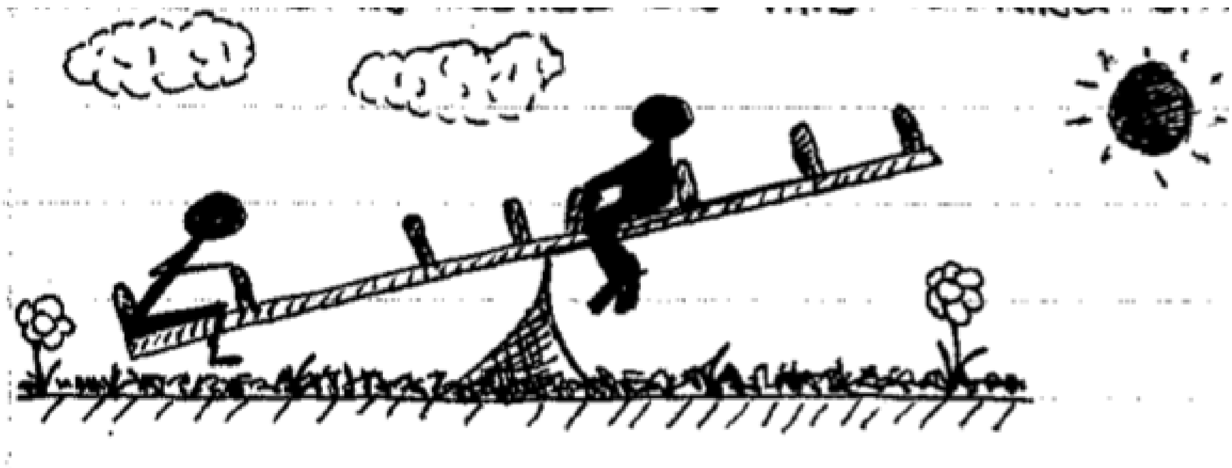
A-Level Applied Mathematics

Year 2
Mechanics

M • O • M • E • N • T • S



1.1 The Moment Of A Force

In Mechanics we often assume that the forces under consideration act upon a point which is typically described as being a particle.

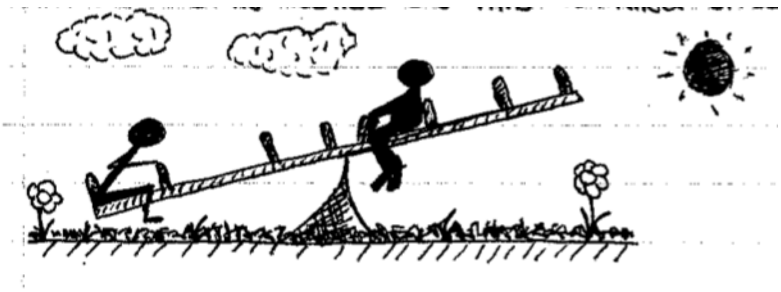
However, in the physical world forces act on "lumps of stuff".

This can result in a "lump of stuff" rotating.

In this course we are primarily concerned with either preventing such rotation or in using the fact that an object is not rotating to deduce information about some of the forces applying.

The "lumps of stuff" are called *rigid bodies* and the turning effect of a force about a pivot is termed a *moment*. If an object is described as being in equilibrium then the overall turning effect upon it must be zero. We say that "the sum of the moments is zero" or equivalently that "the sum of the clockwise moments must equal the sum of the anti-clockwise moments" (about any given pivot).

A see-saw makes use of moments and, in particular, the fact that a lighter person, a child, can approximately balance a heavier person, its mother, if the lighter person is further away from the pivot that the heavier person.



The sea-saw makes it clear that moments depend upon two factors;

(i) The magnitude of the force applied

(ii) The distance of that force from the pivot

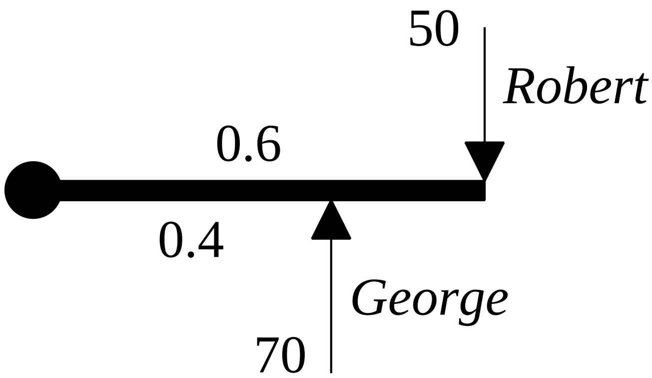
The mathematical relationship is as simple as could be hoped for;

$$moment = force\ applied \times distance\ from\ pivot$$

1.2 Example

Robert and George are trying to push in opposite directions through a door.
Robert applies a force of 50 N at a distance of 0.6 m from the door's hinge.
George applies a force of 70 N at a distance of 0.4 m from the door's hinge.

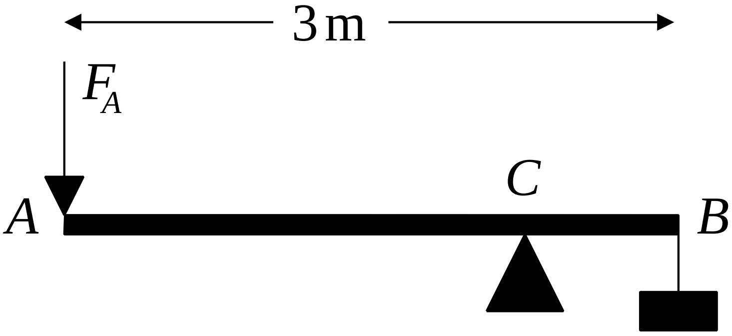
A bird's eye view from above the door is shown.



Who will win this pushing battle ?
Justify your answer.
Teaching video : http://NumberWonder.co.uk/Video/v9069(1a).mp4

1.3 Example

A lever consists of a uniform steel rod *AB* of mass 6 kg and length 3 m, which rests on a small smooth pivot at a point *C*. A load of mass 8 kg is suspended from the end B of the rod by a light rope. The lever is held in equilibrium in a horizontal position by a vertical force applied at end *A*, as shown in the diagram.
Given that *BC* = 0.75 m find the magnitude of the force, F_A, applied at *A*.



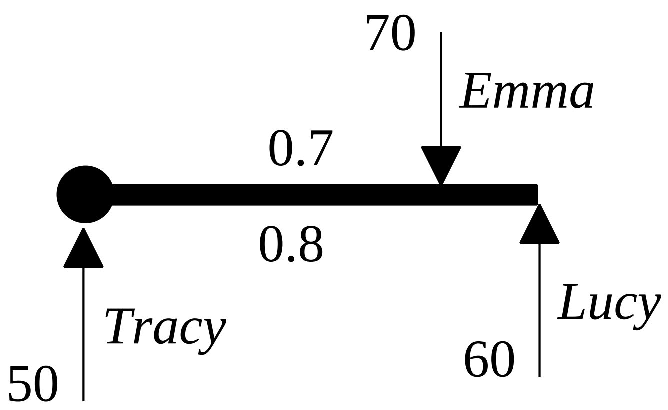
Teaching Video : http://www.NumberWonder.co.uk/Video/v9069(1b).mp4

1.4 Exercise

Question 1

Emma, Lucy and Lucy's friend Tracy (who's not very bright) are trying to push in opposite directions through a door.
Emma applies a force of 70 N at a distance of 0.8 m from the door's hinge.
Lucy applies a force of 60 N at a distance of 0.7 m from the door's hinge assisted by Tracy who applies a force of 50 N at a distance of 0 m from the door's hinge.

A bird's eye view from above the door is shown.



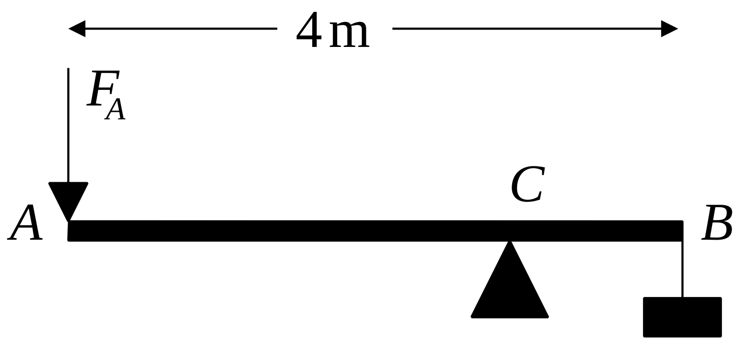
(i) Who will win this pushing battle, Emma or Lucy & Tracy ?
Justify your answer.

(ii) If Emma and Lucy push as before, where would Tracy have to push in order for the door to be in equilibrium ?

Question 2

A lever consists of a uniform steel rod AB of mass 7 kg and length 4 m, which rests on a small smooth pivot at a point C. A load of mass 12 kg is suspended from the end B of the rod by a light rope. The lever is held in equilibrium in a horizontal position by a vertical force applied at end A, as shown in the diagram.

Given that BC = 1.25 m find the magnitude of the force, F_A, applied at A.



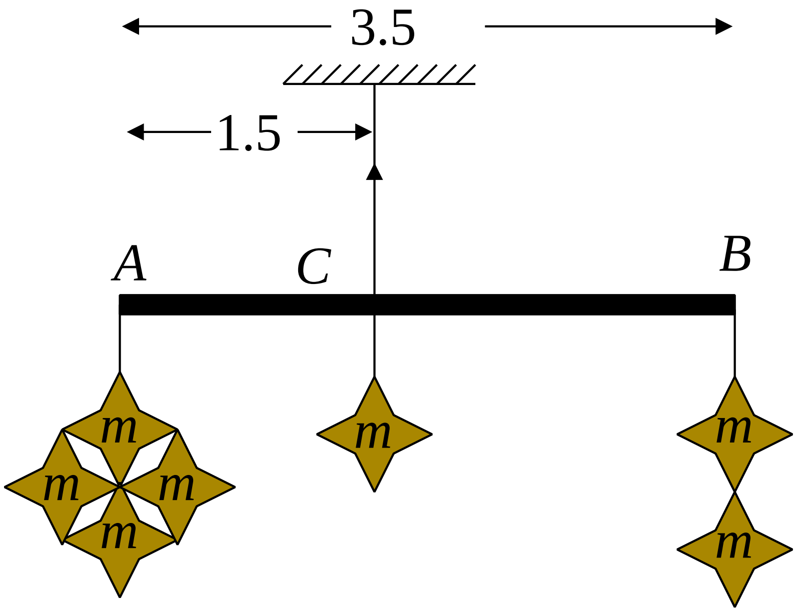
Question 3

A large Christmas decoration is made from a uniform rod, *AB*, of mass 3 kg and length 3.5 metres.

It is suspended at *C* where length *AC* is 1.5 metres.

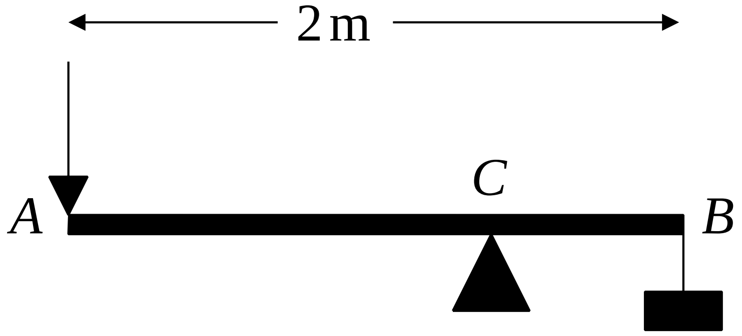
Cut out stars hang from the rod each of mass *m*; Four stars at *A*, one from *C* and two from *B*.

Given that the decoration is in equilibrium, calculate the mass of a star, *m*



Question 4

A-Level Examination Question from January 2004, M1, Q2



A lever consists of a uniform steel rod *AB*, of weight 100 N and length 2 m, which rests on a small smooth pivot at a point *C* of the rod. A load of weight 2200 N is suspended from the end *B* of the rod by a rope. The lever is held in equilibrium in a horizontal position by a vertical force applied at the end *A*, as shown. The rope is modelled as a light string.

Given that *BC* = 0.2 m,

(a) find the magnitude of the force applied at *A*

[4 marks]

The position of the pivot is changed so that the rod remains in equilibrium when the force at *A* has magnitude 120 N.

(b) Find, the the nearest cm, the new distance of the pivot from *B*.

[5 marks]

1.5 Answers

1.5.1 Solution to 1.2 Example

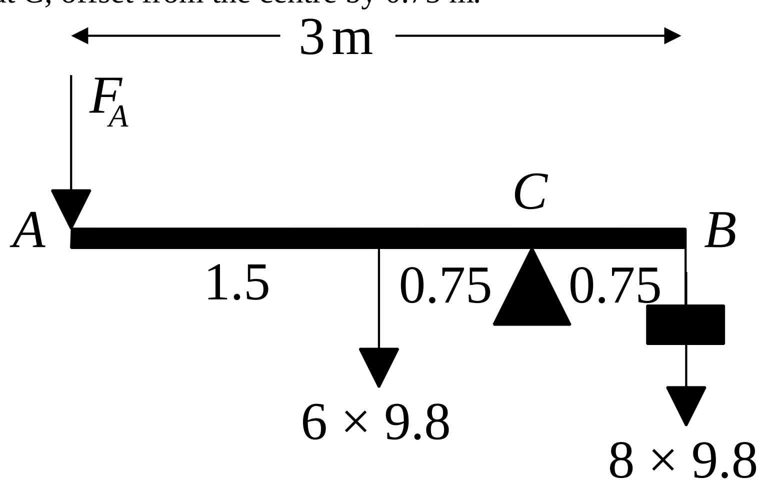
$$Moment_{ROBERT} = 0.6 \times 50 = 30 \text{ Nm} \qquad Moment_{GEORGE} = 0.4 \times 70 = 28 \text{ Nm}$$

$\therefore$ As $30 > 28$, Robert wins the pushing battle

1.5.2 Solution to 1.3 Example

To begin with questions involving moments will feature a single long, straight beam.
It might be made out of wood, metal or plastic and may be described as a rod, a plank, a pipe or a pole, for example.
It will be assumed that it does not bend or break under load.
A key idea, used repeatedly, is that when the beam is described as *uniform* all of its mass can be considered to act half way along its length.

