**Twenty-One Today**
**N U M B E R 4**
**Solutions**

**Answer 1**

$$\frac{\sqrt{2}+3}{5+\sqrt{2}} = \frac{\sqrt{2}+3}{5+\sqrt{2}} \times \frac{5-\sqrt{2}}{5-\sqrt{2}}$$

$$= \frac{5\sqrt{2}-2+15-3\sqrt{2}}{25-5\sqrt{2}+5\sqrt{2}-2}$$

$$= \frac{13+2\sqrt{2}}{23}$$

**Answer 2**

$$\int_2^a x^2\,dx = 111\frac{2}{3}$$

$$\left[\frac{x^3}{3}\right]_2^a = 111\frac{2}{3}$$

$$\frac{a^3}{3} - \frac{2^3}{3} = 111\frac{2}{3}$$

multiply through by 3

$$a^3 - 2^3 = 335$$

$$a^3 = 343$$

$$a = 7$$

**Answer 3**

If $(x-1)$ is a factor then $f(1)$ will equal zero by the factor theorem.

$$f(1) = 5 \times 1^6 + 8 \times 1^5 - 13$$

$$= 0$$

$\therefore$ $(x-1)$ is a factor of $f(x)$

**Answer 4**

The radius of the circle is 8

$$C = 2\pi r$$

$$= 2\pi 8$$

$$= 16\pi$$

**Answer 5**

$$4x - \frac{7}{2} = -\frac{x}{4} + 5$$

multiply through by 4

$$16x - 14 = -x + 20$$

$$17x = 34$$

$$x = 2$$

The point of intersection is $( 2, \; 4.5 )$

**Answer 6**

**( i )** 5 **( ii )** 21 **( iii )** $\frac{5}{21}$

**Answer 7**

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

**Answer 8**

$$n = 6 \qquad r = 3 \qquad {}^{6}C_{3} = 20$$

**Answer 9**

$$\begin{aligned}
\left( \frac{3}{x} - \frac{x^2}{2} \right)^4 &= 1 \times \left( \frac{3}{x} \right)^4 \times \left( - \frac{x^2}{2} \right)^0 \\
&+ 4 \times \left( \frac{3}{x} \right)^3 \times \left( - \frac{x^2}{2} \right)^1 \\
&+ 6 \times \left( \frac{3}{x} \right)^2 \times \left( - \frac{x^2}{2} \right)^2 \\
&+ 4 \times \left( \frac{3}{x} \right)^1 \times \left( - \frac{x^2}{2} \right)^3 \\
&+ 1 \times \left( \frac{3}{x} \right)^0 \times \left( - \frac{x^2}{2} \right)^4 \\
&= \frac{81}{x^4} - \frac{54}{x} + \frac{27\,x^2}{2} - \frac{3\,x^5}{2} + \frac{x^8}{16}
\end{aligned}$$

**Answer 10**

To obtain the midpoint,

$$\left( \frac{3 + 8}{2}, \frac{5 + (-5)}{2} \right) = (5.5, \ 0)$$

Gradient of line between the given points,

$$m = \frac{\Delta Y}{\Delta X}$$

$$= \frac{5 - (-5)}{3 - 8}$$

$$= -2$$

The gradient of the perpendicular bisector will be the sign changed reciprocal,

$$m_p = \frac{1}{2}$$

To obtain the equation of the perpendicular bisector through ( 5.5, 0 ), gradient 0.5,

