Twenty-One Today
N U M B E R 3
Solutions

Answer 1

$$y = 4x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 2x^{-\frac{1}{2}}$$
$$\therefore \frac{2}{\sqrt{x}} = 8$$
$$x = \frac{1}{16} \quad \text{or} \quad 0.0625$$
$$\therefore y = 4 \times \sqrt{\frac{1}{16}}$$
$$= 1$$

The coordinates of the only point on the curve with gradient 8 are (0.0625, 1)

Answer 2

By the remainder theorem, the remainder is given by $g(-2)$

$$g(-2) = (-2)^3 - 7 \times (-2)^2 - (-2) + 23$$
$$= -8 - 28 + 2 = 23$$
$$= -11$$

Alternatively, the long division could be carried out.

Answer 3

Method 1

$$log_a b = c \quad \Leftrightarrow \quad a^c = b$$

This "jump out of logs" manoeuvre gives

$$log_e (11x - 5) = 3 \Leftrightarrow e^3 = 11x - 5$$
$$x = \frac{e^3 + 5}{11}$$

Method 2

$$ln(11x - 5) = 3$$

Exponentiate both sides

$$e^{ln(11x-5)} = e^3$$
$$e^3 = 11x - 5$$
$$x = \frac{e^3 + 5}{11}$$

Answer 4

$$Distance = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$
$$65 = (x - (-2))^2 + (6 - 10)^2$$
$$65 = (x + 2)^2 + (-4)^2$$
$$65 = x^2 + 4x + 20$$
$$x^2 + 4x - 45 = 0$$
$$(x + 9)(x - 5) = 0$$
$$x = -9 \text{ or } x = 5$$

Answer 5

(i) Discriminant of $ax^2 + bx = c = 0$ is $D = b^2 - 4ac$

$$\therefore \text{ for } f(x),\ D = 1 - 4 \times 1 \times 2$$
$$= -7$$

(ii) As $D < 0$, $f(x)$ has no roots

Answer 6

$$cos^2\,\theta + sin^2\,\theta = 1$$
$$cos^2\,\theta + \left(\frac{8}{17}\right)^2 = 1$$
$$cos^2\,\theta = \frac{289 - 64}{289}$$
$$= \frac{225}{289}$$
$$\therefore\ cos\,\theta = \frac{15}{17}$$

$$tan\,\theta = \frac{sin\,\theta}{cos\,\theta}$$
$$= \frac{8}{15}$$

Answer 7

$$\frac{101\,!}{99\,!} = \frac{101 \times 100 \times 99\,!}{99\,!}$$
$$= 10100$$

Answer 8

$$\begin{aligned} f(x) &= \frac{x^3 + 9x}{3x^2} \\ &= \frac{x^3}{3x^2} + \frac{9x}{3x^2} \\ &= \frac{x}{3} + 3x^{-1} \\ \text{So, } f'(x) &= \frac{1}{3} - 3x^{-2} \\ &= \frac{1}{3} - \frac{3}{x^2} \end{aligned}$$

Answer 9

$$\begin{aligned} \int f(x)\, dx &= \int \frac{x^4 + 5x^2}{x}\, dx \\ &= \int \frac{x^4}{x} + \frac{5x^2}{x}\, dx \\ &= \int x^3 + 5x\, dx \\ &= \frac{x^4}{4} + \frac{5x^2}{2} + c \end{aligned}$$

Answer 10

$$\begin{aligned} 2\cos\theta &= \sqrt{3} \\ \cos\theta &= \frac{\sqrt{3}}{2} \\ \theta &= 30°,\ 330° \end{aligned}$$