In this question you could think of the uniform steel rod as being a see-saw, with its pivot at *C*, offset from the centre by 0.75 m.



Take moments about *C*

$$6 \times 9.8 \times 0.75 + F_A \times 2.25 = 8 \times 9.8 \times 0.75$$

$$44.1 + 2.25\,F_A = 58.8$$

$$F_A = \frac{14.7}{2.25}$$

$$= 6.53 \text{ N}$$

1.5.3 Solutions to 1.4 Exercise

Answer 1

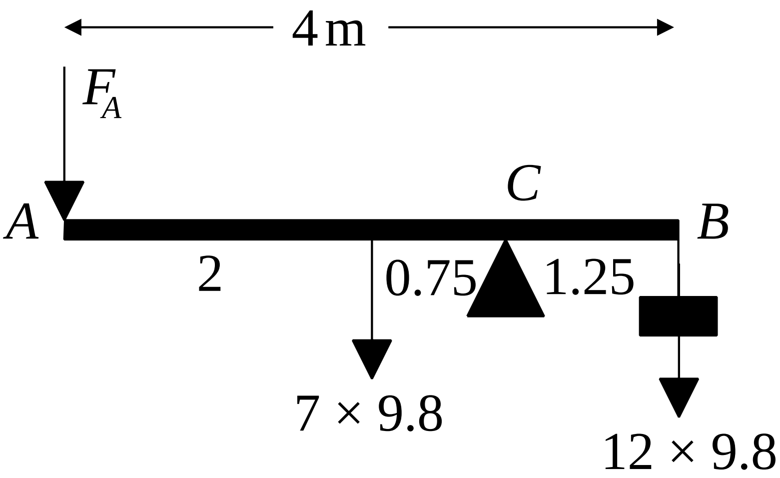
(i) Emma wins because,

$$Moment_{EMMA} > Moment_{LUCY} + Moment_{TRACY}$$
$$0.7 \times 70 > 0.8 \times 60 + 50 \times 0$$
$$49 > 48$$

(ii) Let x be the distance from the pivot at which Tracy applies her force,

$$0.7 \times 70 = 0.8 \times 60 + 50 \times x$$
$$49 = 48 + 50x$$
$$50x = 1$$
$$x = 0.02 \text{ metres}$$
$$i.e \ \ 2 \text{ cm in from the pivot (Hinge)}$$

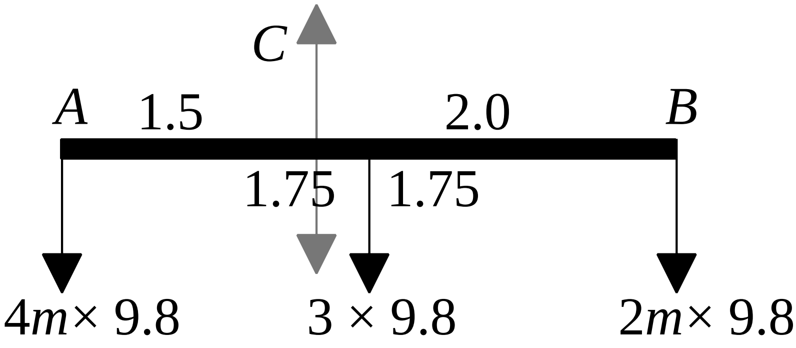
Answer 2



Take moments about C

$$12 \times 9.8 \times 1.25 = 7 \times 9.8 \times 0.75 + F_A \times 2.75$$
$$147 = 51.45 + 2.75\,F_A$$
$$F_A = \frac{95.55}{2.75}$$
$$= 34.7 \text{ N}$$

Answer 3



Take moments about C

The star hanging from C can be ignored as it is 0 m from C

The force in the string attached at C can be ignored for the same reason

$$4m \times 9.8 \times 1.5 = 3 \times 9.8 \times 0.25 + 2m \times 9.8 \times 2.0$$

$$58.8m = 7.35 + 39.2m$$

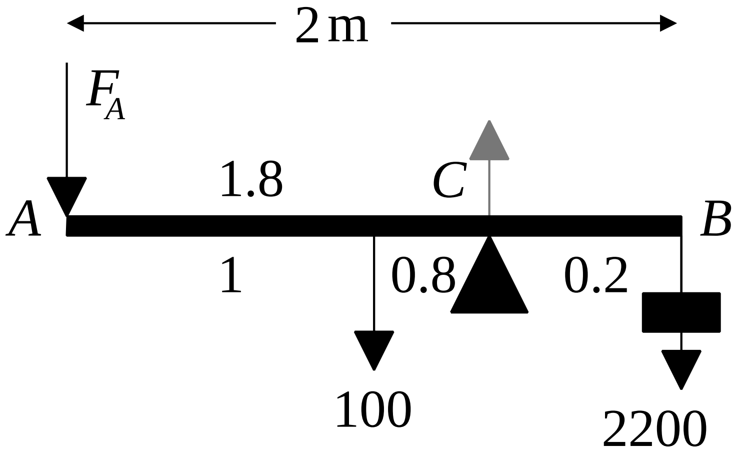
$$19.6m = 7.35$$

$$m = \frac{7.35}{19.6}$$

$$= 0.375 \text{ kg}$$

Answer 4

(a)



Take moments about C

The force acting on the pivot at C can be ignored as it is 0 m from C

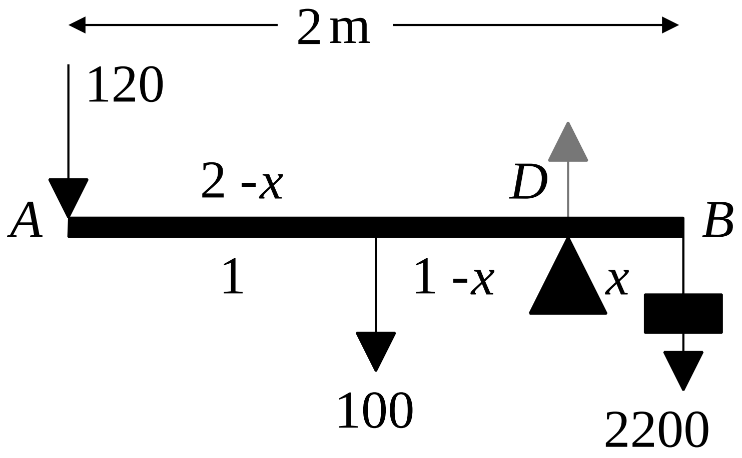
$$2200 \times 0.2 = 100 \times 0.8 + F_A \times 1.8$$

$$440 = 80 + 1.8\,F_A$$

$$F_A = \frac{360}{1.8}$$

$$= 200 \text{ N}$$

(b)



Let D be the new location of the pivot at a distance x from B,

$$\begin{aligned}
2200 \times x &= 100\,(1 - x) + 120\,(2 - x) \\
2200x &= 100 - 100x + 240 - 120x \\
2420x &= 340 \\
x &= \frac{340}{2420} \\
&= 0.14 \text{ metres}
\end{aligned}$$

i.e. 14 cm

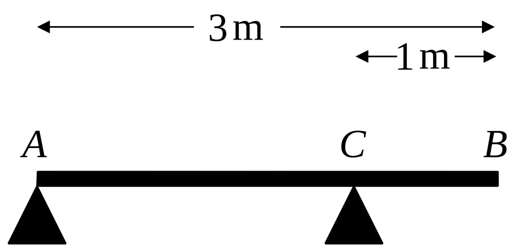
2.1 Equilibrium

When a rigid body is in equilibrium, the resultant force in any direction is zero and the resultant moment about any point is zero.
In practice, these two general facts are typically is applied as the more specific;

◇ The sum of the forces 'up' must balance the sum of the forces 'down'
◇ The sum of the moments anticlockwise about any pivot equal those clockwise

2.2 Example

An M1 Examination Question from June 2005, Q6



A uniform beam *AB* has mass 12 kg and length 3 m.
The beam rests in equilibrium in a horizontal position, resting on two smooth supports.
One support is at end *A*, the other at a point *C* on the beam, where *BC* = 1 m.
The beam is modelled as a uniform rod.

(a) Find the reaction on the beam at *C*
Teaching video : http://www.NumberWonder.co.uk/Video/v9069(2a).mp4

[3 marks]

A woman of mass 48 kg stands on the beam at the point *D*.
The beam remains in equilibrium.
The reactions on the beam at *A* and *C* are now equal.

(b) Find the distance *AD*
Teaching Video : http://www.NumberWonder.co.uk/Video/v9069(2b).mp4

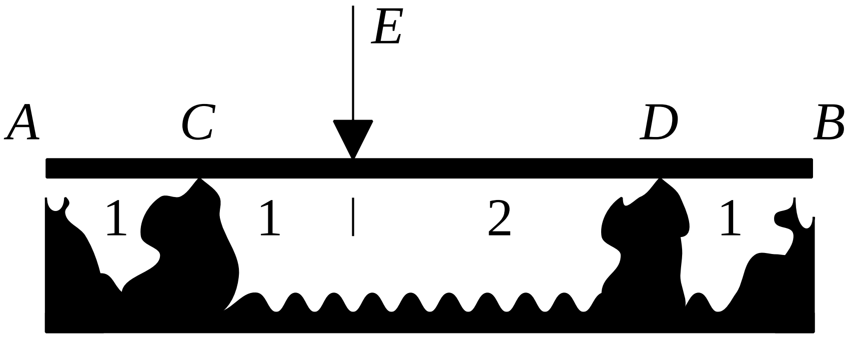
[7 marks]

2.3 Exercise

Question 1

A uniform plank of wood *AB* of length 5 m and mass 30 kg is used to form a bridge across a stream in a forest. The plank of wood rests on two rocks, *C* and *D*, which are each 1 m in from the ends of the plank, as shown.

A lumberjack of mass 75 kg stands on the bridge at *E* where *AE* = 2 m.

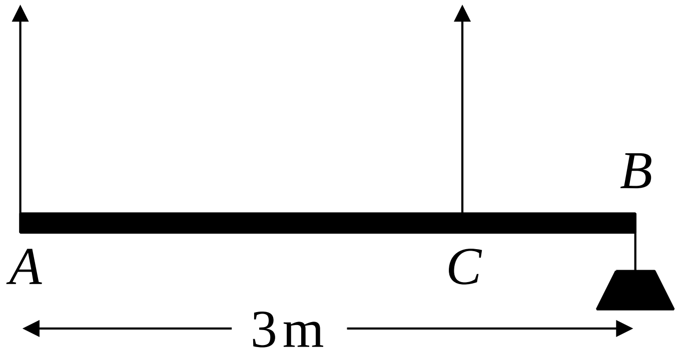


(a) By taking moments about *C*, determine the magnitude of the reaction force at *D*.

(b) By equating the total force upwards with the total force downward determine the reaction force at *C*.

Question 2

M1 Exam question, 12th January 2005, Q2



A plank AB has mass 40 kg and length 3 m. A load of mass 20 kg is attached to the plank at B. The loaded plank is held in equilibrium, with AB horizontal, by two vertical ropes attached at A and C, as shown. The plank is modelled as a uniform rod and the load as a particle. Given that the tension in the rope at C is three times the tension in the rope at A, calculate

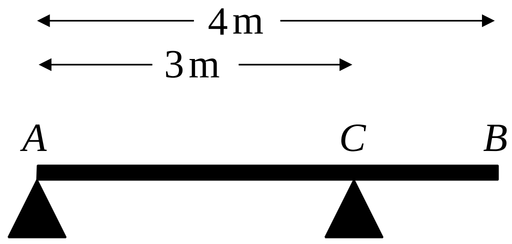
(a) the tension in the rope at C

[2 marks]

(b) the distance CB

[5 marks]

Question 3
M1 Examination Question, 19th May 2003, Q1



A uniform plank AB has mass 40 kg and length 4 m. It is supported in a horizontal position by two smooth pivots, one at the end A, the other at the point C of the plank where AC = 3 m.
A man of mass 80 kg stands on the plank which remains in equilibrium.
The magnitudes of the reactions at the two pivots are each equal to R newtons.
By modelling the plank as a rod and the man as a particle, find

(a) the value of R

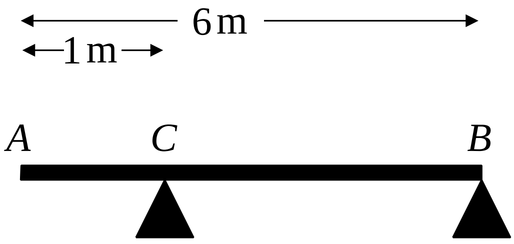
[2 marks]

(b) the distance of the man from A

[4 marks]

Question 4

M1 Examination Question, 19th January 2011, Q3



A uniform beam *AB* has mass 20 kg and length 6 m. The beam rests in equilibrium in a horizontal position on two smooth supports. One support is at *C*, where *AC* = 1 m, and the other is at end *B*. The beam is modelled as a rod.

(a) Find the magnitudes of the reactions on the beam at *B* and at *C*

[5 marks]

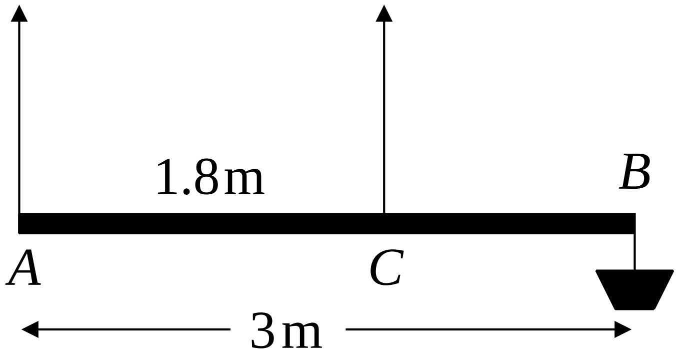
A boy of mass 30 kg stands on the beam at the point *D*.
The beam remains in equilibrium.
The magnitudes of the reactions on the beam at *B* and at *C* are now equal.
The boy is modelled as a particle.

