Answer 1

(i)

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y\frac{dy}{dx} = -x \qquad \text{separating the variables}$$

integrate both sides with respect to x

$$\int y\ dy = -\int x\ dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$y^2 + x^2 = r^2 \qquad \text{for some constant, } r$$

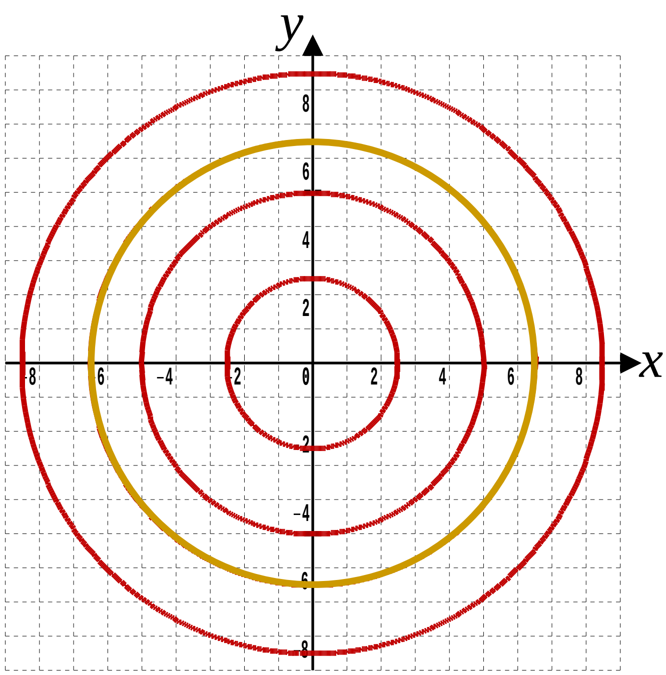
[3 marks]

(ii) As $y = 6$ when $x = 2.5$

$$y^2 + x^2 = 6.5^2 \qquad \text{is the particular solution}$$

[1 mark]

(iii) The relevant circle is highlighted in gold.



[1 mark]

Answer 2

$$\frac{dy}{dx} = \frac{3}{y\ cos^2 x}$$

Separate the variables...

$$y\frac{dy}{dx} = \frac{3}{cos^2 x}$$

Integrate both sides with respect to *x*...

$$\int y\,dy = 3\int sec^2 x\,dx$$

$$\frac{y^2}{2} = 3\,tan\,x + c$$

Using the fact that when $y = 2,\ x = \frac{\pi}{4}$...

$$2 = 3 + c \Rightarrow c = -1$$

$$\frac{y^2}{2} = 3\,tan\,x - 1$$

$$y^2 = 6\,tan\,x - 2$$

[5 marks]

Answer 3

(a) The volume of a sphere is given by, $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{4\pi r^2} \times (-c) \quad \text{as told } V \text{ is decreasing at a constant rate}$$

$$= -\frac{k}{r^2} \quad \text{where } k \text{ is a positive constant}$$

[3 marks]

(b)

$$r^2 \frac{dr}{dt} = -k \quad \text{separating the variables}$$

$$\int r^2\, dr = -k \int 1\, dt \quad \text{integrating both sides with respect to } t$$

$$\frac{r^3}{3} = -kt + c$$

When $t = 0,\ r = 40,\ \text{so } c = \frac{40^3}{3} = \frac{64000}{3}$

When $t = 5,\ r = 20,$

$$\frac{20^3}{3} = -5k + \frac{64000}{3}$$

$$5k = \frac{56000}{3} \Leftrightarrow k = \frac{11200}{3}$$

$$\frac{r^3}{3} = -\frac{11200}{3}t + \frac{64000}{3}$$

$$r^3 = -11200\,t + 64000$$

[5 marks]

(c) From the fact that "the volume of the balloon continues to decrease at a constant rate until the balloon is empty" we need to work out what value of t corresponds to the radius being zero.

$$0^3 = -11200\,t + 64000$$

$$t = \frac{64000}{11200}$$

$$= \frac{40}{7} \quad \text{(About 5.71 seconds)}$$

[2 marks]

Answer 4

(a)

$$\int (4y+3)^{-\frac{1}{2}}\,dy = \frac{1}{4}\int 4(4y+3)^{-\frac{1}{2}}\,dy$$

$$= \frac{1}{4}\times 2(4y+3)^{\frac{1}{2}} + k$$

$$= \frac{1}{2}(4y+3)^{\frac{1}{2}} + k$$

[2 marks]

(b)

$$\frac{dy}{dx} = \frac{\sqrt{(4y+3)}}{x^2}$$

$$(4y+3)^{-\frac{1}{2}}\frac{dy}{dx} = x^{-2}$$ separating the variables

$$\int (4y+3)^{-\frac{1}{2}}\,dy = \int x^{-2}\,dx$$ integrating both sides with respect to x

$$\frac{1}{2}(4y+3)^{\frac{1}{2}} = -x^{-1} + c$$ from part (a)

$$\frac{1}{2}(9)^{\frac{1}{2}} = \frac{1}{2} + c \Rightarrow c = 1$$

$$\frac{1}{2}(4y+3)^{\frac{1}{2}} = 1 - \frac{1}{x}$$

$$4y + 3 = 4\left(1 - \frac{1}{x}\right)^2$$

$$4y + 3 = 4\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)$$

$$4y = 4 - \frac{8}{x} + \frac{4}{x^2} - 3$$

$$y = \frac{1}{4} - \frac{2}{x} + \frac{1}{x^2}$$

[6 marks]

Answer 5

(a)

$$\frac{5}{[\,(x-1)(3x+2)\,]} = \frac{A}{(x-1)} + \frac{B}{(3x+2)}$$

Multiply through by the red bracket

$$\frac{5\,[...]}{[\,(x-1)(3x+2)\,]} = \frac{A\,[...]}{(x-1)} + \frac{B\,[...]}{(3x+2)}$$

$$5 = A(3x+2) + B(x-1)$$

Let $x = 1$: $5 = A(5) \Leftrightarrow A = 1$

Let $x = 0$: $5 = 2 - B \Leftrightarrow B = -3$

$$\therefore \frac{5}{(x-1)(3x+2)} = \frac{1}{x-1} - \frac{3}{3x+2}$$

[3 marks]

(b) $$\int \frac{5}{(x-1)(3x+2)}\,dx = \int \frac{1}{x-1}dx - \int \frac{3}{3x+2}\,dx, \quad x > 1$$

$$= \ln(x-1) - \ln(3x+2) + c$$

[3 marks]

(c) $$(x-1)(3x+2)\frac{dy}{dx} = 5y$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{5}{(x-1)(3x+2)} \qquad \text{separating the variables}$$

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{5}{(x-1)(3x+2)}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{5}{(x-1)(3x+2)}\,dx$$

$$\ln|y| = \ln|x-1| - \ln|3x+2| + c$$

$$\ln|y| = \ln\left|\frac{x-1}{3x+2}\right| + c$$

$$|y| = B\left(\frac{x-1}{3x+2}\right) \qquad \text{for } B \in \mathbb{R},\ B > 0 \text{ and } x > 1$$

$$y = \frac{A(x-1)}{(3x+2)} \qquad \text{for } A \in \mathbb{R},\ x > 1$$

Making use of the fact that $y = 8$ at $x = 2$ yields,

$$8 = \frac{A \times 1}{8} \Leftrightarrow A = 64$$

$$y = \frac{64(x-1)}{(3x+2)} \qquad \text{for } x > 1$$

[6 marks]