8.2 Answers

Answer 1

(i)

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \qquad \text{separating the variables}$$

integrate both sides with respect to x

$$ln|y| = ln|x| + c$$

exponentiate both sides

$$e^{ln|y|} = e^{ln|x|+c}$$

$$|y| = e^{ln|x|}e^{c}$$

$$= B|x| \qquad \text{for } B \in \mathbb{R},\ B > 0$$

$$y = Ax \qquad \text{for } A \in \mathbb{R} \qquad \text{is the general solution}$$

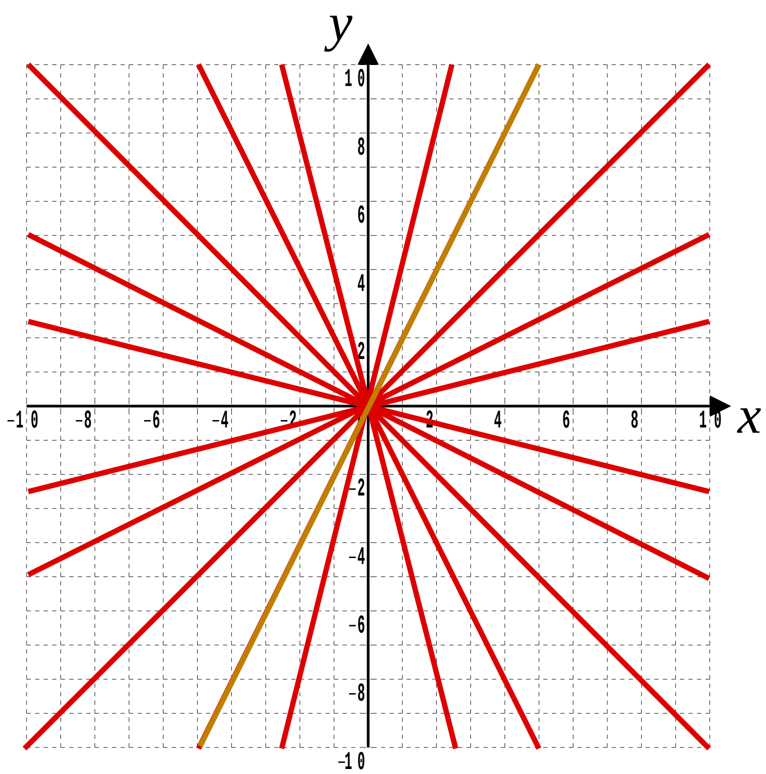
[3 marks]

(ii) As $y = 8$ when $x = 4$, $A = 2$

$$y = 2x \qquad \text{is the particular solution}$$

[1 mark]

(iii) The relevant line is highlighted in gold.



[1 mark]

Answer 2

(a)

$$\frac{dH}{dt} = \frac{H\cos(0.25t)}{40}$$

$$\frac{1}{H}\frac{dH}{dt} = \frac{1}{40}\cos\left(\frac{t}{4}\right) \qquad \text{separating the variables}$$

$$\int \frac{1}{H}\,dH = \frac{1}{40} \times 4 \int \frac{1}{4}\cos\left(\frac{t}{4}\right)dt \qquad \text{integrating both sides with respect to } t$$

$$ln|H| = \frac{1}{40} \times 4\sin\left(\frac{t}{4}\right) + c \qquad \text{as } H > 0,\ \text{modulus signs dropped}$$

$$ln\,H = \frac{1}{10}\sin(0.25t) + c \qquad c = ln\,5 \text{ from initial conditions}$$

$$ln\,H - ln\,5 = 0.1\sin(0.25t)$$

$$ln\left(\frac{H}{5}\right) = 0.1\sin(0.25t) \qquad \text{exponentiate both sides}$$

$$\frac{H}{5} = e^{0.1\sin(0.25t)}$$

$$H = 5\,e^{0.1\sin(0.25t)} \quad \square$$

[5 marks]

(b) Passenger maximum height will occur when $sin(0.25t) = 1$ in which case $H_{MAX} = 5\,e^{0.1}$ (About 5.53 metres)

[1 mark]

(c) Maximum height for the first time will be when $0.25t = \frac{\pi}{2}$ and for the second time when,

$$0.25T = \frac{\pi}{2} + 2\pi$$

$$\frac{T}{4} = \frac{5\pi}{2}$$

$$T = 10\pi \text{ (About 31.4 seconds)}$$

[2 marks]