(b) Find the distance *AD*

[5 marks]

Question 5

M1 Exam question, 15th January 2010, Q4



A pole *AB* has length 3 m and weight *W* newtons. The pole is held in a horizontal position in equilibrium by two vertical ropes attached to the pole at the points *A* and *C* where *AC* = 1.8 m, as shown. A load of weight 20 N is attached to the rod at *B*. The pole is modelled as a uniform rod, the ropes as light inextensible strings and the load as a particle.

(a) Show that the tension in the rope attached to the pole at *C* is

$$\left(\frac{5}{6}W + \frac{100}{3}\right)N$$

[4 marks]

(b) Find, in terms of W, the tension in the rope attached to the pole at A.

[3 marks]

Given that the tension in the rope attached to the pole at C is eight times the tension in the rope attached to the pole at A,

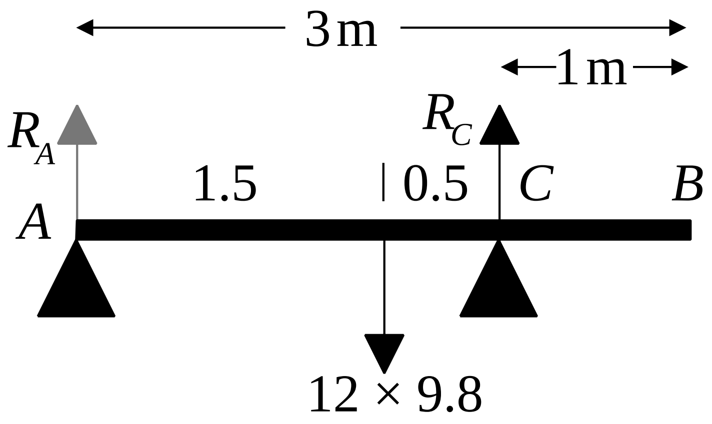
(c) find the value of W.

[3 marks]

2.4 Answers

2.4.1 Solutions to 2.2 Example

(a)

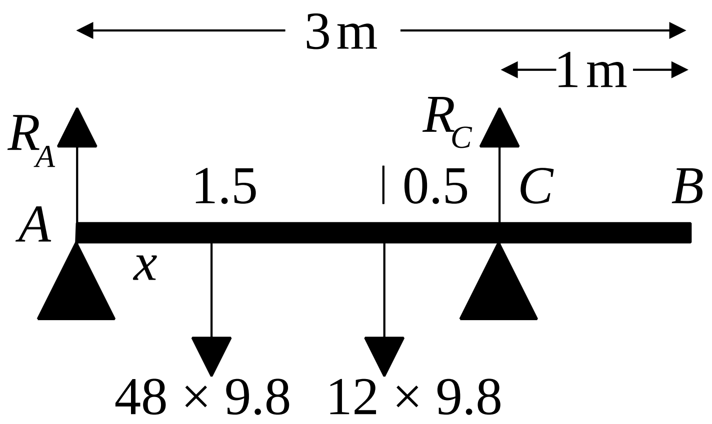


Take moments about A,

$$12 \times 9.8 \times 1.5 = R_C \times 2$$

$$R_C = 88.2 \text{ N}$$

(b)



Balancing linear forces vertically,

$$R_A + R_C = 48 \times 9.8 + 12 \times 9.8$$

$$2\,R_A = 2\,R_C = 588$$

$$R_A = R_C = 294\,TN$$

Take moments about A,

$$R_C \times 2 = 48 \times 9.8 \times x + 12 \times 9.8 \times 1.5$$

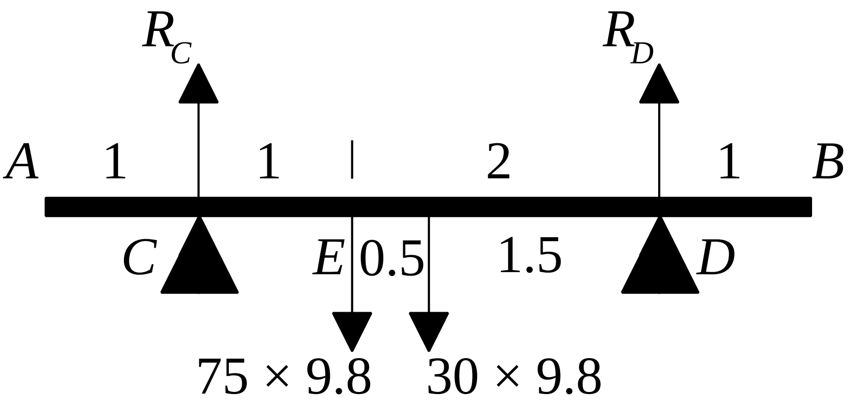
$$294 \times 2 = 470x + 176$$

$$x = 0.88 \text{ metres}$$

2.4.2 Answers to 2.3 Exercise

Answer 1

A clear working diagram is an enormous help in understanding the question.



(a) Taking moments about C,

$$R_D \times 3 = 75 \times 9.8 \times 1 + 30 \times 9.8 \times 1.5$$

$$3\,R_D = 735 + 441$$

$$R_D = 392\text{ N}$$

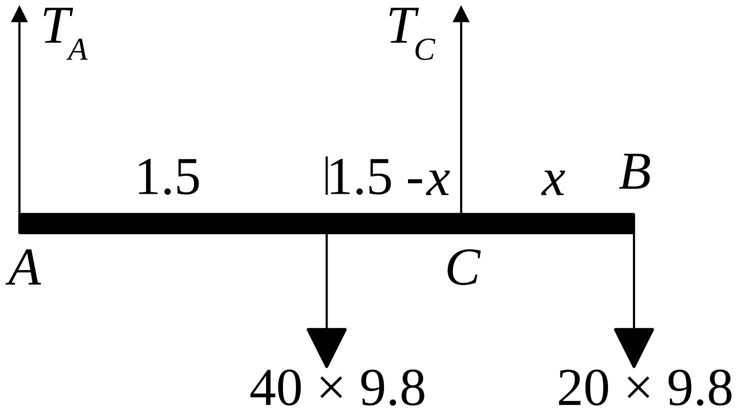
(b) Balancing forces vertically,

$$R_C + R_D = 75 \times 9.8 + 30 \times 9.8$$

$$R_C + 392 = 735 + 294$$

$$R_C = 637\text{ N}$$

Answer 2



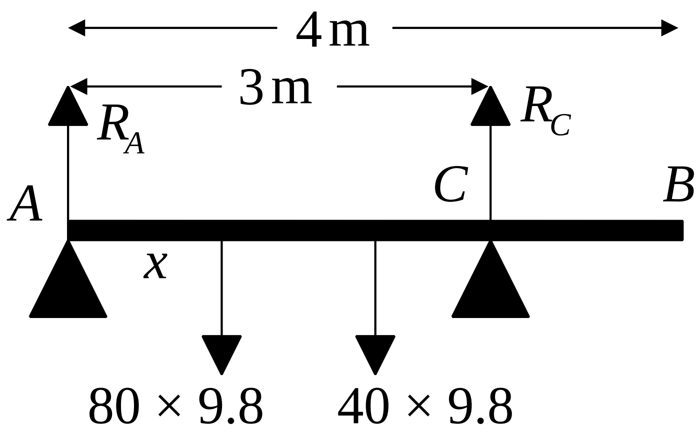
(a) Balancing forces vertically,

$$
\begin{aligned}
T_A + T_C &= 40 \times 9.8 + 20 \times 9.8 \\
T_A + 3\,T_A &= 392 + 196 \\
4\,T_A &= 588 \\
T_A &= 147 \text{ N} \\
\therefore\ T_C &= 441 \text{ N}
\end{aligned}
$$

(b) Taking moments about B,

$$
\begin{aligned}
T_C \times x + T_A \times 3 &= 40 \times 9.8 \times 1.5 \\
441x + 3 \times 147 &= 588 \\
441x &= 147 \\
x &= \frac{1}{3} \text{ metre}
\end{aligned}
$$

Answer 3



(a) Balancing forces vertically,

$$R_A + R_C = 80 \times 9.8 + 40 \times 9.8$$

$$2\,R = 784 + 392$$

$$R_A = R_C = 588\ \text{N}$$

(b) It's strategically unsound to take moments about any point other than the end-points of the plank when an unknown distance is involved.

Taking moments about *A*,

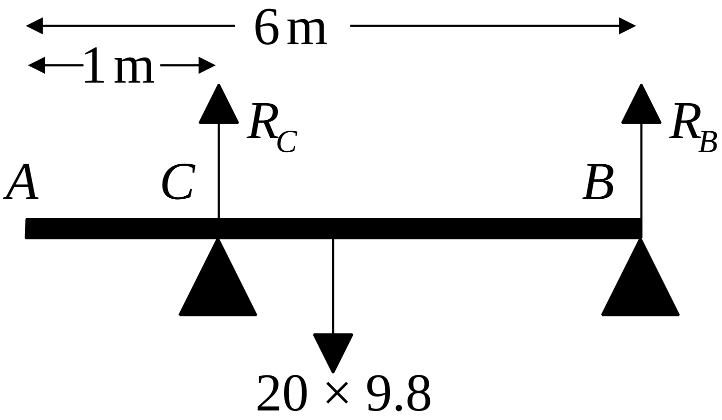
$$R_C \times 3 = 80 \times 9.8 \times x + 40 \times 9.8 \times 2$$

$$3 \times 588 = 784x + 784$$

$$784x = 980$$

$$x = 1.25\ \text{metres}$$

Answer 4



(a) Take moments about C is valid although taking moments about B is marginally easier to get correct. So, taking moments about B,

$$R_C \times 5 = 20 \times 9.8 \times 3$$

$$R_C = 118\text{ N}$$

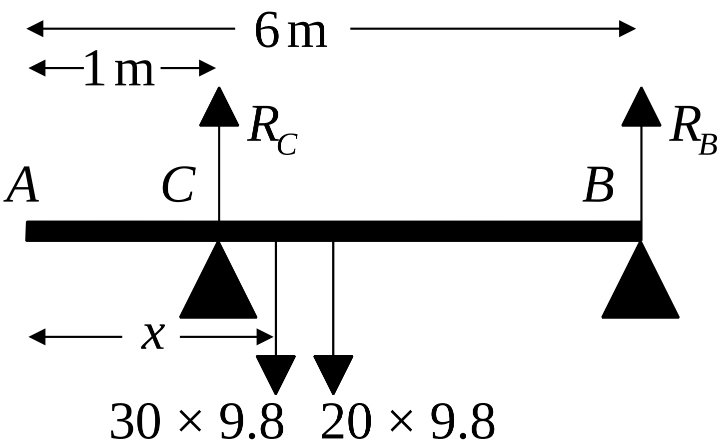
Balancing forces vertically,

$$R_B + R_C = 20 \times 9.8$$

$$R_B = 196 - 118$$

$$= 78\text{ N}$$

(b)



Balancing forces vertically,

$$R_C + R_B = 30 \times 9.8 + 20 \times 9.8$$

$$2\,R = 294 + 196$$

$$R_C = R_B = 245\text{ N}$$

Taking moments about B,

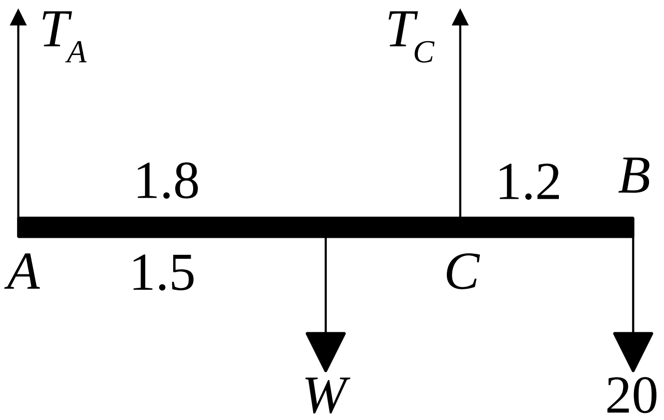
$$R_C \times 5 = 30 \times 9.8 \times (6 - x) + 20 \times 9.8 \times 3$$

$$1225 = 1764 - 294x + 588$$

$$294x = 1127$$

$$x = 3.83\text{ metres}$$

Answer 5



(a) Take moments about A,

$$T_C \times 1.8 = W \times 1.5 + 20 \times 3$$

$$T_C = \frac{1.5\,W}{1.8} + \frac{60}{1.8}$$

$$= \frac{15}{18}\,W + \frac{600}{18}$$

$$= \frac{5}{6}\,W + \frac{100}{3} \qquad \square$$

(b) Balancing forces vertically,

$$T_A + T_C = W + 20$$

$$T_A = W + 20 - \frac{5}{6}\,W - \frac{100}{3}$$

$$= \frac{1}{6}\,W - \frac{40}{3}$$

(c)

$$T_C = 8\,T_A$$

$$\frac{5}{6}\,W + \frac{100}{3} = 8\left(\frac{1}{6}\,W - \frac{40}{3}\right)$$

Multiply through by 6

$$5W + 200 = 8\,(\,W - 80\,)$$

$$200 + 640 = 8W - 5W$$

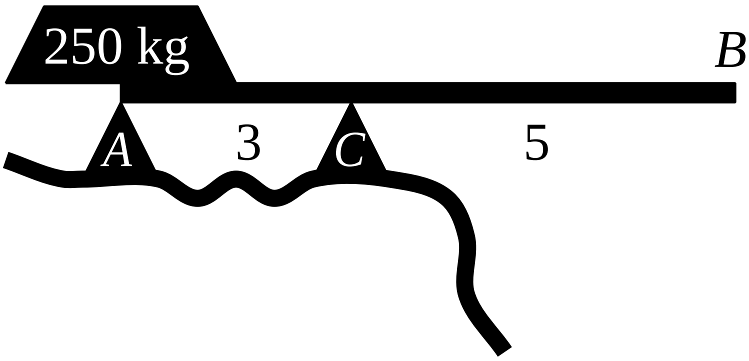
$$3W = 840$$

$$W = 280\text{ N}$$

3.1 On the Edge Of Tipping

Example

A 250 kg mass is bolted onto the end of a uniform plank of wood *AB* of length 8 m and mass 45 kg.
The plank rests on two supports at *A* and *C* where *AC* = 3 m.
The plank is in equilibrium and horizontal.



