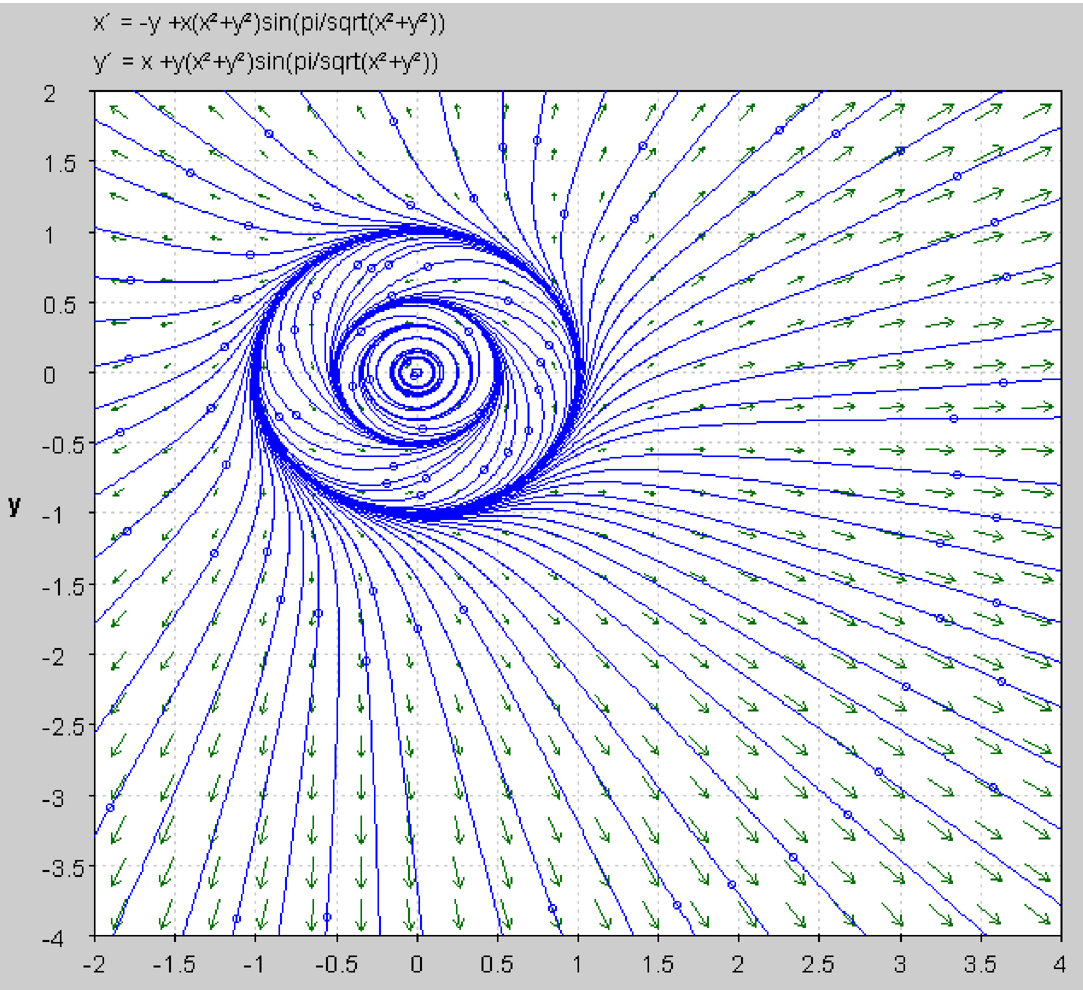
A-Level Pure Mathematics
Year 2

DIFFERENTIAL EQUATIONS I



Differential Equations I

Lesson 1

1.1 Type One

Three types of differential equation are to be considered starting in this lesson with the most straight forward.

A Type One differential equation is of the form

$$\frac{dy}{dx} = f(x)$$

These have been encountered before, in the Year 1 pure mathematics course.

Example

A-Level Examination Question from January 2013, Paper C1, Q8 (Edexcel)

$$\frac{dy}{dx} = -x^3 + \frac{4x - 5}{2x^3}, \qquad x \neq 0$$

Given that $y = 7$ at $x = 1$, find y in terms of x, giving each term in its simplest form.

Teaching Video : http://www.NumberWonder.co.uk/Video/v9066(1).mp4

[6 marks]

The Year 2 version of such Type One questions will draw on the fact that it is now known how to integrate many more functions, using many more techniques.

1.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Solve the differential equation,

$$\frac{dy}{dx} = \cos x \qquad \text{given that} \quad y = 0.5 \quad \text{when} \quad x = \frac{\pi}{2}$$

[4 marks]

Question 2

Solve the differential equation

$$\frac{dy}{dx} = \sec^2 x \qquad \text{given that} \quad y = 1 \quad \text{when} \quad x = \frac{3\pi}{4}$$

[5 marks]

Question 3

(i) Find the displacement, s cm, from O of a particle at time t s, if its velocity, v cm.s^{-1}, is given by the differential equation

$$v = \frac{ds}{dt} = t^2 - t + 4$$

and the displacement is 100 cm at time 6 s.

[3 marks]

(ii) What will be the particle's displacement when $t = 3$ s ?

[2 marks]

Question 4

Solve the differential equation,

$$\frac{dy}{dx} = x^2 \left(x^3 + 5\right)^4 \text{ given that } y = 209 \text{ when } x = 0$$

Hint : Chain rule backwards.

[5 marks]

Question 5

(i) Find the general solution of the differential equation

$$5x \frac{dy}{dx} - 1 = 0$$

[4 marks]

(ii) Given that $y = 0$ when $x = 3$, find the particular solution.
Give your answer in an elegant a form as possible.

[3 marks]

Question 6

(i) Find the general solution of the differential equation

$$sec\,x\,\frac{dy}{dx} - x = 0$$

Hint : Integration by parts.

[4 marks]

(ii) Given that $y = \pi$ when $x = \frac{\pi}{2}$, find the particular solution.

[3 marks]

Question 7

Solve the differential equation,

$$\frac{dy}{dx} = \frac{1}{9 + x^2} \qquad \text{given that } y = \frac{\pi}{9} \text{ when } x = \sqrt{3}$$

Hint : Let $x = 3 \tan u$

[8 marks]

Question 8

(i) Find the displacement s m of a particle t s after leaving O, where

$$t\,\frac{ds}{dt} = t^2 + 4$$

[5 marks]

(ii) Given that $s = 4\ln 2$ when $t = 2$, and $s = a + b\ln 2$ when $t = 4$, find a and b.

[4 marks]

1.3 Answers

1.3.1 Solution to Example

The key idea is (as always) to solve this equation by doing the same to both sides, in this case integrating both sides with respect to *x*.

$$\frac{dy}{dx} = -x^3 + \frac{4x - 5}{2x^3}$$

$$\int \frac{dy}{dx}\,dx = \int -x^3 + \frac{4x - 5}{2x^3}\,dx$$

$$y = \int -x^3 + \frac{4x}{2x^3} - \frac{5}{2x^3}\,dx$$

$$= \int -x^3 + 2x^{-2} - \frac{5x^{-3}}{2}\,dx$$

$$= -\frac{x^4}{4} - 2x^{-1} + \frac{5x^{-2}}{4} + c$$

Given that $y = 7$ at $x = 1$,

$$7 = -\frac{1}{4} - 2 + \frac{5}{4} + c$$

$$\therefore\ c = 8$$

Thus, $$y = -\frac{x^4}{4} - 2x^{-1} + \frac{5x^{-2}}{4} + 8$$

[6 marks]

1.3.2 Solutions to Exercise

Answer 1

$$\frac{dy}{dx} = \cos x$$

Integrate both sides wrt x

$$\int \frac{dy}{dx}\,dx = \int \cos x\,dx$$

$$y = \sin x + c$$

Given that $y = 0.5$ at $x = \frac{\pi}{2}$,

$$0.5 = \sin\left(\frac{\pi}{2}\right) + c$$

$$0.5 = 1 + c$$

$$c = -0.5$$

Thus;

$$y = \sin x - 0.5$$

[4 marks]

Answer 2

$$\frac{dy}{dx} = sec^2 x$$

Integrate both sides with respect to x

$$\int \frac{dy}{dx}\,dx = \int sec^2 x\ dx$$

$$y = tan\,x + c$$

Given that $y = 1$ at $x = \frac{3\pi}{4}$,

$$1 = tan\left(\frac{3\pi}{4}\right) + c$$

$$1 = -1 + c$$

$$c = 2$$

Thus;

$$y = tan\,x + 2$$

[5 marks]

Answer 3

(i)

$$v = \frac{ds}{dt} = t^2 - t + 4$$

Integrate both sides with respect to t

$$\int \frac{ds}{dt}\,dt = \int t^2 - t + 4\ dt$$

$$s = \frac{t^3}{3} - \frac{t^2}{2} + 4t + c$$

Given that $s = 100$ at $t = 6$,

$$100 = 72 - 18 + 24 + c$$

$$c = 22$$

Thus;

$$s = \frac{t^3}{3} - \frac{t^2}{2} + 4t + 22$$

[3 marks]

(ii)

When $t = 3$ s,

$$s = 9 - 4.5 + 12 + 22$$

$$s = 38.5\,\text{cm}$$

[2 marks]

Answer 4

$$\frac{dy}{dx} = x^2 \left(x^3 + 5 \right)^4$$

Integrate both sides with respect to x

$$\int \frac{dy}{dx}\, dx = \frac{1}{3} \int 3x^2 \left(x^3 + 5 \right)^4 \, dx$$

$$y = \frac{1}{3} \times \frac{\left(x^3 + 5 \right)^5}{5} + c$$

Given that $y = 8$ at $x = 0$,

$$209 = \frac{5^5}{15} + c$$

$$c = \frac{2}{3}$$

Thus;

$$y = \frac{\left(x^3 + 5 \right)^5}{15} + \frac{2}{3}$$

[5 marks]

Answer 5

(i)

$$5x \frac{dy}{dx} - 1 = 0$$

$$5x \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{5x}$$

Integrate both sides with respect to x

$$\int \frac{dy}{dx}\, dx = \frac{1}{5} \int \frac{1}{x}\, dx$$

$$y = \frac{1}{5}\, ln|x| + c$$

[4 marks]

(ii) Given that $y = 0$ at $x = 3$,

$$0 = \frac{1}{5}\, ln\, 3 + c$$

$$\therefore\ c = -\frac{1}{5}\, ln\, 3$$

Thus,

$$y = \frac{1}{5}\, ln|x| - \frac{1}{5}\, ln\, 3$$

$$y = \frac{1}{5}\, ln \left| \frac{x}{3} \right|$$

[3 marks]

Answer 6

(i)

$$sec\,x\,\frac{dy}{dx} - x = 0$$

$$sec\,x\,\frac{dy}{dx} = x$$

$$\frac{dy}{dx} = \frac{x}{sec\,x}$$

$$\frac{dy}{dx} = x\,cos\,x$$

Integrate both sides wrt x

$$\int \frac{dy}{dx}\,dx = \int x\,cos\,x\;dx$$

$$\boldsymbol{L \quad I \qquad D \quad I}$$

$$y = x\,sin\,x - \int sin\,x\;dx$$

$$= x\,sin\,x + cos\,x + c$$

[4 marks]

(ii)

$$\pi = \frac{\pi}{2}\,sin\left(\frac{\pi}{2}\right) + cos\left(\frac{\pi}{2}\right) + c$$

$$\pi = \frac{\pi}{2} + c$$

$$\therefore\; c = \frac{\pi}{2}$$

Thus;

$$y = x\,sin\,x + cos\,x + \frac{\pi}{2}$$

[3 marks]

Answer 7

$$\text{If } x = 3\tan u \text{ then } \frac{dx}{du} = 3\sec^2 u$$

$$\frac{dy}{dx} = \frac{1}{9 + x^2}$$

Integrate both sides wrt x

$$\int \frac{dy}{dx}\,dx = \int \frac{1}{9 + (3\tan u)^2}\; 3\sec^2 u\; du$$

$$y = \int \frac{1}{9(1 + \tan^2 u)}\; 3\sec^2 u\; du$$

$$y = \frac{1}{3}\int \frac{1}{\sec^2 u}\sec^2 u\; du$$

$$y = \frac{1}{3}\int 1\, du$$

$$y = \frac{1}{3}u + c$$

Given that $y = \frac{\pi}{9}$ at $x = \sqrt{3}$ when $u = \frac{\pi}{6}$,

$$\frac{\pi}{9} = \frac{\pi}{18} + c$$

$$c = \frac{\pi}{18}$$

Thus;

$$y = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + \frac{\pi}{18}$$

[8 marks]

Answer 8
(i)

$$t\,\frac{ds}{dt} \;=\; t^2 + 4$$

$$\frac{ds}{dt} \;=\; t + \frac{4}{t}$$

Integrate both sides wrt t

$$\int \frac{ds}{dt}\,dt \;=\; \int \; t + \frac{4}{t} \;\; dt$$

$$s \;=\; \frac{t^2}{2} + 4\,ln\left|\,t\,\right| + c$$

[5 marks]

(ii) Using the given boundary that $s = 4\,ln\,2$ when $t = 2$, yields;

$$4\,ln\,2 \;=\; \frac{2^2}{2} + 4\,ln\,2 + c$$

$$c \;=\; -2$$

$$s \;=\; \frac{t^2}{2} + 4\,ln\left|\,t\,\right| - 2$$

So, when $t = 4$

$$s \;=\; \frac{t^2}{2} + 4\,ln\left|\,t\,\right| - 2$$

$$=\; \frac{4^2}{2} + \;4\,ln\,4 - 2$$

$$=\; 8 + 4\,ln\,2^2 - 2$$

$$=\; 6 + 8\,ln\,2$$

That is,

$$a \;=\; 6 \qquad \text{and} \qquad b \;=\; 8$$

[4 marks]

2.1 Type Two

A Type One differential equation is of the form

$$\frac{dy}{dx} = f(x)$$

A Type Two differential equation is of the form

$$\frac{dy}{dx} = f(y)$$

The solution technique to apply to Type Two differential equation problems involves inverting both sides and then integrating with respect to y
Often, partial fractions arise and the tricky rearranging of a formula.

