\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{24\sqrt{5}}{5} \text{ m}^3\text{ h}^{-1} \quad (= \ 10.73 \text{ m}^3\text{ h}^{-1})$$

[3 marks]

(ii) Told that $\dfrac{dV}{dt} = 8 \text{ m}^3\text{ h}^{-1}$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{5}{24\sqrt{5}} \times 8$$

$$= \frac{\sqrt{5}}{3}$$

$$= 0.75 \text{ m s}^{-1}$$

[3 marks]

Answer 6

$V = \dfrac{4}{3}\pi r^3 \Rightarrow \dfrac{dV}{dr} = 4\pi r^2$ Told $\dfrac{dV}{dt} = 10 \text{ cm}^3\text{ s}^{-1}$

$S = 4\pi r^2 \Rightarrow \dfrac{dS}{dr} = 8\pi r$

Also $r = \sqrt{\dfrac{S}{4\pi}}$ so when $S = 32\pi$, $r = \sqrt{\dfrac{32\pi}{4\pi}} = \sqrt{8} = 2\sqrt{2}$

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dV}{dt} \times \frac{dr}{dV}$$

$$= 8\pi r \times 10 \times \frac{1}{4\pi r^2}$$

$$= \frac{20}{r}$$

$$= \frac{20}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= 5\sqrt{2} \text{ cm}^2\text{ s}^{-1} \qquad (= \ 7.07 \text{ cm}^2\text{ s}^{-1})$$

[5 marks]

Answer 7

$V = x^3 \Rightarrow \dfrac{dV}{dx} = 3x^2$

$S = 6x^2 \Rightarrow \dfrac{dS}{dx} = 12x$

Told that $\dfrac{dS}{dt} = 0.45 \text{ cm}^2\text{ s}^{-1}$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dS} \times \frac{dS}{dt}$$

$$= 3x^2 \times \frac{1}{12x} \times 0.45$$

$$= 0.1125x$$

When $x = 8$, $\dfrac{dV}{dt} = 0.9 \text{ cm}^3\text{ s}^{-1}$

[5 marks]