Teaching Video : http://www.NumberWonder.co.uk/Video/v9069(3).mp4

(a) By taking moments about *A* determine the reaction force at *C*.

(b) By equating the total force upwards with the total force downward determine the reaction force at *A*.

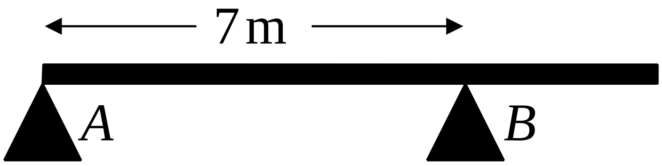
The plank of wood is being used as a diving board.

(c) By taking moments about *C*, determine the maximum mass that can be at *B* with the plank remaining in equilibrium ?

(d) In the light of your previous answers, what advice would you give regarding the safe use of this diving board ?

3.2 Exercise

Question 1



A uniform plank of mass 100 kg and length 10 m rests horizontally on two smooth supports, *A* and *B*, as shown. A man of mass 80 kg starts walking from one end of the plank, *A*, to the other end. When he is a distance x past *B* the plank starts to tip.

(i) Draw a diagram showing all significant forces on the plank when it is on the verge of tipping.

[2 marks]

(ii) When the plank is about to tip, what will be the reaction at *A* ?

[1 mark]

(iii) When the plank is about to tip, what will be the reaction at *B* ?

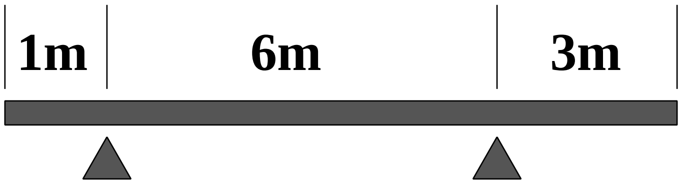
[1 mark]

(iv) Find the distance, x, the man can walk past *B* before the plank starts to tip.

[4 marks]

Question 2

A uniform rod of mass 14 kg rests on two supports as shown below.



(a) The reaction at the left hand support is 75 newtons.
What, in newtons, is the reaction at the other support ?

(b) A mass, *m*, is now placed an the right hand end of the rod, such that the rod is on the point of tipping about the right hand support.

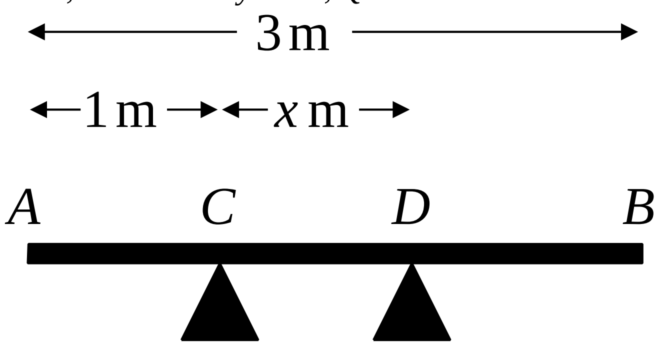
(i) What, now, is the reaction at the left hand support ?

(ii) What, in kg, is the mass, *m* ?

(iii) What, in N, is the reaction at the right hand support ?

Question 3

M1 Exam question, 12th January 2007, Q2



A uniform plank *AB* has weight 120 N and length 3 m.
The plank rests horizontally in equilibrium on two smooth supports *C* and *D*, where *AC* = 1 m and *CD* = x m.
The reaction of the support on the plank at *D* has magnitude 80 N.
Modelling the plank as a rod,

(a) show that $x = 0.75$

[3 marks]

A rock is now placed at *B* and the plank is on the point of tilting about *D*
Modelling the rock as a particle, find

(b) the weight of the rock

[4 marks]

(c) the magnitude of the reaction of the support on the plank at *D*

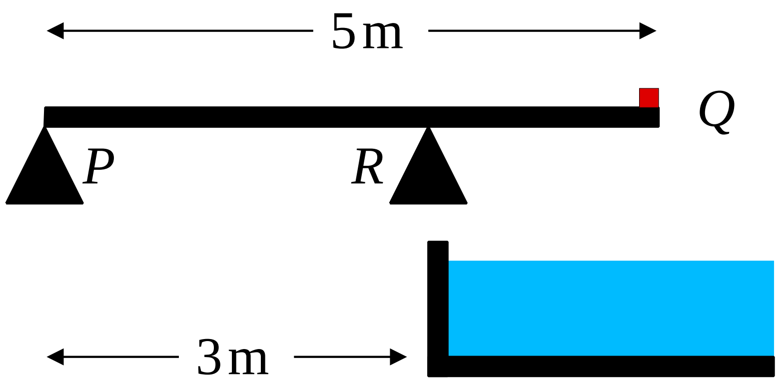
[2 marks]

(d) State how you have used the model of the rock as a particle

[1 mark]

Question 4

A-Level Examination Question from January 2019, IAL, M1, Q4



A boy sees a box on the end Q of a plank PQ which overhangs a swimming pool. The plank has mass 30 kg, is 5 m long and rests in a horizontal position on two bricks. The bricks are modelled as smooth supports, one acting on the plank at P and one acting on the plank at R, where PR is 3 m. The support at R is on the edge of the swimming pool, as shown. The boy has a mass 40 kg and the box has mass 2.5 kg. The plank is modelled as a uniform rod and the boy and the box as particles.

The boy steps on to the plank at P and begins to walk slowly along the plank towards the box.

(a) Find the distance he can walk along the plank from P before the plank starts to tilt.

[4 marks]

(b) State how you have used, in your working, the fact that the box is modelled as a particle.

[1 mark]

A rock of mass M kg is placed on the plank at P. The boy is then able to walk slowly along the plank to the box at the end Q without the plank tilting.
The rock is modelled as a particle.

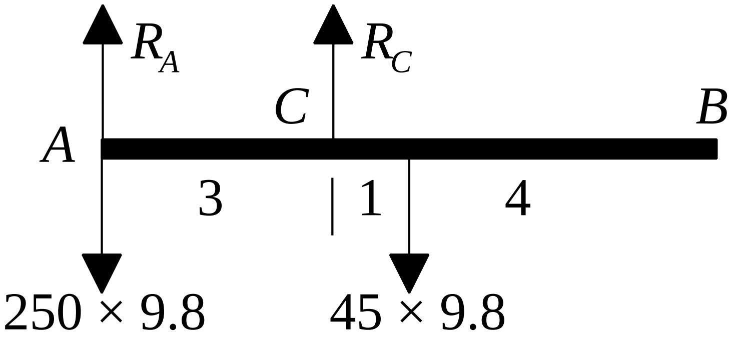
(c) Find the smallest possible value of M.

[4 marks]

3.3 Answers

3.3.1 Solution to the Introductory Example

As the rod is *uniform*, the 45 kg mass of the plank can be assumed to act at the midpoint of the *rod*, at it's *centre of mass*.
Thus, the working diagram becomes;



(a) Taking moments about A (as forces at A then have moments of zero)

$$45 \times 9.8 \times 4 = 3 \times R_C$$

$$R_C = 588\,N$$

(b) Balancing forces vertically;

$$R_A + R_C = 250 \times 9.8 + 45 \times 9.8$$

$$= 2891$$

$$\therefore\ R_A = 2303\,N$$

(c) At tipping point, reaction at A will be zero
Taking moments about C

$$250 \times 9.8 \times 3 = 45 \times 9.8 \times 1 + M \times 9.8 \times 5$$

$$M = 141\,kg$$

(d) An adult typically has a mass of 70 kg
A skinny man typically has a mass of about 48 kg
An obese man typically has a mass of 141 kg

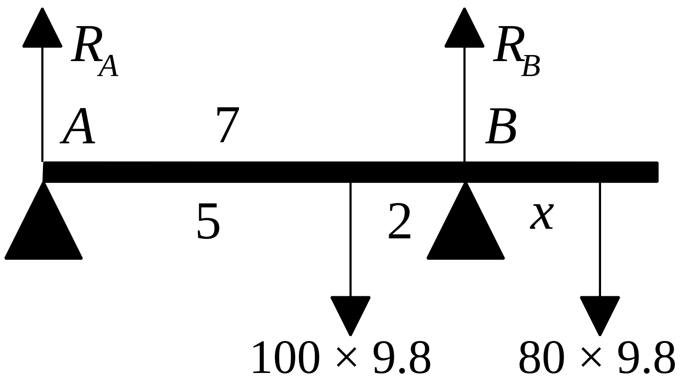
$\therefore$ Don't let move than one person at a time use the diving board !

Also : Jumping at the end, as divers do, considerably adds to the maximum force, the impulse, being exerted on the board.

3.3.2 Solutions to 3.2 Exercise

Answer 1

(i) In the diagram R_A could be omitted as it will be 0 N on the point of tipping.



(ii) When the beam is on the point of tipping, $R_A = 0$ N

(iii) Balancing forces vertically,

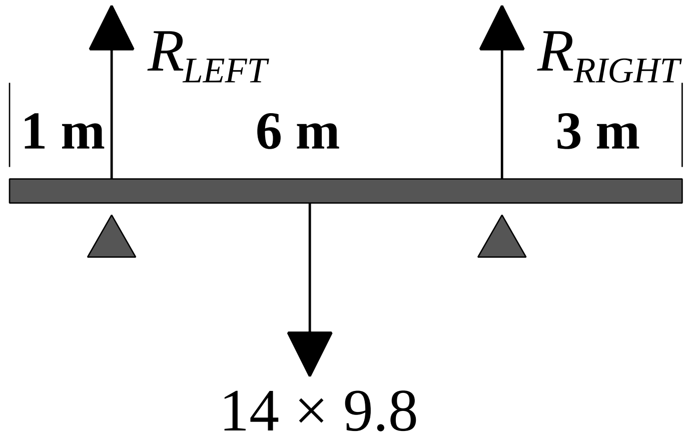
$$R_A + R_B = 100 \times 9.8 + 80 \times 9.8$$
$$0 + R_B = 980 + 784$$
$$R_B = 1764 \text{ N}$$

(iv) Taking moments about B,

$$80 \times 9.8 \times x = 100 \times 9.8 \times 2$$
$$784x = 1960$$
$$x = 2.5 \text{ metres}$$

Answer 2

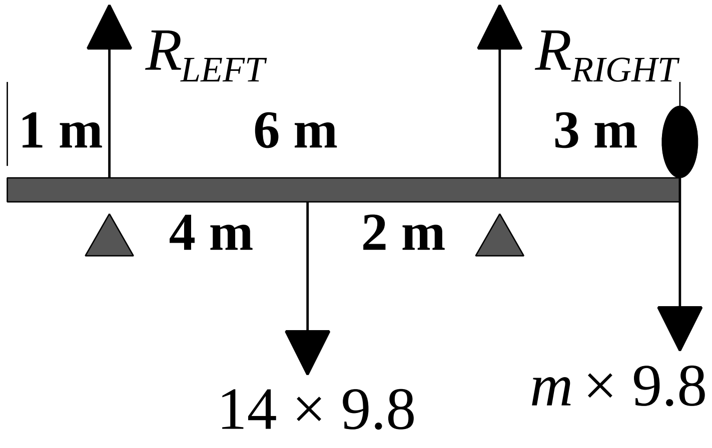
(a)



Balancing forces vertically,

$$\begin{aligned} R_{LEFT} + R_{RIGHT} &= 14 \times 9.8 \\ R_{RIGHT} &= 137 - 75 \\ &= 62 \text{ N} \end{aligned}$$

(b)



(i) 0 N

(ii) Taking moments about the right hand support,

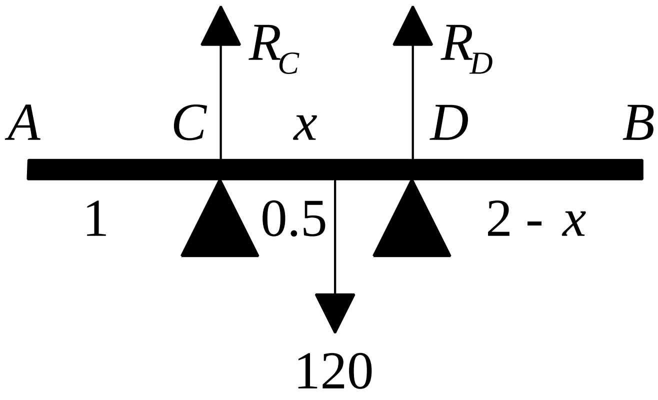
$$\begin{aligned} R_{LEFT} \times 6 + m \times 9.8 \times 3 &= 14 \times 9.8 \times 2 \\ 0 + 29.4m &= 274 \\ m &= 9.33 \text{ kg} \end{aligned}$$

(iii) Balancing forces vertically,

$$\begin{aligned} R_{LEFT} + R_{RIGHT} &= 14 \times 9.8 + m \times 9.8 \\ 0 + R_{RIGHT} &= 137 + 9.33 \times 9.8 \\ R_{RIGHT} &= 229 \text{ N} \end{aligned}$$

Answer 3

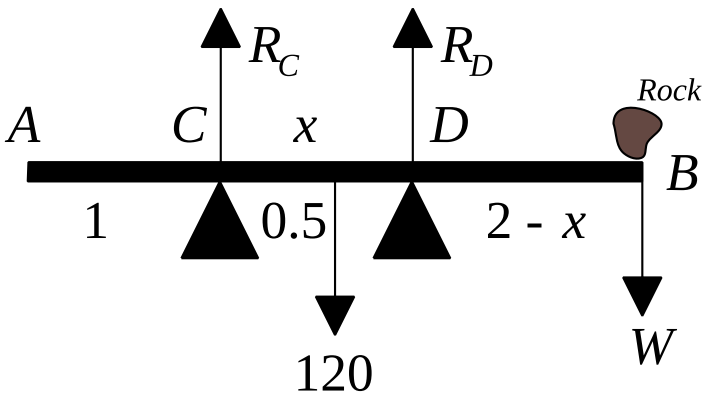
(a)



Taking moments about *C*,

$$R_D \times x = 0.5 \times 120$$
$$80x = 60$$
$$x = 0.75 \text{ metres}$$

(b)



Taking moments about *D*, and ignoring R_C as it is equal to zero,

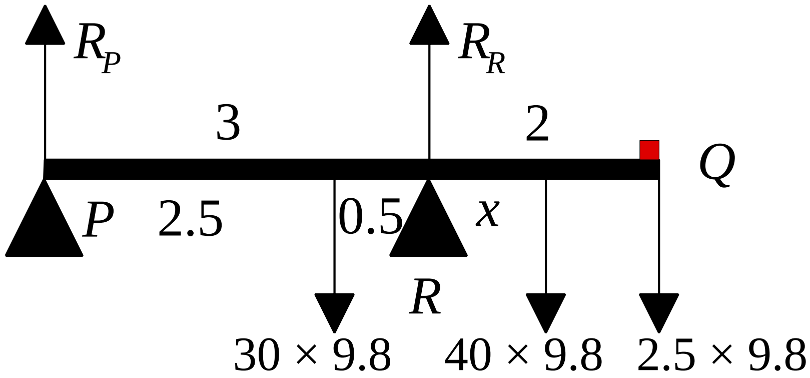
$$W \times 1.25 = 120 \times 0.25$$
$$W = 24 \text{ N}$$

(c) Balancing forces vertically,

$$R_D = 120 + 24$$
$$= 144 \text{ N}$$

(d) The weight of the rock is assumed to act precisely at *B*.
In practice a slight vibration might cause the rock to fall off the plank.