Example

Solve the following differential equation,

$$\frac{dy}{dx} = \frac{1}{y^4}$$

given that $y = 3$ when $x = 49$
Present your solution in the form $y = f(x)$
Teaching video : http://www.NumberWonder.co.uk/Video/v9066(2).mp4

[6 marks]

2.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Solve the following differential equation,

$$\frac{dy}{dx} = \frac{1}{y}$$

given that $y = 3$ when $x = 4$

Present your solution in the form $y = f(x)$

[5 marks]

Question 2

Find the general solution to the differential equation,

$$\frac{dy}{dx} = y$$

Present your solution as elegantly as possible and in the form $y = f(x)$

[5 marks]

Question 3

Find the general solution to the differential equation,

$$\frac{dy}{dx} = y + 2$$

Present your solution as elegantly as possible and in the form $y = f(x)$

[5 marks]

Question 4

Solve the differential equation

$$\frac{dy}{dx} = cos^2 y \qquad \text{given that } y = \frac{\pi}{4} \text{ when } x = 7$$

Present your solution in the form $y = f(x)$

[7 marks]

Question 5

Solve the following differential equation,

$$\frac{dy}{dx} = \frac{1}{2y - 8}$$

given that $y = 5$ when $x = 4$

(i) Present your solution in the form $x = f(y)$

[4 marks]

(ii) By completing the square on your part (i) answer, present your solution in the form $y = f(x)$

[3 marks]

Question 6

A-Level Examination Question from June 2018, Paper C34, Q13 (Edexcel)

(a) Express $\dfrac{1}{(4-x)(2-x)}$ in partial fractions.

[2 marks]

The mass, x grams, of a substance at time t seconds after a chemical reaction is modelled by the differential equation,

$$\frac{dx}{dt} = k(4-x)(2-x), \qquad t \geqslant 0, \ 0 \leqslant x < 2$$

where k is a constant.

Given that when $t = 0$, $x = 0$

(b) solve the differential equation and show that the solution can be written as,

$$x = \frac{4 - 4e^{2kt}}{1 - 2e^{2kt}}$$

[7 marks]

Given that $k = 0.1$

(c) find the value of t when $x = 1$

Giving your answer, in seconds, to 3 significant figures.

[2 marks]

2.3 Answers

2.3.1 Solution to Example

$$\frac{dy}{dx} = \frac{1}{y^4}$$

Invert both sides

$$\frac{dx}{dy} = y^4$$

Integrate both sides with respect to y

$$\int \frac{dx}{dy}\, dy = \int y^4\, dy$$

$$x = \frac{y^5}{5} + c$$

$$5x = y^5 + 5c$$

$$y^5 = 5x - 5c$$

$$y^5 = 5x + A$$

$$y = (5x + A)^{\frac{1}{5}} \qquad \text{is the general solution}$$

Given that $y = 3$ when $x = 49$,

$$3^5 = 5 \times 49 + A$$

$$\therefore A = -2$$

Thus;

$$y = (5x - 2)^{\frac{1}{5}} \qquad \text{is particular solution}$$

[6 marks]

2.3.2 Solutions to Exercise

Answer 1

$$\frac{dy}{dx} = \frac{1}{y}$$

Invert both sides

$$\frac{dx}{dy} = y$$

Integrate both sides with respect to y

$$\int \frac{dx}{dy}\, dy = \int y\ dy$$

$$x = \frac{y^2}{2} + c$$

$$y = \sqrt{2x + A} \quad \text{is the general solution}$$

Given that $y = 3$ when $x = 4$,

$$y = \sqrt{2x + 1} \quad \text{is the particular solution}$$

[5 marks]

Answer 2

$$\frac{dy}{dx} = y$$

Invert both sides

$$\frac{dx}{dy} = \frac{1}{y}$$

Integrate both sides with respect to y

$$\int \frac{dx}{dy}\, dy = \int \frac{1}{y}\ dy$$

$$x = ln|y| + c$$

$$|y| = e^{x-c}$$

$$|y| = e^{-c} e^{x}$$

$$|y| = Be^{x} \quad \text{for } B \in \mathbb{R},\ B > 0$$

$$y = Ae^{x} \quad \text{for } A \in \mathbb{R}$$

[5 marks]

Answer 3

$$\frac{dy}{dx} = y + 2$$

Invert both sides

$$\frac{dx}{dy} = \frac{1}{y + 2}$$

Integrate both sides with respect to y

$$\int \frac{dx}{dy}\, dy = \int \frac{1}{y + 2}\, dy$$

$$x = ln\left| y + 2 \right| + c$$

$$\left| y + 2 \right| = e^{x - c}$$

$$= e^{-c} e^{x}$$

$$= B e^{x} \qquad \text{for } B \in \mathbb{R}, B > 0$$

$$y + 2 = A e^{x} \qquad \text{for } A \in \mathbb{R}$$

$$y = A e^{x} - 2$$

[5 marks]

Answer 4

$$\frac{dy}{dx} = cos^2 y$$

Invert both sides

$$\frac{dx}{dy} = \frac{1}{cos^2 y}$$

$$\frac{dx}{dy} = sec^2 y$$

Integrate both sides with respect to y

$$\int \frac{dx}{dy}\, dy = \int sec^2 y\; dy$$

$$x = tan\, y + c$$

$$x - c = tan\, y$$

$$\therefore \quad y = arctan\,(x + A) \qquad \text{is the general solution}$$

Given that $y = \frac{\pi}{4}$ when $x = 7$

$$tan\left(\frac{\pi}{4}\right) = 7 + A$$

$$A = -6$$

$$\therefore \quad y = arctan\,(x - 6) \quad \text{is the particular solution}$$

[7 marks]

Answer 5

(i)

$$\frac{dy}{dx} = \frac{1}{2y - 8}$$

Invert both sides

$$\frac{dx}{dy} = 2y - 8$$

Integrate both sides with respect to y

$$\int \frac{dx}{dy}\, dy = \int 2y - 8 \ dy$$

$$x = y^2 - 8y + c \qquad \text{is the general solution}$$

Given that $y = 5$ when $x = 4$,

$$4 = 5^2 - 8 \times 5 + c$$

$$\therefore \ c = 19$$

$$x = y^2 - 8y + 19 \qquad \text{is the particular solution}$$

[4 marks]

(ii)

$$x = y^2 - 8y + 19$$

$$x = (y - 4)^2 - 16 + 19$$

$$x - 3 = (y - 4)^2$$

$$y = 4 + \sqrt{x - 3}$$

[3 marks]

Note that $y = 4 - \sqrt{x - 3}$ cannot be a particular solution

Answer 6

(a)

$$\frac{1}{[(4-x)(2-x)]} = \frac{A}{4-x} + \frac{B}{2-x}$$

Multiply through by the red bracket

$$\frac{1\ [\ \ldots\ldots\ ]}{[(4-x)(2-x)]} = \frac{A\ [\ \ldots\ldots\ ]}{4-x} + \frac{B\ [\ \ldots\ldots\ ]}{2-x}$$

$$1 = A(2-x) + B(4-x)$$

$$\text{Let } x = 2\ :\ 1 = 2B \Rightarrow B = \frac{1}{2}$$

$$\text{Let } x = 4\ :\ 1 = -2A \Rightarrow A = -\frac{1}{2}$$

$$\therefore \frac{1}{(4-x)(2-x)} = -\frac{1}{2(4-x)} + \frac{1}{2(2-x)}$$

[2 marks]

(b)

$$\frac{dx}{dt} = k(4-x)(2-x)$$

$$\frac{dt}{dx} = \frac{1}{k\,(4-x)(2-x)}$$

$$\int \frac{dt}{dx}\,dx = \int \frac{1}{k\,(4-x)(2-x)}\,dx$$

$$\int dt = \frac{1}{2k}\int -\frac{1}{(4-x)} + \frac{1}{(2-x)}\,dx$$

$$2k\int dt = \int (-1)\,\frac{1}{4-x}\,dx - \int (-1)\,\frac{1}{2-x}\,dx$$

$$2kt + c = ln|4-x| - ln|2-x|$$

$$2kt + c = ln\left|\frac{4-x}{2-x}\right|$$

As $0 \leqslant x < 2$ the modulus signs are not needed.

$$e^{2kt+c} = \frac{4-x}{2-x} \Rightarrow e^{2kt}e^{c} = \frac{4-x}{2-x} \Rightarrow A\,e^{2kt} = \frac{4-x}{2-x}$$

Given that when $t = 0$, $x = 0$, the constant A can be found; $A = 2$

$$2\,e^{2kt}(2-x) = 4-x$$

$$4\,e^{2kt} - 2x\,e^{2kt} = 4-x$$

$$x - 2x\,e^{2kt} = 4 - 4\,e^{2kt}$$

$$x\left(1 - 2\,e^{2kt}\right) = 4 - 4\,e^{2kt}$$

$$x = \frac{4 - 4\,e^{2kt}}{1 - 2\,e^{2kt}} \qquad \square$$

[7 marks]

(c) With $A = 2$, $k = 0.1$ and $x = 1$,

$$A\,e^{2kt} = \frac{4 - x}{2 - x}$$

becomes,

$$2\,e^{0.2\,t} = 3$$

$$e^{0.2\,t} = \frac{3}{2}$$

take the natural log of both sides,

$$0.2\,t = ln\left(\frac{3}{2}\right)$$

$$t = 5\,ln\left(\frac{3}{2}\right)$$

$$= 2.03 \text{ seconds}$$

[2 marks]

Lesson 3

3.1 Type Three

A Type Three differential equation is of the form

$$\frac{dy}{dx} = f(x)\, g(y)$$

The solution technique is called *separating the variables*.

By this is meant rearranging the given equation into the form

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x)$$

and then integrating both sides with respect to x.

Example

Solve the following differential equation,

$$\frac{dy}{dx} = \frac{y^2}{x}$$

given that $y = 4$ when $x = 1$

Present your solution as elegantly as possible and in the form $y = f(x)$

[6 marks]

3.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Solve the following differential equation,

$$\frac{dy}{dx} = 6x\,y^2$$

given that $y = 0.2$ when $x = 1$

Present your solution as elegantly as possible and in the form $y = f\,(x)$

[6 marks]

Question 2

Solve the following differential equation,

$$cos^2 x \ \frac{dy}{dx} - \frac{1}{y^2} = 0$$

given that $y = 2$ when $x = \dfrac{\pi}{4}$

Present your solution as elegantly as possible and in the form $y = f(x)$

[6 marks]

Question 3

Consider the differential equation;

$$\frac{dy}{dx} = \frac{3x^2}{y} \qquad \text{where } y = 3 \text{ when } x = 2$$

By separating the variables, show that this has solution

$$y^2 = Ax^3 + B$$

where A and B are integers that you should determine.

[4 marks]

Question 4

Consider the differential equation;

$$\frac{dy}{dx} = e^{y+x} \qquad \text{where } y = 0 \text{ when } x = 0$$

By separating the variables, show that this has solution

$$\frac{1}{e^y} + e^x = k$$

where k is an integer that you should determine.

[4 marks]

Question 5

The differential equation

The differential equation $\frac{dv}{dt} = 10 - 0.2v$ models the motion of a body falling vertically subject to air resistance, where v is the downward vertical speed in m/s and the time, t, is in seconds.

(i) Does $\frac{dv}{dt}$ represent displacement, velocity or acceleration ?

[1 mark]

(ii) The body is dropped from rest. What is the initial acceleration ?