Answer 11

$$\begin{aligned} \overrightarrow{AB} &= b - a \\ &= \begin{pmatrix} 8 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 8 \end{pmatrix} \end{aligned}$$

i.e. $2\boldsymbol{i} + 8\boldsymbol{j}$

Answer 12

$$y = 11 - 4x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -2x^{-\frac{1}{2}}$$

$$= -\frac{2}{\sqrt{x}}$$

$$\text{Inserting } x = 4 \text{ gives } \frac{dy}{dx} = -1$$

$$\therefore \text{ Normal's gradient } = 1$$

$$\text{Also } y = 11 - 4\sqrt{4}$$

$$\text{So } (4, 3) \text{ is on the normal}$$

$$y = mx + c$$

$$3 = 1 \times 4 + c$$

$$c = -1$$

$$\text{Normal is } y = x - 1$$

Answer 13

$$(3 - 2x)^3 = 1 \times (3)^3 \times (-2x)^0$$
$$+ 3 \times (3)^2 \times (-2x)^1$$
$$+ 3 \times (3)^1 \times (-2x)^2$$
$$+ 1 \times (3)^0 \times (-2x)^3$$
$$= 27 - 54x + 36x^2 - 8x^3$$

Answer 14

Method 1

$$log_a b = c \quad \Leftrightarrow \quad a^c = b$$

This “jump out of logs” manoeuvre gives

$$log_3 x = -\frac{1}{2} \Leftrightarrow 3^{-\frac{1}{2}} = x$$

$$x = \frac{1}{\sqrt{3}}$$

Method 2

$$log_3 x = -\frac{1}{2}$$

Raise both sides to the power 3

$$3^{log_3 x} = 3^{-\frac{1}{2}}$$

$$x = -\frac{1}{\sqrt{3}}$$

Answer 15

$$\log_7(y+3) + \log_7(2y+1) = 1 \qquad y \in R,\ \ y > -\frac{1}{2}$$

$$\log_7((y+3)(2y+1)) = 1$$

$$\log_7\left(2y^2 + 7y + 3\right) = 1$$

$$2y^2 + 7y + 3 = 7^1$$

$$2y^2 + 7y - 4 = 0$$

$$(2y-1)(y+4) = 0$$

$$y = \frac{1}{2} \qquad \text{or} \qquad y = -4$$

$$\text{But } y > -\frac{1}{2}$$

$$\text{so the only solution is } \ y = \frac{1}{2}$$

Answer 16

$$9^x - 3^{x+1} - 18 = 0$$

$$\left(3^x\right)^2 - 3\left(3^x\right) - 18 = 0$$

$$\left(3^x - 6\right)\left(3^x + 3\right) = 0$$

$$3^x = 6 \qquad or \qquad 3^x = -3 \text{ (no solution)}$$

$$x = \frac{\ln 6}{\ln 3}$$

$$= 1.631$$

Answer 17

The circle will cross the x-axis when $y = 0$ in which case,

$$(x-3)^2 + (-4)^2 = 25$$

$$(x-3)^2 = 9$$

square root both sides

$$x - 3 = \pm 3$$

$$x = 0 \qquad \text{or} \qquad x = 6$$

$$Points\ :\ (0,\ 0),\ \ (6,\ 0)$$

Answer 18

Use the 'backwards' version of the cosine rule,

$$\cos\theta = \frac{5^2 + 5^2 - 6^2}{2 \times 5 \times 5}$$

$$= 0.28$$

$$\therefore\ \theta = \arccos 0.28$$

$$= 73.7°$$

Answer 19

$$12 + 4x > x^2$$

$$x^2 - 4x - 12 < 0$$

$$(x - 6)(x + 2) < 0$$

critical values are $x = -2$ and $x = 6$

Via a quick sketch graph,

$$-2 < x < 6$$

Answer 20

$$\int_1^2 24x^{-3}\,dx = \left[\frac{12}{x^2} \right]_1^2$$

$$= 9$$

Answer 21

$$2\cos^2\theta + \sin\theta - 1 = 0$$

$$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$$

$$2\sin^2\theta - \sin\theta - 1 = 0$$

$$(\sin\theta - 1)(2\sin\theta + 1) = 0$$

either or

$$\sin\theta = 1 \qquad \sin\theta = -\frac{1}{2}$$

$$\theta = 90°,\ 210°,\ 330°$$