Answer 4



(a) Let x be the distance between the boy and support R when the plank is on the point of tipping. This choice of what to call x is based on the experience gained in Question 1.

When on the point of tipping, R_P will be 0 N
Taking moments about R,

$$40 \times 9.8 \times x + 2.5 \times 9.8 \times 2 = 30 \times 9.8 \times 0.5$$

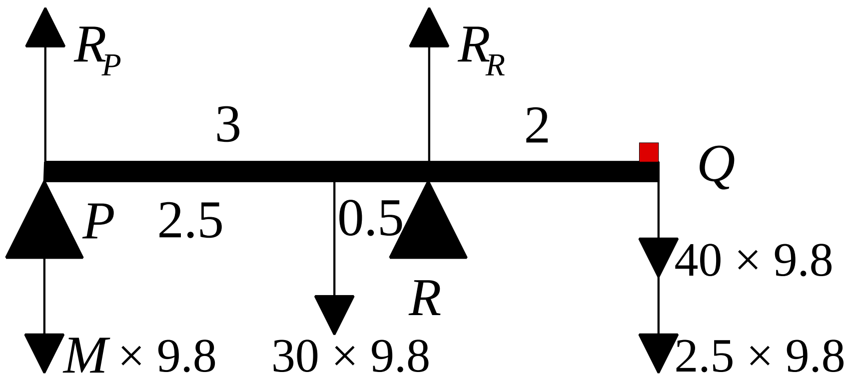
$$392x + 49 = 147$$

$$x = 0.25\,\text{m}$$

From P this will be a distance of 3.25 metres along the plank.

(b) I've assumed that the weight of the box is all at the very end of the plank whereas in the diagram the box is depicted in a manner that suggests the centre of mass of the box is a little in from the end of the plank.

(c)



Again, on the point of tipping, R_P will be 0 N
Taking moments about R,

$$M \times 9.8 \times 3 + 30 \times 9.8 \times 0.5 = 40 \times 9.8 \times 2 + 2.5 \times 9.8 \times 2$$

$$29.4M + 147 = 784 + 49$$

$$M = 23.3\,\text{kg}$$

4.1 The Nonuniform Rod

The assumption that a rod is uniform allowed for its weight to be considered as acting half way along its length. Given a sawn scaffolder's plank of wood it is reasonable to model it as a uniform rod. However, when a tree trunk is used as a bridge, for example, one end of the tree trunk will typically have less circumference than the other; it will have a centre of mass that is more towards the end with the greater circumference. In modelling such a tree trunk with a rod, the rod would be described as *nonuniform*. The centre of mass of the rod is still the place where all of the rod's weight can be considered to act, but that centre of mass is now at an unknown distance along the rod.

4.2 Example

A nonuniform log AB of mass 15 kg and length 8 metres is used as a bridge across a river. The log is supported at P and Q, where AP is 2 metres and QB is 1.5 metres. The reaction at P is twice the reaction at Q.
Find the distance of the centre of mass of the beam from A.
Teaching Video : http://www.NumberWonder.co.uk/Video/v9069(4).mp4

4.3 Exercise

Question 1

A nonuniform bar *AB* of length 5 m has end supports at *A* and *B*
When a man of mass 65 kg stands at the centre of the bar the reactions at supports *A* and *B* are 525 N and 700 N respectively.

(i) Draw a diagram marking on the key forces and lengths.

(ii) By considering all vertical forces, determine the mass of the bar

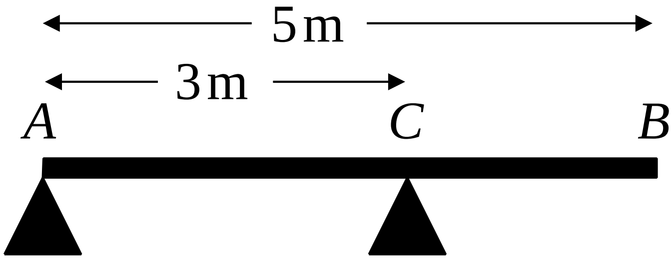
(iii) By taking moments about *A*, find the distance of the centre of mass of the bar from *A*

Question 2

Molly and Mac are sitting on a nonuniform log *AB* of mass 260 kg and length 4 m.
The plank is pivoted at *P*, the midpoint of *AB.*
The centre of mass of *AB* is at *C*, where *AC* is 1.95 m.
Molly has mass 24 kg and sits at *A.*
Mac has a mass of 38 kg.
Where must Mac sit for the log to be in horizontal equilibrium ?

Question 3

A-Level Examination Question from October 2018, IAL, M1, Q2



A nonuniform plank AB has weight 60 N and length 5 m. The plank rests horizontally in equilibrium on two smooth supports at A and C, where AC = 3 m, as shown.

A parcel of weight 12 N is placed on the plank at B and the plank remains horizontal and in equilibrium. The magnitude of the reaction of the support at A on the plank is half the magnitude of the reaction of the support at C on the plank.

By modelling the plank as a nonuniform rod and the parcel as a particle,

(a) find the distance of the centre of mass of the plank from A

[6 marks]

(b) State briefly how you have used the modelling assumption

(i) that the parcel is a particle

[1 mark]

(ii) that the plank is a rod

[1 mark]

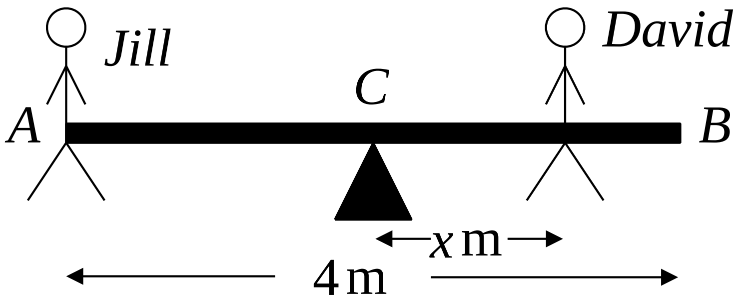
Question 4

A-Level Question from January 2014, IAL, M1, Q3

A beam *AB* has length 15 m and mass 25 kg. The beam is smoothly supported at the point *P*, where *AP* = 8 m. A man of mass 100 kg stands on the beam at a distance of 2 m from *A* and another man stands on the beam at a distance of 1 m from *B*.
The beam is modelled as a nonuniform rod and the men are modelled as particles.
The beam is in equilibrium in a horizontal position with the reaction on the beam at *P* having magnitude 2009 N. Find the distance of the centre of mass of the beam from *A*.

[5 marks]

Question 5
A-Level Question from January 2006, M1, Q3



A seesaw in a playground consists of a beam AB of length 4 m which is supported by a smooth pivot at its centre C. Jill has a mass 25 kg and sits on the end A. David has mass 40 kg and sits at a distance x metres from C, as shown. The beam is initially modelled as a uniform rod.

Using this model,

(a) find the value of x for which the seesaw can rest in equilibrium in a horizontal position

[3 marks]

(b) State what is implied by the modelling assumption that the beam is uniform.

[1 mark]

David realises that the beam is not uniform as he finds he must sit at a distance 1.4 m from *C* for the seesaw to rest horizontally in equilibrium. The beam is now modelled as a nonuniform rod of mass 15 kg.

Using this model,

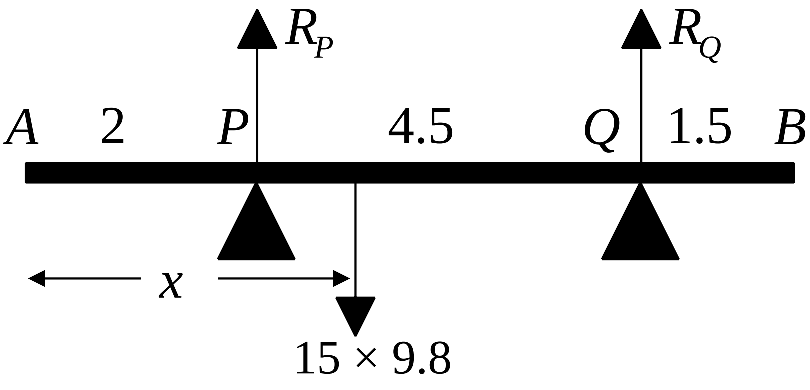
(c) find the distance of the centre of mass of the beam from *C*.

[4 marks]

4.4 Answers

4.4.1 Solution to 4.2 Example

The question does not provide any diagram, so the first task is to remedy that shortcoming,



Balancing forces vertically,

$$
\begin{aligned}
R_P + R_Q &= 15 \times 9.8 \\
2\,R_Q + R_Q &= 147 \\
R_Q &= 49 \text{ N} \\
\therefore R_P &= 98 \text{ N}
\end{aligned}
$$

It may be obvious that the centre of mass is somewhere between the two supports *P* and *Q* in which case a marginally easier calculation would take moments about *P*. However, with other weights around such as a man standing on the log, it may not be possible to decide in advance which side of *P* the centre of mass lies. The consequence of this would be to not know if the moment of the rod's weight is clockwise or anticlockwise about *P*.

It's thus, in general, easier to simply take moments about one of the plank's endpoints.

So, taking moments about *A*,

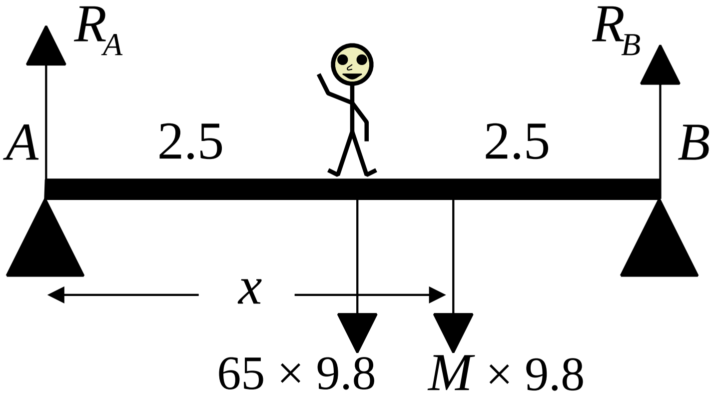
$$
\begin{aligned}
15 \times 9.8 \times x &= R_P \times 2 + R_Q \times 6.5 \\
147x &= 98 \times 2 + 49 \times 6.5 \\
147x &= 196 + 319 \\
x &= 3.5 \text{ metres}
\end{aligned}
$$

As the log was 8 metres long, the centre of mass is not is the middle of the log.