[1 mark]

(iii) Find the terminal velocity which occurs when $\frac{dv}{dt} = 0$

[1 mark]

(iv) Show that the differential equation can be written as, $\frac{1}{50 - v} \frac{dv}{dt} = 0.2$

[2 marks]

(v) Remembering that the body was dropped from rest, show that the differential equation can be solved to give that, $v = 50\left(1 - e^{-0.2t}\right)$

[5 marks]

Question 6

(i) Find values of A and B for which, $\dfrac{1}{v(1+v)} = \dfrac{A}{v} + \dfrac{B}{1+v}$

where $v > 0$

[2 marks]

(ii) Show that the differential equation $\dfrac{dv}{dt} = -\left(v + v^2\right)$ where $v = 1$

when $t = 0$ has solution, $\dfrac{2v}{1+v} = e^{-t}$ for $v > 0$

[6 marks]

(iii) Show how the part (ii) answer can be rearranged to give

$$v = \frac{1}{2e^{t} - 1}$$

[2 marks]

3.3 Answers

3.3.1 Solution to Example

$$\frac{dy}{dx} = \frac{y^2}{x}$$

Separate the variables

$$\frac{1}{y^2}\,\frac{dy}{dx} = \frac{1}{x}$$

Integrate both sides with respect to x

$$\int y^{-2}\,\frac{dy}{dx}\,dx = \int \frac{1}{x}\,dx$$

$$-y^{-1} = ln|x| + c$$

$$-y = \frac{1}{ln|x| + c}$$

$$y = \frac{1}{-ln|x| - c}$$

$$y = \frac{1}{A - ln|x|} \qquad \text{is the general solution}$$

Given that $y = 4$ when $x = 1$

$$4 = \frac{1}{A - ln\,1}$$

$$\therefore\ A = 0.25$$

Thus;

$$y = \frac{1}{0.25 - ln|x|} \qquad \text{is the particular solution}$$

[6 marks]

3.3.2 Solutions to Exercise

Answer 1

$$\frac{dy}{dx} = 6xy^2$$

Separate the variables

$$\frac{1}{y^2}\frac{dy}{dx} = 6x$$

Integrate both sides with respect to x

$$\int y^{-2}\frac{dy}{dx}\,dx = 6\int x\,dx$$

$$-y^{-1} = 3x^2 + c$$

$$-y = \frac{1}{3x^2 + c}$$

$$y = \frac{1}{-3x^2 - c}$$

$$y = \frac{1}{A - 3x^2} \qquad \text{is the general solution}$$

Given that $y = 0.2$ when $x = 1$

$$0.2 = \frac{1}{A - 3}$$

$$\therefore\ A = 8$$

Thus;

$$y = \frac{1}{8 - 3x^2} \qquad \text{is the particular solution}$$

[6 marks]

Answer 2

$$\cos^2 x \;\frac{dy}{dx} - \frac{1}{y^2} = 0$$

Separate the variables

$$\cos^2 x \;\frac{dy}{dx} = \frac{1}{y^2}$$

$$y^2 \frac{dy}{dx} = \frac{1}{\cos^2 x}$$

Integrate both sides with respect to x

$$\int y^2 \frac{dy}{dx}\, dx = \int \sec^2 x \; dx$$

$$\int y^2 \; dy = \int \sec^2 x \; dx$$

$$\frac{y^3}{3} = \tan x + c$$

$$y^3 = 3 \tan x + 3c$$

$$y = (3 \tan x + A)^{\frac{1}{3}}$$ is the general solution

Given that $y = 2$ when $x = \frac{\pi}{4}$

$$2 = \left(3 \tan\left(\frac{\pi}{4}\right) + A\right)^{\frac{1}{3}}$$

$$\therefore\; A = 5$$

Thus $y = (3 \tan x + 5)^{\frac{1}{3}}$ is the particular solution

[6 marks]

Answer 3

$$\int y \; dy = 3 \int x^2\, dx$$

$$\frac{y^2}{2} = \frac{3\,x^3}{3} + c$$

$$y^2 = 2\,x^3 + k$$

But $y = 3$ when $x = 2$, $\quad\therefore\; k = -7$

$$y^2 = 2\,x^3 - 7$$

[4 marks]

Answer 4

$$\frac{dy}{dx} = e^y \, e^x$$

$$\frac{1}{e^y} \frac{dy}{dx} = e^x$$

$$e^{-y} \frac{dy}{dx} = e^x$$

$$\int e^{-y} \, dy = \int e^x \, dx$$

$$-e^{-y} = e^x + c$$

$$\text{But } y = 0 \text{ when } x = 0, \quad \therefore \; c = -2$$

$$e^{-y} + e^x = 2$$

[4 marks]

Answer 5

(i) Acceleration

[1 mark]

(ii) 10 ms^{-2}

[1 mark]

(iii) 50 ms^{-1}

[1 mark]

(iv)

$$\frac{dv}{dt} = 0.2 \, (50 - v)$$

$$\frac{1}{50 - v} \frac{dv}{dt} = 0.2$$

[2 marks]

(v)

$$\int \frac{1}{50 - v} \, dv = \int 0.2 \, dt$$

$$-ln \left| 50 - v \right| = 0.2 \, t + c$$

$$ln \left| 50 - v \right| = -0.2 \, t + k$$

As the terminal velocity is 50 ms^{-1} from part (iii), v is always less than 50 and so the modulus signs can be dropped.

$$50 - v = A \, e^{-0.2 \, t}$$

$$v = 50 - A \, e^{-0.2 \, t}$$

$$\text{But } v = 0 \text{ when } t = 0, \quad \therefore \; A = 50$$

$$v = 50 - 50 \, e^{-0.2 \, t}$$

$$v = 50 \left(1 - e^{-0.2 \, t} \right)$$

[5 marks]

Answer 6

(i) $$\frac{1}{v\,(1+v)} = \frac{1}{v} - \frac{1}{1+v}$$

[2 marks]

(ii)
$$\frac{dv}{dt} = -\left(v + v^2\right)$$
$$\frac{1}{v+v^2}\frac{dv}{dt} = -1$$
$$\int \frac{1}{v(1+v)}\,dv = -\int 1\,dt$$
$$\int \frac{1}{v} - \frac{1}{1+v}\,dv = -\int 1\,dt$$
$$ln|v| - ln|1+v| = -t + c$$
$$ln\left|\frac{v}{1+v}\right| = -t + c$$

As we're told $v > 0$ the modulus signs can be dropped

$$\frac{v}{1+v} = e^{-t+c}$$
$$= e^{-t}e^{c}$$
$$= Ae^{-t}$$

But $v = 1$ when $t = 0$, $\therefore\ A = \frac{1}{2}$

$$\frac{2v}{1+v} = e^{-t}$$

[6 marks]

(iii)
$$2v = e^{-t} + ve^{-t}$$
$$2v - ve^{-t} = e^{-t}$$

Multiply through by e^{t}

$$2ve^{t} - v = 1$$
$$v\left(2e^{t} - 1\right) = 1$$
$$v = \frac{1}{2e^{-t} - 1}$$

[2 marks]

Lesson 4

4.1 Type One, Two and Three Mash Up

In this Lesson, the three types of differential equation, previously considered separately, will occur in random order.
Additionally, the letters in use will not always be x and y.

Example

During the decay of a radioactive substance, the rate at which mass is lost is proportional to the mass present at that instant.
A radioactive chemical has a rate of decay given by;

$$\frac{dm}{dt} = -3m$$

and $m = 12$ when $t = 0$.
Show that this has a solution of the form

$$m = Ae^{-kt}$$

where A and k are numbers that you should determine.

[6 marks]

4.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1
Solve the following differential equation,

$$\frac{dy}{dx} = \frac{e^{2x}}{y} \qquad \text{given that } y = 3 \text{ when } x = 0$$

Present your solution as elegantly as possible and in the form $y^2 = f(x)$

[6 marks]

Question 2

Solve the following differential equation,

$$\frac{dy}{dx} + csc^2 3x = 0 \qquad \text{given that } y = 1 \text{ when } x = \frac{\pi}{4}$$

Present your solution as elegantly as possible and in the form $y = f(x)$

[6 marks]

Question 3

The population of a colony of rabbits increases at a rate proportional to the population. When first observed, the population is 50 rabbits, increasing at a rate of 5 rabbits per month.

(i) The differential equation that models this situation is of the form

$$\frac{dP}{dt} = K\,P$$

Determine the value of the constant of the proportionality, *K*.

[2 marks]

(ii) Solve the part (i) differential equation to obtain an expression of the form;

$$P = A\,e^{k\,t}$$

where *A* and *k* are numbers you have determined.

[6 marks]

(iii) What will be the population two years after first observed ?

[1 mark]

Question 4

A-Level Examination Question from January 2012, Paper C4, Q8 (Edexcel)

(a) Express in partial fractions;

$$\frac{1}{P(5-P)}$$

[3 marks]

A team of conservationists is studying the population of meerkats on a nature reserve. The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{15}P(5-P), \qquad t \geqslant 0$$

where *P*, in thousands, is the population of meerkats and *t* is the time measured in years since the study began.

Given that when $t = 0$, $P = 1$,

(b) solve the differential equation, giving your answer in the form,

$$P = \frac{a}{b + c\,e^{-\frac{1}{3}t}}$$

where a, b and c are integers.

[8 marks]

(c) Hence show that the population cannot exceed 5000

[1 mark]

Question 5

(i) Using partial fractions, find

$$\int \frac{1}{4 - x^2} \, dx$$

[4 marks]

(ii) The amount of chemical present in a particular reaction at time t is x.

$$\frac{dx}{dt} = 10\left(4 - x^2\right) \qquad \text{with } x = 0 \text{ when } t = 0$$

Given that $0 \leqslant x < 2$, solve this to express x in terms of t.

[6 marks]

Question 6

A-Level Examination Question from June 2019, Paper 2, Q14 (Edexcel)

(a) Use the substitution $u = 4 - \sqrt{h}$ to show that,

$$\int \frac{dh}{4 - \sqrt{h}} = -8\, ln\left|4 - \sqrt{h}\right| - 2\sqrt{h} + k$$

where k is a constant.

[6 marks]

A team of scientists is studying a species of slow growing tree. The rate of change in height of a tree in this species is modelled by the differential equation,

$$\frac{dh}{dt} = \frac{t^{0.25}\left(4 - \sqrt{h}\right)}{20}$$

where h is the height in metres and t is the time, in years, after the tree is planted.

(b) Find, according to the model, the range in height of trees in this species.

[2 marks]

One of these trees is one metre high when it is first planted.
According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

[7 marks]

4.3 Answers

4.3.1 Solution to Example

$$\frac{dm}{dt} = -3m$$

This is a Type Two differential equation with the *y* replaced with *m*, and the *x* with *t*.

$$\frac{dm}{dt} = -3m$$

Invert both sides

$$\frac{dt}{dm} = -\frac{1}{3m}$$

Integrate both sides with respect to *m*

$$\int \frac{dt}{dm}\, dm = -\frac{1}{3} \int \frac{1}{m}\, dm$$

$$t = -\frac{1}{3} \ln|m| + c$$

As mass cannot be negative the modulus signs can be dropped.

$$-3t = \ln m - 3c$$

$$\ln m = -3t + 3c$$

$$m = e^{-3t+3c}$$

$$m = e^{-3t}\, e^{3c}$$

$$m = A e^{-3t} \qquad \text{is the general solution.}$$

But $m = 12$ when $t = 0$, $\therefore A = 12$

$$m = 12\, e^{-3t} \qquad \text{is the particular solution.}$$

That is, $A = 12$, $k = 3$

[6 marks]

4.3.2 Solutions to Exercise

Answer 1

$$\frac{dy}{dx} = \frac{e^{2x}}{y}$$

Type 3 : Separate the variables

$$y\frac{dy}{dx} = e^{2x}$$

Integrate both sides with respect to x

$$\int y\frac{dy}{dx}\,dx = \int e^{2x}\,dx$$

$$\int y\ dy = \int e^{2x}\,dx$$

$$\frac{y^2}{2} = \frac{e^{2x}}{2} + c$$

$$y^2 = e^{2x} + A \quad \text{is the general solution}$$

Given that $y = 3$ when $x = 0 \Rightarrow 3^2 = 1 + A \qquad \therefore\ A = 8$

$$y^2 = e^{2x} + 8 \quad \text{is the particular solution}$$

[6 marks]

Answer 2

$$\frac{dy}{dx} + csc^2\,3x = 0$$

$$\frac{dy}{dx} = -\,csc^2\,3x$$

Type 1 : Integrate both sides with respect to x

$$\int \frac{dy}{dx}\,dx = \frac{1}{3}\int 3\left(-\,csc^2\,3x\right)\,dx$$

$$\int 1\,dy = \frac{1}{3}\int 3\left(-\,csc^2\,3x\right)\,dx$$

$$y = \frac{1}{3}\cot 3x + c \qquad \text{is the general solution}$$

Given that $y = 1$ when $x = \frac{\pi}{4} \Rightarrow$

$$1 = \frac{1}{3\tan\left(\frac{3\pi}{4}\right)} + c \Rightarrow 1 = \frac{1}{-3} + c \qquad \therefore\ c = \frac{4}{3}$$

$$y = \frac{1}{3}\cot 3x + \frac{4}{3} \qquad \text{is the particular solution}$$

[6 marks]

Answer 3

(i)

$$\frac{dP}{dt} = K\,P$$

$$5 = K \times 50 \qquad \therefore\ K = 0.1$$

$$\frac{dP}{dt} = 0.1\,P$$

[2 marks]

(ii)

$$\frac{dP}{dt} = 0.1\,P$$

Type 2 : Invert both sides

$$\frac{dt}{dP} = \frac{10}{P}$$

Integrate both sides with respect to P

$$\int \frac{dt}{dP}\, dP = \int \frac{10}{P}\, dP$$

$$\int 1\, dt = 10 \int \frac{1}{P}\, dP$$

$$t = 10\, ln\left| P \right| + c$$

$$0.1\, t = ln\, P + 0.1\, c \qquad P,\ \text{population, is always positive}$$

$$ln\, P = 0.1\, t - 0.1\, c$$

$$P = e^{0.1\, t - 0.1\, c}$$

$$P = e^{0.1\, t}\ e^{-0.1\, c}$$

$$P = A\, e^{0.1\, t} \quad \text{is the general solution}$$

But $P = 50$ when $t = 0$, $\therefore\ A = 50$

$$P = 50\, e^{0.1\, t} \quad \text{is the particular solution}$$

[6 marks]

(iii) When $t = 24$, $P = 551$ rabbits.