Answer 3

$$\frac{dy}{dx} = x\,e^{2y}$$

$$e^{-2y}\frac{dy}{dx} = x \qquad \text{separating the variables}$$

$$\int e^{-2y}\,dy = \int x\,dx \qquad \text{integrate both sides with respect to } x$$

$$-\frac{1}{2}\int(-2)\,e^{-2y}\,dy = \int x\,dx \qquad \text{setting up a chain rule backwards}$$

$$-\frac{1}{2}e^{-2y} = \frac{x^2}{2} + c \qquad \text{use } y = 0 \text{ when } x = 0 \text{ to get } c = -\frac{1}{2}$$

$$-\frac{1}{2}e^{-2y} = \frac{x^2}{2} - \frac{1}{2} \qquad \text{multiply through by } (-2)$$

$$e^{-2y} = -x^2 + 1 \qquad \text{take the } ln \text{ of both sides}$$

$$-2y = ln\left|1 - x^2\right| \qquad \text{can remove modulus signs as } |x| < 1$$

$$y = -\frac{1}{2}ln\left(1 - x^2\right)$$

$$y = ln\left(1 - x^2\right)^{-\frac{1}{2}}$$

$$y = ln\left(\frac{1}{\sqrt{1 - x^2}}\right) \qquad |x| < 1 \qquad \square$$

[4 marks]

Answer 4

(a)

$$\frac{d\theta}{dt} = \lambda\,(120 - \theta\,)$$

$$\frac{1}{(\,120 - \theta\,)}\,\frac{d\theta}{dt} = \lambda \qquad \text{separate the variables}$$

$$\int \frac{1}{(\,120 - \theta\,)}\,d\theta = \int \lambda\,dt \qquad \text{integrate both sides with respect to } t$$

$$(-1)\int(-1)\,\frac{1}{(\,120 - \theta\,)}\,d\theta = \int \lambda\,dt \qquad \text{setting up a chain rule backwards}$$

$$-\,ln\,|\,120 - \theta\,| = \lambda t + c \qquad \theta = 20 \text{ when } t = 0 \Rightarrow c = -\,ln\,100$$

$$-\,\lambda t = ln\,|\,120 - \theta\,| - ln\,100$$

$$-\,\lambda t = ln\left(\frac{120 - \theta}{100}\right) \qquad \text{As } \theta \leqslant 100 \text{ modulus sign removed}$$

$$e^{-\lambda t} = \frac{120 - \theta}{100}$$

$$100\,e^{-\lambda t} = 120 - \theta$$

$$\theta = 120 - 100\,e^{-\lambda t} \qquad \square$$

[8 marks]

(b) The equation that needs to be solved is,

$$100 = 120 - 100\,e^{-0.01\,t}$$

$$100\,e^{-0.01\,t} = 20$$

$$e^{-0.01\,t} = 0.2$$

$$-\,0.01\,t = ln\,0.2$$

$$t = 161 \text{ seconds}$$

[3 marks]

Answer 5

(a) Using the red bracket method,

$$\frac{2x - 1}{[\,(x-1)(2x-3)\,]} = \frac{A}{(x-1)} + \frac{B}{(2x-3)}$$

$$\frac{(2x-1)\,[...]}{[\,(x-1)(2x-3)\,]} = \frac{A\,[...]}{(x-1)} + \frac{B\,[...]}{(2x-3)}$$

$$2x - 1 = A(2x-3) + B(x-1)$$

$$\text{Let } x = 1 \;:\; 1 = A(-1) \Leftrightarrow A = -1$$

$$\text{Let } x = 0 \;:\; -1 = (-1)(-3) + B(-1) \Leftrightarrow B = 4$$

$$\therefore \frac{2x-1}{(x-1)\,(2x-3)} = -\frac{1}{(x-1)} + \frac{4}{(2x-3)}$$

[3 marks]

(b)

$$(2x-3)(x-1)\frac{dy}{dx} = (2x-1)y$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x-1}{(2x-3)(x-1)} \qquad \text{separating variables}$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{(x-1)} + \frac{4}{(2x-3)} \qquad \text{using part (a)}$$

$$\int \frac{1}{y}\,dy = -\int \frac{1}{x-1}\,dx + 2\int \frac{2}{2x-3}\,dx$$

$$ln|y| = -ln|x-1| + 2\,ln|2x-3| + c$$

$$= -ln(x-1) + ln(2x-3)^2 + c$$

The RHS modulus signs removed as told that $x \geqslant 2$

$$= ln\left(\frac{(2x-3)^2}{x-1}\right) + c$$

$$|y| = \frac{B(2x-3)^2}{x-1} \qquad \text{for } B \in \mathbb{R},\; B > 0$$

$$y = \frac{A(2x-3)^2}{x-1} \qquad \text{for } A \in \mathbb{R}$$

[5 marks]

(c) Using the fact that $y = 10$ at $x = 2$,

$$10 = \frac{A \times 1}{1} \Leftrightarrow A = 10$$

$$y = \frac{10(2x-3)^2}{x-1}$$

[4 marks]