4.4.2 Solution to 4.3 Exercise

Answer 1

(i) $R_A = 525$ N and $R_B = 700$ N



(ii) Balancing forces vertically,

$$R_A + R_B = 65 \times 9.8 + M \times 9.8$$

$$525 + 700 = 637 + 9.8M$$

$$9.8M = 588$$

$$M = 60 \text{ kg}$$

(iii) Taking moments about A,

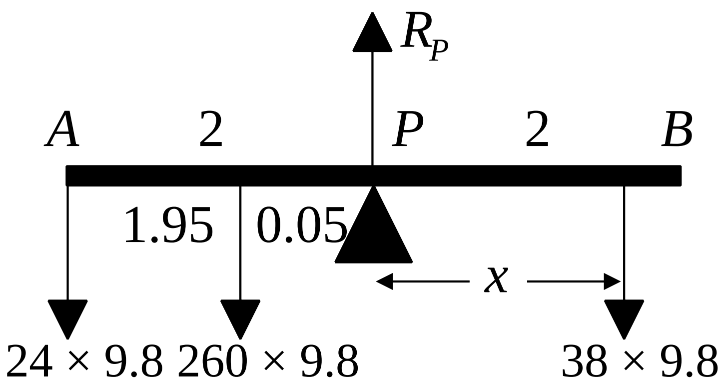
$$R_B \times 5 = 65 \times 9.8 \times 2.5 + M \times 9.8 \times x$$

$$700 \times 5 = 1593 + 588x$$

$$588x = 1907$$

$$x = 3.24 \text{ metres}$$

Answer 2



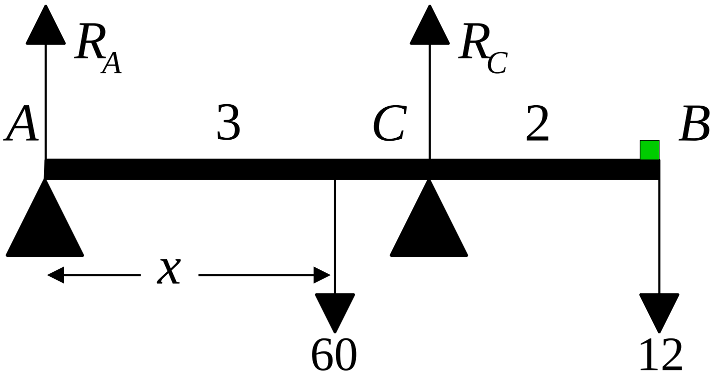
Take moments about P,

$$38 \times 9.8 \times x = 260 \times 9.8 \times 0.05 + 24 \times 9.8 \times 2$$

$$372x = 127 + 470$$

$$x = 1.60 \text{ metres}$$

Answer 3



(a) Balancing forces vertically,

$$R_A + R_C = 60 + 12$$
$$0.5\,R_C + R_C = 72$$
$$R_C = 48\text{ N}$$
$$\therefore\ R_A = 24\text{ N}$$

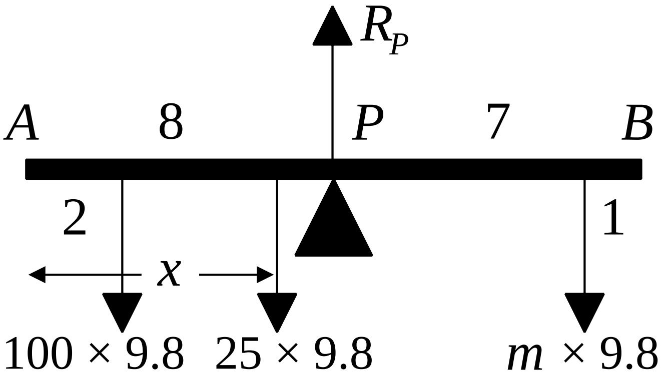
Taking moments about A,

$$60 \times x + 12 \times 5 = R_C \times 3$$
$$60x + 60 = 48 \times 3$$
$$60x = 84$$
$$x = 1.4\text{ metres}$$

(b) **(i)** Assumed that all of the parcel's weight acted at B
In reality, the parcel's centre of mass would be a few centimetres in from B as it wold otherwise fall off the end as the parcel was "placed" rather than fixed in some way at B

(ii) Assumed that the rod does not bend.
As it's long for its weight, it is likely to have some bend in reality.

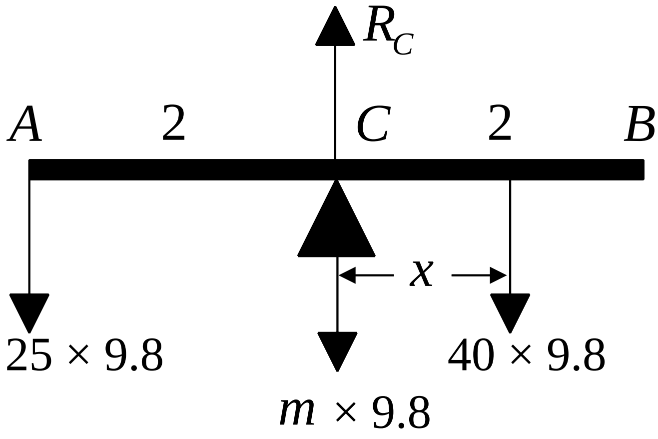
Answer 4



As the mass of the second man, near B is not known, and not requested, the direct method of doing this question is to take moments about where that second man is standing,

$$\begin{aligned} 100 \times 9.8 \times 12 + 25 \times 9.8 \times (14 - x) &= R_P \times 6 \\ 11760 + 245(14 - x) &= 2009 \times 6 \\ 11760 + 3430 - 245x &= 12054 \\ 245x &= 3136 \\ x &= 12.8 \text{ metres} \end{aligned}$$

Answer 5

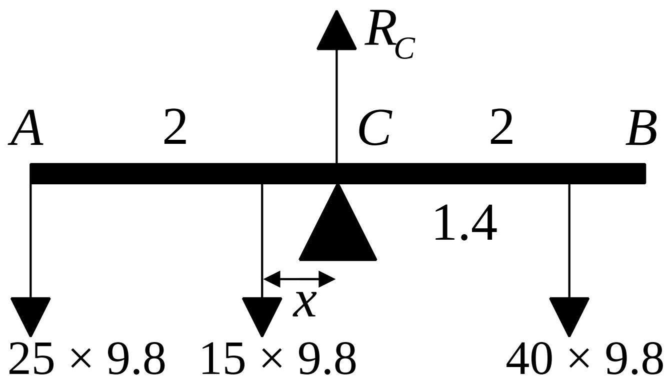


(a) As the mass of the uniform beam is not known, take moments about C,

$$40 \times 9.8 \times x = 25 \times 9.8 \times 2$$
$$392x = 490$$
$$x = 1.25 \text{ metres}$$

(b) That the mass of the beam acts at the midpoint of the beam

(c)



Taking moments about C,

$$25 \times 9.8 \times 2 + 15 \times 9.8 \times x = 40 \times 9.8 \times 1.4$$
$$490 + 147x = 549$$
$$x = 0.4 \text{ metres}$$

5.1 Rods and Simultaneous Equations

The more difficult questions on moments typically involve more algebra, the result of being told less information in a question and so having to work with more unknowns. Often a mass is placed at one location on a rod, an observation made, the mass then moved to another location and another observation made.
The resulting two equations in two unknowns may then be solved either by the method of substitution or as simultaneous equations. Often a question will provide guidance on forming the two equations, but not always.

The two key techniques remain those employed previously, namely that when a rigid body is in equilibrium, the resultant force in any direction is zero (usually the vertical direction) and the resultant moment about any point is zero. Picking the best point to be thought of as a pivot is a skill that develops with experience.
So, when not sure how to proceed, start trying to use the standard techniques.
They remain that,

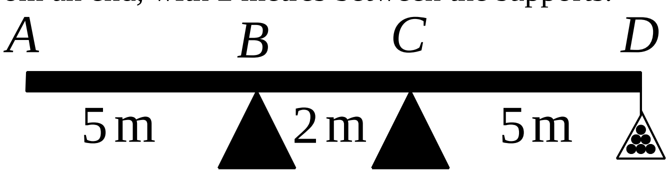
◇ The sum of the forces 'up' must balance the sum of the forces 'down'

◇ The sum of the moments anticlockwise about any pivot equal those clockwise

5.2 How to Weigh a Rod

Surprisingly, the following practical method of determining the mass of a rod without any fancy equipment does not seem to be widely known but provides an excellent introductory example for this lesson.

An aluminium yacht mast *AD* is resting horizontally on two supports *B* and *C* in a boatyard. The mast is 12 metres long. By lifting one end and moving a support, then lifting the other end and moving the other support is is arranged that each support is 5 metres in from an end, with 2 metres between the supports.



A large bag is slung from end *D* and recycled 2 litre fizzy drink bottles are placed one by one into the bag. Once six of these bottles, each full of water, have been added the mast is on the verge of tilting about *C*, as shown.
The large bag is now moved to the other end of the mast. When slung from *A* it takes 18 bottles to cause the mast to be on the verge of tilting about *B*.

(i) How is it immediately known that the mast is nonuniform ?

(ii) What is the mass of the mast ?

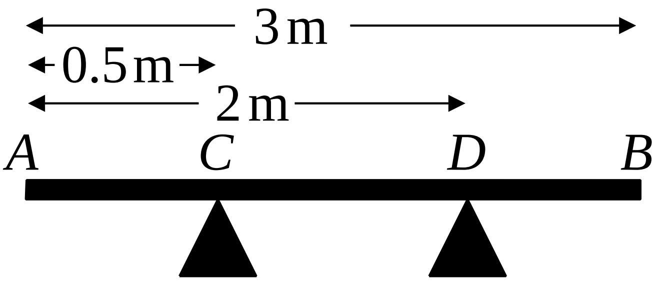
(iii) How far from end *D* is the centre of mass of the mast ?

Over the page, have a go at answering this question before watching the video.

Teaching Video : http://www.NumberWonder.co.uk/Video/v9069(5).mp4

5.3 Exercise

Question 1



A uniform rod *AB* of length 3 m and weight 120 N rests in equilibrium in a horizontal position, smoothly supported at points *C* and *D* where *AC* = 0.5 m and *AD* = 2 m. When a particle of weight *W* newtons is attached at a point *E*, where *AE* = *x* m metres, the rod remains in equilibrium with the reaction at *C* now twice the reaction at *D*.

(i) Draw a diagram marking on all significant forces and lengths.

[2 marks]

(ii) By taking moments about *A*, show that $3R_D = Wx + 180$

[2 marks]

(iii) By balancing forces vertically, show that $3R_D = 120 + W$

[2 marks]

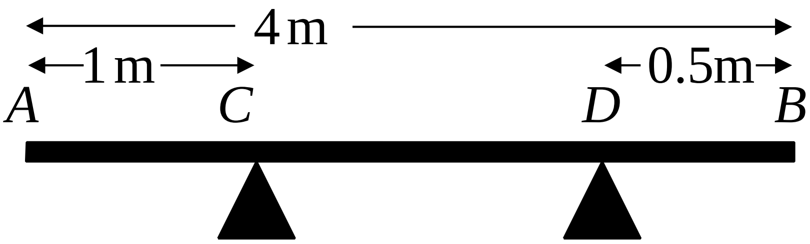
(iv) Hence show that $W = \dfrac{60}{1 - x}$

[2 marks]

(v) Hence deduce the range of possible values of x

[2 marks]

Question 2



A nonuniform rod AB has length 4 m and weight 150 N. The rod rests horizontally in equilibrium on two smooth supports C and D, where AC = 1 m and DB = 0.5 m. The centre of mass of AB is x metres from A. R_C and R_D are the magnitudes of the reactions at C and D respectively. A particle of weight W N is placed on the rod at A. The rod remains in equilibrium with R_C then 100 N.

(i) Draw a diagram marking on all significant lengths and forces.

[2 marks]

(ii) By balancing forces vertically show that $R_D = W + 50$

[1 mark]

(iii) By taking moments about A and then using your part (i) answer, show that $300x = 550 + 7W$

[4 marks]

The particle is now removed from A and placed on the rod at B. The rod remains in equilibrium and the reaction of C on the rod, R_C', now has magnitude 52 N.
Note that the reaction at D, R_D' will also now have changed.

(iv) Repeat steps (i), (ii) and (iii) to show that $300x = 790 - \quad W$

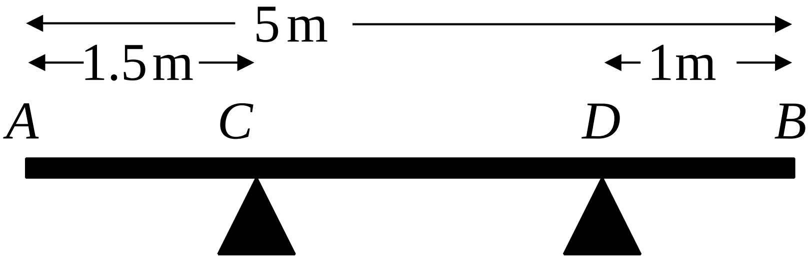
[4 marks]

(v) Combine the equations obtained previously to calculate the value of x and the value of W

[3 marks]

Question 3

M1 Examination Question from November 2003, Q6



A nonuniform rod AB has length 5 m and weight 200 N. The rod rests horizontally in equilibrium on two smooth supports C and D, where AC = 1.5 m and DB = 1 m. The centre of mass of AB is x metres from A.

A particle of weight W N is placed on the rod at A. The rod remains in equilibrium and the magnitude of the reaction of C on the rod is 160 N.

(a) Show that $50x - W = 100$

[5 marks]

The particle is now removed from A and placed on the rod at B. The rod remains in equilibrium and the reaction of C on the rod now has magnitude 50 N

(b) Obtain another equation connecting W and x

[3 marks]

(c) Calculate the value of x and the value of W

[4 marks]

Question 4

M1 Examination Question from June 2016, Q6

A nonuniform plank AB has length 6 m and mass 30 kg. The plank rests in equilibrium in a horizontal position on supports at the points S and T of the plank where AS = 0.5 m and TB = 2 m.

When a block of mass M kg is placed on the plank at A, the plank remains horizontal and in equilibrium and the plank is on the point of tipping about S.

When the block is moved to B, the plank remains horizontal and in equilibrium and the plank is on the point of tipping about T.

The distance of the centre of mass of the plank from A is d metres.
The block is modelled as a particle and the plank is modelled as a nonuniform rod.

Find **(i)** the value of d

(ii) the value of M

[7 marks]

5.4 Answers

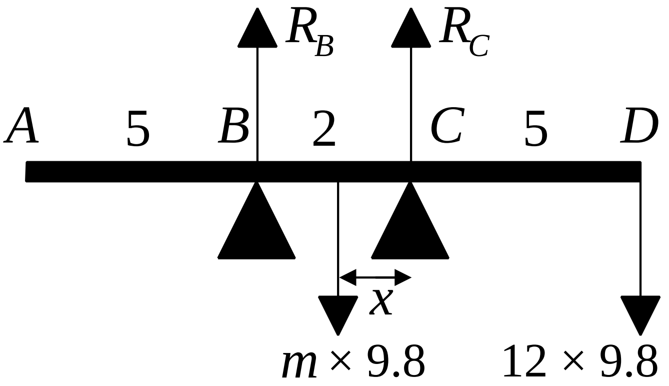
5.4.1 Solution to 5.2 Example

(i) By symmetry, if the mast could be modelled by a rod that was uniform, the number of bottles of water to be on the verge of tilting would be the same when they were at end *A* as at end *D*.

(ii) The centre of mass has to be between the supports as if it wasn't one end of the mast would be on the ground.