[1 mark]

Answer 4

(a)

$$\frac{1 \times [\ \dots\ ]}{[\ P\,(\,5 - P\,)\ ]} = \frac{A \times [\ \dots\ ]}{P} + \frac{B \times [\ \dots\ ]}{5 - P}$$

$$1 = A\,(\,5 - P\,) + B\,(\,P\,)$$

Let $P = 0 \Rightarrow A = \dfrac{1}{5}$ and let $P = 5 \Rightarrow B = \dfrac{1}{5}$

$$\therefore\ \frac{1}{P\,(\,5 - P\,)} = \frac{1}{5}\left(\frac{1}{P} + \frac{1}{5 - P} \right)$$

[3 marks]

(b)

$$\frac{dP}{dt} = \frac{1}{15} P (5 - P)$$

Type 2 : Invert both sides

$$\frac{dt}{dP} = 15 \left(\frac{1}{P (5 - P)} \right)$$

$$\frac{dt}{dP} = 15 \times \frac{1}{5} \left(\frac{1}{P} + \frac{1}{5 - P} \right)$$

Integrate both sides with respect to P

$$\int \frac{dt}{dP}\, dP = 3 \int \frac{1}{P} + \frac{1}{5 - P}\, dP$$

$$\int 1\, dt = 3 \int \frac{1}{P} + \frac{1}{5 - P}\, dP$$

$$\frac{1}{3} t = ln|P| - ln|5 - P| + K$$

$$-\frac{1}{3} t = ln (5 - P) - ln (P) - K \qquad P \text{ is always positive}$$

$$K - \frac{1}{3} t = ln\left(\frac{5 - P}{P} \right)$$

$$e^{K - \frac{1}{3} t} = \frac{5 - P}{P}$$

$$e^{K} e^{-\frac{1}{3} t} = \frac{5}{P} - 1$$

$$\frac{5}{P} = 1 + e^{K} e^{-\frac{1}{3} t}$$

$$\frac{P}{5} = \frac{1}{1 + e^{K} e^{-\frac{1}{3} t}}$$

$$P = \frac{5}{1 + c\, e^{-\frac{1}{3} t}} \quad \text{is the general solution}$$

But $P = 1$ when $t = 0$, $\therefore\ c = 4$

$$P = \frac{5}{1 + 4 e^{-\frac{1}{3} t}}$$

[8 marks]

(c)

As $t \to \infty$, $e^{-\frac{1}{3} t} \to 0$ and so $P \to 5$

As P is in thousands the population $\to 5000$ □

[1 mark]

Answer 5

(a)

$$\frac{1}{4 - x^2} = \frac{1}{(2 + x)(2 - x)}$$

$$\frac{1 \times [\ldots]}{[(2 + x)(2 - x)]} = \frac{A \times [\ldots]}{(2 + x)} + \frac{B \times [\ldots]}{(2 - x)}$$

$$1 = A(2 - x) + B(2 + x)$$

Let $P = 2$ to get $B = \dfrac{1}{4}$

Let $P = -2$ to get $A = \dfrac{1}{4}$

$$\int \frac{1}{4 - x^2}\, dx = \frac{1}{4} \int \left(\frac{1}{2 + x} + \frac{1}{2 - x} \right) dx$$

$$= \frac{1}{4} \left(ln|2 + x| - ln|2 - x| \right) + c$$

$$= \frac{1}{4}\, ln \left| \frac{2 + x}{2 - x} \right| + c$$

[4 marks]

(b)

$$\frac{dx}{dt} = 10\left(4 - x^2\right)$$

Type 2 : Invert both sides

$$\frac{dt}{dx} = \frac{1}{10(4 - x^2)}$$

Integrate both sides with respect to x

$$\int 1\ dt = \frac{1}{10} \int \frac{1}{4 - x^2}\, dx$$

$$t = \frac{1}{40}\, ln \left| \frac{2 + x}{2 - x} \right| + \frac{c}{10}$$

As $0 \leqslant x < 2$ the modulus signs can be dropped

$$40\, t = ln\left(\frac{2 + x}{2 - x} \right) + 4c$$

$$e^{40\, t} = \frac{A\, (2 + x)}{(2 - x)}$$

As $x = 0$ when $t = 0$, $A = 1$

$$e^{40\, t}\, (2 - x) = 2 + x$$

$$2\, e^{40\, t} - 2 = x\, e^{40\, t} + x$$

$$x = \frac{2\left(e^{40\, t} - 1\right)}{\left(e^{40\, t} + 1\right)}$$

[6 marks]

Answer 6

(a) $u = 4 - \sqrt{h} \Leftrightarrow u = 4 - h^{\frac{1}{2}}$ $\left(\text{Also} \sqrt{h} = 4 - u \right)$

$$\Rightarrow \frac{du}{dh} = -\frac{1}{2} h^{-\frac{1}{2}}$$
$$= -\frac{1}{2\sqrt{h}}$$
$$= -\frac{1}{2(4 - u)}$$
$$= \frac{1}{2(u - 4)}$$

$$\int \frac{1}{4 - \sqrt{h}} \, dh = \int \frac{1}{u} 2(u - 4) \, du$$
$$= 2 \int \frac{u - 4}{u} \, du$$
$$= 2 \int 1 - \frac{4}{u} \, du$$
$$= 2 [u - 4 \ln|u|] + c$$
$$= 2u - 8 \ln|u| + c$$
$$= -8 \ln|4 - \sqrt{h}| + 2\left(4 - \sqrt{h}\right) + c$$
$$= -8 \ln|4 - \sqrt{h}| + 8 - 2\sqrt{h} + c$$
$$= -8 \ln|4 - \sqrt{h}| - 2\sqrt{h} + k \qquad \square$$

[6 marks]

(b) It's in the nature of a tree to increase in height so the rate of change of height will need to always be positive.

$$\therefore \frac{dh}{dt} \geqslant 0$$

$$\frac{t^{0.25} (4 - \sqrt{h})}{20} \geqslant 0$$

The time t can never be negative so we require that,

$$\left(4 - \sqrt{h} \right) \geqslant 0$$
$$-\sqrt{h} \geqslant -4$$
$$\sqrt{h} \leqslant 4$$
$$0 \leqslant h \leqslant 16 \text{ metres}$$

[2 marks]

(c) Solving the differential equation,

$$\frac{dh}{dt} = \frac{t^{0.25}\,(4 - \sqrt{h}\,)}{20}$$

Separate the variables, keeping part (a) in mind,

$$\frac{1}{4 - \sqrt{h}}\,\frac{dh}{dt} = \frac{1}{20}\,t^{0.25}$$

Integrate both sides with respect to t

$$\int \frac{1}{4 - \sqrt{h}}\,dh = \frac{1}{20}\int t^{0.25}\,dt$$

$$-\,8\,ln\left|4 - \sqrt{h}\,\right| - 2\sqrt{h} + k_1 = \frac{1}{20}\,\frac{t^{1.25}}{1.25} + k_2$$

$$t^{1.25} = -\,200\,ln\left|4 - \sqrt{h}\,\right| - 50\sqrt{h} + c$$

Using the initial conditions, that $h = 1$ when $t = 0$,

$$0 = -\,200\,ln\,3 - 50 + c$$

$$c = 200\,ln\,3 + 50$$

$$\therefore\ t^{1.25} = -\,200\,ln\left|4 - \sqrt{h}\,\right| - 50\sqrt{h} + 200\,ln\,3 + 50$$

When $h = 12$, $t^{1.25} = -\,200\,ln\left|4 - \sqrt{12}\,\right| - 50\sqrt{12} + 200\,ln\,3 + 50$

$$= 221.280$$

$$\therefore\ t = 75.2 \text{ years}$$

[7 marks]

Lesson 5

5.1 Rates Of Change (without integration)

Differentiation is often used by physicists to model a *rate of change*. Some rates of change occur so often that they are given names. For example, when considering a moving object, the rate of change of its displacement with respect to time is termed its velocity. This fact can be written in mathematics as,

$$v = \frac{ds}{dt}$$

Economists also make use of *rates of change*. For example, the rate at which the price of goods in shops are increasing with respect to time is termed inflation. This time we can write,

$$I = \frac{dp}{dt}$$

Many rates of change are interconnected as the following example will show.

5.2 The Coffee Stain Example

The photograph shows a coffee stain on a carpet as it slowly spreads outward. (The biscuit is there to give a sense of scale)

Photograph by Martin Hansen

The biscuit has since been eaten (before you ask)

Careful measurements of the photographs reveal that the radius of a coffee stain (which will be modelled as a circle) is increasing at a steady 0.04 mm.s^{-1}.

(a) Write down the well known formulae for

(i) The circumference of a circle in terms of its radius.

[1 mark]

(ii) The area of a circle, also in terms of its radius.

[1 mark]

(b) Find the rate at which the circumference is increasing.
Give your answer correct to 3 significant figures and state the units.

[3 marks]

(c) Find the rate at which the area is increasing when the radius is 8 mm.
Give your answer correct to 3 significant figures and state the units.

[3 marks]

(d) Identify a major underlying assumption within this model, and comment on the resulting limitations of this model.
How might the model be improved ?

[3 marks]

5.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1
The melting of an ice cube in a cool room is modelled by assuming it is melting at a constant rate of 2700 mm^3 per hour.

(i) Find the rate at which the length of one side of the cube is decreasing when the volume of the ice cube is 8000 mm^3.
Give an exact answer and state the units of your answer.

HINT: $$\frac{dL}{dt} = \frac{dL}{dV} \times \frac{dV}{dt}$$

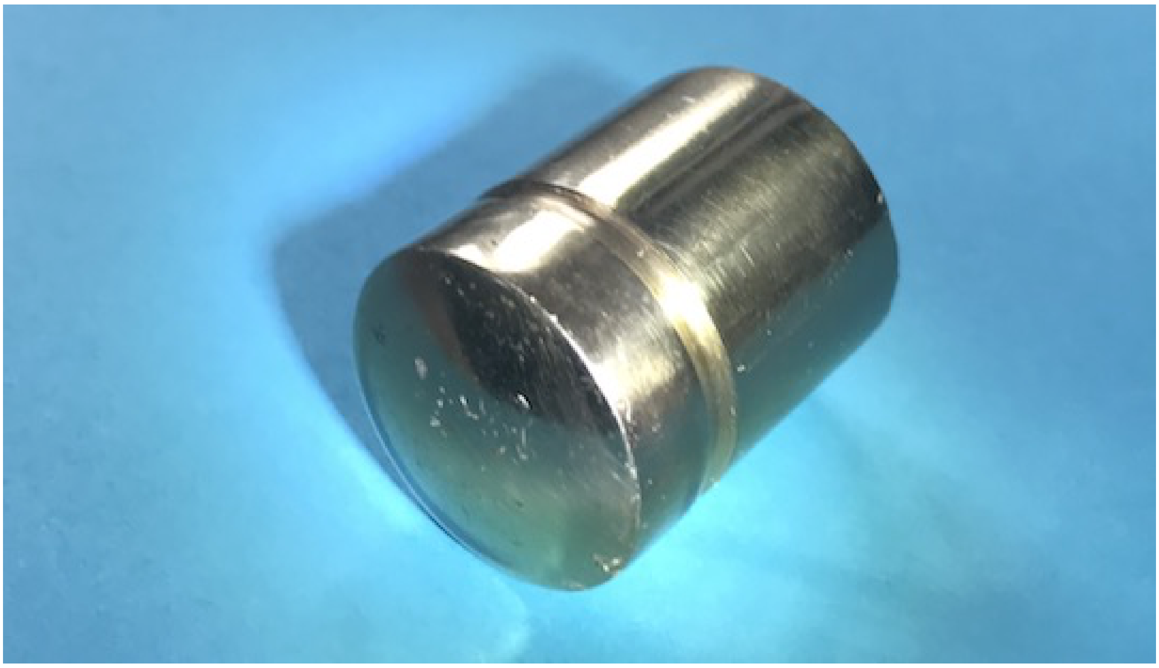
where L is the length of a side, V is the volume and t is time.

[4 marks]

(ii) According to the model, how long will it take for the ice cube to melt completely ? Give your answer in hours and minutes the nearest minute.

[1 mark]

Question 2



Photograph by Martin Hansen

The solid brass right circular cylindrical rod shown in the photograph is to be heated.
After t seconds, the radius of the rod is x cm and the length of the rod is $1.5x$ cm.
The cross-sectional area of the rod is increasing at the constant rate of $0.032\ \text{cm}^2\ \text{s}^{-1}$

(i) Find $\dfrac{dx}{dt}$ when the radius of the rod is 3 cm.
Give your answer to 3 significant figures and state the units of your answer.

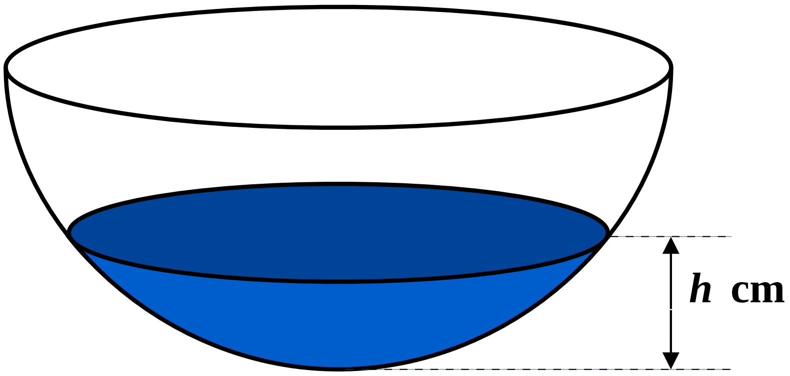
[4 marks]

(ii) Find the rate of increase of the volume of the rod when x is 3 cm.
Give your answer to 3 significant figures and state the units of your answer.

