6.4 Answers

6.4.1 Solution to 6.2 Will The Sink Overflow ?

$$\frac{dV}{dt} = rate\ in \ - \ rate\ out$$

$$\frac{dV}{dt} = 3 - 0.25V \qquad \text{Type Two}$$

$$\frac{dt}{dV} = \frac{1}{3 - 0.25V}$$

Integrate both sides with respect to V

$$\int \frac{dt}{dV}\, dV = \int \frac{1}{3 - 0.25V}\, dV$$

$$\int 1\, dt = \frac{1}{(-0.25)} \int (-0.25) \times \frac{1}{3 - 0.25V}\, dV$$

$$t + c = \frac{1}{(-0.25)} ln|3 - 0.25V|$$

$$-0.25t - 0.25c = ln|3 - 0.25V|$$

$$|3 - 0.25V| = e^{-0.25t} e^{-0.25c}$$

$$= B e^{-0.25t} \qquad \text{for } B \in \mathbb{R},\ B > 0$$

$$3 - 0.25V = C e^{-0.25t} \qquad \text{for } C \in \mathbb{R}$$

$$0.25V = 3 - C e^{-0.25t}$$

$$V = 12 - A e^{-0.25t} \qquad \text{for } A \in \mathbb{R} \text{ is the general solution}$$

As $t \to \infty$, $e^{-0.25t} \to 0$ and so $V \to 12$ litres.

As the sink has a capacity of only 11 litres, it will overflow.

[8 marks]

6.4.2 Solutions to 6.3 Exercise

Answer 1

(a) Volume of water is given by, $V = 3 \times 8 \times h$

$$V = 24h \Rightarrow \frac{dV}{dh} = 24$$

$$\frac{dV}{dt} = rate\ in - rate\ out$$

$$= 0.48 - 0.1h$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{24} \times (0.48 - 0.1h)$$

Multiply both sides by 1200

$$1200\frac{dh}{dt} = 50(0.48 - 0.1h)$$

$$1200\frac{dh}{dt} = 24 - 5h \qquad \Box$$

[4 marks]

(b) Type 2, so invert both sides and integrate with respect to h

$$\frac{1}{1200}\int \frac{dt}{dh}\,dh = \int \frac{1}{24 - 5h}\,dh$$

$$\frac{1}{1200}\int 1\,dt = \left(-\frac{1}{5}\right)\int (-5) \times \frac{1}{24 - 5h}\,dh$$

$$\frac{t}{1200} = \left(-\frac{1}{5}\right) ln|24 - 5h| + c_1$$

$$-\frac{t}{240} = ln|24 - 5h| + c_2$$

$$e^{-\frac{t}{240}} = B|24 - 5h| \quad \text{for } B \in \mathbb{R},\ B > 0$$

$$e^{-\frac{t}{240}} = A(24 - 5h) \quad \text{for } A \in \mathbb{R}$$

This is a good moment to use the fact that $h = 2$ when $t = 0$

$$1 = A(24 - 10)$$

$$A = \frac{1}{14}$$

$$14e^{-\frac{t}{240}} = 24 - 5h$$

$$h = \frac{24}{5} - \frac{14}{5}e^{-\frac{t}{240}}$$

$$h = 4.8 - 2.8e^{-\frac{t}{240}}$$

[6 marks]

(c) As $t \to \infty$, $e^{-\frac{t}{240}} \to 0$ and so $h \to 4.8$ metres (from below).

As the tank is 5 metres high, it can never become full.

[2 marks]

Answer 2

(a) $\dfrac{dh}{dt} = k(h - 9)^{\frac{1}{2}} \Rightarrow -1.1 = k(130 - 9)^{\frac{1}{2}} \quad \therefore \quad k = -0.1$

[2 marks]

(b) $\dfrac{dh}{dt} = -0.1(h - 9)^{\frac{1}{2}}$

Type 2, so invert both sides then integrate both sides with respect to h

$$\int \frac{dt}{dh}\, dh = -10 \int (h - 9)^{-\frac{1}{2}}\, dh$$

$$t = -10 \times 2 \times (h - 9)^{\frac{1}{2}} + c$$

$$t = -20\sqrt{h - 9} + c \qquad \text{General Solution}$$

Told that when $t = 0$, $h = 200$

$$0 = -20 \times \sqrt{191} + c$$

$$c = 20\sqrt{191}$$

$$t = -20\sqrt{h - 9} + 20\sqrt{191} \qquad \text{Particular Solution}$$

So, when $h = 50$,

$$t = -20\sqrt{41} + 20\sqrt{191}$$

$$= 148.34 \text{ minutes}$$

[6 marks]

Answer 3

(a) Various methods are possible; here's the pedantic way of getting it...

$$\frac{dx}{dt} = k$$

Invert both sides and integrate both sides with respect to x

$$\int \frac{dt}{dx}\,dx = \frac{1}{k}\int 1\,dx$$

$$t = \frac{x}{k} + c$$

But when $t = 0,\ x = 0 \ \therefore c = 0$

$$t = \frac{x}{k}$$

But when $t = 2,\ x = 1.5 \ \therefore k = 0.75$

$$t = \frac{4x}{3}$$

[2 marks]

(b) $t = 4$ hours

[1 mark]