Let x be the distance of the centre of mass from support *C*.

When the 6 × 2 litre bottles, mass 12 kg, are suspended from *D*,

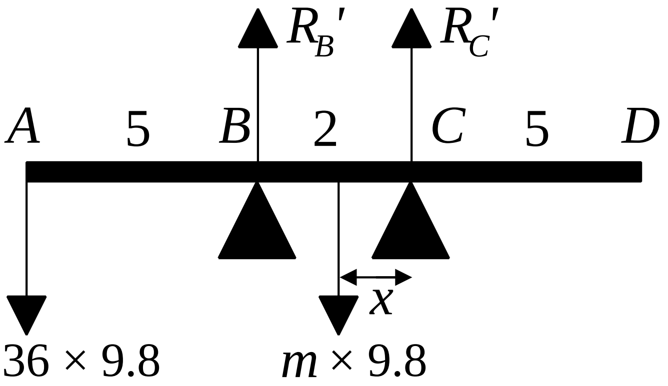


On the verge of tilting, $R_B = 0$ N. Taking moments about *C*,

$$m \times 9.8 \times x = 12 \times 9.8 \times 5$$

$$mx = 60$$

When the 18 × 2 litre bottles, mass 36 kg, are suspended from *A*,



On the verge of tilting, $R_C' = 0$ N. Taking moments about *B*,

$$m \times 9.8 \times (2 - x) = 36 \times 9.8 \times 5$$

$$2m - mx = 180$$

Combining equations: $$2m - 60 = 180$$

$$2m = 240$$

$$m = 120 \text{ kg}$$

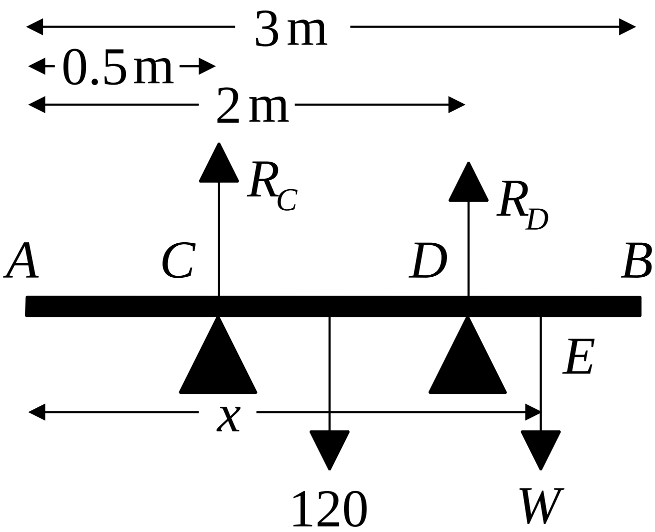
(iii) Substituting the value of m back into the equation $mx = 60$ gives, $x = 0.5$

Thus the centre of mass is 5.5 metres from end *D*

5.4.2 Solutions to 5.3 Exercise

Answer 1

(i)



Location of E not known

(ii)

Taking moments about *A*,

$$Wx + 120 \times 1.5 = R_C \times 0.5 + R_D \times 2$$
$$Wx + 180 = 2 \times R_D \times 0.5 + 2\,R_D$$
$$Wx + 180 = 3\,R_D$$

(iii)

Balancing forces vertically,

$$R_C + R_D = 120 + W$$
$$2 \times R_D + R_D = 120 + W$$
$$3\,R_D = 120 + W$$

(iv) Equating previous answers,

$$\therefore\; Wx + 180 = 120 + W$$
$$60 = W - Wx$$
$$60 = W\,(1 - x)$$
$$W = \frac{60}{1 - x} \qquad \square$$

(v) As a length cannot be negative, $x \geqslant 0$

A weight *W* also cannot be negative, so $x < 1$

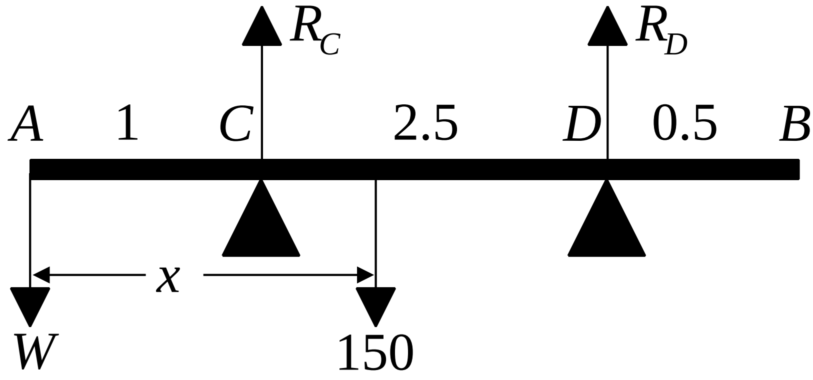
$\therefore\; 0 \leqslant x < 1$

The minimum weight that could be used is thus 60 N.

There is no upper limit, according to this model.

Answer 2

(i)



(ii) Balancing forces vertically,

$$R_C + R_D = W + 150$$

$$100 + R_D = W + 150$$

$$R_D = W + 50 \quad \square$$

(iii) Taking moments about A,

$$150x = R_C \times 1 + R_D \times 3.5$$

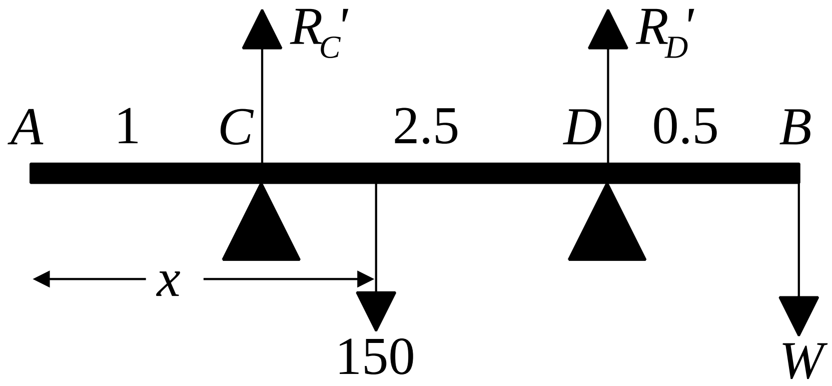
$$150x = 100 \times 1 + (W + 50) \times 3.5$$

$$150x = 100 + 3.5W + 175$$

$$150x = 275 + 3.5W$$

$$300x = 550 + 7W \quad \square$$

(iv) Let R_C' and R_D' be the different reactions at C and D.



Balancing forces vertically,

$$R_C' + R_D' = W + 150$$

$$52 + R_D' = W + 150$$

$$R_D' = W + 98 \quad \square$$

Taking moments about A,

$$150 \times x + W \times 4 = R_C' \times 1 + R_D' \times 3.5$$

$$150x + 4W = 52 \times 1 + (W + 98) \times 3.5$$

$$150x + 4W = 52 + 3.5W + 343$$

$$150x = 395 - 0.5W$$

$$300x = 790 - W$$

(v)

$$550 + 7W = 790 - W$$

$$8W = 240$$

$$W = 30 \text{ N}$$

Thus,

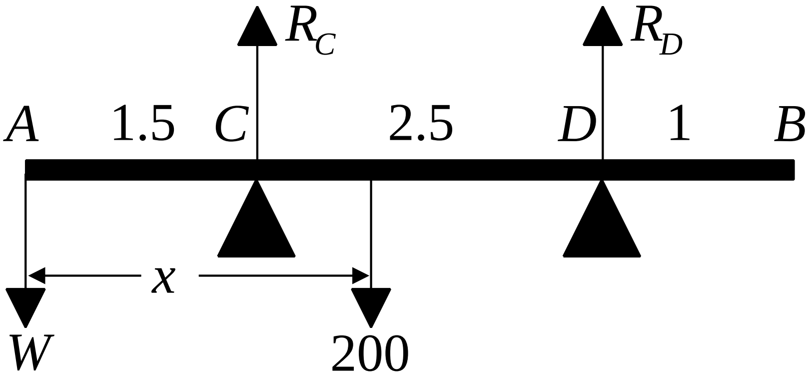
$$300x = 790 - W$$

$$300x = 790 - 30$$

$$x = 2.53 \text{ metres}$$

Answer 3

(a)



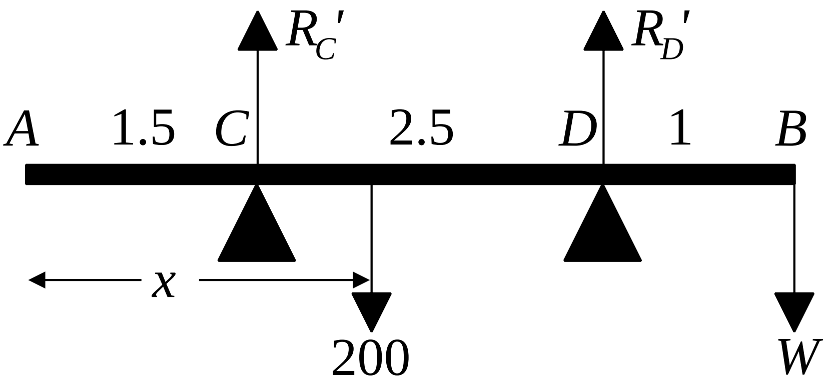
Balancing forces vertically,

$$R_C + R_D = W + 200$$
$$160 + R_D = W + 200$$
$$R_D = W + 40$$

Taking moments about A,

$$200x = R_C \times 1.5 + R_D \times 4$$
$$200x = 160 \times 1.5 + (W + 40) \times 4$$
$$200x = 240 + 4W + 160$$
$$200x = 400 + 4W$$
$$50x = 100 + W$$
$$50x - W = 100 \quad \square$$

(b) Let R_C' and R_D' be the different reactions at C and D.



Balancing forces vertically,

$$R_C' + R_D' = W + 200$$
$$50 + R_D' = W + 200$$
$$R_D' = W + 150 \quad \square$$

Taking moments about A,

$$200 \times x + W \times 5 = R_C' \times 1.5 + R_D' \times 4$$

$$200x + 5W = 50 \times 1.5 + (W + 150) \times 4$$

$$200x + 5W = 75 + 4W + 600$$

$$200x + W = 675$$

Solving the two equations simultaneously,

$$50x - W = 100$$

$$200x + W = 675$$

ADD

$$250x = 775$$

$$x = 3.10 \text{ metres}$$

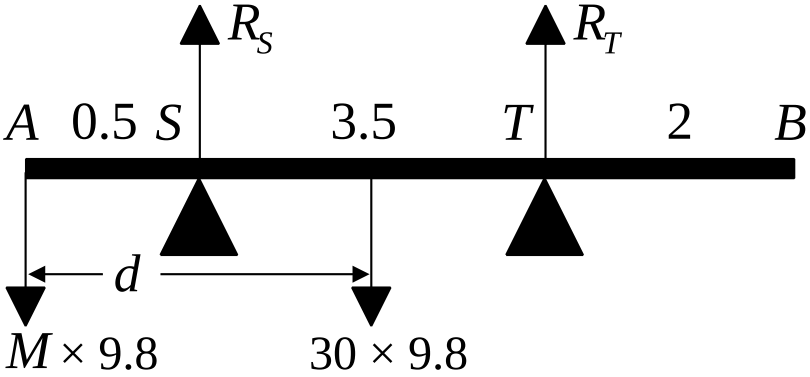
Thus,

$$200x + W = 675$$

$$W = 675 - 200 \times 3.10$$

$$= 55 \text{ N}$$

Answer 4



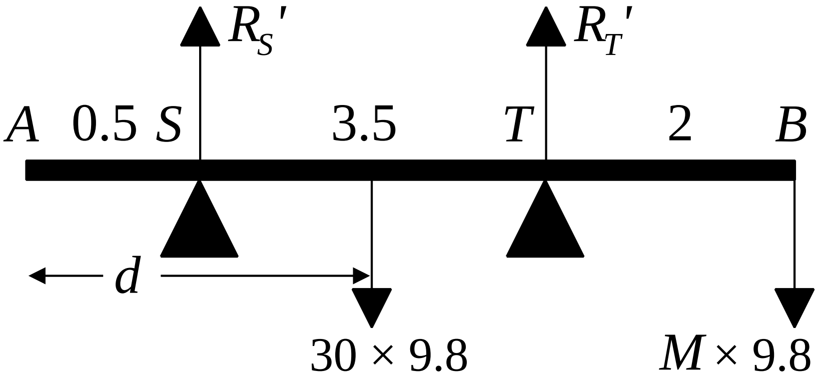
When about to tilt about S, $R_T = 0$ N

Taking moments about S,

$$30 \times 9.8 \times (d - 0.5) = M \times 9.8 \times 0.5$$

$$30d - 15 = 0.5M$$

$$M = 60d - 30$$



When about to tilt about T, $R_S' = 0$ N

Taking moments about T,

$$M \times 9.8 \times 2 = 30 \times 9.8 \times (4 - d)$$

$$2M = 120 - 30d$$

$$M = 60 - 15d$$

Combining the two equations,

$$60d - 30 = 60 - 15d$$

$$75d = 90$$

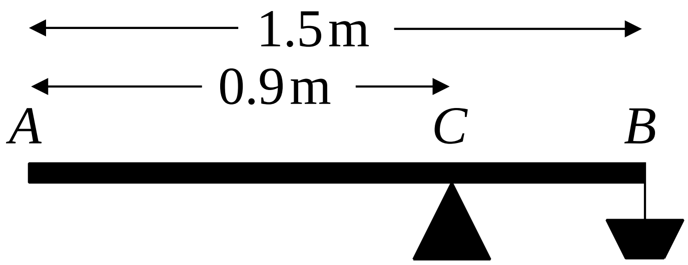
$$d = 1.2$$

$$\therefore\ M = 42 \text{ N}$$

6.1 Test

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40
Where a numerical value of g is required, assume it is 9.8 m s^{-2}

Question 1



A uniform rod AB has length 1.5 m and mass 8 kg.
A particle of mass m kg is attached to the rod at B.
The rod is supported at the point C, where AC = 0.9 m and the system is in equilibrium with AB horizontal.