[4 marks]

Question 3

A-Level Examination Question from June 2018, Paper C34, Q7 (Edexcel)



The diagram is of a hemispherical bowl.

Water is flowing into the bowl at a constant rate of 180 $cm^3 s^{-1}$

When the height of the water is h cm, the volume of water V cm^3 is given by

$$V = \frac{1}{3}\pi h^2(90 - h), \qquad 0 \leqslant h \leqslant 30$$

Find the rate of change of the height of the water, in cm s^{-1}, when $h = 15$

Give your answer to 2 significant figures.

[5 marks]

Question 4

A-Level Examination Question from June 2014, Paper C4(R), Q5 (Edexcel)

At time t seconds the radius of a sphere is r cm, its volume is V cm^3 and its surface area is S cm^2.

You are given that $V = \frac{4}{3}\pi r^3$ and that $S = 4\pi r^2$

The volume of the sphere is increasing uniformly at a constant rate of 3 cm^3 s^{-1}

(a) Find $\frac{dr}{dt}$ when the radius of the sphere is 4 cm.

Give your answer to 3 significant figures.

[4 marks]

(b) Find the rate at which the surface area of the sphere is increasing when the radius is 4 cm.

[2 marks]

Question 5

A-Level Examination Question from January 2008, Q4 (OCR)

Earth is being added to a pile so that, when the height of the pile is h metres, its volume is V cubic metres, where

$$V = \left(h^6 + 16\right)^{\frac{1}{2}} - 4$$

(i) Find the value of $\dfrac{dV}{dh}$ when $h = 2$

[3 marks]

(ii) The volume of the pile is increasing at a constant rate of 8 cubic metres per hour. Find the rate, in metres per hour, at which the height of the pile is increasing at the instant when $h = 2$.
Give your answer correct to 2 significant figures.

[3 marks]

Question 6

In a tennis ball manufacturing process, rubber spheres are produced in such a way that the volume, V, of a sphere increases at a constant rate of 10 cm^3 per second. Find the rate of change of the surface area, A, of a sphere at the moment when the surface area is equal to 32π cm^2.

You are given that $V = \frac{4}{3}\pi r^3$ and that $S = 4\pi r^2$

[5 marks]

Question 7
The surface area A, of a metallic cube of side length x, is increasing at the constant rate of 0.45 $\text{cm}^2\,\text{s}^{-1}$
Find the rate at which the volume of the cube is increasing, when the cube's side length is 8 cm.

[5 marks]

5.4 Answers

5.4.1 Solution to 5.2 The Coffee Stain example

(a) **(i)** $C = 2\pi r$ **(ii)** $A = \pi r^2$

[2 marks]

(b) The question is asking for $\frac{dC}{dt}$ which is not immediately accessible.

Strategy : Find $\frac{dC}{dr}$ and then use $\frac{dC}{dt} = \frac{dC}{dr} \times \frac{dr}{dt}$

$C = 2\pi r \Rightarrow \frac{dC}{dr} = 2\pi$

$$\begin{aligned}\frac{dC}{dt} &= \frac{dC}{dr} \times \frac{dr}{dt} \\ &= 2\pi \times 0.04 \\ &= 0.251 \text{ mm s}^{-1}\end{aligned}$$

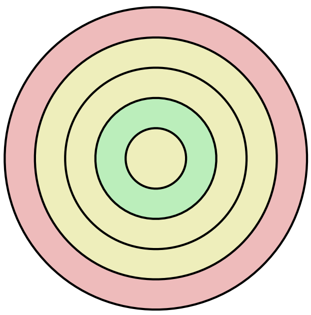
[3 marks]

(c) $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

$$\begin{aligned}\frac{dA}{dt} &= \frac{dA}{dr} \times \frac{dr}{dt} \\ &= 2\pi r \times 0.04 \\ &= 2\pi \times 8 \times 0.04 \qquad \text{when } r = 8 \text{ mm} \\ &= 2.01 \text{ mm}^2 \text{ s}^{-1}\end{aligned}$$

[3 marks]

(d) The major assumption is that the radius is growing at a constant rate.
Each increase creates a concentric ring of the same width as previous rings but with an ever greater area, requiring ever larger amounts of coffee to stain.
Compare the red area with the green in this diagram.



The modelling assumption may be realistic over a restricted time span, say $4 \leqslant t \leqslant 8$ seconds. Making the rate of increase in area the constant may be a more realistic modelling assumption or the rate of increase of radius inversely proportional to the radius better still.

[3 marks]

5.4.2 Solutions to 5.3 Exercise

Answer 1

(i) $V = L^3 \Rightarrow \dfrac{dV}{dL} = 3L^2$

Also, $L = \sqrt[3]{V}$ and so when $V = 8000$, $L = \sqrt[3]{8000} = 20$ mm

$$\frac{dL}{dt} = \frac{dL}{dV} \times \frac{dV}{dt}$$
$$= \frac{1}{3L^2} \times 2700$$
$$= \frac{1}{3 \times 20^2} \times 2700$$
$$= 2.25 \text{ mm h}^{-1}$$

Note that this rate is not constant, it's only the rate when $V = 8000$ mm^3

[4 marks]

(ii) Told to assume the volume melting is constant at 2700 mm^3 per hour

$$0 = 8000 - 2700\,t$$
$$t = 2.963$$
$$t = 2 \text{ hours and } 58 \text{ minutes}$$

[1 mark]

Answer 2

(i) Let A be the cross sectional area (a circle of radius x), in which case,

$A = \pi x^2 \Rightarrow \dfrac{dA}{dx} = 2\pi x$

Furthermore, $\dfrac{dx}{dt} = \dfrac{dx}{dA} \times \dfrac{dA}{dt}$

$$= \frac{1}{2\pi x} \times 0.032$$
$$= \frac{1}{2\pi \times 3} \times 0.032$$
$$= 0.00170 \text{ cm s}^{-1}$$

[4 marks]

(ii) Let V be volume of rod (a cylinder, radius x, length $1.5x$), in which case,

$V = \pi x^2 \times 1.5x = 1.5\pi x^3 \Rightarrow \dfrac{dV}{dx} = 4.5\pi x^2$

Furthermore, $\dfrac{dV}{dt} = \dfrac{dV}{dx} \times \dfrac{dx}{dt}$

$$= 4.5\pi \times 3^2 \times 0.00170$$
$$= 0.216 \text{ cm}^3 \text{ s}^{-1}$$

[4 marks]

Answer 3

Assembling the pieces...

$$V = \frac{1}{3}\pi h^2(90 - h)$$

$$= 30\pi h^2 - \frac{1}{3}\pi h^3$$

$$\frac{dV}{dh} = 60\pi h - \pi h^2 \qquad \text{Told } \frac{dV}{dt} = 180 \text{ cm}^3 \text{ s}^{-1}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{60\pi h - \pi h^2} \times 180$$

$$= \frac{1}{60\pi \times 15 - \pi 15^2} \times 180 \quad \text{when } h = 15 \text{ cm}$$

$$= 0.085 \text{ cm s}^{-1}$$

[5 marks]

Answer 4

(a) $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2 \qquad$ Told $\frac{dV}{dt} = 3 \text{ cm}^3 \text{ s}^{-1}$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{4\pi r^2} \times 3$$

$$= \frac{1}{4\pi \times 4^2} \times 3$$

$$= 0.0149 \text{ cm s}^{-1}$$

[4 marks]

(b) $S = 4\pi r^2 \Rightarrow \frac{dS}{dr} = 8\pi r$

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$$

$$= 8\pi r \times 0.0149$$

$$= 8\pi \times 4 \times 0.0149$$

$$= 1.500 \text{ cm}^2 \text{ s}^{-1}$$

[2 marks]

Answer 5

(i) $V = \left(h^6 + 16\right)^{\frac{1}{2}} - 4 \Rightarrow \dfrac{dV}{dh} = \dfrac{1}{2}\left(h^6 + 16\right)^{-\frac{1}{2}} 6h^5$

$$= \frac{3h^5}{\sqrt{h^6 + 16}}$$

When $h = 2$, $\dfrac{dV}{dh} = \dfrac{3 \times 2^5}{\sqrt{2^6 + 16}}$

$$= \frac{96}{4\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{24\sqrt{5}}{5} \text{ m}^3\text{ h}^{-1} \quad (= \ 10.73 \text{ m}^3\text{ h}^{-1})$$

[3 marks]

(ii) Told that $\dfrac{dV}{dt} = 8 \text{ m}^3\text{ h}^{-1}$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{5}{24\sqrt{5}} \times 8$$

$$= \frac{\sqrt{5}}{3}$$

$$= 0.75 \text{ m s}^{-1}$$

[3 marks]

Answer 6

$V = \dfrac{4}{3}\pi r^3 \Rightarrow \dfrac{dV}{dr} = 4\pi r^2$ Told $\dfrac{dV}{dt} = 10 \text{ cm}^3\text{ s}^{-1}$

$S = 4\pi r^2 \Rightarrow \dfrac{dS}{dr} = 8\pi r$

Also $r = \sqrt{\dfrac{S}{4\pi}}$ so when $S = 32\pi$, $r = \sqrt{\dfrac{32\pi}{4\pi}} = \sqrt{8} = 2\sqrt{2}$

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dV}{dt} \times \frac{dr}{dV}$$

$$= 8\pi r \times 10 \times \frac{1}{4\pi r^2}$$

$$= \frac{20}{r}$$

$$= \frac{20}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= 5\sqrt{2} \text{ cm}^2\text{ s}^{-1} \qquad (= \ 7.07 \text{ cm}^2\text{ s}^{-1})$$

[5 marks]

Answer 7

$V = x^3 \Rightarrow \dfrac{dV}{dx} = 3x^2$

$S = 6x^2 \Rightarrow \dfrac{dS}{dx} = 12x$

Told that $\dfrac{dS}{dt} = 0.45$ cm^2 s^{-1}

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dS} \times \frac{dS}{dt}$$

$$= 3x^2 \times \frac{1}{12x} \times 0.45$$

$$= 0.1125x$$

When $x = 8$, $\dfrac{dV}{dt} = 0.9$ cm^3 s^{-1}

[5 marks]

Lesson 6

6.1 Rates Of Change (with integration)

Situations involving *rates of change* often result in a differential equation. There is a skill in setting up the differential equation that effectively models the physical situation, and another skill in solving it (if it's solvable!).

6.2 Will The Sink Overflow ?

Photograph by Martin Hansen

A bathroom sink has a maximum capacity of 11 litres.
A small child has left a tap running and water is entering the sink at a constant rate of 3 litres per minute. Fortunately the plug has been left out.
Given a volume of water, V, in the sink, the rate at which water can exit is $0.25V$.
Form a differential equation and obtain its general solution.
Use the general solution to determine if the sink will overflow or not.

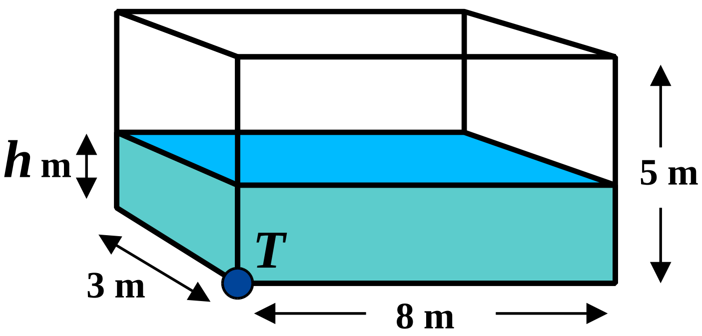
[8 marks]

6.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 65

Question 1

A-Level Examination Question from October 2021, Paper 2, Q14 (Edexcel)



Water flows at a constant rate into a large tank.
The tank is a cuboid, with all sides of negligible thickness.
The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.
There is a tap at a point T at the bottom of the tank, as shown.
At time t minutes after the tap has been opened,

- the depth of the water in the tank is h metres
- water is flowing into the tank at a constant rate of 0.48 m^3 per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h$ m^3 per minute

(a) Show that, according to the model,

$$1200\frac{dh}{dt} = 24 - 5h$$

[4 marks]

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + Be^{-kt}$$

where A, B and k are constants to be found.

[6 marks]

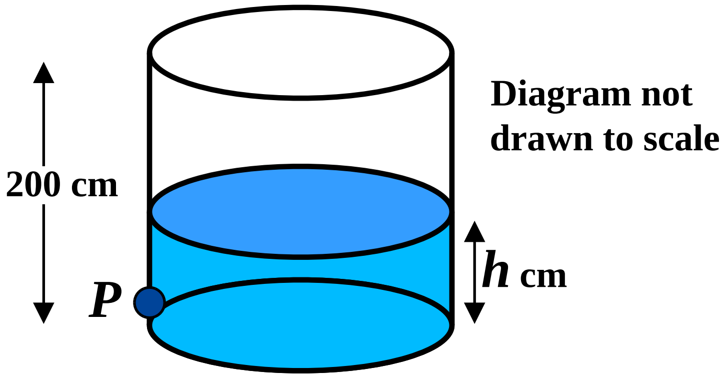
Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

[2 marks]

Question 2

A-Level Examination Question from June 2017, Paper C4, Q7 (Edexcel)



The diagram shows a vertical cylindrical tank of height 200 cm containing water.
Water is leaking from a hole P on the side of the tank.
At time t minutes after the leaking starts, the height of water in the tank is h cm.
The height h cm of the water in the tank satisfies the differential equation,

$$\frac{dh}{dt} = k(h-9)^{\frac{1}{2}}, \quad 9 < h \leqslant 200 \quad \text{where } k \text{ is a constant.}$$

When $h = 130$, the height of the water is falling at a rate of 1.1 cm per minute.