(c) Type 2, invert, then integrate both sides with respect to x

$$\frac{dx}{dt} = \frac{\lambda}{(2x + 1)}$$

$$\int \frac{dt}{dx}\,dx = \frac{1}{\lambda}\int 2x + 1\ dx$$

$$t = \frac{1}{\lambda}\left(x^2 + x\right) + c \qquad \text{General Solution}$$

We know that $t = 0$ when $x = 0 \ \therefore c = 0$

$$t = \frac{x^2 + x}{\lambda} \qquad \text{Particular Solution}$$

[3 marks]

(d) Use the fact that when $t = 2,\ x = 1.5$

$$\lambda = \frac{1.5^2 + 1.5}{2}$$

$$= \frac{15}{8}$$

[1 mark]

(e) $t = \dfrac{8\left(3^2 + 3\right)}{15} = 6.4$ hours

Time is thus 10:24 pm (= 22:24)

[2 marks]

Answer 4

(a) The question tells us that $\dfrac{dS}{dt} = 8 \text{ cm}^2\text{ s}^{-1}$

For a cube with edge length x, $S = 6x^2 \Rightarrow \dfrac{dS}{dx} = 12x$

$$\frac{dx}{dt} = \frac{dx}{dS} \times \frac{dS}{dt}$$

$$= \frac{1}{12x} \times 8$$

$$= \frac{2}{3x} \quad \therefore\ k = \frac{2}{3} \qquad \square$$

[4 marks]

(b) Also, $V = x^3 \Rightarrow \dfrac{dV}{dx} = 3x^2$ and $x = V^{\frac{1}{3}}$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$= 3x^2 \times \frac{2}{3x}$$

$$= 2x$$

$$= 2V^{\frac{1}{3}} \qquad \square$$

[4 marks]

(c) Type 2, invert both sides and integrate both sides with respect to V

$$\int \frac{dt}{dV}\, dV = \frac{1}{2}\int V^{-\frac{1}{3}}\, dV$$

$$t = \frac{3V^{\frac{2}{3}}}{4} + c \qquad \text{The general solution}$$

Using the fact that $V = 8$ when $t = 0$ gives $c = -3$

$$t = \frac{3V^{\frac{2}{3}}}{4} - 3 \qquad \text{The particular solution}$$

When $V = 16\sqrt{2} = (2\sqrt{2})^3$

$$t = \frac{3 \times 8}{4} - 3$$

$$= 3 \text{ seconds}$$

[7 marks]

Answer 5

(a)

$$\frac{d\theta}{dt} = \frac{(3-\theta)}{125}$$

$$\frac{1}{3-\theta}\frac{d\theta}{dt} = 0.008$$

Integrate both sides with respect to t

$$(-1)\int \frac{(-1)}{3-\theta}\,d\theta = \int 0.008\,dt$$

$$-\ln|3-\theta| = 0.008t + c$$

$$\ln|3-\theta| = -0.008t - c$$

$$|3-\theta| = Be^{-0.008t} \qquad \text{for } B \in \mathbb{R},\ B > 0$$

$$3-\theta = De^{-0.008t-c} \qquad \text{for } D \in \mathbb{R}$$

$$\theta = -De^{-0.008t} + 3$$

$$\theta = Ae^{-0.008t} + 3 \quad \text{for } A \in \mathbb{R} \qquad \square$$

[4 marks]

(b) When $t = 0,\ \theta = 16\,°\text{C}$

$$16 = Ae^{-0.008\times 0} + 3$$

$$A = 16 - 3$$

$$= 13$$

$$\therefore\ \theta = 13e^{-0.008t} + 3$$

When $\theta = 10\,°\text{C}$,

$$10 = 13e^{-0.008t} + 3$$

$$e^{-0.008t} = \frac{7}{13}$$

$$-0.008t = \ln\left(\frac{7}{13}\right)$$

$$t = 77 \text{ minutes}$$

[5 marks]

Answer 6

(a) $$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$$

[1 mark]

(b) $$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{4\pi r^2} \times \frac{9000\pi}{(t + 81)^{\frac{5}{4}}} \qquad t \geqslant 0$$

$$= \frac{2250}{r^2 (t + 81)^{\frac{5}{4}}}$$

[2 marks]

(c) Separate the variables...

$$r^2 \frac{dr}{dt} = 2250 (t + 81)^{-\frac{5}{4}}$$

And integrate both sides with respect to *t*...

$$\int r^2 \, dr = 2250 \int (t + 81)^{-\frac{5}{4}} \, dt$$

$$\frac{r^3}{3} = 2250 \times \frac{4}{-1} (t + 81)^{-\frac{1}{4}} + c$$

$$r^3 = -27000 (t + 81)^{-\frac{1}{4}} + k$$

We're told that when $t = 0$, $r = 3$,

$$27 = -\frac{27000}{\sqrt[4]{81}} + k$$

$$k = 9027$$

$$r = \sqrt[3]{9027 - 27000 (t + 81)^{-\frac{1}{4}}}$$

[6 marks]

(d) When $t = 175$ in the part (c) formula, $r = 13.2$ cm

[1 mark]

(e) When $t = 175$ in the part (b) formula,

$$\frac{dr}{dt} = \frac{2250}{13.2^2 (175 + 81)^{\frac{5}{4}}}$$

$$= 0.0126 \quad \text{(0.0127 if more accurate } r \text{ used)}$$

[2 marks]