(i) Draw a diagram showing all significant lengths and forces acting on the rod.

[2 marks]

(ii) Find the mass, m, in kg of the particle attached at B.

[3 marks]

Question 2

Inspired by an A-Level Examination Question from June 2017, M1, Q3

A plank AB has length 6 m and mass 30 kg
The point C is on the plank with CB = 2 m
The plank rests in equilibrium in a horizontal position on supports at A and C
Two people, each of mass 75 kg, stand on the plank.
One person stands at the point P of the plank, where $AP = x$ metres, and the other person stands at the point Q of the plank, where $AQ = 2x$ metres
The plank remains horizontal and in equilibrium with the magnitude of the reaction at C five times the magnitude of the reaction at A
The plank is modelled as a uniform rod and each person is modelled as a particle.

(i) Draw a diagram showing the significant lengths and forces acting on the rod.

[3 marks]

(ii) By balancing forces vertically determine, in Newtons, the reaction on the rod at A and the reaction on the rod at C clearly stating which is which.

[3 marks]

(iii) By taking moments about A, find the value of x

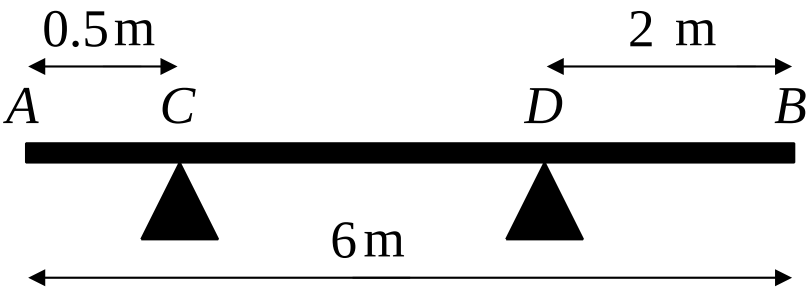
[4 marks]

(iv) State two ways in which you have used the assumptions made in modelling the plank as a uniform rod

[2 marks]

Question 3

A-Level Question from October 2017, M1, Q2



A metal girder *AB*, of weight 1080 N and length 6 m, rests in equilibrium in a horizontal position on two supports, one at *C* and one at *D*, where *AC* = 0.5 m and *BD* = 2 m, as shown. A boy of weight 400 N stands on the girder at *B* and the girder remains horizontal and in equilibrium.

The boy is modelled as a particle and the girder is modelled as a uniform rod.

(a) Find,

(i) the magnitude of the reaction on the girder at *C*,

[3 marks]

(ii) the magnitude of the reaction on the girder at D

[3 marks]

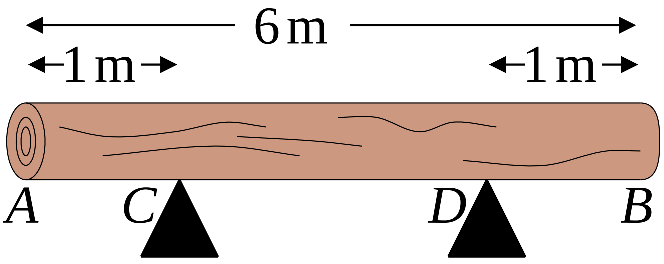
The boy now stands at a point E on the girder, where $AE = x$ metres, and the girder remains in equilibrium. Given that the magnitude of the reaction on the girder at D is now 520 N greater than the magnitude of the reaction on the girder at C,

(b) find the value of x

[5 marks]

Question 4

M1 Exam Question from June 2001, Q5



A large log AB is 6 m long. It rests in a horizontal position on two smooth supports C and D, where AC = 1m and BD = 1 m, David needs an estimate of the weight of the log, but the log is too heavy to lift off both supports. When David applies a force of magnitude 1500 N vertically upwards to the log at A, the log is about to tilt about D.

(a) State the value of the reaction on the log at C for this case.

[1 mark]

David initially models the log as a uniform log.

(b) Using this model estimate the weight of the log

[3 marks]

The shape of the log convinces David that his initial modelling assumption is too simple. He removes the force at A and applies a force acting vertically upwards at B. He finds that the log is about to tilt about C when this force has magnitude 1000 N. David now models the log as a nonuniform rod, with the distance of the centre of mass of the log from C as x metres.

Using this model find,

(c) a new estimate for the weight of the log

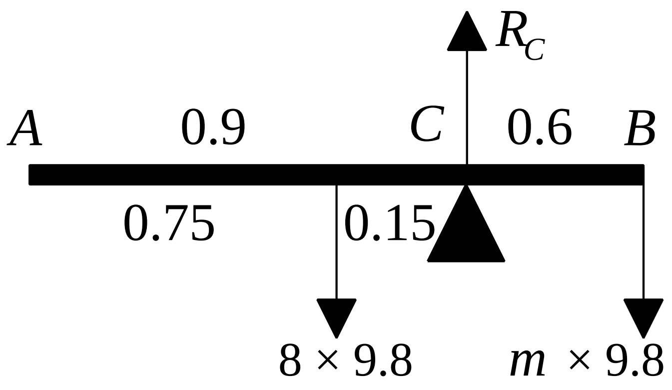
[6 marks]

(d) the value of x

[2 marks]

Answer 1

(i)



(ii) Take moments about C,

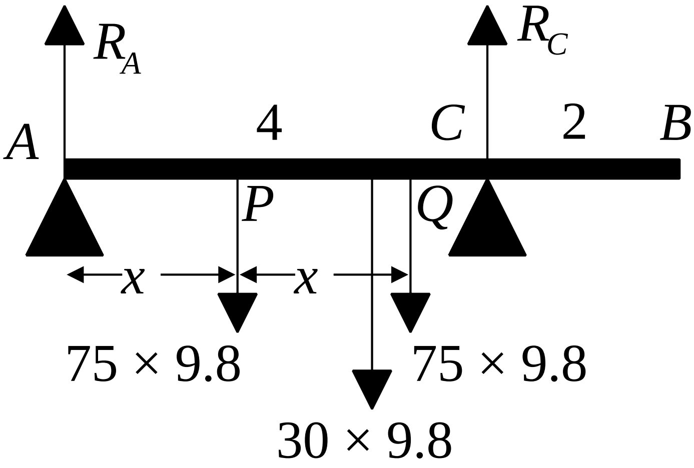
$$m \times 9.8 \times 0.6 = 8 \times 9.8 \times 0.15$$

$$0.6m = 1.2$$

$$m = 2 \text{ kg}$$

Answer 2

(i)



Told that : $R_C = 5R_A$

(ii) Balancing forces vertically,

$$R_A + R_C = 30 \times 9.8 + 75 \times 9.8 + 75 \times 9.8$$
$$R_A + 5R_A = 1764$$
$$R_A = 294 \text{ N}$$
$$\therefore R_C = 1470 \text{ N}$$

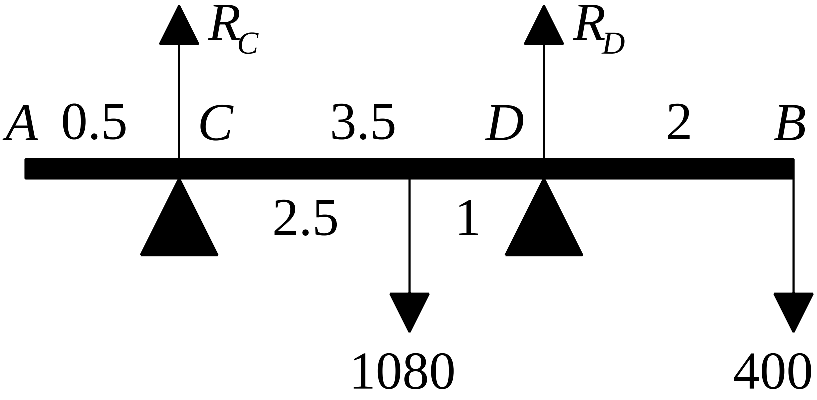
(iii) Taking moments about A,

$$75 \times 9.8 \times x + 30 \times 9.8 \times 3 + 75 \times 9.8 \times 2x = R_C \times 4$$
$$735x + 882 + 1470x = 1470 \times 4$$
$$2205x = 4998$$
$$x = 2.27 \text{ metres}$$

(iv) The fact that the rod was described as uniform allowed the assumption that the rod's weight all acted at its midpoint to be made.

It was also assumed that the rod did not bend under either load or, indeed, its own weight.

Answer 3



(a) **(i)** Take moments about *D*,

$$R_C \times 3.5 + 400 \times 2 = 1080 \times 1$$

$$3.5\,R_C = 280$$

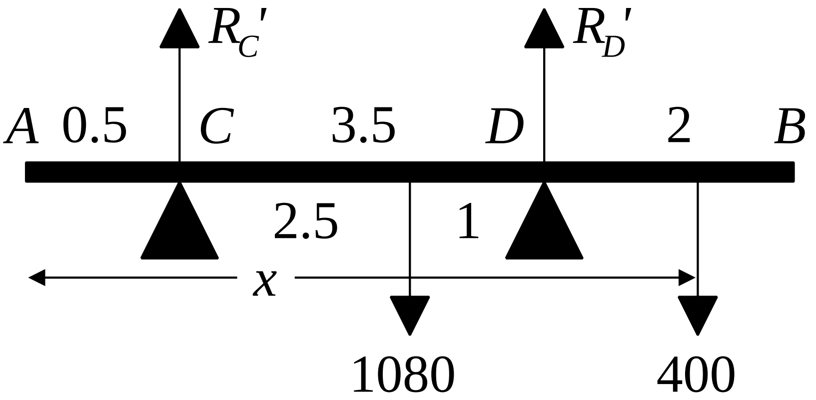
$$R_C = 80\text{ N}$$

(ii) Balance forces vertically,

$$R_C + R_D = 1080 + 400$$

$$80 + R_D = 1480$$

$$R_D = 1400\text{ N}$$



(b) Balance forces vertically,

$$R_C' + R_D' = 1080 + 400$$

$$R_C' + (R_C' + 520) = 1480$$

$$2\,R_C' = 960$$

$$R_C' = 480\text{ N} \text{ and } R_D' = 1000\text{ N}$$

Take moments about *A*,

$$R_C \times 0.5 + R_D \times 4 = 1080 \times 3 + 400 \times x$$

$$480 \times 0.5 + 1000 \times 4 = 3240 + 400x$$

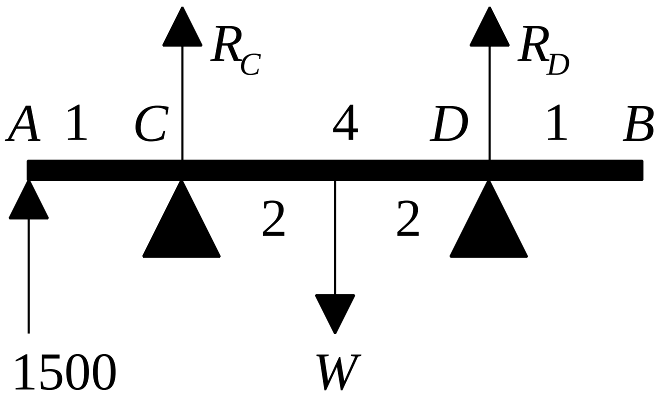
$$400x = 240 + 4000 - 3240$$

$$x = 2.5\text{ metres}$$

Answer 4

(a) $R_C = 0$ N

(b)

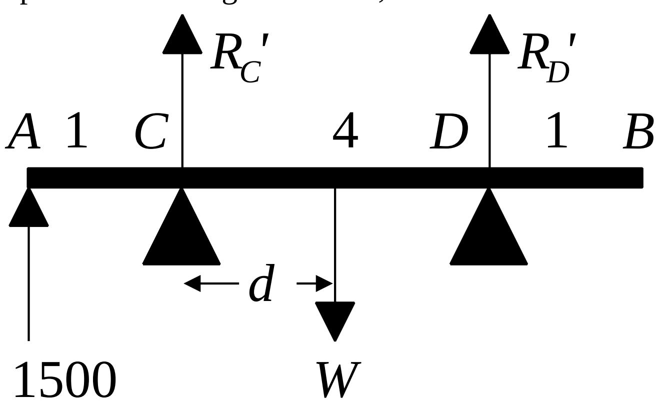


Take moments about R_D,

$$W \times 2 = 1500 \times 5$$

$$W = 3750 \text{ N}$$

(c) The previous information still holds, so rerun part (b) without the assumption that the log is uniform,

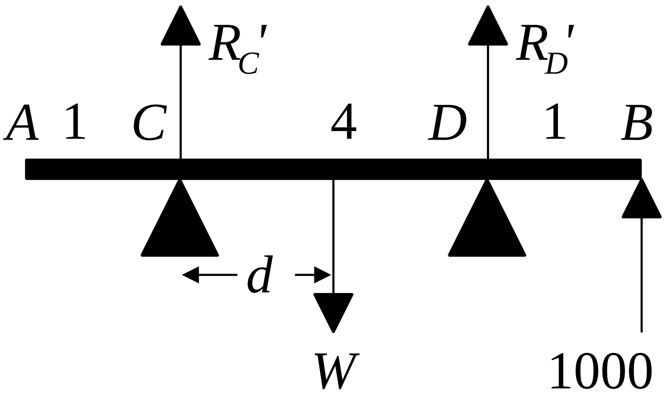


Take moments about R_D,

$$W \times (4 - d) = 1500 \times 5$$

$$4W - Wd = 7500$$

Now to make use of the additional information,



This time, it's R_D that will be zero due to the tilt,
Take moments about R_C,

$$W \times d = 1000 \times 5$$
$$Wd = 5000$$

Combine the two equations by addition,

$$4W - Wd = 7500$$
$$Wd = 5000$$

ADD

$$4W = 12500$$
$$W = 3125\,\text{N}$$

(d)

$$Wd = 5000$$
$$3125d = 5000$$
$$d = 1.6$$