(a) Find the value of k

[2 marks]

Given that the tank was full of water when the leaking started,

(b) solve the differential equation with your value of k, to find the value of t when $h = 50$

[6 marks]

Question 3

A-Level Examination Question from January 2017, Paper C34, Q12

In freezing temperatures, ice forms on the surface of the water in a barrel.

At time t hours after the start of freezing, the thickness of the ice formed is x mm.

You may assume the thickness of the ice is uniform across the surface of the water.

At 4 pm there is no ice on the surface, and freezing begins.

At 6 pm, after two hours of freezing, the ice is 1.5 mm thick.

In a simple model, the rate of increase of x, in mm per hour, is assumed to be constant for a period of 20 hours.

Using this simple model,

(a) express t in terms of x,

[2 marks]

(b) find the value of t when $x = 3$

[1 mark]

In a second model, the rate of increase of x, in mm per hour, is given by,

$$\frac{dx}{dt} = \frac{\lambda}{(2x + 1)} \text{ where } \lambda \text{ is a constant and } 0 \leqslant t \leqslant 20$$

Using this second model,

(c) solve the differential equation and express t in terms of x and λ

[3 marks]

(d) find the exact value for λ,

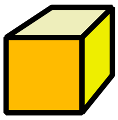
[1 mark]

(e) find at what time the ice is predicted to be 3 mm thick.

[2 marks]

Question 4

A-Level Examination Question from June 2006, Paper C4, Q7 (Edexcel)



At time t seconds the length of the side of a cube is x cm, the surface area of the cube is S cm^2, and the volume of the cube is V cm^3.

The surface area of the cube is increasing at a constant rate of 8 cm^2 s^{-1}

Show that,

(a) $\dfrac{dx}{dt} = \dfrac{k}{x}$, where k is a constant to be found,

[4 marks]

(b) $\dfrac{dV}{dt} = 2V^{\frac{1}{3}}$

[4 marks]

(c) Given that $V = 8$ when $t = 0$ solve the differential equation in part (b), and find the value of t when $V = 16\sqrt{2}$

[7 marks]

Question 5

A-Level Examination Question from January 2013, Paper C4, Q8 (Edexcel)

A bottle of water is put into a refrigerator. The temperature inside the refrigerator remains constant at 3 °C and t minutes after the bottle is placed in the refrigerator the temperature of the water in the bottle is θ °C

The rate of change of the temperature of the water in the bottle is modelled by the differential equation, $\dfrac{d\theta}{dt} = \dfrac{(3 - \theta)}{125}$

(a) By solving the differential equation show that, $\theta = A e^{-0.008t} + 3$ where A is a constant.

[4 marks]

Given that the temperature of the water in the bottle when it was put in the refrigerator was 16 °C,

(b) find the time taken for the temperature of the water in the bottle to fall to 10 °C, giving your answer to the nearest minute.

[5 marks]

Question 6

A-Level Examination Question from January 2018, Paper C34, Q14 (Edexcel)

The volume of a spherical balloon of radius r cm is V cm^3, where $V = \dfrac{4}{3}\pi r^3$

(a) Find $\dfrac{dV}{dr}$

[1 mark]

The volume of the balloon increases with time t seconds according to the formula,

$$\frac{dV}{dt} = \frac{9000\pi}{(t + 81)^{\frac{5}{4}}} \qquad t \geqslant 0$$

(b) Using the chain rule, or otherwise, show that

$$\frac{dr}{dt} = \frac{k}{r^n (t + 81)^{\frac{5}{4}}} \qquad t \geqslant 0$$

where k and n are constants to be found.

[2 marks]

Initially, the radius of the balloon is 3 cm.

(c) Using the values of k and n found in part (b), solve the differential equation

$$\frac{dr}{dt} = \frac{k}{r^n (t + 81)^{\frac{5}{4}}} \qquad t \geqslant 0$$

to obtain a formula for r in terms of t.

[6 marks]

(d) Hence find the radius of the balloon when t = 175, giving your answer to 3 significant figures.

[1 mark]

(e) Find the rate of increase of the radius of the balloon when t = 175
Give your answer to 3 significant figures.

[2 marks]

6.4 Answers

6.4.1 Solution to 6.2 Will The Sink Overflow ?

$$\frac{dV}{dt} = rate\ in \ - \ rate\ out$$

$$\frac{dV}{dt} = 3 - 0.25V \qquad \text{Type Two}$$

$$\frac{dt}{dV} = \frac{1}{3 - 0.25V}$$

Integrate both sides with respect to V

$$\int \frac{dt}{dV}\, dV = \int \frac{1}{3 - 0.25V}\, dV$$

$$\int 1\, dt = \frac{1}{(-0.25)} \int (-0.25) \times \frac{1}{3 - 0.25V}\, dV$$

$$t + c = \frac{1}{(-0.25)} ln\left|3 - 0.25V\right|$$

$$-0.25t - 0.25c = ln\left|3 - 0.25V\right|$$

$$\left|3 - 0.25V\right| = e^{-0.25t}\, e^{-0.25c}$$

$$= B\, e^{-0.25t} \qquad \text{for } B \in \mathbb{R},\ B > 0$$

$$3 - 0.25V = C\, e^{-0.25t} \qquad \text{for } C \in \mathbb{R}$$

$$0.25V = 3 - C\, e^{-0.25t}$$

$$V = 12 - A\, e^{-0.25t} \qquad \text{for } A \in \mathbb{R} \text{ is the general solution}$$

As $t \to \infty$, $e^{-0.25t} \to 0$ and so $V \to 12$ litres.

As the sink has a capacity of only 11 litres, it will overflow.

[8 marks]

6.4.2 Solutions to 6.3 Exercise

Answer 1

(a) Volume of water is given by, $V = 3 \times 8 \times h$

$$V = 24h \Rightarrow \frac{dV}{dh} = 24$$

$$\frac{dV}{dt} = \textit{rate in} - \textit{rate out}$$

$$= 0.48 - 0.1h$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{24} \times (0.48 - 0.1h)$$

Multiply both sides by 1200

$$1200\frac{dh}{dt} = 50(0.48 - 0.1h)$$

$$1200\frac{dh}{dt} = 24 - 5h \qquad \square$$

[4 marks]

(b) Type 2, so invert both sides and integrate with respect to h

$$\frac{1}{1200}\int \frac{dt}{dh}\,dh = \int \frac{1}{24 - 5h}\,dh$$

$$\frac{1}{1200}\int 1\,dt = \left(-\frac{1}{5}\right)\int (-5) \times \frac{1}{24 - 5h}\,dh$$

$$\frac{t}{1200} = \left(-\frac{1}{5}\right) ln|24 - 5h| + c_1$$

$$-\frac{t}{240} = ln|24 - 5h| + c_2$$

$$e^{-\frac{t}{240}} = B|24 - 5h| \quad \text{for } B \in \mathbb{R},\ B > 0$$

$$e^{-\frac{t}{240}} = A(24 - 5h) \quad \text{for } A \in \mathbb{R}$$

This is a good moment to use the fact that $h = 2$ when $t = 0$

$$1 = A(24 - 10)$$

$$A = \frac{1}{14}$$

$$14e^{-\frac{t}{240}} = 24 - 5h$$

$$h = \frac{24}{5} - \frac{14}{5}e^{-\frac{t}{240}}$$

$$h = 4.8 - 2.8e^{-\frac{t}{240}}$$

[6 marks]

(c) As $t \to \infty$, $e^{-\frac{t}{240}} \to 0$ and so $h \to 4.8$ metres (from below).

As the tank is 5 metres high, it can never become full.

[2 marks]

Answer 2

(a) $\dfrac{dh}{dt} = k(h - 9)^{\frac{1}{2}} \Rightarrow -1.1 = k(130 - 9)^{\frac{1}{2}} \quad \therefore \quad k = -0.1$

[2 marks]

(b) $\dfrac{dh}{dt} = -0.1(h - 9)^{\frac{1}{2}}$

Type 2, so invert both sides then integrate both sides with respect to h

$$\int \frac{dt}{dh}\, dh = -10 \int (h - 9)^{-\frac{1}{2}}\, dh$$

$$t = -10 \times 2 \times (h - 9)^{\frac{1}{2}} + c$$

$$t = -20\sqrt{h - 9} + c \qquad \text{General Solution}$$

Told that when $t = 0$, $h = 200$

$$0 = -20 \times \sqrt{191} + c$$

$$c = 20\sqrt{191}$$

$$t = -20\sqrt{h - 9} + 20\sqrt{191} \qquad \text{Particular Solution}$$

So, when $h = 50$,

$$t = -20\sqrt{41} + 20\sqrt{191}$$

$$= 148.34 \text{ minutes}$$

[6 marks]

Answer 3

(a) Various methods are possible; here's the pedantic way of getting it...

$$\frac{dx}{dt} = k$$

Invert both sides and integrate both sides with respect to *x*

$$\int \frac{dt}{dx}\,dx = \frac{1}{k}\int 1\,dx$$

$$t = \frac{x}{k} + c$$

But when $t = 0,\ x = 0\ \therefore c = 0$

$$t = \frac{x}{k}$$

But when $t = 2,\ x = 1.5\ \therefore k = 0.75$

$$t = \frac{4x}{3}$$

[2 marks]

(b) $t = 4$ hours

[1 mark]

(c) Type 2, invert, then integrate both sides with respect to *x*

$$\frac{dx}{dt} = \frac{\lambda}{(2x + 1)}$$

$$\int \frac{dt}{dx}\,dx = \frac{1}{\lambda}\int 2x + 1\ dx$$

$$t = \frac{1}{\lambda}\left(x^2 + x\right) + c \qquad \text{General Solution}$$

We know that $t = 0$ when $x = 0\ \therefore\ c = 0$

$$t = \frac{x^2 + x}{\lambda} \qquad \text{Particular Solution}$$

[3 marks]

(d) Use the fact that when $t = 2,\ x = 1.5$

$$\lambda = \frac{1.5^2 + 1.5}{2}$$

$$= \frac{15}{8}$$

[1 mark]

(e) $t = \dfrac{8\left(3^2 + 3\right)}{15} = 6.4$ hours

Time is thus 10:24 pm (= 22:24)

[2 marks]

Answer 4

(a) The question tells us that $\dfrac{dS}{dt} = 8$ cm^2 s^{-1}

For a cube with edge length x, $S = 6x^2 \Rightarrow \dfrac{dS}{dx} = 12x$

$$\frac{dx}{dt} = \frac{dx}{dS} \times \frac{dS}{dt}$$

$$= \frac{1}{12x} \times 8$$

$$= \frac{2}{3x} \quad \therefore \; k = \frac{2}{3} \qquad \square$$

[4 marks]

(b) Also, $V = x^3 \Rightarrow \dfrac{dV}{dx} = 3x^2$ and $x = V^{\frac{1}{3}}$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$= 3x^2 \times \frac{2}{3x}$$

$$= 2x$$

$$= 2V^{\frac{1}{3}} \qquad \square$$

[4 marks]

(c) Type 2, invert both sides and integrate both sides with respect to V

$$\int \frac{dt}{dV}\, dV = \frac{1}{2} \int V^{-\frac{1}{3}}\, dV$$

$$t = \frac{3V^{\frac{2}{3}}}{4} + c \qquad \text{The general solution}$$

Using the fact that $V = 8$ when $t = 0$ gives $c = -3$

$$t = \frac{3V^{\frac{2}{3}}}{4} - 3 \qquad \text{The particular solution}$$

When $V = 16\sqrt{2} = (2\sqrt{2})^3$

$$t = \frac{3 \times 8}{4} - 3$$

$= 3$ seconds

[7 marks]

Answer 5

(a)

$$\frac{d\theta}{dt} = \frac{(3-\theta)}{125}$$

$$\frac{1}{3-\theta}\,\frac{d\theta}{dt} = 0.008$$

Integrate both sides with respect to t

$$(-1)\int \frac{(-1)}{3-\theta}\,d\theta = \int 0.008\,dt$$

$$-\ln|3-\theta| = 0.008t + c$$

$$\ln|3-\theta| = -0.008t - c$$

$$|3-\theta| = Be^{-0.008t} \quad \text{for } B \in \mathbb{R},\ B > 0$$

$$3-\theta = De^{-0.008t-c} \quad \text{for } D \in \mathbb{R}$$

$$\theta = -De^{-0.008t} + 3$$

$$\theta = Ae^{-0.008t} + 3 \quad \text{for } A \in \mathbb{R} \qquad \square$$

[4 marks]

(b) When $t = 0,\ \theta = 16\,°\mathrm{C}$

$$16 = Ae^{-0.008\times 0} + 3$$

$$A = 16 - 3$$

$$= 13$$

$$\therefore\ \theta = 13e^{-0.008t} + 3$$

When $\theta = 10\,°\mathrm{C}$,

$$10 = 13e^{-0.008t} + 3$$

$$e^{-0.008t} = \frac{7}{13}$$

$$-0.008t = \ln\left(\frac{7}{13}\right)$$

$$t = 77 \text{ minutes}$$

[5 marks]