$$y = mx + c$$

$$0 = \frac{1}{2} \times 5.5 + c$$

$$\therefore \ c = -2.75$$

The equation of the line is thus,

$$y = \frac{1}{2} x - 2.75$$

**Answer 11**

$$\int y \, dx = \int 1.6 x^4 + 0.9 x^2 \, dx$$

$$= 0.32 x^5 + 0.3 x^3 + c$$

**Answer 12**

**( i )** The discriminant, $D$, is the piece under the square root.

i.e. $D = b^2 - 4ac$

**( ii )** If $D < 0$ then the graph of the quadratic does not cross or touch the $x$-axis because it has no roots.

**Answer 13**

$$\frac{dy}{dx} = 40x^3$$

Integrate both sides wrt $x$

$$y = 10x^4 + c$$

Now to use the fact that this curve passes through $(1,\ 13)$

$$13 = 10 \times 1^4 + c$$

$$c = 3$$

$$\therefore\ y = 10x^4 + 3$$

**Answer 14**

$$x^2 + y^2 - 10x + 4y = 7$$

$$\left[x^2 - 10x\right] + \left[y^2 + 4y\right] = 7$$

$$\left[(x-5)^2 - 25\right] + \left[(y+2)^2 - 4\right] = 7$$

$$(x-5)^2 + (y+2)^2 = 36$$

$\therefore$ centre $(5, -2)$, radius 6

**Answer 15**

$$x^2 - 3 \geqslant 1$$

$$x^2 - 2^2 \geqslant 0$$

$$(x+2)(x-2) \geqslant 0$$

critical values at $\pm 2$

and so, from a sketch graph,

$x \leqslant -2$ or $x \geqslant 2$

**Answer 16**

$$128\sqrt{2} = 2^7 \times 2^{0.5}$$

$$= 2^{7.5}$$

**Answer 17**
The crucial aspect of this question is to avoid a division by *sin x* because this would be dividing by zero, which is not defined in mathematics.

**Method 1**

$$\tan^2 x + \cos^2 x = 1$$

$$\frac{\sin^2 x}{\cos^2 x} + \cos^2 x = 1$$

multiply through by $\cos^2 x$

$$\sin^2 x + \cos^4 x = \cos^2 x$$

$$(1 - \cos^2 x) + \cos^4 x = \cos^2 x$$

$$\cos^4 x - 2\cos^2 x + 1 = 0$$

$$(\cos^2 x - 1)^2 = 0$$

square root both sides

$$\cos^2 x - 1 = 0$$

$$\cos^2 x = 1$$

square root both sides

$$\cos x = \pm 1$$

$$\therefore x = 0°,\ 180°,\ 360°$$

**Method 2**

$$\tan^2 x + \cos^2 x = 1$$

$$\frac{\sin^2 x}{\cos^2 x} = \sin^2 x$$

$$\sin^2 x = \sin^2 x \cos^2 x$$

$$\sin^2 x - \sin^2 x \cos^2 x = 0$$

$$\sin^2 x (1 - \cos^2 x) = 0$$

$$\sin^2 x \sin^2 x = 0$$

$$\sin^4 x = 0$$

$$\sin x = 0$$

$$\therefore x = 0°,\ 180°,\ 360°$$

**Answer 18**

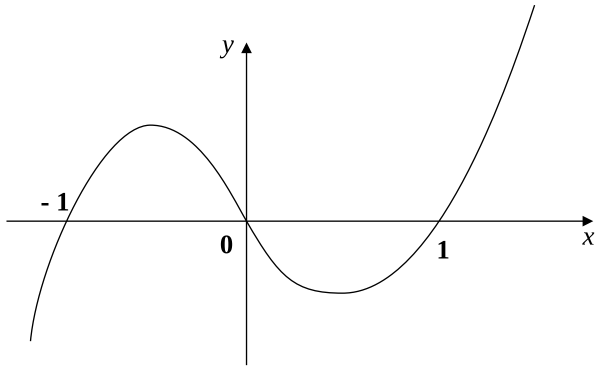
**( i )**

$$f(x) = x^3 - x$$
$$= x(x^2 - 1)$$
$$= x(x+1)(x-1)$$

**( ii )**

$$(1,\ 0),\ (0,\ 0),\ (-1,\ 0)$$

**Answer 19**




**Answer 20**

$$y = x^3 - x$$
$$\frac{dy}{dx} = 3x^2 - 1$$

from which it's apparent that the gradient when $x = 1$ is $2$

$$y = mx + c$$
$$0 = 2 \times 1 + c$$
$$c = -2$$
$$\therefore \ \text{Tangent} \ : \ y = 2x - 2$$

**Answer 21**

Integration will see any area underneath the *x*-axis as negative, so the previous questions have been raising awareness that this curve has this issue.

In my solution I've gone for the left hand part of the curve, which will have a positive area of 0.25, and then used the symmetry to get the area requested by doubling that answer.

$$
\begin{aligned}
Area &= 2 \times \int_{-1}^{0} x^3 - x \; dx \\
&= 2\left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^{0} \\
&= \left[ \frac{x^4}{2} - x^2 \right]_{-1}^{0} \\
&= [\,0\,] - \left[ \frac{1}{2} - 1 \right] \\
&= \frac{1}{2}
\end{aligned}
$$