Answer 6

(a) $$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$$

[1 mark]

(b) $$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{4\pi r^2} \times \frac{9000\pi}{(t + 81)^{\frac{5}{4}}} \qquad t \geqslant 0$$

$$= \frac{2250}{r^2 (t + 81)^{\frac{5}{4}}}$$

[2 marks]

(c) Separate the variables...

$$r^2 \frac{dr}{dt} = 2250 (t + 81)^{-\frac{5}{4}}$$

And integrate both sides with respect to t...

$$\int r^2 \, dr = 2250 \int (t + 81)^{-\frac{5}{4}} \, dt$$

$$\frac{r^3}{3} = 2250 \times \frac{4}{-1} (t + 81)^{-\frac{1}{4}} + c$$

$$r^3 = -27000 (t + 81)^{-\frac{1}{4}} + k$$

We're told that when $t = 0$, $r = 3$,

$$27 = -\frac{27000}{\sqrt[4]{81}} + k$$

$$k = 9027$$

$$r = \sqrt[3]{9027 - 27000 (t + 81)^{-\frac{1}{4}}}$$

[6 marks]

(d) When $t = 175$ in the part (c) formula, $r = 13.2$ cm

[1 mark]

(e) When $t = 175$ in the part (b) formula,

$$\frac{dr}{dt} = \frac{2250}{13.2^2 (175 + 81)^{\frac{5}{4}}}$$

$$= 0.0126 \quad \text{(0.0127 if more accurate } r \text{ used)}$$

[2 marks]

Lesson 7

7.1 Revision

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

(i) Determine the general solution to the differential equation, $\dfrac{dy}{dx} = -\dfrac{x}{y}$

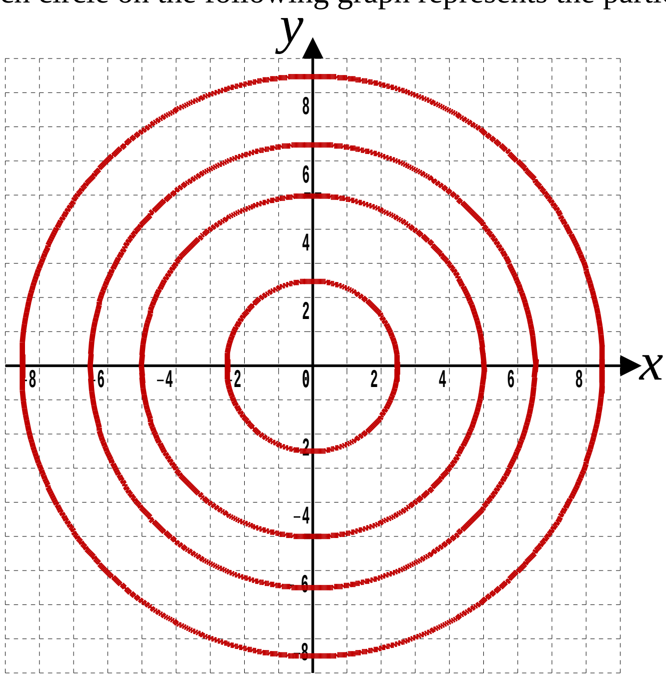
[3 marks]

(ii) The general solution is represented by all possible circles centred on the origin. Given that $y = 6$ when $x = 2.5$ find the particular solution.

Present your solution in the form $x^2 + y^2 = r^2$ where r is a constant.

[1 mark]

(iii) Which circle on the following graph represents the particular solution ?



[1 mark]

Question 2

A-Level Examination Question from June 2012, Paper C4, Q4 (Edexcel)

Given that $y = 2$ at $x = \frac{\pi}{4}$, solve the differential equation,

$$\frac{dy}{dx} = \frac{3}{y\ cos^2 x}$$

[5 marks]

Question 3

A-Level Examination Question from October 2020, Paper 1, Q14 (Edexcel)

A large spherical balloon is deflating.

At time t seconds the balloon has radius r cm and volume V cm^3

The volume of the balloon is modelled as decreasing at a constant rate.

(a) Using this model, show that $\frac{dr}{dt} = -\frac{k}{r^2}$ where k is a positive constant.

[3 marks]

Given that

- the initial radius of the balloon is 40 cm
- after 5 seconds the radius of the balloon is 20 cm
- the volume of the balloon continues to decrease at a constant rate until the balloon is empty

(b) solve the differential equation to find a complete equation linking r and t

[5 marks]

(c) Find the limitation on the values of t for which the equation in part (b) is valid.

[2 marks]

Question 4

A-Level Examination Question from June 2011, Paper C4, Q8 (Edexcel)

(a) Find $\int (4y + 3)^{-\frac{1}{2}}\, dy$

[2 marks]

(b) Given that $y = 1.5$ at $x = -2$, solve the differential equation,

$$\frac{dy}{dx} = \frac{\sqrt{(4y + 3)}}{x^2}$$

giving your answer in the form $y = f(x)$

[6 marks]

Question 5

A-Level Examination Question from January 2011, Paper C4, Q3 (Edexcel)

(a) Express in partial fractions;

$$\frac{5}{(x-1)(3x+2)}$$

[3 marks]

(b) Hence find;

$$\int \frac{5}{(x-1)(3x+2)}\,dx, \qquad x > 1$$

[3 marks]

(c) Find the particular solution of the differential equation

$$(x - 1)(3x + 2)\frac{dy}{dx} = 5y, \qquad x > 1$$

for which $y = 8$ at $x = 2$

Give your answer in the form $y = f(x)$

[6 marks]

7.2 Answers

Answer 1

(i)

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y\frac{dy}{dx} = -x \qquad \text{separating the variables}$$

integrate both sides with respect to x

$$\int y\ dy = -\int x\ dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$y^2 + x^2 = r^2 \qquad \text{for some constant, } r$$

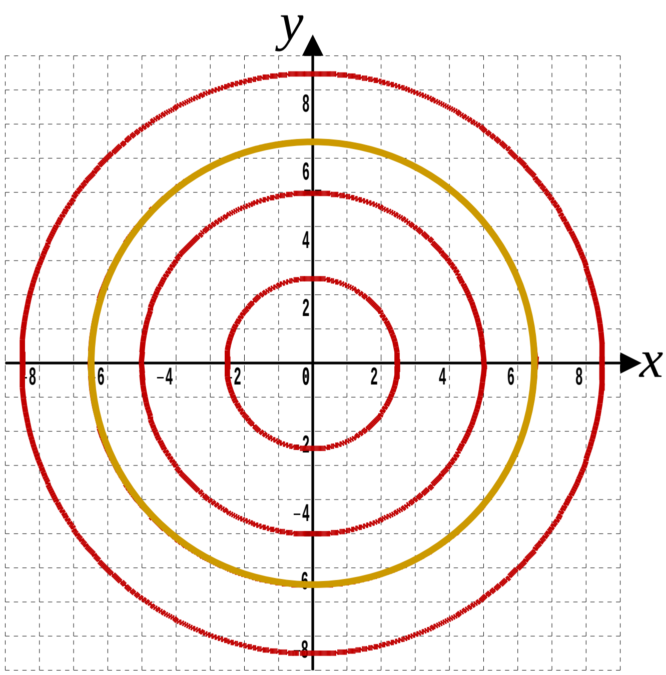
[3 marks]

(ii) As $y = 6$ when $x = 2.5$

$$y^2 + x^2 = 6.5^2 \qquad \text{is the particular solution}$$

[1 mark]

(iii) The relevant circle is highlighted in gold.



[1 mark]

Answer 2

$$\frac{dy}{dx} = \frac{3}{y \; cos^2 x}$$

Separate the variables...

$$y \frac{dy}{dx} = \frac{3}{cos^2 x}$$

Integrate both sides with respect to *x*...

$$\int y \, dy = 3 \int sec^2 x \, dx$$

$$\frac{y^2}{2} = 3 \, tan \, x + c$$

Using the fact that when $y = 2, \; x = \frac{\pi}{4}$...

$$2 = 3 + c \Rightarrow c = -1$$

$$\frac{y^2}{2} = 3 \, tan \, x - 1$$

$$y^2 = 6 \, tan \, x - 2$$

[5 marks]

Answer 3

(a) The volume of a sphere is given by, $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{4\pi r^2} \times (-c) \qquad \text{as told } V \text{ is decreasing at a constant rate}$$

$$= -\frac{k}{r^2} \qquad \text{where } k \text{ is a positive constant}$$

[3 marks]

(b)

$$r^2 \frac{dr}{dt} = -k \qquad \text{separating the variables}$$

$$\int r^2\, dr = -k \int 1\, dt \qquad \text{integrating both sides with respect to } t$$

$$\frac{r^3}{3} = -kt + c$$

When $t = 0,\ r = 40,$ so $c = \frac{40^3}{3} = \frac{64000}{3}$

When $t = 5,\ r = 20,$

$$\frac{20^3}{3} = -5k + \frac{64000}{3}$$

$$5k = \frac{56000}{3} \Leftrightarrow k = \frac{11200}{3}$$

$$\frac{r^3}{3} = -\frac{11200}{3}t + \frac{64000}{3}$$

$$r^3 = -11200\,t + 64000$$

[5 marks]

(c) From the fact that “the volume of the balloon continues to decrease at a constant rate until the balloon is empty” we need to work out what value of t corresponds to the radius being zero.

$$0^3 = -11200\,t + 64000$$

$$t = \frac{64000}{11200}$$

$$= \frac{40}{7} \qquad \text{(About 5.71 seconds)}$$

[2 marks]

Answer 4

(a)

$$\int (4y+3)^{-\frac{1}{2}}\,dy = \frac{1}{4}\int 4(4y+3)^{-\frac{1}{2}}\,dy$$

$$= \frac{1}{4}\times 2(4y+3)^{\frac{1}{2}} + k$$

$$= \frac{1}{2}(4y+3)^{\frac{1}{2}} + k$$

[2 marks]

(b)

$$\frac{dy}{dx} = \frac{\sqrt{(4y+3)}}{x^2}$$

$$(4y+3)^{-\frac{1}{2}}\frac{dy}{dx} = x^{-2}$$ separating the variables

$$\int (4y+3)^{-\frac{1}{2}}\,dy = \int x^{-2}\,dx$$ integrating both sides with respect to x

$$\frac{1}{2}(4y+3)^{\frac{1}{2}} = -x^{-1} + c$$ from part (a)

$$\frac{1}{2}(9)^{\frac{1}{2}} = \frac{1}{2} + c \Rightarrow c = 1$$

$$\frac{1}{2}(4y+3)^{\frac{1}{2}} = 1 - \frac{1}{x}$$

$$4y+3 = 4\left(1-\frac{1}{x}\right)^2$$

$$4y+3 = 4\left(1-\frac{2}{x}+\frac{1}{x^2}\right)$$

$$4y = 4 - \frac{8}{x} + \frac{4}{x^2} - 3$$

$$y = \frac{1}{4} - \frac{2}{x} + \frac{1}{x^2}$$

[6 marks]

Answer 5

(a)

$$\frac{5}{[\,(x-1)(3x+2)\,]} = \frac{A}{(x-1)} + \frac{B}{(3x+2)}$$

Multiply through by the red bracket

$$\frac{5\,[\ldots]}{[\,(x-1)(3x+2)\,]} = \frac{A[\ldots]}{(x-1)} + \frac{B\,[\ldots]}{(3x+2)}$$

$$5 = A(3x+2) + B(x-1)$$

Let $x = 1$: $5 = A(5) \Leftrightarrow A = 1$

Let $x = 0$: $5 = 2 - B \Leftrightarrow B = -3$

$$\therefore \frac{5}{(x-1)(3x+2)} = \frac{1}{x-1} - \frac{3}{3x+2}$$

[3 marks]

(b) $\displaystyle \int \frac{5}{(x-1)(3x+2)}\,dx = \int \frac{1}{x-1}dx - \int \frac{3}{3x+2}\,dx,\ x > 1$

$$= ln(x-1) - ln(3x+2) + c$$

[3 marks]

(c) $\displaystyle (x-1)(3x+2)\frac{dy}{dx} = 5y$

$$\frac{1}{y}\frac{dy}{dx} = \frac{5}{(x-1)(3x+2)}$$ separating the variables

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{5}{(x-1)(3x+2)}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{5}{(x-1)(3x+2)}\,dx$$

$$ln|y| = ln|x-1| - ln|3x+2| + c$$

$$ln|y| = ln\left|\frac{x-1}{3x+2}\right| + c$$

$$|y| = B\left(\frac{x-1}{3x+2}\right) \quad \text{for } B \in \mathbb{R},\ B > 0 \text{ and } x > 1$$

$$y = \frac{A(x-1)}{(3x+2)} \quad \text{for } A \in \mathbb{R},\ x > 1$$

Making use of the fact that $y = 8$ at $x = 2$ yields,

$$8 = \frac{A \times 1}{8} \Leftrightarrow A = 64$$

$$y = \frac{64(x-1)}{(3x+2)} \quad \text{for } x > 1$$

[6 marks]

8.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

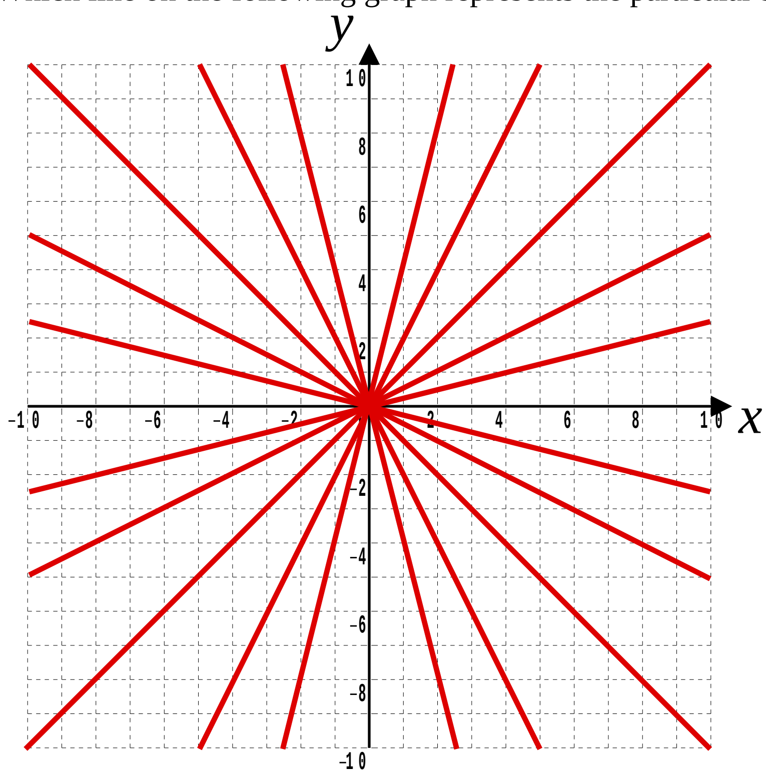
(i) Determine the general solution to the differential equation, $\dfrac{dy}{dx} = \dfrac{y}{x}$

[3 marks]

(ii) The general solution is represented by all possible straight lines through the origin. Given that $y = 8$ when $x = 4$ find the particular solution. Present your solution as elegantly as possible and in the form $y = f(x)$.

[1 mark]

(iii) Which line on the following graph represents the particular solution ?



[1 mark]

Question 2

A-Level Examination Question from June 2018, Paper 1, Q10 (Edexcel)

The height above ground, H metres, of a passenger on a roller coaster can be modelled by the differential equation,

$$\frac{dH}{dt} = \frac{H\cos(0.25t)}{40}$$

where t is the time, in seconds, from the start of the ride.

Given that the passenger is 5 m above the ground at the start of the ride,

(a) Show that $H = 5e^{0.1\sin(0.25t)}$

[5 marks]

(b) State the maximum height of the passenger above the ground.

[1 mark]

The passenger reaches the maximum height, for the second time, T seconds after the start of the ride.

(c) Find the value of T

[2 marks]

Question 3

Show that the differential equation,

$$\frac{dy}{dx} = x\,e^{2y} \qquad \text{where} \qquad y = 0 \quad \text{when} \quad x = 0$$

has solution,

$$y = ln\left(\frac{1}{\sqrt{1 - x^2}}\right) \qquad\qquad |x| < 1$$

[4 marks]

Question 4

A-Level Examination Question from June 2013, Paper C4, Q6 (Edexcel)

Water is being heated in a kettle.

At time t seconds, the temperature of the water is θ °C.

The rate of increase of the temperature of the water at any time t is modelled by the differential equation

$$\frac{d\theta}{dt} = \lambda(120 - \theta), \qquad \theta \leqslant 100$$

where λ is a positive constant.

Given that $\theta = 20$ when $t = 0$

(a) solve this differential equation to show that

$$\theta = 120 - 100e^{-\lambda t}$$

[8 marks]

When the temperature of the water reaches 100 °C, the kettle switches off.

(b) Given that $\lambda = 0.01$, find the time, to the nearest second, when the kettle switches off.

[3 marks]

Question 5

A-Level Examination Question from January 2007, Paper C4, Q4 (Edexcel)

(a) Express in partial fractions;

$$\frac{2x - 1}{(x - 1)(2x - 3)}$$

[3 marks]

(b) Given that $x \geqslant 2$, find the general solution of the differential equation

$$(2x - 3)(x - 1)\frac{dy}{dx} = (2x - 1)y$$

[5 marks]

(c) Hence find the particular solution of this differential equation that satisfies $y = 10$ at $x = 2$, giving your answer in the form $y = f(x)$

[4 marks]

8.2 Answers

Answer 1

(i)

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \qquad \text{separating the variables}$$

integrate both sides with respect to x

$$ln|y| = ln|x| + c$$

exponentiate both sides

$$e^{ln|y|} = e^{ln|x|+c}$$

$$|y| = e^{ln|x|}e^{c}$$

$$= B|x| \qquad \text{for } B \in \mathbb{R},\ B > 0$$

$$y = Ax \qquad \text{for } A \in \mathbb{R} \qquad \text{is the general solution}$$

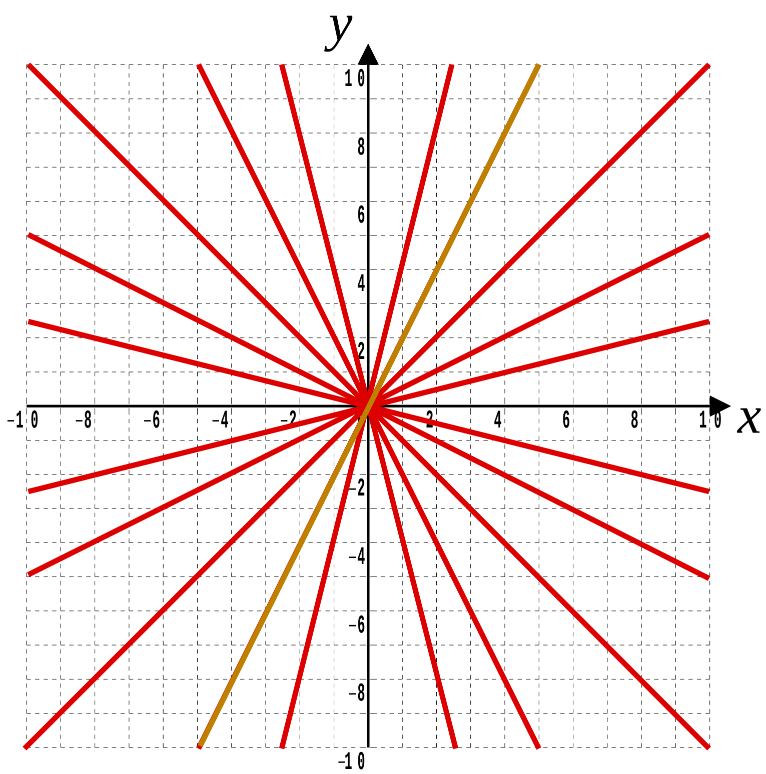
[3 marks]

(ii) As $y = 8$ when $x = 4$, $A = 2$

$$y = 2x \qquad \text{is the particular solution}$$

[1 mark]

(iii) The relevant line is highlighted in gold.



[1 mark]

Answer 2

(a) $\dfrac{dH}{dt} = \dfrac{H\cos(0.25t)}{40}$

$\dfrac{1}{H}\dfrac{dH}{dt} = \dfrac{1}{40}\cos\left(\dfrac{t}{4}\right)$ separating the variables

$\displaystyle\int \frac{1}{H}\, dH = \frac{1}{40} \times 4 \int \frac{1}{4}\cos\left(\frac{t}{4}\right) dt$ integrating both sides with respect to t

$ln|H| = \dfrac{1}{40} \times 4\sin\left(\dfrac{t}{4}\right) + c$ as $H > 0$, modulus signs dropped

$ln\,H = \dfrac{1}{10}\sin(0.25t) + c$ $c = ln\,5$ from initial conditions

$ln\,H - ln\,5 = 0.1\sin(0.25t)$

$ln\left(\dfrac{H}{5}\right) = 0.1\sin(0.25t)$ exponentiate both sides

$\dfrac{H}{5} = e^{0.1\sin(0.25t)}$

$H = 5e^{0.1\sin(0.25t)}$ □

[5 marks]

(b) Passenger maximum height will occur when $\sin(0.25t) = 1$ in which case $H_{MAX} = 5e^{0.1}$ (About 5.53 metres)

[1 mark]

(c) Maximum height for the first time will be when $0.25t = \dfrac{\pi}{2}$ and for the second time when,

$$0.25T = \frac{\pi}{2} + 2\pi$$

$$\frac{T}{4} = \frac{5\pi}{2}$$

$$T = 10\pi \text{ (About 31.4 seconds)}$$

[2 marks]

Answer 3

$$\frac{dy}{dx} = x\,e^{2y}$$

$$e^{-2y}\frac{dy}{dx} = x \qquad \text{separating the variables}$$

$$\int e^{-2y}\,dy = \int x\,dx \qquad \text{integrate both sides with respect to } x$$

$$-\frac{1}{2}\int(-2)\,e^{-2y}\,dy = \int x\,dx \qquad \text{setting up a chain rule backwards}$$

$$-\frac{1}{2}e^{-2y} = \frac{x^2}{2} + c \qquad \text{use } y = 0 \text{ when } x = 0 \text{ to get } c = -\frac{1}{2}$$

$$-\frac{1}{2}e^{-2y} = \frac{x^2}{2} - \frac{1}{2} \qquad \text{multiply through by } (-2)$$

$$e^{-2y} = -x^2 + 1 \qquad \text{take the } ln \text{ of both sides}$$

$$-2y = ln\left|1 - x^2\right| \qquad \text{can remove modulus signs as } |x| < 1$$

$$y = -\frac{1}{2}\,ln\left(1 - x^2\right)$$

$$y = ln\left(1 - x^2\right)^{-\frac{1}{2}}$$

$$y = ln\left(\frac{1}{\sqrt{1 - x^2}}\right) \qquad |x| < 1 \qquad \square$$

[4 marks]

Answer 4

(a)

$$\frac{d\theta}{dt} = \lambda\,(120 - \theta)$$

$$\frac{1}{(120 - \theta)}\frac{d\theta}{dt} = \lambda \qquad \text{separate the variables}$$

$$\int \frac{1}{(120 - \theta)}\,d\theta = \int \lambda\,dt \qquad \text{integrate both sides with respect to } t$$

$$(-1)\int (-1)\frac{1}{(120 - \theta)}\,d\theta = \int \lambda\,dt \qquad \text{setting up a chain rule backwards}$$

$$-\ln|120 - \theta| = \lambda t + c \qquad \theta = 20 \text{ when } t = 0 \Rightarrow c = -\ln 100$$

$$-\lambda t = \ln|120 - \theta| - \ln 100$$

$$-\lambda t = \ln\left(\frac{120 - \theta}{100}\right) \qquad \text{As } \theta \leqslant 100 \text{ modulus sign removed}$$

$$e^{-\lambda t} = \frac{120 - \theta}{100}$$

$$100\,e^{-\lambda t} = 120 - \theta$$

$$\theta = 120 - 100\,e^{-\lambda t} \qquad \square$$

[8 marks]

(b) The equation that needs to be solved is,

$$100 = 120 - 100\,e^{-0.01\,t}$$

$$100\,e^{-0.01\,t} = 20$$

$$e^{-0.01\,t} = 0.2$$

$$-0.01\,t = \ln 0.2$$

$$t = 161 \text{ seconds}$$

[3 marks]

Answer 5

(a) Using the red bracket method,

$$\frac{2x - 1}{[\,(x-1)(2x-3)\,]} = \frac{A}{(x-1)} + \frac{B}{(2x-3)}$$

$$\frac{(2x-1)\,[\ldots]}{[\,(x-1)(2x-3)\,]} = \frac{A\,[\ldots]}{(x-1)} + \frac{B\,[\ldots]}{(2x-3)}$$

$$2x - 1 = A(2x-3) + B(x-1)$$

$$\text{Let } x = 1 \;:\; 1 = A(-1) \Leftrightarrow A = -1$$

$$\text{Let } x = 0 \;:\; -1 = (-1)(-3) + B(-1) \Leftrightarrow B = 4$$

$$\therefore \quad \frac{2x-1}{(x-1)\,(2x-3)} = -\frac{1}{(x-1)} + \frac{4}{(2x-3)}$$

[3 marks]

(b)

$$(2x-3)(x-1)\frac{dy}{dx} = (2x-1)y$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x-1}{(2x-3)(x-1)} \qquad \text{separating variables}$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{(x-1)} + \frac{4}{(2x-3)} \qquad \text{using part (a)}$$

$$\int \frac{1}{y}\,dy = -\int \frac{1}{x-1}\,dx + 2\int \frac{2}{2x-3}\,dx$$

$$ln|y| = -ln|x-1| + 2\,ln|2x-3| + c$$

$$= -ln(x-1) + ln(2x-3)^2 + c$$

The RHS modulus signs removed as told that $x \geqslant 2$

$$= ln\left(\frac{(2x-3)^2}{x-1}\right) + c$$

$$|y| = \frac{B(2x-3)^2}{x-1} \qquad \text{for } B \in \mathbb{R},\; B > 0$$

$$y = \frac{A(2x-3)^2}{x-1} \qquad \text{for } A \in \mathbb{R}$$

[5 marks]

(c) Using the fact that $y = 10$ at $x = 2$,

$$10 = \frac{A \times 1}{1} \Leftrightarrow A = 10$$

$$y = \frac{10(2x-3)^2}{x-1}$$

[4 marks]